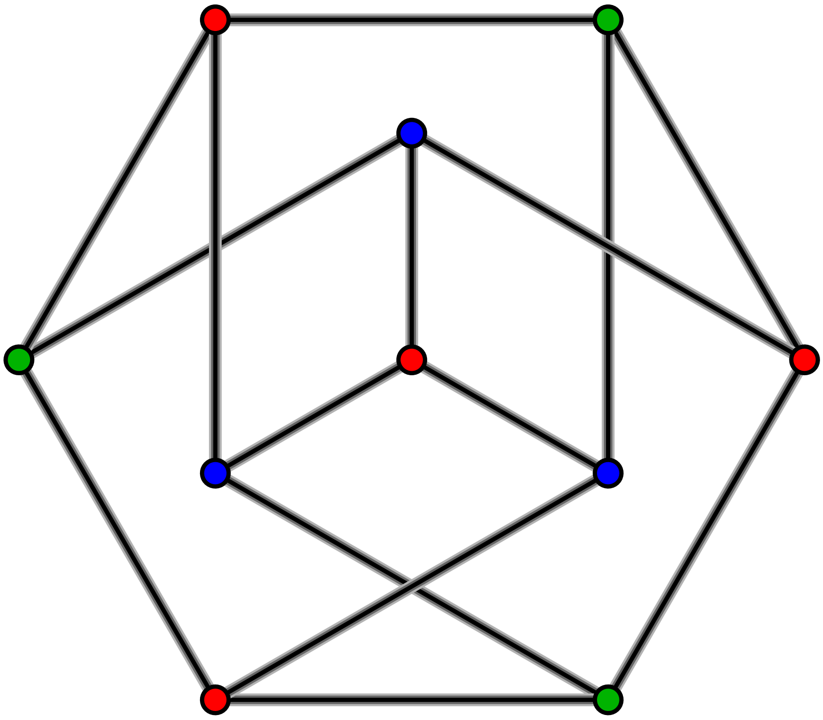
Graph Theory



Motiejus Valiunas

(based on the Part II course on Graph Theory at the University of Cambridge,
and its implementation by Paul Russell in 2014/15)

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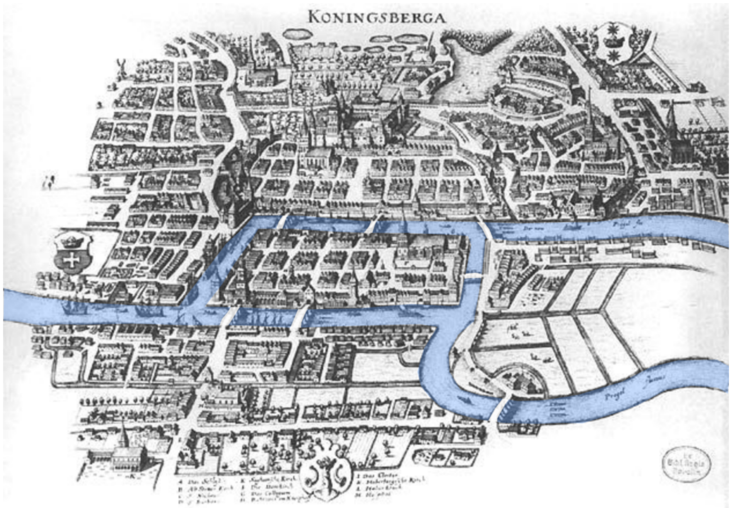
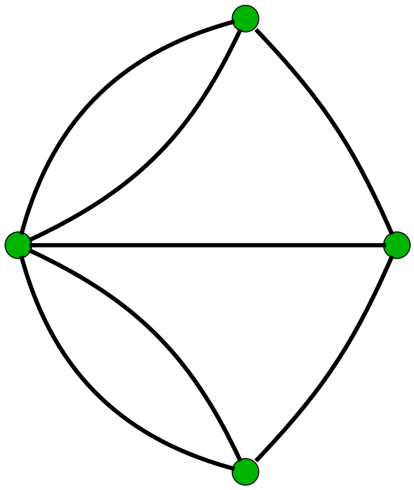
Motivation

A graph is a collection of “vertices” joined together by “edges”.

Example 0.1 (Seven Bridges of Königsberg)**.** A folklore problem asked if there was a way to walk around the city of Königsberg in Prussia (now Kaliningrad, Russia—a.k.a. Królewiec) by crossing each of its seven bridges exactly once (see Figure 0.1a). A negative answer to this question was given in 1735 by Swiss mathematician Leonhard Euler, using what would now be known as graph-theoretic methods (see Figure 0.1b). This has led to development of the branch of mathematics now known as graph theory.



(a) A 17th century map of Königsberg.



(b) A “graph” representing this.

Figure 0.1: The seven bridges of Königsberg.

Example 0.2 (Simultaneous representation of cosets)**.** Let G be a finite group, and let $H \leq G$ be a subgroup. We know that we can express G as a disjoint union of left H-cosets,

$$G = a_1 H \sqcup a_2 H \sqcup \cdots \sqcup a_k H,$$

where $k = \frac{|G|}{|H|}$. Similarly, we can write G as a disjoint union of right H-cosets,

$$G = H b_1 \sqcup H b_2 \sqcup \cdots \sqcup H b_k.$$

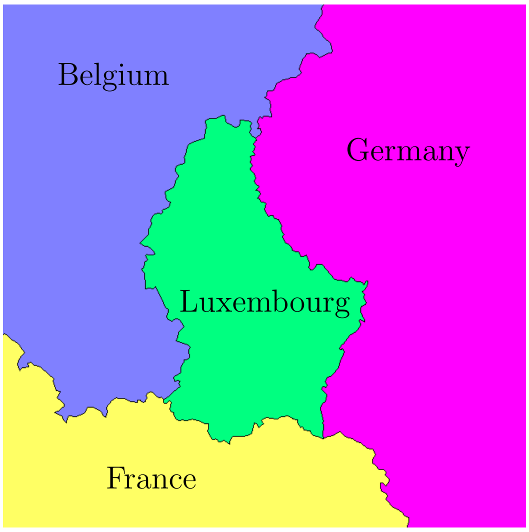
But can we choose the same representatives, that is, can we have $a_i = b_i$ for $1 \leq i \leq k$? Hall’s Marriage Theorem, which we will prove in this course, will tell us that the answer is “yes”.

Example 0.3 (Map colouring problem)**.** Suppose we have a political map of some place, and we want to colour it so that

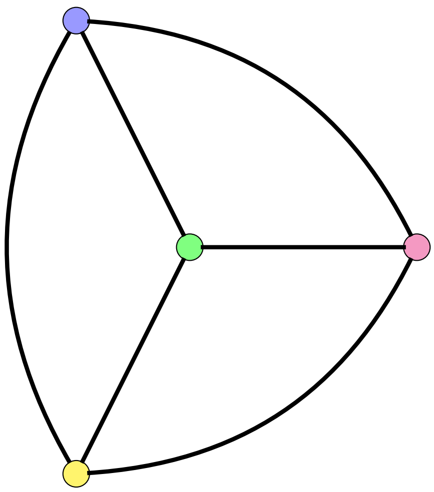
(i) any given contiguous region (country/voivodeship/etc) uses a single colour; and

(ii) no two regions sharing a border are coloured by the same colour.

How many colours we need to do this? In general, we will need at ≥ 4 colours, as we can verify by visiting Luxembourg (see Figure 0.2). In fact, four colours are always enough—this is the so-called Four Colour Theorem. After many incorrect proof attempts spanning more than a century, a computer-assisted proof of this fact was finally given by Kenneth Appel and Wolfgang Haken in 1976. The Four Colour Theorem is beyond the scope of our course, but we will prove that we can always colour a map if we are given *five* colours.



(a) Luxembourg and its neighbours.



(b) Rephrased in terms of graphs.

Figure 0.2: The map colouring problem. There are four countries and each of them has borders with all the others, so we need four colours to colour them all.

Example 0.4 (Fermat's Last Theorem modulo p)**.** Let $n > 2$ be an integer. A well-known theorem, stated by Pierre de Fermat in 1637 and finally proved by Andrew Wiles in 1994, says that there are no integer solutions to the equation $x^n + y^n = z^n$ with $x, y, z \neq 0$. But we can ask the same question modulo primes: given a prime number p, do there exist $x, y, z \in \mathbb{Z}$ such that $x^n + y^n \equiv z^n \pmod p$ but $x, y, z \not\equiv 0 \pmod p$? An argument by Issai Schur from 1916 shows that the answer is "yes" for all sufficiently large primes p, and the proof goes as follows.

Let $G = (\mathbb{Z}/p\mathbb{Z})^\times$, the multiplicative group of integers modulo p. Consider the subgroup $H = \{g^n \mid g \in G\}$. Given $h \in H$, the polynomial $X^n - h \in \mathbb{Z}/p\mathbb{Z}[X]$ has degree n and so it must have $\leq n$ roots, implying that there are at most n elements $g \in G$ such that $g^n = h$. It follows that $|H| \geq \frac{|G|}{n}$, and so H has $\leq n$ left cosets in G. Suppose we have $a, b, c \in gH$ with $a + b = c$. Then $g^{-1}a + g^{-1}b = g^{-1}c$ with $g^{-1}a, g^{-1}b, g^{-1}c \in H$, meaning that $g^{-1}a = x^n$, $g^{-1}b = y^n$ and $g^{-1}c = z^n$ for some $x, y, z \in G$. It is therefore enough to show that some left coset of H in G contains elements a, b, c with $a + b = c$. In particular, it is enough to show the following:

For any sufficiently large $k \in \mathbb{Z}$, if the set $\{1, \ldots, k-1\}$ is partitioned into n parts, one of these parts must contain some x, y and z such that $x + y = z$. $(*)$

We will prove $(*)$ using methods of graph theory.

1

Structural properties

In this chapter we study the main structural properties of graphs. We start by introducing some definitions.

1.1 Some basic concepts

Definition (graphs, vertices, edges)**.** A *graph* is an ordered pair $G = (V, E)$, where

- $V = V(G)$ is a set, called the set of *vertices*, and
- $E = E(G)$ is a set of unordered pairs $\{v, w\}$, where $v, w \in V$ and $v \neq w$, called the set of *edges*.

We write $v \in G$ to mean $v \in V$, and we denote by vw an edge $\{v, w\} \in E$; we call v and w the *endpoints* of the edge vw, and we will say the edge vw is *incident* to v (or to w). Unless specified otherwise, we will always assume that the set V is finite; a graph $G = (V, E)$ with $|V| = \infty$ will be called an *infinite graph*.

Remark. There are several notions of a graph appearing in the literature. To be specific, our graphs could be called "undirected simple graphs"; here, "undirected" means that the edges $\{v, w\}$ are *unordered* pairs, and "simple" means that $E(G)$ is a set (as opposed to a multiset) and does not contain pairs of the form $\{v, v\}$.

We may "draw" a graph to make it easier to understand. For each vertex v we put a dot labelled v on a plane, and for each edge vw we join dots labelled v and w by an arc. An example of such a drawing is displayed in Figure 1.1.

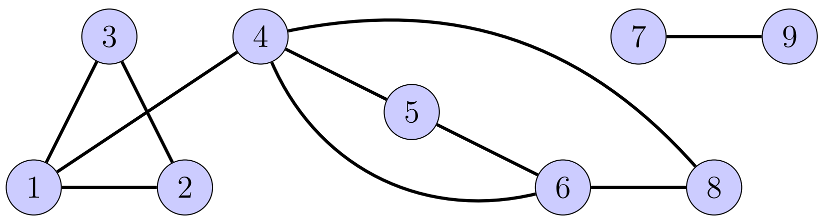


Figure 1.1: A drawing of the graph $G = (V, E)$ with vertex set $V = \{1, 2, \ldots, 9\}$ and edge set $E = \{12, 13, 14, 23, 45, 46, 48, 56, 68, 79\}$.

Given a graph $G = (V, E)$, we write $|G|$ for $|V|$, and $e(G)$ for $|E|$. We call $|G|$ the *order* or G, and we call $e(G)$ the *size* of G.

We also want to discuss when a graph is 'contained' in some bigger graph. Formally, this is done as follows.

Definition (isomorphisms, subgraphs)**.** Let $G = (V, E)$ and $G' = (V', E')$ be two graphs.

- We say that G and G' are *isomorphic*, written $G \cong G'$, if there exists a bijection $\varphi\colon V \to V'$ such that for all $v, w \in V$ we have $vw \in E$ if and only if $\varphi(v)\varphi(w) \in E'$.
- We say that G' is a *subgraph* of G, written $G' \leq G$, if $V' \subseteq V$ and $E' \subseteq E$.
- If H and G are graphs such that G has no subgraphs isomorphic to H, we will say that G is *H-free*.

If H, H', G are graphs such that $H \cong H'$ and $H' \leq G$, we will abuse the terminology slightly to say that H is a subgraph of G and write $H \leq G$.

We now introduce some specific graphs that will appear throughout the course. Given an integer $n \geq 1$, we will write $[n]$ for the set $\{1, \ldots, n\}$.

Definition (paths, cycles)**.** Let $n \geq 1$.

- The *path* of length $n-1$, denoted P_{n-1}, is a graph with $V(P_{n-1}) = [n]$ and $E(P_{n-1}) = \{i(i+1) \mid 1 \leq i \leq n-1\}$.
- If $n \geq 3$, then the *cycle* of length n, denoted C_n, is a graph with $V(C_n) = [n]$ and $E(C_n) = \{i(i+1) \mid 1 \leq i \leq n-1\} \cup \{1n\}$.
- If P_{n-1} (respectively C_n) is a subgraph of a graph G with vertices $v_1, \ldots, v_n$ and edges $v_1v_2, \ldots, v_{n-1}v_n$ (respectively $v_1v_2, \ldots, v_{n-1}v_n, v_1v_n$), then we will denote such a subgraph by $v_1v_2 \cdots v_n$ (respectively $v_1v_2 \cdots v_nv_1$).

Example 1.1. We have the following.

- For $n \leq m$, P_n is a subgraph of P_m and of C_{m+1}.
- The graph G displayed in Figure 1.1 has as subgraphs a cycle 1231 of length 3, a cycle 45684 of length 4, and a path 2314568 of length 6. The sequence 12314 is not a path in G as we cannot have a vertex appearing more than once in a path.

We now introduce notions of induced subgraphs and connected graphs.

Definition (induced subgraphs, connected components)**.** Let $G = (V, E)$ be a graph.

- If $A \subseteq V$, the subgraph of G *induced* by A is the subgraph $G[A] = (A, E_A)$, where $E_A = \{vw \in E \mid v, w \in A\}$. We write $G - A$ for the graph $G[V \setminus A]$. Similarly, given $F \subseteq E$, we write $G - F$ for the subgraph $H \leq G$ with $V(H) = V$ and $E(H) = E \setminus F$.
- Given $v, w \in V$, write $v \approx w$ if there exists a path $v \cdots w$ in G. Then $\approx$ is an equivalence relation on V (see Problem 1.2). A *connected component* of G is a subgraph $G[W] \leq G$, where $W \subseteq V$ is an equivalence class under $\approx$.
- We say G is *connected* if $v \approx w$ for every $v, w \in V$ (equivalently, if G has at most one connected component).

For instance, the graph G displayed in Figure 1.1 is not connected, and its connected components are $G[\{1,2,3,4,5,6,8\}]$ and $G[\{7,9\}]$.

Finally, we will need to study adjacency in graphs, so we introduce the following terminology.

Definition (neighbourhoods, degree, regular graphs)**.** Let $G = (V, E)$ be a graph.

- Let $v, w \in G$. If $vw \in E(G)$, then we will say that v and w are *adjacent* in G (or that w is a *neighbour* of v), and we will write $v \sim w$.
- Let $v \in G$. The *neighbourhood* of v is $N_G(v) := \{w \in G \mid v \sim w\}$, and the *degree* of v is $d_G(v) := |N_G(v)|$. We write $N(v)$ for $N_G(v)$ and $d(v)$ for $d_G(v)$ if the graph G is clear.
- Let $A \subseteq V$. The *neighbourhood* of A is $N_G(A) := \bigcup_{v \in A} N(v)$. We write $N(A)$ for $N_G(A)$ if the graph G is clear.
- We define the *minimal degree* $\delta(G)$, the *maximal degree* $\Delta(G)$ and the *average degree* $d(G)$ of G as

$$\delta(G) = \min_{v \in G} d(v), \qquad \Delta(G) = \max_{v \in G} d(v) \qquad \text{and} \qquad d(G) = \frac{\sum_{v \in G} d(v)}{|G|},$$

  respectively.
- Note that $\delta(G) \leq d(G) \leq \Delta(G)$. If we have an equality—that is, if there exists $r \geq 0$ such that $d(v) = r$ for all $v \in G$—then we say that G is r*-regular*. We say G is *regular* if it is r-regular for some r.

Lemma 1.2 (Handshaking Lemma)**.** *For any graph G we have $e(G) = \frac{1}{2} \sum_{v \in G} d(v) = \frac{|G|}{2} d(G)$.*

Proof. Let $A = \{(e, v) \mid v \in V(G), e \in E(G), v \in e\}$. For each $e = vw \in E(G)$, we have $(e, u) \in A$ if and only if $u \in \{v, w\}$, and therefore $|A| = 2e(G)$. On the other hand, for each $v \in V(G)$ we have $(e, v) \in A$ if and only if $e = vw$ for some $w \in N(v)$, and therefore $|A| = \sum_{v \in G} d(v) = |G| \cdot d(G)$. □

Example 1.3. We have the following.

- For $n \geq 3$, C_n is 2-regular, but P_{n-1} is not regular as it has both vertices of degree 1 and vertices of degree 2.
- The graph G displayed in Figure 1.1 has minimal degree $\delta(G) = d(7) = 1$, maximal degree $\Delta(G) = d(4) = 4$, and we may compute that its average degree is $d(G) = \frac{20}{9}$.
- For the graph G displayed in Figure 1.1, we have neighbourhoods $N(1) = \{2, 3, 4\}$, $N(\{1, 5\}) = \{2, 3, 4, 6\}$ and $N(\{4, 5\}) = \{1, 2, 3, 4, 5, 6, 8\}$.

1.2 Hall's Marriage Theorem

In this section we study the class of bipartite graphs, defined as follows.

Definition (bipartite graphs)**.** We say a graph $G = (V, E)$ is *bipartite* (*with vertex classes* U *and* W) if we can partition the vertex set as $V = U \sqcup W$ so that every edge has the form uw for some $u \in U$ and $w \in W$.

We have the following characterisation of bipartite graphs.

Proposition 1.4. *A graph $G = (V, E)$ is bipartite if and only if it has no cycles of odd length.*

Proof.

($\Rightarrow$) Let U and W be the vertex classes, and let $v_1 v_2 \cdots v_n v_1$ be a cycle in G. Without loss of generality, suppose that $v_1 \in U$. Then $v_2 \in W$ as $v_1 \sim v_2$, $v_3 \in U$ as $v_2 \sim v_3$, etc; specifically, we have $v_i \in U$ if i is odd and $v_i \in W$ if i is even. But we have $v_n \sim v_1 \in U$ so $v_n \in W$, implying that n is even.

($\Leftarrow$) Suppose G has no cycles of odd length. Without loss of generality, assume that $V(G) \neq \varnothing$ and that G is connected (as G will be bipartite if all its connected components are bipartite). Fix some $v \in G$, and for every $w \in G$ define the *distance* $\operatorname{dist}(v, w)$ from v to w to be the smallest $n \geq 0$ such that there exists a path $v \cdots w$ in G of length n. Let $V_n := \{w \in G \mid \operatorname{dist}(v, w) = n\}$, and set $U := V_0 \sqcup V_2 \sqcup V_4 \sqcup \cdots$ and $W := V_1 \sqcup V_3 \sqcup V_5 \sqcup \cdots$. We aim to show that there are no edges in G of the form $v'v''$ with either $v', v'' \in U$ or $v', v'' \in W$.

Suppose $v'v'' \in E(G)$ with $v' \in V_m$, $v'' \in V_n$ and $m \leq n$. Then there exists a path $v \cdots v'v''$ in G of length $m+1$, implying that $n \in \{m, m+1\}$. Suppose that $n = m$, and let $v'_0 v'_1 \cdots v'_m$ and $v''_0 v''_1 \cdots v''_m$ be paths in G with $v = v'_0 = v''_0$, $v' = v'_m$ and $v'' = v''_m$. Note that $v'_i, v''_i \in V_i$ for $0 \leq i \leq m$. Let $k \geq 0$ be largest such that $v'_k = v''_k$, and note that $k \leq m-1$ (as $v' \neq v''$). Then $v'_k v'_{k+1} \cdots v'_m v''_m v''_{m-1} \cdots v''_k$ is a cycle in G of length $2(m-k)+1$, contradicting the fact that G has no cycles of odd length.

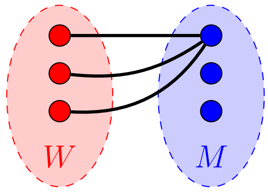
Therefore, we must have $n = m+1$. But then exactly one of n and m is even, meaning that exactly one of v' and v'' is in U, as required. □

We now give a criterion for a bipartite graph to have a "matching".

Definition (matchings)**.** Let G be a bipartite graph with vertex classes W and M. Given $W' \subseteq W$, a *partial matching* in G from W' to M is a subset $\{wv_w \mid w \in W'\} \subseteq E(G)$ for some vertices $v_w \in M$ (where $w \in W'$) such that $v_w \neq v_{w'}$ when $w \neq w'$. A partial matching in G from W to M is called a *matching*.

It is traditional to use the so-called "marriage terminology": we think of W as a set of women, M as a set of men, and we draw an edge wv for $w \in W$ and $v \in M$ if w and v form a "suitable" couple. The question about existence of a matching then becomes, can we marry all the women to suitable husbands?

An obvious necessary condition is for every woman to have a suitable husband. However, this condition is not sufficient, as the situation on the right demonstrates. A better necessary condition is to say that for any n women, there are n men such that each of these men is suitable to at least one of the n women. This condition can be expressed by saying that $|N(A)| \geq |A|$ for every $A \subseteq W$, and it turns out that this condition is in fact sufficient.



Theorem 1.5 (Hall's Marriage Theorem)**.** *Let G be a bipartite graph with vertex classes W and M. Then G contains a matching from W to M if and only if (G, W) satisfies* Hall's condition*: $|N(A)| \geq |A|$ for every $A \subseteq W$.*

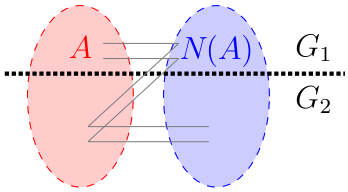
Proof.

(⇒) Given a matching $\{wv_w \mid w \in W\}$ and a subset $A \subseteq W$, the collection $\{v_w \mid w \in A\}$ is contained in $N(A)$ and has cardinality $|A|$.

(⇐) We use induction on $|W|$. The cases $|W| = 0$ and $|W| = 1$ are clear, so we may assume that $|W| \geq 2$.

Suppose first that $|N(A)| > |A|$ for every non-empty subset $A \subsetneq W$. Pick any $w \in W$ and $v \in N(w)$, and let $G_0 = G - \{w, v\}$. For any non-empty $B \subseteq W \setminus \{w\}$, we have $N_{G_0}(B) = N_G(B) \setminus \{v\}$ and therefore $|N_{G_0}(B)| \geq |N_G(B)| - 1 \geq |B|$, implying that $(G_0, W \setminus \{w\})$ satisfies Hall's condition. By the inductive hypothesis, there is a matching P in G_0 from $W \setminus \{w\}$ to $M \setminus \{v\}$. Then $P \sqcup \{wv\}$ is a matching in G from W to M.

Suppose now that $|N(A)| = |A|$ for some non-empty subset $A \subsetneq W$. Let $G_1 = G[A \cup N(A)]$ and $G_2 = G[(W \setminus A) \cup (M \setminus N(A))]$. We aim to show that (G_1, A) and $(G_2, W \setminus A)$ both satisfy Hall's condition.



G_1: For any $B \subseteq A$, we have $N_G(B) \subseteq N_G(A) \subseteq V(G_1)$ and therefore

$$|N_{G_1}(B)| = |N_G(B)| \geq |B|.$$

G_2: For any $B \subseteq W \setminus A$, we have $N_{G_2}(B) = N_G(B) \setminus N_G(A) = N_G(A \cup B) \setminus N_G(A)$ and therefore

$$\begin{aligned} |N_{G_2}(B)| &= |N_G(A \cup B) \setminus N_G(A)| \geq |N_G(A \cup B)| - |N_G(A)| \\ &\geq |A \cup B| - |A| = |A| + |B| - |A| = |B|. \end{aligned}$$

Therefore, both (G_1, A) and $(G_2, W \setminus A)$ satisfy Hall's condition, as claimed. By the inductive hypothesis, it then follows that there exists a matching P_1 in G_1 from A to $N_G(A)$, and a matching P_2 in G_2 from $W \setminus A$ to $M \setminus N_G(A)$. The union $P_1 \cup P_2$ is then a matching in G from W to M. □

We are now ready to give an answer to the question posed in Example 0.2.

Corollary 1.6. *Let G be a finite group and let $H \leq G$ be a subgroup with $\frac{|G|}{|H|} = k$. Then we can write*

$$g_1H \sqcup \cdots \sqcup g_kH = G = Hg_1 \sqcup \cdots \sqcup Hg_k$$

for some $g_1, \ldots, g_k \in G$.

Proof. Let $L = \{a_1H, \ldots, a_kH\}$ and $R = \{Hb_1, \ldots, Hb_k\}$ be the sets of left and right cosets (respectively) of H in G. Let K be a bipartite graph with vertex classes L and R, where $a_iH \sim Hb_j$ in K if and only if $a_iH \cap Hb_j \neq \varnothing$ in G. Given any $A \subseteq L$, we have $\left|\bigcup_{U \in A} U\right| = |A| \cdot |H|$ as subsets of G; as $|V| = |H|$ for every $V \in R$, it follows that $\bigcup_{U \in A} U$ has non-trivial intersection with at least $|A|$ elements of R and therefore $|N_K(A)| \geq |A|$. Thus, by Theorem 1.5, there exists a matching P in K from L to R. The result follows by taking g_i to be any element in $a_iH \cap Hb_j$ for the edge $(a_iH)(Hb_{j_i})$ of P (for $1 \leq i \leq k$): indeed, we then have $a_iH = g_iH$ and $Hb_{j_i} = Hg_i$. □

Finally, we use Hall's Marriage Theorem to deduce a couple of its variations. We will prove these results using "marriage terminology".

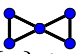
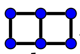
Corollary 1.7 (Hall's Missing Soulmate Theorem)**.** *Let G be a bipartite graph with vertex classes W and M, and let $d \geq 1$. Then G contains a partial matching from W' to M for some $W' \subseteq W$ with $|W'| \geq |W| - d$ if and only if $|N(A)| \geq |A| - d$ for every $A \subseteq W$.*

Proof. The ($\Rightarrow$) direction is clear. For ($\Leftarrow$), introduce d imaginary perfect men that are suitable husbands to every woman. Then Hall's condition is satisfied, so we can marry all women to suitable (real or imaginary) husbands. In real life, at most d women are left unmarried. □

Corollary 1.8 (Hall's Polygamous Marriage Theorem)**.** *Let G be a bipartite graph with vertex classes W and M, and let $d \geq 1$. Then G contains a subgraph H with $W \subseteq V(H)$ in which each $w \in W$ has degree d and each $v \in M \cap V(H)$ has degree 1 if and only if $|N(A)| \geq d|A|$ for every $A \subseteq W$.*

Proof. The ($\Rightarrow$) direction is clear. For ($\Leftarrow$), clone each woman $d - 1$ times. Then Hall's condition is satisfied, so we can marry all women (originals and clones) to suitable husbands. Now merge the clones with the originals. □

1.3 Menger's Theorem

Recall, we say that a graph G is connected if for each $v, w \in V(G)$ there is a path $v \cdots w$ in G. However, some connected graphs look "more connected" than others: consider $H =$  and $K =$ . The graph H has a *cut vertex*, i.e. a vertex v such that $H - \{v\}$ is not connected, whereas K does not. This motivates the following definition.

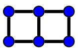
Definition (k-connected graphs)**.** Let G be a graph, and let k be an integer with $k \geq 0$. We say G is *k-connected* if $G - A$ is connected for any $A \subseteq V(G)$ with $|A| < k$.

Remark. One has to be slightly careful with terminology here, as there is an unrelated definition of a "k-connected space" appearing in topology. However, our notion is standard in graph theory.

We will also need to consider the following special class of graphs.

Definition (complete graphs)**.** A graph G is *complete* if $v \sim w$ in G for every $v, w \in G$ with $v \neq w$.

Example 1.9. Let G be a graph.

- G is 0-connected.
- G is 1-connected if and only if G is connected.
- G is 2-connected if and only if G is connected and has no cut vertices.
- The graph $K =$  is 2-connected but not 3-connected.
- If G is k-connected, then for every $A \subseteq V(G)$ with $|A| \leq k$ the graph $G - A$ is $(k - |A|)$-connected.
- If G is k-connected for some $k \geq |G| - 1$ then G is complete. Indeed, if $v, w \in G$ are such that $v \neq w$ and $v \nsim w$ then $G - A$ is disconnected, where $A = V(G) \setminus \{v, w\}$.

Our aim is to relate the notion of k-connectedness to the notion of independent paths, defined as follows.

Definition ((A, B)-paths, (A, B)-cuts, independent paths)**.** Let $G = (V, E)$ be a graph.

- Let $A, B \subseteq V$. An (A, B)*-path* is a path in G of the form $a \cdots b$ for some $a \in A$ and $b \in B$. An (A, B)*-cut* in G is a subset $C \subseteq V$ such that $G - C$ contains no $(A \setminus C, B \setminus C)$-paths.
- Let $a, b \in V$. For simplicity, we call an $(\{a\}, \{b\})$-path in G an (a, b)*-path*. A collection $P^{(1)}, \ldots, P^{(k)}$ of (a, b)-paths in G are said to be *independent* if $P^{(i)} - \{a, b\}$ and $P^{(j)} - \{a, b\}$ have no vertices in common for $i \neq j$.

Note that for any graph G and any $A, B, C \subseteq V(G)$, if either $A \subseteq C$ or $B \subseteq C$ then C is an (A, B)-cut, and conversely, if C is an (A, B)-cut then $A \cap B \subseteq C$.

We aim to show that a graph is k-connected if and only if for every a and b there is a collection of k independent (a, b)-paths. The key ingredient to this is the following result.

Lemma 1.10. *Let G be a graph, $A, B \subseteq V(G)$, and $k \geq 0$. Suppose that $|C| \geq k$ for every (A, B)-cut C in G. Then G contains a collection of k vertex-disjoint (A, B)-paths.*

Proof. We use induction on $e(G)$. As the base case, consider the situation when $e(G) = 0$: then $A \cap B$ is an (A, B)-cut and so $k \leq |A \cap B|$, but every vertex of $A \cap B$ is an (A, B)-path (of length 0) and all these paths are vertex-disjoint, as required.

Suppose now that $e(G) \geq 1$, pick an edge $e \in E(G)$, and let $H = G - \{e\}$. If every (A, B)-cut in H has order $\geq k$, then by the inductive hypothesis there are k vertex-disjoint (A, B)-paths in H and therefore in G, so we are done.

Therefore, without loss of generality, assume that H has an (A, B)-cut C with $|C| < k$. Then C is not an (A, B)-cut in G, so $G - C$ contains an (A, B)-path of the form $a \cdots vw \cdots b$ for some $a \in A$ and $b \in B$, where $v, w \in G$ are the endpoints of e. Moreover, every (A, B)-path in $G - C$ contains the vertex v, implying that $C' = C \cup \{v\}$ is an (A, B)-cut in G, and in particular that $|C| + 1 = |C'| \geq k$. Thus in fact $|C| = k - 1$, and we can write $C = \{c_1, \dots, c_{k-1}\}$.

Now since $v \in C'$, any (A, C')-cut D in H is also an (A, C')-cut in G; as every (A, B)-path in G contains a vertex of C', it follows that D is also an (A, B)-cut in G and so $|D| \geq k$. Therefore, by the inductive hypothesis, there exist vertex-disjoint (A, C')-paths $P^{(1)}, \dots, P^{(k-1)}, P^{(k)}$ in H ending at $c_1, \dots, c_{k-1}, v$, respectively. Similarly, there exist vertex-disjoint (C'', B)-paths $Q^{(1)}, \dots, Q^{(k-1)}, Q^{(k)}$ in H starting at $c_1, \dots, c_{k-1}, w$, respectively, where $C'' = C \cup \{w\}$. Moreover, as C' is an (A, B)-cut in G, no $P^{(i)}$ and $Q^{(j)}$ can share a vertex u except when $i = j \leq k - 1$ and $u = c_i$. This implies that $P^{(1)} \cdot Q^{(1)}, \dots, P^{(k-1)} \cdot Q^{(k-1)}, P^{(k)} \cdot e \cdot Q^{(k)}$ are k vertex-disjoint (A, B)-paths in G (where $P \cdot Q$ denotes the concatenation of P and Q), as required. □

Remark. We may deduce Hall's Marriage Theorem from Lemma 1.10. Indeed, let G be a bipartite graph with vertex classes W and M, and suppose that (G, W) satisfies Hall's condition. Let C be a (W, M)-cut in G. Then $N(W \setminus C) \subseteq M \cap C$, and therefore

$$|C| = |W \cap C| + |M \cap C| \geq |W \cap C| \cap |N(W \setminus C)| \geq |W \cap C| + |W \setminus C| = |W|.$$

Thus, by Lemma 1.10, G contains $|W|$ vertex-disjoint (W, M)-paths. Each of these paths must have length 1 (i.e. must be an edge), implying that this collection of paths is actually a matching.

We now deduce a criterion for a graph to be k-connected, as follows.

Theorem 1.11 (Menger's Theorem). *Let G be an incomplete graph, and let $k \geq 0$. Then G is k-connected if and only if for every $a, b \in G$ with $a \neq b$, there exists a collection of k independent (a, b)-paths in G.*

Proof.

($\Leftarrow$) Let $C \subseteq V(G)$, and suppose $G - C$ is disconnected. Pick $a, b \in G - C$ belonging to different connected components of $G - C$. By our assumption, G contains k independent (a, b)-paths. Each of these paths must have a vertex in C, but no two of these paths share a common vertex apart from a and b. Thus $|C| \geq k$, as required.

($\Rightarrow$) We use induction on k. The base case, $k = 0$, is trivial, so suppose $k \geq 1$, and let $a, b \in G$ with $a \neq b$.

Suppose first that $a \nsim b$. Let $A = N(a)$ and $B = N(b)$. The graphs $G - A$ and $G - B$ are disconnected (as they do not have any paths $a \cdots b$), implying that $|A| \geq k$ and $|B| \geq k$. If C is an (A, B)-cut in G, then $G - C$ has no paths from an element of

$A \setminus C$ to an element of $B \setminus C$, so either $A \subseteq C$, or $B \subseteq C$, or $G - C$ is disconnected. In either case, we have $|C| \geq k$, so by Lemma 1.10, G has k vertex-disjoint (A, B)-paths: $a_1 \cdots b_1, \ldots, a_k \cdots b_k$, say. Then $aa_1 \cdots b_1 b, \ldots, aa_k \cdots b_k b$ are k independent (a, b)-paths, as required.

Suppose now that $a \sim b$, and let $H = G - \{ab\}$. We claim that H is $(k-1)$-connected. Indeed, suppose not, and let $C \subseteq V(H)$ be such that $|C| < k - 1$ and $H - C$ is disconnected. Since G is k-connected, $G - C$ is connected and has no cut vertices, implying that $H - C$ has exactly two connected components, each containing just a single vertex (a or b). But then $|G| = |H| = 2 + |C| \leq k$, so G is a k-connected graph with $|G| \leq k$, contradicting the assumption that G is not complete.

Thus H must be $(k-1)$-connected, and therefore, by the inductive hypothesis, contains $k - 1$ independent (a, b)-paths. Together with the edge ab these paths form a collection of k independent (a, b)-paths in G, as required. □

1.4 Menger's Theorem (edge version)

We now consider a concept closely related to k-connected graphs: k'-edge connected graphs.

Definition (k'-edge-connected graphs)**.** Let G be a graph, and let k' be an integer with $k' \geq 0$. We say G is k'*-edge-connected* if $G - F$ is connected for every $F \subseteq E(G)$ with $|F| < k'$.

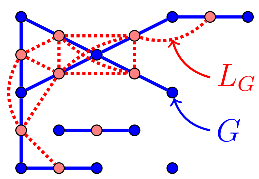
Example 1.12. Let G be a graph.

- G is 0-edge-connected.
- G is 1-edge-connected if and only if G is connected.
- G is 2-edge-connected if and only if G is connected and has no bridges (here a *bridge* is an edge of a connected graph G whose removal disconnects G).

We now use Lemma 1.10 to deduce a characterisation of k'-edge-connected graphs, as follows.

Theorem 1.13 (Menger's Theorem, edge version)**.** *Let G be graph, and let $k' \geq 0$. Then G is k'-edge-connected if and only if for every $a, b \in G$ with $a \neq b$, there exists a collection of k' edge-disjoint (a, b)-paths in G.*

Proof.



($\Rightarrow$) Let L_G be the *line graph* of G, defined as follows: we set $V(L_G) = E(G)$, and for $e, f \in L_G$ with $e \neq f$ we have $e \sim f$ in L_G if and only if e and f have a common endpoint in G.

Let $a, b \in G$ with $a \neq b$. Let $A = \{av \in E(G) \mid v \in N_G(a)\}$ and $B = \{bv \in E(G) \mid v \in N_G(b)\}$, and let C be an (A, B)-cut in L_G, so that $C \subseteq E(G)$. Then there is no (a, b)-path in $G - C$, implying that $|C| \geq k'$. Therefore, by Lemma 1.10, there exist k' vertex-disjoint (A, B)-paths in L_G, and so k' edge-disjoint (a, b)-paths in G.

($\Leftarrow$) Let $F \subseteq E(G)$, and suppose $G - F$ is disconnected. Pick $a, b \in G - F$ belonging to different connected components of $G - F$. By our assumption, G contains k' edge-disjoint (a, b)-paths, and each of these paths must have an edge in F. Thus $|F| \geq k'$, as required. □

Remark. In fact, we can deduce the ($\Rightarrow$) direction of Theorem 1.13 from the max-flow min-cut theorem. Indeed, we can replace each edge vw by a pair of directed edges $v \to w$ and $w \to v$ and, in the optimisation terminology, we can give each edge capacity 1. The fact that G is k'-edge-connected then tells us that any a-b cut for vertices $a \neq b$ of G has capacity $\geq k'$, and so we have an a-b flow of value k'. Moreover, since all edge capacities are integers, we have such a flow taking integer values on each edge. This implies that there are k' edge-disjoint (a, b)-paths, as required.

2

Extremal problems

In this chapter, we deal with so-called *extremal problems*: how large can we make some parameter of a graph G before G is forced to have a certain property? Here, a "parameter" is often the ratio $\frac{e(G)}{\binom{|G|}{2}}$, and a "property" is usually "containing a subgraph isomorphic to H" for some graph H.

2.1 Complete subgraphs

We now introduce complete graphs and (complete) r-partite graphs, as follows.

Definition (complete, r-partite, complete r-partite graphs)**.** Let $r \geq 1$.

- A *complete graph* of order r, denoted K_r, is a graph with $V(K_r) = [r]$ and $E(K_r) = \{ij \mid 1 \leq i < j \leq r\}$. We call K_3 a *triangle*.
- A graph G is called *r-partite* with vertex classes $V_1, \ldots, V_r$ if there exists a partition $V(G) = V_1 \sqcup \cdots \sqcup V_r$ such that for every edge $vw \in E(G)$ with $v \in V_i$ and $w \in V_j$ we have $i \neq j$. Such a graph G is called *complete r-partite* if in addition $vw \in E(G)$ for every $v \in V_i$, $w \in V_j$ and $i \neq j$.
- If $r = 2$ and G is a complete 2-partite graph with vertex classes of orders $|V_1| = m$ and $|V_2| = n$, we call G a *complete bipartite graph* and denote it by $K_{m,n}$.

We have already encountered some of these graphs before: indeed, $K_1 \cong P_0$, $K_2 \cong K_{1,1} \cong P_1$, $K_{1,2} \cong P_3$, $K_3 \cong C_3$, and $K_{2,2} \cong C_4$. See Figure 2.1 for further examples.

Let $n > r \geq 1$. We aim to answer the following question: if G is a graph of order n, how big $e(G)$ needs to be in order to force K_{r+1} to appear as a subgraph of G? It turns out this question has an exact answer. But first, here are some ideas:

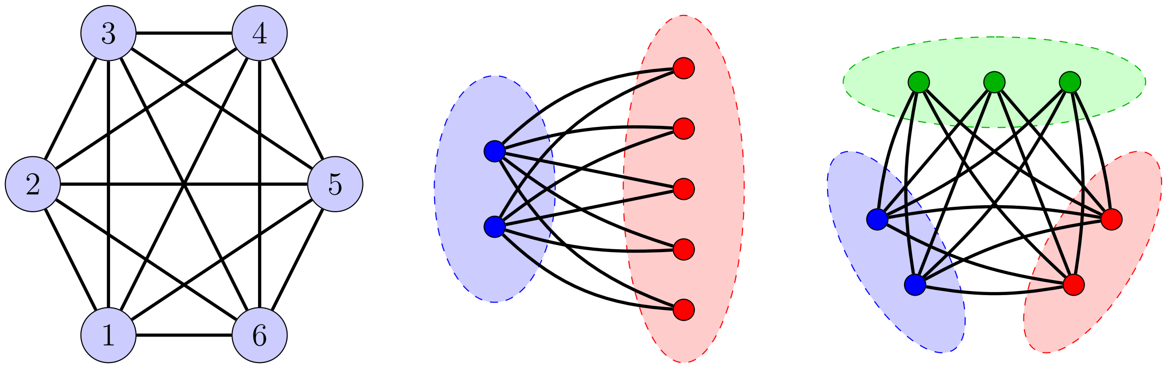
- Given $r \geq 1$, an obvious sufficient condition for a graph G to be K_{r+1}-free is for G to be r-partite, as any subgraph $H \leq G$ with $|H| = r + 1$ will have two vertices from the same vertex class and therefore will not be complete.
- Given $n \geq r$, out of all r-partite graphs with n vertices, the one with most edges is clearly a complete r-partite graph.
- Suppose G is a complete r-partite graph with vertex classes $V_1, \ldots, V_r$. If $|V_i| \geq |V_j| + 2$ for some $i \neq j$, then we may choose a vertex $v \in V_j$, and consider a graph G' obtained from G by removing edges of the form vv_i for $v_i \in V_i$ and adding an edge

vv_j for every $v_j \in V_j \setminus \{v\}$. Then G' is complete r-partite, and we have $|G'| = |G|$ and $e(G') = e(G) - |V_i| + |V_j| - 1 > e(G)$. Thus the r-partite graph with n vertices and the most edges will have vertex classes "as equal in size as possible".

These observations motivate the following definition.

Definition (Turán graphs)**.** Let $n \geq r \geq 1$. The *Turán graph* $T_r(n)$ is a complete r-partite graph of order n with all vertex classes of size $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$. We write $t_r(n)$ for $e(T_r(n))$.

It is clear from the definition that if r divides n then all vertex classes in $T_r(n)$ have the same size, whereas otherwise $T_r(n)$ has "large" and "small" vertex classes, with any large class having one more vertex than any small one.



(a) K_6; (b) $K_{2,5}$; (c) $T_3(7)$.

Figure 2.1: Some examples of complete and complete r-partite graphs.

Observation 2.1. Let $n > r \geq 1$.

(i) If G is a graph obtained by adding an edge to $T_r(n)$ (that is, $T_r(n) \cong G - \{e\}$ for some $e \in E(G)$), then G is not K_{r+1}-free.

(ii) If r divides n, then we have $\delta(T_r(n)) = d(T_r(n)) = \Delta(T_r(n) = n - \frac{n}{r}$. Otherwise, vertices in the large classes have minimal degree, $\delta(T_r(n)) = n - \lceil \frac{n}{r} \rceil$, and vertices in the small classes have maximal degree, $\Delta(T_r(n)) = n - \lfloor \frac{n}{r} \rfloor$. This implies that in any case, we have $\delta(T_r(n)) = \lfloor d(T_r(n)) \rfloor$ and $\Delta(T_r(n)) = \lceil d(T_r(n)) \rceil$.

(iii) We have $T_r(n-1) \cong T_r(n) - \{v\}$, where $v \in T_r(n)$ is a vertex of minimal degree (that is, any vertex if r divides n, and a vertex from one of the large classes otherwise). In particular, $t_r(n-1) = t_r(n) - \delta(T_r(n))$.

(iv) Suppose we would like to add one vertex v and m edges to $T_r(n-1)$, so that m is as large as possible while the resulting graph G is K_{r+1}-free. Then v cannot be adjacent in G to a vertex in every class, so we have $m = d_G(v) \leq n - 1 - \lfloor \frac{n-1}{r} \rfloor = n - \lceil \frac{n}{r} \rceil$, with equality if and only if G is complete r-partite, obtained by adding v to any vertex class of $T_r(n-1)$ if r divides $n-1$, or to one of the small classes otherwise. This yields $G \cong T_r(n)$ and $m = n - \lceil \frac{n}{r} \rceil = \delta(T_r(n))$.

We are now ready to state our main result of this section: namely, out of all K_{r+1}-free graphs G with n vertices, the unique graph maximising $e(G)$ is $G \cong T_r(n)$.

Theorem 2.2 (Turán's Theorem)**.** *Let $n \geq r \geq 1$, and let G be a K_{r+1}-free graph with $|G| = n$ and $e(G) \geq t_r(n)$. Then $G \cong T_r(n)$.*

Proof. We prove the result by induction on n. If $n = r$, then $T_r(n) \cong K_r$ and therefore $\binom{n}{2} = t_r(n) \leq e(G) \leq \binom{n}{2}$, so the result follows.

Suppose now that $n > r$. Pick a subset $E' \subseteq E(G)$ such that $|E'| = e(G) - t_r(n)$, and let $H = G - E'$, so that $e(H) = t_r(n)$. We then have $d(H) = \frac{2e(H)}{n} = \frac{2t_r(n)}{n} = d(T_r(n))$ by Lemma 1.2, and therefore $\delta(H) \leq \lfloor d(H) \rfloor = \lfloor d(T_r(n)) \rfloor = \delta(T_r(n))$, where the last equality follows from Observation 2.1(ii).

Now pick a vertex $v \in H$ with $d(v) = \delta(H)$, and let $K = H - \{v\}$. Then K is K_{r+1}-free, $|K| = n - 1$, and

$$e(K) = e(H) - d_H(v) = t_r(n) - \delta(H) \geq t_r(n) - \delta(T_r(n)) = t_r(n-1),$$

where the last equality follows from Observation 2.1(iii). Therefore, by the inductive hypothesis it follows that $K \cong T_r(n-1)$. In particular, this implies that $e(K) = t_r(n-1)$ and therefore $d_H(v) = \delta(T_r(n))$, so it follows from Observation 2.1(iv) that $H \cong T_r(n)$.

Finally, since $V(H) = V(G)$ and $E(H) = E(G) \setminus E' \subseteq E(G)$, and since G is K_{r+1}-free, it follows from Observation 2.1(i) that $|E'| = 0$ and so $G \cong H \cong T_r(n)$, as required. □

2.2 Complete bipartite subgraphs

Let $t \geq 2$ and $n \geq 2t$. We now look at the following question: if G is a $K_{t,t}$-free graph of order n, how big $e(G)$ can be? Unlike the analogous question for K_{r+1}-free graphs (see Theorem 2.2), the exact answer to this question is not known, but we will find some bounds.

First, recall the following notation.

Notation. Let $f\colon \mathbb{N} \to (0, \infty)$ and $g\colon \mathbb{N} \to (0, \infty)$ be functions. We write:

- $f(n) = O(g(n))$ if $f(n) < C \cdot g(n)$ for some constant $C < \infty$ (for n large enough);
- $f(n) = \Omega(g(n))$ if $f(n) > c \cdot g(n)$ for some constant $c > 0$ (for n large enough);
- $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$;
- $f(n) = o(g(n))$ if $\frac{f(n)}{g(n)} \to 0$ as $n \to \infty$;
- $f(n) = \omega(g(n))$ if $\frac{f(n)}{g(n)} \to \infty$ as $n \to \infty$;
- $f(n) \sim g(n)$ if $\frac{f(n)}{g(n)} \to 1$ as $n \to \infty$.

We will also use the following well-known inequality from analysis.

Lemma 2.3 (Jensen's Inequality)**.** *Let $a < b$ be real numbers and $f\colon [a,b] \to \mathbb{R}$ a convex function. Then $\frac{1}{n}\sum_{i=1}^{n} f(x_i) \geq f(\frac{1}{n}\sum_{i=1}^{n} x_i)$ for all $x_1, \ldots, x_n \in [a,b]$.* □

A particular case of Jensen's Inequality that we will use is when $f = b_t$, where

$$b_t(x) = \begin{cases} \binom{x}{t} = \frac{1}{t!}x(x-1)\cdots(x-t+1) & \text{if } x \geq t-1, \\ 0 & \text{otherwise,} \end{cases}$$

for some $t \in \mathbb{N}$. It is easy to verify that b_t is convex on $\mathbb{R}$; moreover, we have $b_t(x) = \binom{x}{t}$ for any $x \in \mathbb{N}$.

We now give an upper bound for the number of edges that a graph of order n not containing $K_{2,2} \cong C_4$ as a subgraph can have.

Example 2.4. Let G be a C_4-free graph with $|G| = n \geq 1$, and let k be the number of 2-paths in G. We will estimate k in two different ways.

First, note that each vertex $v \in G$ is the middle vertex of exactly $\binom{d(v)}{2} = b_2(d(v))$ 2-paths in G. We therefore have

$$k = \sum_{v \in G} b_2(d(v)) \geq n \cdot b_2(\tfrac{1}{n}\textstyle\sum_{v\in G} d(v)) = n \cdot b_2(2e(G)/n) \geq n \cdot \dbinom{2e(G)/n}{2}, \tag{2.1}$$

where the first inequality and the second equality follow from Lemmas 2.3 and 1.2, respectively, and the last inequality follows because $b_2(x) = \max\{\binom{x}{2}, 0\}$ for any $x \geq 0$. On the other hand, as G is C_4-free, any pair of distinct vertices in G are the endpoints of at most one 2-path in G. This implies that

$$k \leq \binom{n}{2} = n\frac{n-1}{2}. \tag{2.2}$$

Combining (2.1) and (2.2) gives us $n - 1 \geq \frac{2e(G)}{n}\left(\frac{2e(G)}{n} - 1\right)$, or equivalently,

$$4 \cdot e(G)^2 - 2n \cdot e(G) - n^2(n-1) \leq 0.$$

The roots of the polynomial $4x^2 - 2nx - n^2(n-1)$ are $x_\pm = \frac{n}{4}\left(1 \pm \sqrt{4n-3}\right)$, implying that

$$e(G) \leq \frac{n}{4}\left(1 + \sqrt{4n-3}\right) < \frac{n}{4} \cdot 2\sqrt{4n} = n\sqrt{n}.$$

We may use essentially the same argument to bound the number of edges in a $K_{t,t}$-free graph for any $t \geq 2$.

Theorem 2.5. *For any $t \geq 2$, there exists a function $f = f_t\colon \mathbb{N} \to (0,\infty)$ with $f(n) = O(n^{2-\frac{1}{t}})$, such that if G is a $K_{t,t}$-free graph with $|G| = n$ then $e(G) \leq f(n)$.*

Proof. Let G be a $K_{t,t}$-free graph with $|G| = n \geq 1$ and $e(G) = m$. We call a subgraph $H \leq G$ a *t-fan* if $H \cong K_{1,t}$. Let k be the number of t-fans in G.

On the one hand, each vertex of G is the degree-t vertex of exactly $\binom{d(v)}{t} = b_t(d(v))$ t-fans in G, implying that

$$k = \sum_{v \in G} b_t(d(v)) \geq n \cdot b_t(\tfrac{1}{n} \textstyle\sum_{v \in G} d(v)) = n \cdot b_t(2m/n), \tag{2.3}$$

where the middle inequality and the last equality follow from Lemmas 2.3 and 1.2, respectively. On the other hand, as G is $K_{t,t}$-free, any collection of t distinct vertices in G are the degree-1 vertices of at most $(t-1)$ t-fans in G. This implies that

$$k \leq \binom{n}{t} \cdot (t-1) \leq \frac{n^t}{t!} \cdot t. \tag{2.4}$$

Now since $tn = O(n^{2-\frac{1}{t}})$, we may without loss of generality assume that $m \geq tn$ and therefore $\frac{2m}{n} \geq \frac{m}{n} + t \geq t$. Then (2.3) implies that

$$k \geq n \cdot \binom{2m/n}{t} \geq \frac{n \cdot \left(\frac{2m}{n} - t + 1\right)^t}{t!} > \frac{n}{t!} \left(\frac{m}{n}\right)^t = \frac{m^t}{n^{t-1} \cdot t!}.$$

Combining this with (2.4) gives $m^t \leq n^{2t-1} \cdot t$, that is, $m \leq \sqrt[t]{t} \cdot n^{2-\frac{1}{t}}$. Therefore, the function $f_t(n) = \max\{tn, \sqrt[t]{t} \cdot n^{2-\frac{1}{t}}\}$ satisfies the conclusion of the Theorem. □

Remark. Theorem 2.5 is similar to the *Zarankiewicz problem*, asking the following: given $n \geq t \geq 2$, what is the smallest number $z_t(n)$ such that any bipartite $K_{t,t}$-free graph G with n vertices in each class has $e(G) \leq z_t(n)$? The numbers $z_t(n)$ are called *Zarankiewicz numbers*, and Theorem 2.5 implies that $z_t(n) \leq f_t(2n) = O(n^{2-\frac{1}{t}})$ for a fixed t. We will come back to these numbers later in the course to give lower asymptotic bounds as well. In the literature, one may often see $z(n;t)$ instead of $z_t(n)$.

2.3 Arbitrary subgraphs

We now direct our attention to the general *forbidden subgraph problem*: given a graph H, how many edges can an H-free graph of order n have?

Notation. Let H be a graph with $e(H) \geq 1$, and let $n \geq 1$. We write

$$\mathrm{ex}(n; H) := \max\{e(G) \mid G \text{ an } H\text{-free graph with } |G| = n\}.$$

We would like to analyse the asymptotic behaviour of $\mathrm{ex}(n; H)$ as $n \to \infty$. So far, we have seen the following.

- Theorem 2.2 implies that $\mathrm{ex}(n; K_{r+1}) = t_r(n)$. It follows from Observation 2.1(ii) that $|d(T_r(n)) - n(1 - \frac{1}{r})| < 1$ and therefore $|t_r(n) - \frac{n^2}{2}(1 - \frac{1}{r})| < \frac{n}{2}$ by Lemma 1.2, implying that $\mathrm{ex}(n; K_{r+1}) \sim \frac{n^2}{2}(1 - \frac{1}{r})$ when $r \geq 2$.

- Theorem 2.5 shows that $\mathrm{ex}(n; K_{t,t}) = O(n^{2-\frac{1}{t}})$.

We first introduce an asymptotic invariant $\mathrm{ex}(H)$ of a graph, as follows.

Proposition 2.6. *Let H be a graph with $e(H) \geq 1$. For $n \geq 2$, let $x_n = \mathrm{ex}(n; H)/\binom{n}{2}$. Then the sequence $(x_n)_{n=2}^{\infty}$ converges.*

Notation. We write $\mathrm{ex}(H) := \lim_{n\to\infty} \mathrm{ex}(n; H)/\binom{n}{2}$.

Proof of Proposition 2.6. The sequence (x_n) is bounded below by zero, so it is enough to show that it is also non-increasing. Let $n \geq 3$, and let G be an H-free graph with $|G| = n$ and $e(G) = \mathrm{ex}(n; H)$. For any $v \in G$, the graph $G - \{v\}$ is H-free and has order $n - 1$, implying that $e(G - \{v\}) \leq \mathrm{ex}(n - 1; H)$. On the other hand, a given edge $uw \in E(G)$ appears in precisely $n - 2$ graphs $G - \{v\}$ for $v \in G$, namely those with $v \notin \{u, w\}$. This implies that $(n - 2)e(G) = \sum_{v\in G} e(G - \{v\})$, and therefore

$$x_n = \frac{\mathrm{ex}(n; H)}{\binom{n}{2}} = \frac{2e(G)}{n(n-1)} = \sum_{v\in G} \frac{2e(G - \{v\})}{n(n-1)(n-2)} \leq \frac{2\,\mathrm{ex}(n-1; H)}{(n-1)(n-2)} = x_{n-1},$$

implying that the sequence (x_n) is non-increasing, as required. □

The question now is, can we determine $\mathrm{ex}(H)$ for a given graph H? It will turn out that there is a way to do this. Some specific cases are as follows.

- The fact that $\mathrm{ex}(n; K_{r+1}) = t_r(n)$ implies that $\mathrm{ex}(K_{r+1}) = 1 - \frac{1}{r}$.
- We have $\mathrm{ex}(n; K_{t,t}) = o(n^2)$, implying that $\mathrm{ex}(K_{t,t}) = 0$.
- If H is *any* bipartite graph, we have $H \leq K_{t,t}$ for some t and therefore $\mathrm{ex}(H) = 0$.

The following definition will turn out to allow us to determine $\mathrm{ex}(H)$ exactly.

Definition (chromatic number)**.** The *chromatic number* of a graph H, denoted $\chi(H)$, is the smallest integer $r \geq 1$ such that H is r-partite.

For example, we have $\chi(K_r) = r$, $\chi(T_r(n)) = r$ for $n \geq r$, and if H is a bipartite graph with $e(H) \geq 1$ then $\chi(H) = 2$.

Remark. One may consider a colouring of vertices in a graph H with $r \geq 1$ colours such that every edge has endpoints of different colours. Such a colouring is possible if and only if H is r-partite. That explains the name "chromatic number" (from Ancient Greek *χρῶμα = colour*). We will return to this viewpoint later in the course.

The following is the main result of this section, which will allow us (among other things) to exactly determine $\mathrm{ex}(H)$ for a given graph H.

Theorem 2.7 (Erdős–Stone Theorem)**.** *Let k, r be integers with $k - 1 \geq r \geq 1$, and let $\varepsilon > 0$. Then there exists an integer N such that for all $n \geq N$, if G is a graph with $|G| = n$ and $e(G) \geq (1 - \frac{1}{r} + \varepsilon)\binom{n}{2}$, then $T_{r+1}(k) \leq G$.*

We postpone the proof of Theorem 2.7 for later. First, we state a couple of corollaries.

Corollary 2.8. *Let H be a graph with $e(H) \geq 1$. Then $\mathrm{ex}(H) = 1 - \frac{1}{\chi(H)-1}$.*

Proof. Let $r = \chi(H) - 1$, choose k such that $H \le T_{r+1}(k)$ (for instance, we could take $k = (r+1)|H|$), and let $\varepsilon > 0$. Let N be the integer appearing in Theorem 2.7. Then for any $n \ge N$ and any H-free graph G with $|G| = n$, we know that G is also $T_r(k)$-free and therefore $e(G) < (1 - \frac{1}{r} + \varepsilon)\binom{n}{2}$. This shows that $\mathrm{ex}(n; H) < (1 - \frac{1}{r} + \varepsilon)\binom{n}{2}$ for all $n \ge N$, and therefore $\mathrm{ex}(H) \le 1 - \frac{1}{r} + \varepsilon$. But $\varepsilon > 0$ was arbitrary, implying that $\mathrm{ex}(H) \le 1 - \frac{1}{r}$.

On the other hand, for any $n \ge r$ the graph $T_r(n)$ is H-free (as H is not r-partite), and we have $t_r(n) \sim (1 - \frac{1}{r})\binom{n}{2}$, implying that $\mathrm{ex}(H) \ge 1 - \frac{1}{r}$. □

Remark. If a graph H is not bipartite, then Corollary 2.8 implies that $\mathrm{ex}(H) > 0$, and so we can completely determine asymptotic behaviour of $\mathrm{ex}(n; H)$: in particular, $\mathrm{ex}(n; H) \sim (1 - \frac{1}{\chi(H)-1})\binom{n}{2}$. However, if H is bipartite then all Corollary 2.8 gives is that $\mathrm{ex}(n; H) = o(n^2)$. In Theorem 2.5, we have shown that $\mathrm{ex}(n; K_{t,t}) = O(n^{2-\frac{1}{t}})$. We have $\mathrm{ex}(n; K_{2,2}) = \Omega(n^{3/2})$ (see Problem 2.7) and thus $\mathrm{ex}(n; K_{2,2}) = \Theta(n^{3/2})$; it can also be shown that $\mathrm{ex}(n; K_{3,3}) = \Theta(n^{5/3})$. It is not known if $\mathrm{ex}(n; K_{4,4}) = \Theta(n^{7/4})$.

Our next application concerns the "density of large finite subgraphs" of an infinite graph, defined as follows.

Definition (upper density)**.** Let G be an infinite graph. The *upper density* of G is defined as

$$\mathrm{ud}(G) = \limsup_{n\to\infty} \max \left\{ \frac{e(H)}{\binom{n}{2}} \;\middle|\; H \le G, |H| = n \right\}.$$

In fact, it turns out that lim sup can be replaced by lim in this definition (see Problem 2.9).

It seems *a priori* that $\mathrm{ud}(G)$ could take any value in $[0, 1]$. However, we have the following consequence of Erdős–Stone Theorem.

Corollary 2.9. *Let G be an infinite graph. Then either* $\mathrm{ud}(G) = 1$ *or* $\mathrm{ud}(G) = 1 - \frac{1}{r}$ *for some integer* $r \ge 1$.

Proof. For $n \ge 2$, let $x_n = \max\{e(H)/\binom{n}{2} \mid H \le G, |H| = n\}$. It is enough to show that for each $r \ge 1$, if $\mathrm{ud}(G) > 1 - \frac{1}{r}$ then actually $\mathrm{ud}(G) \ge 1 - \frac{1}{r+1}$. So assume that $\mathrm{ud}(G) > 1 - \frac{1}{r}$, and pick $\varepsilon > 0$ such that $\mathrm{ud}(G) = \limsup_{n\to\infty} x_n > 1 - \frac{1}{r} + \varepsilon$. Then we can find a sequence $(H_\ell)_{\ell=1}^{\infty}$ of subgraphs of G such that $e(H_\ell) \ge (1 - \frac{1}{r} + \varepsilon)\binom{|H_\ell|}{2}$ for all ℓ and $|H_\ell| \to \infty$ as $\ell \to \infty$. It follows by Theorem 2.7 that $T_{r+1}(n) \le G$ for all $n \ge r+1$; consequently, we have $x_n \ge t_{r+1}(n)/\binom{n}{2}$ for all $n \ge r+1$. This implies that

$$\mathrm{ud}(G) = \limsup_{n\to\infty} x_n \ge \lim_{n\to\infty} \frac{t_{r+1}(n)}{\binom{n}{2}} = 1 - \frac{1}{r+1},$$

as required. □

2.4 Proof of the Erdős–Stone Theorem

Finally, we prove Erdős–Stone Theorem. It turns out to be more convenient to have a condition on $\delta(G)$ rather than $e(G)$. Therefore, we will first prove the following Lemma, and then deduce the full theorem from it.

Lemma 2.10. *Let k, r be integers with $k-1 \geq r \geq 1$, and let $\varepsilon > 0$. Then there exists an integer $N = N(\varepsilon)$ such that for all $n \geq N$, if G is a graph with $|G| = n$ and $\delta(G) \geq (1 - \frac{1}{r} + \varepsilon)n$, then $T_{r+1}(k) \leq G$.*

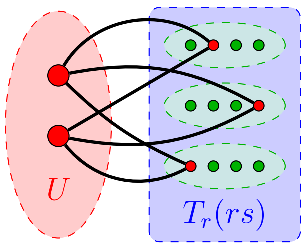
Proof. We prove this by induction on r.

Suppose first that $r = 1$. Then we have

$$e(G) = \frac{\sum_{v \in G} d(v)}{2} \geq \frac{n\delta(G)}{2} \geq \frac{n \cdot \varepsilon n}{2} = \frac{\varepsilon}{2} n^2 = \Omega(n^2).$$

On the other hand, we have $\mathrm{ex}(n; K_{t,t}) = O(n^{2-\frac{1}{t}}) = o(n^2)$ by Theorem 2.5, implying that for any $k \geq 2$ we have $T_2(k) \leq K_{\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil} \leq G$ when n is sufficiently large.

Suppose for contradiction that the result fails for some $r \geq 2$, $k \geq r+1$ and $\varepsilon > 0$. For simplicity, by replacing k with $\lceil \frac{k}{r+1} \rceil (r+1)$ if necessary, we may assume that $k = (r+1)t$ for some $t \in \mathbb{Z}$. We fix a large integer s—for instance, $s > \left(\frac{2}{r\varepsilon}\right)^t r(t-1)$—and let N be large enough so that if $|G| = n \geq N$ and $\delta(G) \geq (1 - \frac{1}{r} + \varepsilon)n$, then $T_r(rs) \leq G$ (such an N exists by the inductive hypothesis). Without loss of generality, suppose moreover that $N \geq \frac{2}{r\varepsilon} t$.



By our assumption on the failure of the result, there exists a $T_{r+1}((r+1)t)$-free graph G with $|G| = n \geq N$ and $\delta(G) \geq (1 - \frac{1}{r} + \varepsilon)n$. We fix a copy of $T_r(rs)$ in G with vertex classes $V_1, \ldots, V_r \subseteq V(G)$; note that $|V_i| = s$ for each i. Now let K be the number of tuples $(U, v_1, \ldots, v_r)$, where $v_i \in V_i$ for $1 \leq i \leq r$, and $U \subseteq \bigcap_{i=1}^r N_G(v_i)$ with $|U| = t$. We aim to give upper and lower bounds for K, which will give us an inequality leading to a contradiction.

For a lower bound, suppose we have already chosen $v_1, \ldots, v_r$; since $|V_i| = s$ for each i, there are s^r ways to do this. Now since $d(v_i) \geq \delta(G) \geq (1 - \frac{1}{r} + \varepsilon)n$, we have $|G - N(v_i)| = n - d(v_i) \leq (\frac{1}{r} - \varepsilon)n$ and therefore

$$\left| \bigcap_{i=1}^r N(v_i) \right| = n - \left| \bigcup_{i=1}^r [V(G) \setminus N(v_i)] \right| \geq n - \sum_{i=1}^r |G - N(v_i)| \geq n - r\left(\frac{1}{r} - \varepsilon\right) n = r\varepsilon n,$$

implying that $K \geq \binom{r\varepsilon n}{t} s^r$.

For an upper bound, suppose we have already chosen U: there are $\binom{n}{t}$ ways to do this. As G is $T_{r+1}((r+1)t)$-free, there exists some $i \in \{1, \ldots, r\}$ such that the vertices of U have $\leq t-1$ common neighbours in V_i, and therefore there are at most $t-1$ options for choosing v_i. This implies that $K \leq \binom{n}{t} r s^{r-1}(t-1)$. In particular, combining the upper and lower bounds on K yields

$$\binom{r\varepsilon n}{t} s^r \leq \binom{n}{t} r s^{r-1} (t-1). \tag{2.5}$$

We now estimate both sides of (2.5). For the left hand side, we have

$$\binom{r\varepsilon n}{t} s^r = \frac{r\varepsilon n(r\varepsilon n - 1)\cdots(r\varepsilon n - t + 1)}{t!} s^r \geq \frac{1}{t!}(r\varepsilon n - t + 1)^t s^r \geq \frac{1}{t!}\left(\frac{r\varepsilon n}{2}\right)^t s^r,$$

where the last inequality follows because $\frac{r\varepsilon n}{2} \geq \frac{r\varepsilon N}{2} \geq t$ by the choice of N. For the right hand side, we have

$$\binom{n}{t} rs^{r-1}(t-1) = \frac{n(n-1)\cdots(n-t+1)}{t!} s^{r-1} r(t-1) \leq \frac{1}{t!} n^t s^{r-1} r(t-1).$$

Thus, combining everything together we get $\frac{1}{t!}\left(\frac{r\varepsilon n}{2}\right)^t s^r \leq \frac{1}{t!} n^t s^{r-1} r(t-1)$, or equivalently, $s \leq \left(\frac{2}{r\varepsilon}\right)^t r(t-1)$. This contradicts our choice of s. □

It is now enough to deduce the Erdős–Stone Theorem from the Lemma 2.10.

Proof of Theorem 2.7. We first claim that if $n \geq N' := 8/\varepsilon$, then G contains a subgraph H with $\delta(H) \geq (1-\frac{1}{r}+\frac{\varepsilon}{2})|H|$ and $|H| \geq \left(\frac{\varepsilon}{2}\right)^{1/2} n$. Indeed, if this was not the case then we could construct a sequence of graphs $G = G_n \geq G_{n-1} \geq \cdots \geq G_\ell$, where $\ell = \left\lfloor \left(\frac{\varepsilon}{2}\right)^{1/2} n \right\rfloor$, by repeatedly removing a vertex of minimal degree: that is, so that for $\ell < j \leq n$ we have $|G_j| = j$ and $G_{j-1} = G_j - \{v_j\}$, where $v_j \in G_j$ satisfies $d_{G_j}(v_j) = \delta(G_j) < (1-\frac{1}{r}+\frac{\varepsilon}{2})j$. Note that we have $8\left[\left(\frac{1}{2}\right)^{1/2} - \left(\frac{1}{3}\right)^{1/2}\right] > 1 > \sqrt{\varepsilon}$, so since $n \geq \frac{8}{\varepsilon}$ we have $n\left[\left(\frac{\varepsilon}{2}\right)^{1/2} - \left(\frac{\varepsilon}{3}\right)^{1/2}\right] > 1$ and therefore $\ell > \left(\frac{\varepsilon}{2}\right)^{1/2} n - 1 > \left(\frac{\varepsilon}{3}\right)^{1/2} n$. We then have

$$\begin{aligned} e(G_\ell) &= e(G) - \sum_{j=\ell+1}^{n} \delta(G_j) > \left(c + \frac{\varepsilon}{2}\right)\binom{n}{2} - \sum_{j=\ell+1}^{n} cj \\ &= \left(c + \frac{\varepsilon}{2}\right)\binom{n}{2} - c\left[\binom{n+1}{2} - \binom{\ell+1}{2}\right] = \frac{\varepsilon}{2}\binom{n}{2} + c\binom{\ell+1}{2} - cn \\ &> \frac{\varepsilon}{2}\binom{n}{2} > \binom{\ell}{2} \end{aligned}$$

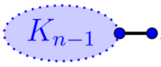
where $c = 1 - \frac{1}{r} + \frac{\varepsilon}{2}$: indeed, the last two inequalities follow since we have $\binom{\ell+1}{2} > \frac{1}{2}\ell^2 > \frac{1}{2}\left[\left(\frac{\varepsilon}{3}\right)^{1/2} n\right]^2 = \frac{\varepsilon}{6} n^2 > \frac{\varepsilon n}{8} n \geq n$ and $\frac{2}{\varepsilon}\binom{\ell}{2} = \frac{\ell^2}{\varepsilon} - \frac{\ell}{\varepsilon} < \frac{n^2}{2} - \frac{n}{\sqrt{3\varepsilon}} < \frac{n^2}{2} - \frac{n}{2} = \binom{n}{2}$. Since clearly $e(G_\ell) \leq e(K_\ell) = \binom{\ell}{2}$, we have a contradiction, proving our claim.

Clearly if G is $T_{r+1}(k)$-free then so is the graph H constructed above. Therefore, if we take $N = N(\varepsilon/2)$ as in Lemma 2.10, then the conclusion of the theorem holds for any $n \geq \max\left\{\left(\frac{2}{\varepsilon}\right)^{1/2} N, N'\right\}$, as required. □

2.5 Hamiltonian and Eulerian graphs

We now consider conditions which force a graph to contain a "large" cycle. This is slightly different to the forbidden subgraph problem we discussed before, as the length of the "forbidden cycle" will vary with the order of the graph.

Definition (Hamiltonian graphs)**.** A *Hamilton cycle* in a graph G is a cycle containing all vertices in G, that is, a cycle of length $|G|$. We say G is *Hamiltonian* if it contains a Hamilton cycle.

It might seem tempting to show, as we did earlier, that if G has "enough edges" then it must be Hamiltonian. However, this turns out not to work too well. Indeed, consider the graph G = 

, that is the graph $G = (V, E)$ with $V = [n]$ and $E = \{ij \mid 1 \leq i < j \leq n-1\} \cup \{(n-1)n\}$. Then $e(G) = \binom{n}{2} - (n-2)$, but G is not Hamiltonian (as it has no cycle containing the vertex n). So it is possible for a graph to contain almost all possible edges without containing a Hamilton cycle.

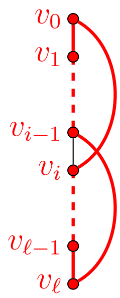
It turns out to be more interesting to impose bounds on the minimal degree. Indeed, we have the following result.

Theorem 2.11 (Dirac's Theorem)**.** *Let G be a graph with $|G| = n \geq 3$ and $\delta(G) \geq \frac{n}{2}$. Then G is Hamiltonian.*

Proof. First observe that G is connected (and in fact, any two vertices of G are endpoints of a path of length ≤ 2). Indeed, if $u, v \in G$ are such that $u \neq v$ and $u \nsim v$, then $N(u) \cup N(v) \subseteq V(G) \setminus \{u, v\}$, and therefore

$$|N(u) \cup N(v)| \leq n - 2 < n \leq 2\delta(G) \leq |N(u)| + |N(v)|,$$

implying that $N(u) \cap N(v) \neq \varnothing$, and so u and v are endpoints of a path of length 2.



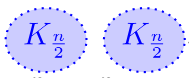
Now let $v_0 v_1 \cdots v_\ell$ be a path of maximal length in G. By maximality of this length, we have $N(v_0) \subseteq \{v_1, \ldots, v_\ell\}$ and $N(v_\ell) \subseteq \{v_0, \ldots, v_{\ell-1}\}$. Now let $A = \{i \in [\ell] \mid v_0 \sim v_i\}$ and $B = \{i \in [\ell] \mid v_\ell \sim v_{i-1}\}$. We then have

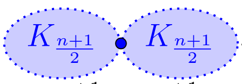
$$|A| + |B| = d(v_0) + d(v_\ell) \geq 2\delta(G) \geq n > \ell \geq |A \cup B|,$$

implying that $A \cap B \neq \varnothing$. Now fix any $i \in A \cap B$. Then $C := v_0 v_1 \cdots v_{i-1} v_\ell v_{\ell-1} \cdots v_i v_0$ is a cycle of length $\ell + 1$ in G.

If $\ell + 1 = n$, then C is a Hamilton cycle and we're done. Otherwise, as G is connected, there exists a vertex $w \in G - V(C)$ such that $w \sim u$ for some $u \in C$. We can relabel the cycle C so that $C = u_0 u_1 \cdots u_\ell u_0$, where $u_0 = u$. Then $w u_0 u_1 \cdots u_\ell$ is a path in G of length $\ell + 1$, contradicting the maximality of the length of $v_0 v_1 \cdots v_\ell$. □

Remark. The bound $\delta(G) \geq \frac{n}{2}$ in Dirac's Theorem is the best possible. Indeed, consider the following graphs:

- if n is even, take G = 

, that is, the graph G with $V(G) = [n]$ and $E(G) = \{ij \mid 1 \leq i < j \leq \frac{n}{2} \text{ or } \frac{n}{2} + 1 \leq i < j \leq n\}$;
- if n is odd, take G = 

, that is, the graph G with $V(G) = [n]$ and $E(G) = \{ij \mid 1 \leq i < j \leq \frac{n+1}{2} \text{ or } \frac{n+1}{2} \leq i < j \leq n\}$.

In either case, we have $\delta(G) = \lceil \frac{n}{2} \rceil - 1$ but G is not Hamiltonian.

However, an inspection of the proof tells us that we may nevertheless slightly weaken the assumption that $\delta(G) \geq \frac{n}{2}$, by replacing it with the assumption that $d_G(u) + d_G(v) \geq n$ for any distinct non-adjacent vertices $u, v \in G$. This stronger statement is known as *Ore's Theorem*.

A Hamilton cycle can be viewed as a "closed walk" on a graph visiting each vertex exactly once. A superficially similar problem asks what happens if we replace "vertex" by "edge". We thus introduce the following terminology.

Definition (walks, trails, Eulerian graphs)**.** Let G be a graph.

- A *walk* in G (of *length* m) is a sequence $v_0v_1\cdots v_m$ of vertices of G such that $v_{i-1}\sim_G v_i$ for $1\leq i\leq m$. Such a walk is called *closed* if additionally $v_0=v_m$.
- A walk $v_0\cdots v_m$ in G is a *trail* if $v_{i-1}v_i\neq v_{j-1}v_j$ (as edges of G) whenever $i\neq j$.
- An *Euler trail* is a trail in G of length $e(G)$.
- We say G is *Eulerian* if it has a closed Euler trail.

It turns out that we can characterise (connected) Eulerian graphs exactly.

Proposition 2.12. *A connected graph G is Eulerian if and only if every vertex of G has even degree.*

Proof.

($\Rightarrow$) Every time a closed Euler trail passes a vertex $v\in G$, it contributes exactly 2 to $d(v)$, implying that $d(v)$ is even. More precisely, if $v_0v_1\cdots v_m$ is a closed Euler trail then we have $N(v)=\bigsqcup_{i\in[m],v=v_i}\{v_{i-1},v_{i+1}\}$ (taking indices modulo m), implying that $d(v)=2|\{i\in[m]\mid v=v_i\}|$.

($\Leftarrow$) We use induction on $e(G)$. The base case, $e(G)=0$, is trivial. Thus, suppose that $e(G)\geq 1$.

Let $v_0v_1\cdots v_k$ be a path of maximal length in G. Then $N(v_0)\subseteq\{v_1,\ldots,v_k\}$; on the other hand, since $d(v_0)$ is even and $v_0\sim v_1$ it follows that $d(v_0)\geq 2$ and therefore $v_0\sim v_i$ for some $i>1$. Then $v_0v_1\cdots v_iv_0$ is a cycle in G, and therefore a closed trail of length >0.

Let $C=v_0v_1\cdots v_m$, where $v_m=v_0$, be a closed trail in G of maximal length, and let $E'=\{v_{i-1}v_i\mid i\in[m]\}\subseteq E(G)$. If $E'=E(G)$, then C is a closed Euler trail and we are done. Otherwise, there exists a connected component H of $G-E'$ with $e(H)>0$. Note that every vertex in H has an even degree: indeed, we have $d_H(v)=d_G(v)-2|\{i\in[m]\mid v=v_i\}|$ for any $v\in H$. Therefore, by the inductive hypothesis, there exists a closed Euler trail $w_0w_1\cdots w_{e(H)}$ (where $w_{e(H)}=w_0$) in H. Since G is connected, the two trails must share some vertex, that is, $v_i=w_j$ for some i and j. Then $v_i\cdots v_mv_1\cdots v_iw_{j+1}\cdots w_{e(H)}w_1\cdots w_j$ is a closed trail in G of length $m+e(H)>m$, contradicting the maximality of the length of C. □

Eulerian graphs are the method used to solve the *Seven Bridges of Königsberg* problem (see Example 0.1). Indeed, we may consider the graph in which each of the islands (as well as the two banks of the river) is represented by a vertex, and for each bridge we draw an edge between the corresponding vertices. The problem then has a positive solution if and only if the resulting "graph" has an Euler trail—and we can check that it does not.

Remark. We may run into a small problem using this method, as the resulting "graph" may not actually be a graph (we actually do run into this problem in Königsberg): for instance, this happens if we have more than one bridge between the same pair of islands. However, this is easily avoidable by *subdivision*: instead of an edge for each bridge, we construct a bipartite graph with vertex classes I and B (identified with the sets of islands and bridges, respectively), where we have an edge vw for $v \in I$ and $w \in B$ if and only if the bridge w connects the island v to another island. This does not affect the existence or non-existence of (closed) Euler trails.

3

Ramsey theory

In this chapter, we consider colourings of edges in a complete graphs K_n. The main idea is that if n is big enough, then under such a colouring K_n will contain a subgraph K_r (for some fixed r) that is "monochromatic"—that is, all of its edges are coloured with the same colour.

3.1 Ramsey's Theorem

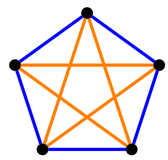
Here we deal with edge colourings of graphs, defined as follows.

Definition (edge colourings, monochromatic subgraphs)**.** Let G be a graph and $k \geq 2$.

- A *k-edge colouring* of G is a map $c\colon E(G) \to [k]$.
- Given a k-edge colouring c of G, a subgraph $H \leq G$ is said to be *monochromatic* if $c|_{E(H)}$ is constant.
- When k is small, we will often identify $[k]$ with actual colours, e.g. if $k = 2$ we can set blue := 1 and orange := 2, refer to a 2-edge colouring as a *blue/orange edge colouring*, and refer to a monochromatic subgraph $H \leq G$ with $c|_{E(H)} = 1$ as a *blue subgraph*.

In this section, we will be dealing with edge colourings of complete graphs. In particular, we will show that for any $k, r \geq 2$, there exists some $n \geq r$ such that for any k-edge colouring of K_n we can find a monochromatic K_r. For a warm-up, let's analyse a couple of examples.

Example 3.1.



(i) There exists a blue/orange edge colouring of K_5 without any monochromatic triangle (i.e. K_3), namely, the one on the right.

(ii) Suppose we are given a blue/orange edge colouring of $G = K_6$, and pick any $v \in G$. Then (at least) 3 of the edges incident to v—vw_1, vw_2 and vw_3, say—are coloured with the same colour: without loss of generality, they are blue. Consider the edges w_1w_2, w_1w_3 and w_2w_3. If at least one of these edges, say w_iw_j, is blue, then $G[\{v, w_i, w_j\}]$ is a blue triangle. Otherwise, all 3 of these edges are orange, and therefore $G[\{w_1, w_2, w_3\}]$ is an orange triangle. Therefore, G must have a monochromatic triangle.

(iii) Suppose we are given a blue/orange edge colouring of $G = K_n$, and we are looking for either a blue triangle, or an orange K_4. Take $n = 10$, and pick any $v \in G$. Then there are 9 edges incident to v, so either ≥ 4 of them are blue, or ≥ 6 of them are orange.

- Suppose the edges $vw_1, \ldots, vw_4$ are all blue, and consider the six edges $\{w_iw_j \mid 1 \leq i < j \leq 4\}$. If at least one of these edges, say w_iw_j, is blue, then $G[\{v, w_i, w_j\}]$ is a blue triangle. Otherwise, all 6 of these edges are orange, and so $G[\{w_1, w_2, w_3, w_4\}]$ is an orange K_4.
- Suppose the edges $vw_1, \ldots, vw_6$ are all orange, and consider the restriction of our edge colouring to the subgraph $G[\{w_1, \ldots, w_6\}] \cong K_6$. As shown above, this subgraph must have a monochromatic triangle: say $H = G[\{w_i, w_j, w_k\}]$ is monochromatic. If H is blue, then H is also a blue triangle in G, and if H is orange, then $G[\{v, w_i, w_j, w_k\}]$ is an orange K_4 in G.

Therefore, in either case G will contain either blue triangle or an orange K_4, as required.

We can generalise the ideas appearing in Example 3.1 to show the existence of the following integers.

Definition (Ramsey numbers)**.** Let $s, t \geq 2$. The *Ramsey number* $R(s,t)$ is the smallest integer $n \geq \max\{s,t\}$ such that every blue/orange edge colouring of K_n contains either a blue K_s or an orange K_t (if such an n exists).

Theorem 3.2 (Ramsey's Theorem)**.** *Let $s, t \geq 2$. Then $R(s,t)$ exists. Moreover, if $s, t > 2$ then we have $R(s,t) \leq R(s-1,t) + R(s,t-1)$.*

Proof. We prove the result by induction on $s + t$. Our base case is when $s = 2$ or $t = 2$. But if $s = 2$, then for every blue/orange edge colouring of K_t either there exists a blue edge (and therefore a blue K_2), or all edges are orange (and so the whole graph is an orange K_t). This shows that $R(2,t) = t$, and in particular that $R(2,t)$ exists. A similar argument shows that $R(s,2) = s$.

Now suppose that $s, t > 2$. By the inductive hypothesis, the numbers $R(s-1,t)$ and $R(s,t-1)$ exist. Let $n = R(s-1,t) + R(s,t-1)$, and consider a blue/orange edge colouring of $G = K_n$. It is then enough to show that G will contain either a blue K_s or an orange K_t.

Let $v \in G$. Then there are $n-1$ edges of G incident to v, so either $\geq R(s-1,t)$ of them are blue, or $\geq R(s,t-1)$ of them are orange. Without loss of generality, suppose the former is true (the argument in the other case is similar). Thus, there exists a subset $W \subseteq V(G) \setminus \{v\}$ with $|W| = R(s-1,t)$ such that all the edges $\{vw \mid w \in W\}$ are blue. Consider the restriction of our edge colouring to the subgraph $G[W]$. By the definition of $R(s-1,t)$, we know that $G[W]$ must contain either a blue $H_1 \cong K_{s-1}$ or an orange $H_2 \cong K_t$. In the former case, $V(H_1) \cup \{v\}$ induces a blue K_s in G, and in the latter case, H_2 is already an orange K_t in G. This completes the proof. □

As the number $R(s,t)$ always exists, we may ask about the value of this number.

Remark. It is clear that $R(2, s) = R(s, 2) = s$ for any $s \geq 2$. In parts (i) and (ii) of Example 3.1, we have also shown that $R(3, 3) = 6$. What about other values? Since obviously $R(s, t) = R(t, s)$ for all $s, t \geq 2$, we may restrict to the case when $2 \leq s \leq t$.

By Example 3.1(iii), we have $R(3, 4) \leq 10$. In fact, $R(3, 4) = 9$ (see Problem 3.1). By Theorem 3.2, we then have $R(4, 4) \leq R(3, 4) + R(4, 3) = 2R(3, 4) = 18$. On the other hand, we also have $R(4, 4) > 17$ (see Problem 3.2), and so $R(4, 4) = 18$.

However, in general surprisingly few exact values are known. In particular, apart from the trivial values $R(2, t) = t$, the only other Ramsey numbers $R(s, t)$ that are known exactly (for $s \leq t$) are $R(3, t)$ for $3 \leq t \leq 9$, $R(4, 4)$ and $R(4, 5)$.

3.2 Variations of Ramsey's Theorem

We now consider k-edge colourings for any $k \geq 2$. We have the following generalisation of the previous definition of Ramsey numbers.

Definition (Ramsey numbers, continued)**.** Let $k, s_1, \ldots, s_k \geq 2$. The *Ramsey number* $R(s_1, \ldots, s_k)$ is the smallest integer $n \geq \max\{s_i \mid i \in [k]\}$, if such an n exists, such that for every k-edge colouring of K_n there exists some $i \in [k]$ and some monochromatic subgraph K_{s_i} of colour i.

By imitating the proof of Theorem 3.2, we can show that, for instance, $R(s, t, u)$ always exists and is not greater than $R(s-1, t, u) + R(s, t-1, u) + R(s, t, u-1)$. However, a slightly easier way to show existence of Ramsey numbers is by induction on k.

Theorem 3.3 (Multicolour Ramsey's Theorem)**.** *Let $k, s_1, \ldots, s_k \geq 2$. Then $R(s_1, \ldots, s_k)$ exists. Moreover, if $k > 2$ then we have $R(s_1, \ldots, s_k) \leq R(s_1, \ldots, s_{k-2}, R(s_{k-1}, s_k))$.*

Proof. We prove this by induction on k. The base case, $k = 2$, is covered by Theorem 3.2.

Now suppose that $k > 2$. By the inductive hypothesis, we know that the number $n := R(s_1, \ldots, s_{k-2}, R(s_{k-1}, s_k))$ exists. Consider a k-edge colouring of $G = K_n$, where the colours $k-1$ and k are light orange and dark orange, respectively, and colours $1, \ldots, k-2$ are not shades of orange. By the choice of n, we know that G contains either a monochromatic K_{s_i} of colour i for some $i \in [k-2]$, or an orange $K_{R(s_{k-1}, s_k)}$—and in the latter case, by the definition of $R(s_{k-1}, s_k)$, G must contain either a light orange $K_{s_{k-1}}$ or a dark orange K_{s_k}. Thus in either case G must contain a monochromatic K_{s_i} of colour i for some $i \in [k]$, as required. □

Now recall Example 0.4, where we've shown that the equation $x^n + y^n = z^n$ has non-trivial solutions modulo a prime number p as long as for every partition of $[p-1]$ into n parts, one of the parts must contain some x, y and z with $x + y = z$. We can now prove this result when the prime p is large enough.

Corollary 3.4 (Schur's Theorem)**.** *Let $n \geq 2$. For all sufficiently large $k \in \mathbb{N}$, if $[k]$ is disjointly partitioned into n parts, then one of these parts must contain some x, y and z with $x + y = z$.*

Proof. We claim that the result holds for all $k \geq R(\overbrace{3, \ldots, 3}^{n}) - 1$; this number is well-defined by Theorem 3.3.

Consider a partition $[k] = A_1 \sqcup \cdots \sqcup A_n$. Let $c\colon E(K_{k+1}) \to [n]$ be an n-edge colouring defined so that $j - i \in A_{c(ij)}$ whenever $1 \leq i < j \leq k+1$. By the choice of k, we must have a monochromatic K_3: that is, the edges ij, $i\ell$ and $j\ell$ are all of the same colour $c' \in [n]$ for some $i, j, \ell \in [k+1]$ with $i < j < \ell$. This means that $x := j - i$, $y := \ell - j$ and $z := \ell - i$ are all in $A_{c'}$, and we have $x + y = z$, as required. □

We now consider edge colourings of the infinite complete graph, defined as follows.

Definition (infinite complete graph)**.** The *infinite complete graph* is an infinite graph K_∞ with vertices $V(K_\infty) = \mathbb{N}$ and edges $E(K_\infty) = \{ij \mid i, j \in \mathbb{N}, i < j\}$.

Since $K_n \leq K_\infty$ for all n, it follows by Theorem 3.3 that for every k-edge colouring of K_∞ we can find a monochromatic K_s for any $s \geq 2$. However, this does not directly imply that K_∞ will contain a monochromatic copy of K_∞. Indeed, consider the blue/orange edge colouring of K_∞ where an edge ij is blue if $m^2 \leq i < j < (m+1)^2$ for some $m \in \mathbb{N}$, and orange otherwise. Then we can find a blue K_s for any integer $s \geq 2$, but there are no blue K_∞'s.

Nevertheless, we have the following result.

Theorem 3.5 (Infinite Ramsey's Theorem)**.** *Let $k \geq 2$ be an integer. Then for any k-edge colouring of K_∞ there exists a monochromatic subgraph isomorphic to K_∞.*

Proof. We choose sequences $v_1, v_2, \ldots \in \mathbb{N}$ and $c_1, c_2, \ldots \in [k]$ of integers, and a sequence $A_0, A_1, A_2, \ldots \subseteq \mathbb{N}$ of infinite subsets, inductively. Let $A_0 = \mathbb{N}$ and, having chosen $A_{n-1} \subseteq \mathbb{N}$ with $|A_{n-1}| = \infty$, we choose v_n, A_n and c_n as follows:

- Let $v_n \in A_{n-1}$ be arbitrary.
- Consider the edges $\{v_n w \mid w \in A_{n-1}, w \neq v_n\}$. These are infinitely many edges coloured by $k < \infty$ colours, and so there exists some infinite subset $A_n \subseteq A_{n-1} \setminus \{v_n\}$ such that all edges $\{v_n w \mid w \in A_n\}$ are coloured by the same colour.
- Let c_n be the colour of any edge $v_n w$ for $w \in A_n$.

Then, by construction, we have $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$, implying that whenever $m < n$ we have $v_n \in A_m$, and therefore $v_m v_n$ has colour c_m by the choice of A_m.

Now since all the colours c_m are in $[k]$, there exists a colour $c' \in [k]$ such that the set $B := \{m \in \mathbb{N} \mid c_m = c'\}$ is infinite. The subgraph of K_∞ induced by $\{v_m \mid m \in B\}$ is then monochromatic of colour c', as required. □

4

Random graphs

In this chapter, we study "random" graphs—these will usually be graphs on a fixed set of vertices, where each pair of vertices are joined by an edge independently at random with some probability p. This method will allow us to obtain some quantitative results that are far beyond what the currently known constructive proofs can tell us.

4.1 Ramsey and Zarankiewicz numbers

We start by recalling some basic notions from probability theory. In this course, we will only consider random variables that take values in some finite subset $S \subset \mathbb{R}$, usually $S = \{0, 1, \ldots, m\}$ for some $m \in \mathbb{N}$.

Definition (expectation, variance)**.** Let X be an S-valued random variable for some finite subset $S \subset \mathbb{R}$.

- The *expectation* of X is $\mathbb{E}X = \sum_{n \in S} n \cdot \mathbb{P}(X = n)$.
- The *variance* of X is $\mathrm{Var}(X) = \mathbb{E}[(X - \mu)^2]$, where $\mu = \mathbb{E}X$.

We also employ the following notation.

Notation. Given an event A, we write $\mathbb{1}(A)$ for the random variable taking value 1 if A takes place and 0 otherwise, so that $\mathbb{P}(A) = \mathbb{E}(\mathbb{1}(A))$.

We now collect some basic results from probability theory that we will use.

Lemma 4.1. *Let X and Y be random variables, let $\mu = \mathbb{E}X$, and let $\lambda > 0$.*

(i) *The expectation is linear: that is,* $\mathbb{E}(X + Y) = \mathbb{E}X + \mathbb{E}Y$ *and* $\mathbb{E}(\pm\lambda X) = \pm\lambda \cdot \mathbb{E}X$.

(ii) *We have* $\mathbb{P}(X \geq \mu) \neq 0$ *and* $\mathbb{P}(X \leq \mu) \neq 0$.

(iii) *We have* $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mu^2$.

(iv) Markov's inequality: *if X takes values in* $[0, \infty)$ *then* $\mathbb{P}(X \geq \lambda) \leq \frac{\mu}{\lambda}$.

(v) Chebyshev's inequality: $\mathbb{P}(|X - \mu| \geq \lambda) \leq \frac{\mathrm{Var}(X)}{\lambda^2}$.

Proof (sketch). Parts (i) and (ii) are easy to verify directly. For part (iii), note that $\mathrm{Var}(X) = \mathbb{E}(X^2 - 2\mu X + \mu^2) = \mathbb{E}(X^2) - 2\mu \cdot \mathbb{E}X + \mu^2 = \mathbb{E}(X^2) - \mu^2$. Part (iv) follows by taking expectations of both sides in the inequality $\lambda \cdot \mathbb{1}(X \geq \lambda) \leq X$. Now if $Z = (X - \mu)^2$, then $\mathrm{Var}(X) = \mathbb{E}Z$ and $\mathbb{P}(|X - \mu| \geq \lambda) = \mathbb{P}(Z \geq \lambda^2)$, so part (v) follows from part (iv). □

We now consider asymptotic bounds on the Ramsey numbers $R(s,s)$. For the upper bounds, we have shown that $R(s,s) = O(4^s)$ (see Problem 3.3). For the lower bounds, we have shown that $R(s,s) = \Omega(s^2)$ (see Problem 3.6), and there is a (much harder) construction showing that $R(s,s) = \Omega(s^k)$ for any $k \geq 2$. This does not even imply that $R(s,s) = \Omega((1+\varepsilon)^s)$; but in fact we have the following result.

Theorem 4.2. $R(s,s) = \Omega\left(\left(\sqrt{2}\right)^s\right)$.

Proof. Fix some $n \geq s \geq 2$, and colour each edge of $G = K_n$ blue or orange independently at random, with each colour equaly likely. Let X be the number of monochromatic subgraphs in G isomorphic to K_s. For each subset $W \subseteq V(G)$ with $|W| = s$, the probability that $G[W]$ is blue (or orange) is $(1/2)^{\binom{s}{2}}$, and therefore the probability that $G[W]$ is monochromatic is $2 \cdot (1/2)^{\binom{s}{2}} = 2^{1-\binom{s}{2}}$. Therefore, as there are $\binom{n}{s}$ subsets of $V(G)$ of cardinality s, it follows that $\mathbb{E}X = \binom{n}{s} 2^{1-\binom{s}{2}}$. Hence,

$$\mathbb{E}X \leq \frac{n^s}{s!} 2^{1-\binom{s}{2}} \leq n^s 2^{-\binom{s}{2}} = n^s \left(\sqrt{2}\right)^{-s(s-1)} = \left(\frac{n}{\left(\sqrt{2}\right)^{s-1}}\right)^s.$$

Now suppose that $n < \left(\sqrt{2}\right)^{s-1}$. Then $\mathbb{E}X < 1$, and therefore, by the Markov's inequality, we have $\mathbb{P}(X \geq 1) \leq \mathbb{E}X < 1$. This implies that $\mathbb{P}(X = 0) > 0$, and so there exists some blue/orange colouring of K_n with no monochromatic K_s. Thus we must have $R(s,s) \geq \left(\sqrt{2}\right)^{s-1} = \Omega\left(\left(\sqrt{2}\right)^s\right)$, as required. □

The "exponential growth factors" given by Problem 3.3 and Theorem 4.2 are the best known—that is, it is not known if $R(s,s) = O((4-\varepsilon)^s)$ or if $R(s,s) = \Omega\left((\sqrt{2}+\varepsilon)^s\right)$ for some $\varepsilon > 0$.

Now recall that given $n \geq t \geq 2$, the *Zarankiewicz number* $z_t(n)$ is the smallest integer m such that any bipartite $K_{t,t}$-free graph with n vertices in each class has at most m edges. By Theorem 2.5, we have $z_t(n) = O(n^{2-\frac{1}{t}})$. What about lower bounds?

The idea is as follows: consider a random bipartite graph G with n vertices in each class, with each pair of vertices in different classes joined by an edge independently at random with (fixed) probability p. Let M and X be the numbers of edges and subgraphs isomorphic to $K_{t,t}$ in G, respectively. If $\mu := \mathbb{E}(M - X) > 0$, then there exists such a graph G in which $M - X \geq \mu$, and we may remove X edges from G (at least one from each $K_{t,t}$) to obtain a $K_{t,t}$-free graph with $\geq \mu$ edges; this technique is known as *modifying a random graph*. By choosing a suitable value of p, we can then ensure that μ is "large enough".

Theorem 4.3. *For any* $t \geq 2$, $z_t(n) = \Omega(n^{2-\frac{2}{t+1}})$.

Proof. Let $p \in (0,1)$ (exact value to be set later) and let $n \geq t$. Let G be a random bipartite graph with n vertices in each class, with each of the n^2 possible edges present in G independently at random with probability p. Let X be the number of subgraphs of G isomorphic to $K_{t,t}$, and let $M = e(G)$. We then have $\mathbb{E}M = n^2 p$ by construction. On the other hand, for each subset of $2t$ vertices in G, with t vertices in each class, the probability that those vertices induce a subgraph isomorphic to $K_{t,t}$ is equal to p^{t^2}, implying that

$$\mathbb{E}X = \binom{n}{t}^2 p^{t^2} \le n^{2t} p^{t^2}.$$

Therefore, $\mathbb{E}(M - X) \ge n^2 p - n^{2t} p^{t^2}$.

Now suppose that $p = \frac{n^{-\frac{2}{1+t}}}{2}$. We then have

$$\mathbb{E}(M-X) \ge n^2 \cdot \frac{n^{-\frac{2}{1+t}}}{2} - n^{2t} \cdot \frac{n^{-\frac{2t^2}{1+t}}}{2^{t^2}} = \frac{1}{2} n^{2-\frac{2}{1+t}} - \frac{1}{2^{t^2}} n^{\frac{2t}{1+t}} = \left(\frac{1}{2} - \frac{1}{2^{t^2}}\right) n^{2-\frac{2}{1+t}} \ge \frac{n^{2-\frac{2}{1+t}}}{4}.$$

In particular, there exists a bipartite graph G with n vertices in each class and X subgraphs isomorphic to $K_{t,t}$ such that $e(G) - X \ge \frac{n^{2-\frac{2}{1+t}}}{4}$.

Let E' be a collection of edges in G obtained by picking one edge in each subgraph of G isomorphic to $K_{t,t}$, and let $G' = G - E'$. Then $|E'| \le X$, and hence G' is a $K_{t,t}$-free bipartite graph with n vertices in each class such that $e(G') \ge \frac{n^{2-\frac{2}{1+t}}}{4}$. Therefore, $z_t(n) \ge \frac{n^{2-\frac{2}{1+t}}}{4} = \Omega(n^{2-\frac{2}{1+t}})$, as required. □

4.2 Chromatic numbers: some constructive bounds

Recall that the *chromatic number* $\chi(G)$ of a graph G is the smallest $r \ge 0$ such that G is r-partite. We start with a couple of observations allowing us to estimate chromatic numbers. In order to give lower bounds on $\chi(G)$, we introduce a couple of invariants of graphs.

Definition (clique number, independence number)**.** Let G be a graph.

- The *clique number* of G is $\omega(G) := \max\{r \ge 1 \mid K_r \le G\}$. For completeness, we set $\omega(G) = 0$ if G has no vertices.
- A subset $W \subseteq V(G)$ is called *independent* if $G[W]$ has no edges. The *independence number* of G is $\alpha(G) := \max\{|W| \mid W \subseteq V(G) \text{ independent}\}$.

We now have the following bounds on $\chi(G)$ for an arbitrary graph G.

- If $K_r \le G$ then clearly $\chi(G) \ge r$, implying that $\chi(G) \ge \omega(G)$.

  However, for each $k \ge 3$ it is possible to construct a triangle-free graph G (that is, a graph G with $\omega(G) = 2$) such that $\chi(G) = k$; see Problem 4.4.

- If G is r-partite then each vertex class is independent and so contains $\le \alpha(G)$ vertices, implying that $\chi(G) \ge \frac{|G|}{\alpha(G)}$.

  However, for any $n \ge 2$, if G is the graph constructed by adding $n^2 - n - 1$ new vertices (and no edges) to K_n, then we have $\frac{|G|}{\alpha(G)} = 1 + \frac{1}{n}$, but $\chi(G) = n$.

- For an upper bound, we can show that $\chi(G) \leq \Delta(G)+1$ using a *greedy algorithm*. In particular, let $|G| = n$ and $V(G) = \{v_1, \dots, v_n\}$. Define an admissible $(\Delta(G)+1)$-colouring $c\colon V(G) \to [\Delta(G)+1]$ inductively, as follows. For each $k \geq 1$, having set $c(v_i)$ for $1 \leq i < k$, the set $\{c(v_i) \mid 1 \leq i < k, v_i \sim v_k\}$ has at most $d(v_k) \leq \Delta(G)$ elements, so some $c_k \in [\Delta(G)+1]$ must be absent in this set; we then set $c(v_k) = c_k$.

  However, for any $n \geq 1$ we have $\Delta(K_{1,n}) = n$ but $\chi(K_{1,n}) = 2$.

4.3 Girth *vs* chromatic number

Recall that for each $r \geq 2$ we have constructed a K_r-free graph G with chromatic number $\chi(G) = r$ (see Problem 2.12), and even, using a more involved construction, a triangle-free graph G with $\chi(G) = r$ (see Problem 4.4). But could we have similar results for graphs that are C_4-free, C_5-free, C_6-free, etc?

Definition. Let G be a graph. The *girth* of G is the largest integer k such that G has no subgraphs isomorphic to C_ℓ for $3 \leq \ell < k$. (For completeness, we say that G has infinite girth if G has no cycles.)

We will show, using the probabilistic method, that there exist graphs with arbitrarily large girth while simultaneously having an arbitrarily high chromatic number. Our strategy is as follows: for a suitably chosen random graph G, we may check that G "probably has few short cycles and no large independent sets of vertices" (recall that $\chi(G) \geq \frac{|G|}{\alpha(G)}$ for any graph G, where $\alpha(G)$ is the independence number of G). We may then form a graph by deleting one vertex from each "short" cycle in G.

Theorem 4.4. *Let $k, r \geq 2$. Then there exists a graph with girth $> k$ and chromatic number $\geq r$.*

Proof. Let $n = 2rs$ for some (sufficiently large) integer $s \geq 2$, and let $p = p_n \in (0,1)$ (exact value to be set later). Let G be a random graph with n vertices, with each of the $\binom{n}{2}$ possible edges present in G independently at random with probability p. Let X be the number of cycles in G of length $\leq k$. Note for any $i \geq 3$ and any i-tuple $(v_1, \dots, v_i)$ of vertices of G (there are $\frac{n!}{(n-i)!}$ such tuples), the probability that $v_1 \cdots v_i v_1$ is a cycle in G is equal to p^i, and we have counted each i-cycle $2i$ times this way (taking into account different starting vertices and orientations). Therefore, we have

$$\mathbb{E}X = \sum_{i=3}^{k} \frac{1}{2i} \cdot \frac{n!}{(n-i)!} \cdot p^i \leq \sum_{i=3}^{k} \frac{1}{4} n^i p^i \leq \frac{1}{4} k (np)^k,$$

provided that $p = p_n$ is chosen so that $np \geq 1$. By Markov's inequality (Lemma 4.1(iv)), we then have

$$\mathbb{P}(X \geq n/2) \leq \frac{\mathbb{E}X}{n/2} \leq \frac{1}{2} k n^{k-1} p^k \leq \frac{1}{2}$$

provided that $p = p_n$ is chosen so that $kn^{k-1}p^k \leq 1$. A choice of $p = p_n$ satisfying both conditions is possible when n is large enough: indeed, if $p = k^{-\frac{1}{k}} n^{\frac{1}{k}-1}$, then we have $kn^{k-1}p^k = 1$, whereas $np = \left(\frac{n}{k}\right)^{\frac{1}{k}} \geq 1$ as long as $n \geq k$. Therefore, after setting $p_n = k^{-\frac{1}{k}} n^{\frac{1}{k}-1}$ we have $\mathbb{P}(X \geq \frac{n}{2}) \leq \frac{1}{2}$.

Now let Y be the number of sets of $s = \frac{n}{2r}$ independent vertices in G. For each set of s vertices (and there are $\binom{n}{s}$ such sets), the probability that these vertices are independent is equal to $(1-p)^{\binom{s}{2}}$. By using the fact that $\exp(x) \geq 1 + x$ for all $x \in \mathbb{R}$, we then get

$$\begin{aligned} \mathbb{E}Y &= \binom{n}{s}(1-p)^{\binom{s}{2}} \leq n^s \exp(-p)^{\binom{s}{2}} = \exp\left[s\ln(n) - p\binom{s}{2}\right] \\ &\leq \exp\left[s\ln(n) - \frac{ps^2}{3}\right] = \exp\left[\frac{1}{2r} n\ln(n) - \frac{k^{-\frac{1}{k}}}{12r^2} n^{1+\frac{1}{k}}\right]. \end{aligned}$$

Since $\ln(n) = o(n^{\frac{1}{k}})$, the expression in the exponent tends to $-\infty$ as $n \to \infty$, implying that $\mathbb{E}Y \to 0$ as $n \to \infty$. By Markov's inequality, we have $\mathbb{P}(Y \neq 0) = \mathbb{P}(Y \geq 1) \leq \mathbb{E}Y$, so for $n \geq k$ sufficiently large (which we fix from now on) we have $\mathbb{P}(Y \neq 0) < \frac{1}{2}$.

Now we have $\mathbb{P}(X \geq \frac{n}{2}$ or $Y \neq 0) \leq \mathbb{P}(X \geq \frac{n}{2}) + \mathbb{P}(Y \neq 0) < 1$, so there exists some graph G with n vertices, $X < \frac{n}{2}$ cycles of length $\leq k$, and no independent sets of s vertices, that is, with independence number $\alpha(G) \leq s = \frac{n}{2r}$. Now let $G' = G - A$, where $A \subseteq V(G)$ is a subset with $|A| = \frac{n}{2}$ containing at least one vertex from each cycle in G of length $\leq k$. Then G' has girth $> k$ and independence number $\alpha(G') \leq \alpha(G) \leq \frac{n}{2r}$, implying that G' has chromatic number $\chi(G') \geq \frac{|G'|}{\alpha(G')} \geq \frac{n/2}{n/2r} = r$. □

4.4 Threshold functions

We now study the structure of random graphs. In particular, we introduce the following probability space.

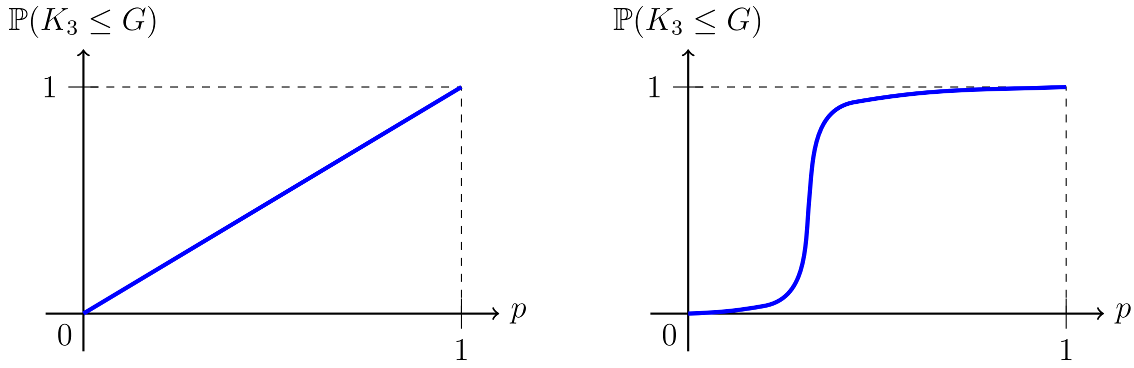
Notation. Let $p\colon \mathbb{N} \to [0,1]$ be a function. We write $\mathcal{G}(n,p)$ for the probability space of all random graphs with vertex set $[n]$ and each of the potential $\binom{n}{2}$ edges appearing independently at random with probability $p = p(n)$. If p is constant, that is, there exists a constant $p_0 \in [0,1]$ such that $p(n) = p_0$ for all n, we also write $\mathcal{G}(n,p_0)$ for $\mathcal{G}(n,p)$.

We can ask how likely it is that $G \in \mathcal{G}(n,p)$ has a certain property. For instance, we may ask about $\mathbb{P}(K_3 \leq G)$. We might expect this probability to grow at roughly constant rate as p increases from 0 to 1. However, it turns out that there is a "sharp transition": the probability $\mathbb{P}(K_3 \leq G)$ for $G \in \mathcal{G}(n,p)$ increases from close to 0 to close to 1 over a narrow interval of p (see Figure 4.1).

In order to formalise such behaviour, we introduce the following terminology. Recall that we write $f(n) = o(g(n))$ (respectively $f(n) = \omega(g(n))$) if $\frac{f(n)}{g(n)} \to 0$ (respectively $\frac{f(n)}{g(n)} \to \infty$) as $n \to \infty$.

Definition (threshold function)**.** Let $\mathcal{P}$ be a property of graphs and $p\colon \mathbb{N} \to [0,1]$.

- We say that *almost every* $G \in \mathcal{G}(n,p)$ has property $\mathcal{P}$ if $\mathbb{P}(G \in \mathcal{G}(n,p)$ has $\mathcal{P}) \to 1$ as $n \to \infty$.



(a) Expectation. (b) Reality.

Figure 4.1: The probability $\mathbb{P}(K_3 \le G)$, where $G \in \mathcal{G}(n,p)$.

- We say $f\colon \mathbb{N} \to [0,1]$ is a *threshold function* for the property $\mathcal{P}$ if almost every $G \in \mathcal{G}(n,p)$ does not have $\mathcal{P}$ whenever $p(n) = o(f(n))$, and almost every $G \in \mathcal{G}(n,p)$ has $\mathcal{P}$ whenever $p(n) = \omega(f(n))$.

A common situation we will consider is as follows. Suppose $A_1, \dots, A_m$ are some events determined by $G \in \mathcal{G}(n,p)$, and define a random variable X as the number of the A_i that occur: that is, $X = \sum_{i=1}^m \mathbb{1}(A_i)$. We want to find a threshold function for some A_i to occur—that is, to have $X \neq 0$—and so we would like to find upper and lower bounds for $\mathbb{P}(X = 0)$. We can use Markov's and Chebyshev's inequalities (see Lemma 4.1): we have

$$\mathbb{P}(X = 0) = 1 - \mathbb{P}(X \ge 1) \ge 1 - \mu,$$

where $\mu = \mathbb{E}X$, by Markov's inequality, and

$$\mathbb{P}(X = 0) \le \mathbb{P}(|X - \mu| \ge \mu) \le \frac{\sigma^2}{\mu^2},$$

where $\sigma^2 = \mathrm{Var}(X)$, by Chebyshev's inequality.

In the particular case when $X = \sum_{i=1}^m \mathbb{1}(A_i)$, we calculate μ and σ^2 as follows. By construction, we have $\mu = \mathbb{E}X = \sum_{i=1}^m \mathbb{P}(A_i)$. On the other hand, we have $\sigma^2 = \mathrm{Var}(X) = \mathbb{E}(X^2) - \mu^2$ by Lemma 4.1(iii), and we can compute that

$$\mathbb{E}(X^2) = \sum_{i=1}^m \sum_{j=1}^m \mathbb{E}(\mathbb{1}(A_i)\mathbb{1}(A_j)) = \sum_{i=1}^m \sum_{j=1}^m \mathbb{E}(\mathbb{1}(A_i \cap A_j)) = \sum_{i=1}^m \sum_{j=1}^m \mathbb{P}(A_i \cap A_j)$$

and

$$\mu^2 = \sum_{i=1}^m \sum_{j=1}^m \mathbb{P}(A_i)\mathbb{P}(A_j),$$

implying that

$$\sigma^2 = \sum_{i=1}^m \sum_{j=1}^m \left[\mathbb{P}(A_i \cap A_j) - \mathbb{P}(A_i)\mathbb{P}(A_j)\right]. \tag{4.1}$$

Note that if the events A_i and A_j are independent then the corresponding term in the sum (4.1) is zero, so we only need to sum over the non-independent pairs (A_i, A_j), including the pairs (A_i, A_i).

Let's start with finding a threshold function for $G \in \mathcal{G}(n,p)$ to contain an edge.

Example 4.5. We claim that $1/n^2$ is a threshold function for $G \in \mathcal{G}(n,p)$ to contain an edge. In order to show this, we need to show that if $p = \omega(1/n^2)$ then almost every $G \in \mathcal{G}(n,p)$ has at least one edge, and if $p = o(1/n^2)$ then almost every $G \in \mathcal{G}(n,p)$ has no edges. Let $A_1, \dots, A_m$, where $m = \binom{n}{2}$, be the events of having edges vw (for each pair of distinct $v, w \in G$), and let $X = \sum_{i=1}^m \mathbb{1}(A_i)$; note that A_i and A_j are independent for $i \neq j$. Let $\mu = \mathbb{E}X$ and $\sigma^2 = \mathrm{Var}(X)$.

Suppose first that $p = o(1/n^2)$. Then we have $\mu = m \cdot p = \binom{n}{2}p$, and therefore, using Markov's inequality, we obtain

$$\mathbb{P}(G \in \mathcal{G}(n,p) \text{ has an edge}) = \mathbb{P}(X \geq 1) \leq \mu = \binom{n}{2}p \leq \frac{1}{2}n^2 p.$$

Now since $p = o(1/n^2)$ we have $n^2 p \to 0$ and therefore $\mathbb{P}(G \in \mathcal{G}(n,p) \text{ has an edge}) \to 0$ as $n \to \infty$, as required.

Now suppose instead that $p = \omega(1/n^2)$. Then, using (4.1) and the fact that A_i and A_j are independent for $i \neq j$, we have $\sigma^2 = m \cdot (p - p^2) = \binom{n}{2}(p-p^2) \leq \binom{n}{2}p = \mu$, implying that $\frac{\sigma^2}{\mu^2} \leq \frac{\mu}{\mu^2} = \frac{1}{\binom{n}{2}p}$. Using Chebyshev's inequality, we then have

$$\mathbb{P}(G \in \mathcal{G}(n,p) \text{ has no edges}) = \mathbb{P}(X = 0) \leq \mathbb{P}(|X - \mu| \geq \mu) \leq \frac{\sigma^2}{\mu^2} \leq \frac{1}{\binom{n}{2}p} \leq \frac{3}{n^2 p}$$

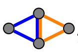
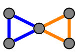
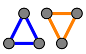
for $n \geq 2$. Now as $p = \omega(1/n^2)$ we have $n^2 p \to \infty$ and so $\mathbb{P}(G \in \mathcal{G}(n,p) \text{ has an edge}) \to 1$ as $n \to \infty$, as required.

We may generalise these ideas to give a threshold function for $G \in \mathcal{G}(n,p)$ to have a triangle, as follows.

Proposition 4.6. *$1/n$ is a threshold function for $G \in \mathcal{G}(n,p)$ to contain a triangle.*

Proof. Let $G \in \mathcal{G}(n,p)$, let X be the number of triangles in G, and let $\mu = \mathbb{E}X$ and $\sigma^2 = \mathrm{Var}(X)$. We can compute, using (4.1), that

$$\mu = \binom{n}{3}p^3 \qquad \text{and} \qquad \sigma^2 = \binom{n}{3} \cdot (p^3 - p^6) + \binom{n}{3} \cdot 3(n-3) \cdot (p^5 - p^6),$$

where the first and the second terms in the expression of σ^2 come from the pairs  and , respectively; note that the pairs  and  are both independent.

Suppose first that $p = o(1/n)$, that is, $np \to 0$ as $n \to \infty$. Then $\mu \leq \frac{1}{6}(np)^3 \to 0$ as $n \to \infty$; using Markov's inequality, this implies that

$$\mathbb{P}(K_3 \leq G) = \mathbb{P}(X \geq 1) \leq \mu \to 0$$

as $n \to \infty$, as required.

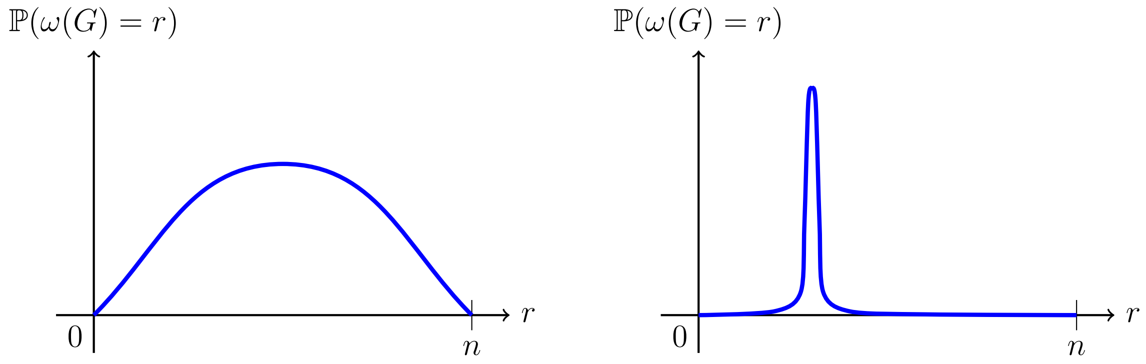
Suppose now that $p = \omega(1/n)$, that is, $np \to \infty$ as $n \to \infty$. We then have $\sigma^2 \leq \frac{1}{6}n^3p^3 + \frac{1}{2}n^4p^5$, whereas since $\mu^2 \sim \frac{n^6p^6}{36}$ we have $\mu^2 \geq \frac{n^6p^6}{37}$ for n large enough. Therefore, using Chebyshev's inequality, for n large we have

$$\mathbb{P}(K_3 \not\leq G) = \mathbb{P}(X = 0) \leq \mathbb{P}(|X - \mu| \geq \mu) \leq \frac{\sigma^2}{\mu^2} \leq \frac{37}{6n^3p^3} + \frac{37}{2n^2p} < \frac{7}{(np)^3} + \frac{19}{(np)n} \to 0$$

as $n \to \infty$, as required. □

4.5 Clique numbers

We now fix $p \in (0,1)$ to be constant, and ask the following question: given $G \in \mathcal{G}(n,p)$, what is the clique number $\omega(G)$? It turns out that for any p, this clique number takes very few—namely, at most two—specific values (that depend on n) for almost every G (see Figure 4.2). We give a proof of this fact (with some calculations omitted) below.



(a) Expectation. (b) Reality.

Figure 4.2: The probability $\mathbb{P}(\omega(G) = r)$, where $G \in \mathcal{G}(n,p)$.

Theorem 4.7. *Let $p \in (0,1)$. Then there exists a function $d\colon \mathbb{N} \to \mathbb{N}$ with $d(n) \sim \frac{2\ln n}{-\ln p}$ such that almost every $G \in \mathcal{G}(n,p)$ has $\omega(G) \in \{d(n)-1, d(n)\}$.*

Proof (sketch). For any $r \geq 1$, let X_r be the number of subgraphs of $G \in \mathcal{G}(n,p)$ isomorphic to K_r. We then have $\mathbb{E}X_r = f(r)$, where $f(r) = \binom{n}{r}p^{\binom{r}{2}}$. We set $d(n)$ to be the largest $r \in [n]$ such that $f(r) \geq n^{-\frac{1}{3}}$.

We first estimate the number $d := d(n)$. Recall the *Stirling's Formula*, which states that $\ln(n!) = n\ln(n) - n + \frac{1}{2}\ln(2\pi n) + o(1)$. Note first that $\ln f(r) \leq \ln(2^n p^{(r-1)^2/2}) = n\ln 2 + \frac{(r-1)^2}{2}\ln p$ for all r, implying that $d = O(\sqrt{n})$. Using Stirling's Formula and the fact that $\ln(1-x) = -x - \frac{x^2}{2} + O(x^3)$ for x small, we can show that for $r = O(\sqrt{n})$ we have $\ln\binom{n}{r} = r\ln n - \frac{r^2}{2n} - r\ln r + r - \frac{1}{2}\ln(2\pi r) + o(1)$, and consequently

$$\ln f(r) = r\left[\ln n - \ln r + 1 + \frac{r-1}{2}\ln p + o(1)\right].$$

This implies that $d = d(n) \sim \frac{2\ln n}{-\ln p}$.

Now suppose that $r = r(n) \sim \frac{2\ln n}{-\ln p}$. Then we have

$$\ln\frac{f(r+1)}{f(r)} = \ln\left(\frac{\binom{n}{r+1}}{\binom{n}{r}}p^{\binom{r+1}{2}-\binom{r}{2}}\right) = \ln\left(\frac{n-r}{r+1}p^r\right) = \ln(n-r) - \ln(r+1) + r\ln p$$
$$\sim \ln n - \ln r - 2\ln n \sim -\ln n,$$

implying that $\frac{f(r+1)}{f(r)} \to 0$ as $n \to \infty$, and in particular that $\frac{f(r+1)}{f(r)} = o(n^{-\frac{2}{3}})$. Now since $f(d+1) < n^{-\frac{1}{3}} \leq f(d)$ by the choice of d, it follows that $f(d+1) \to 0$ and $f(d-1) = \omega(\sqrt[3]{n}) \to \infty$ as $n \to \infty$.

In order to show that $\omega(G) \in \{d-1, d\}$ for almost every $G \in \mathcal{G}(n,p)$, we need to show that $\mathbb{P}(X_{d+1} > 0) \to 0$ and $\mathbb{P}(X_{d-1} = 0) \to 0$ as $n \to \infty$. We do this as follows.

(i) Markov's inequality implies that

$$\mathbb{P}(X_{d+1} > 0) = \mathbb{P}(X_{d+1} \geq 1) \leq \mathbb{E}X_{d+1} = f(d+1),$$

and since $f(d+1) \to 0$ as $n \to \infty$, we have $\mathbb{P}(X_{d+1} > 0) \to 0$ as $n \to \infty$, as required.

(ii) We first calculate $\sigma^2 = \mathrm{Var}(X_r)$, where $r = d-1$. For two subsets $A, B \subseteq [n]$ with $|A| = |B| = r$ and $|A \cap B| = s$, the probability that A (or B) induces a K_r is equal to $p^{\binom{r}{2}}$, and the probability that both A and B induce K_r's is equal to $p^{2\binom{r}{2}-\binom{s}{2}}$. Thus, the contribution of (A,B) to σ^2 is equal to $p^{2\binom{r}{2}-\binom{s}{2}} - p^{2\binom{r}{2}}$. Moreover, there are $\binom{n}{r}\binom{r}{s}\binom{n-r}{r-s}$ such pairs (A,B): there are $\binom{n}{r}$ ways to choose A, $\binom{r}{s}$ ways to choose the subset $A \cap B$ in A, and $\binom{n-r}{r-s}$ ways to choose the remaining $r-s$ vertices of B. This implies that

$$\sigma^2 = \sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}\binom{n-r}{r-s}\left[p^{2\binom{r}{2}-\binom{s}{2}} - p^{2\binom{r}{2}}\right] \leq f(r)\sum_{s=0}^{r} a_s,$$

where $a_s = \binom{r}{s}\binom{n-r}{r-s}p^{\binom{r}{2}-\binom{s}{2}}$.

We now claim that $\max\{a_s \mid 0 \leq s \leq r\} = O((\ln n)^2)$. Indeed, using Stirling's Formula and the asymptotics for $r = d-1$ we can verify that:

- $\ln a_s \sim \frac{(r-s)s}{2}\ln p$ whenever $s = \omega(\ln\ln n)$, and in particular we have $a_s \leq 1$ when $\sqrt{\ln n} \leq s \leq r-1$ and n is sufficiently large;
- $\ln\frac{a_{s-1}}{a_s} \sim \ln n$ whenever $s = o(\ln n)$, and in particular we have $a_s \leq a_{s-1}$ when $1 \leq s \leq \sqrt{\ln n}$ and n is sufficiently large;
- $a_0 = O((\ln n)^2)$ and $a_r = 1$.

Combining these bounds gives $\max\{a_s\} = O((\ln n)^2)$, as claimed. In particular, since $r = O(\ln n)$, it follows that $\sigma^2 = O((\ln n)^3 f(r))$.

Now since $f(r) = f(d-1) = \omega(\sqrt[3]{n})$, Chebyshev's inequality implies that

$$\mathbb{P}(X_{d-1} = 0) \leq \mathbb{P}\Big(|X_{d-1} - f(r)| \geq f(r)\Big) \leq \frac{\sigma^2}{f(r)^2} = o\left(\frac{(\ln n)^3}{\sqrt[3]{n}}\right)$$

and therefore $\mathbb{P}(X_{d-1} = 0) \to 0$ as $n \to \infty$, as required. □

Since $\chi(G) \geq \omega(G)$ for all graphs G, Theorem 4.7 immediately implies that almost every $G \in \mathcal{G}(n,p)$ has $\chi(G) \geq c \ln n$ for some constant $c > 0$. However, we may also deduce a better asymptotic lower bound, namely that $\chi(G) \geq c\frac{n}{\ln n}$ for some $c > 0$, as follows.

Corollary 4.8. *Let $p \in (0,1)$. Then almost every $G \in \mathcal{G}(n,p)$ has chromatic number $\chi(G) \geq \left[-\frac{1}{2}\ln(1-p) + o(1)\right]\frac{n}{\ln n}$.*

Proof. For any $G \in \mathcal{G}(n,p)$ consider its *complement* $\overline{G}$, which has vertices $V(\overline{G}) = [n]$ and edges $E(\overline{G}) = \{ij \mid 1 \leq i < j \leq n, ij \notin E(G)\}$ (see also Problem 1.4). Note that $\overline{G} \in \mathcal{G}(n, 1-p)$ and $\alpha(G) = \omega(\overline{G})$. Therefore, by Theorem 4.7, almost every $G \in \mathcal{G}(n,p)$ has $\alpha(G) = \omega(\overline{G}) \sim \frac{2\ln n}{-\ln(1-p)}$, that is, $\alpha(G) = \left[\frac{2}{-\ln(1-p)} + o(1)\right]\ln n$. This implies that almost every $G \in \mathcal{G}(n,p)$ has $\chi(G) \geq \frac{n}{\alpha(G)} = \left[-\frac{1}{2}\ln(1-p) + o(1)\right]\frac{n}{\ln n}$, as required. □

5

Drawings and colourings

In this chapter, we analyse colourings of graphs so that no two adjacent vertices use the same colour, and relate them to drawings of graphs on the plane and other surfaces.

5.1 Planar graphs

Recall the *map colouring problem* (see Example 0.3). This problem can be rephrased in graph-theoretic terms as follows: find a value of k such that any planar graph has an admissible k-colouring—see the definitions below.

Definition (admissible colourings)**.** Let G be a graph and $k \geq 1$. An *admissible k-colouring* of G is a map $c\colon V(G) \to [k]$ such that $c(v) \neq c(w)$ whenever $v \sim_G w$.

Thus G has an admissible k-colouring if and only if G is k-partite (that is, $\chi(G) \leq k$).

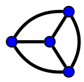
Definition (drawings, planar graphs)**.** Let $G = (V, E)$ be a graph, and let X be a surface. A *drawing of G on X* is an injection $\varphi\colon V \to X$ together with a collection of continuous injections $\gamma_e\colon [0,1] \to X$ for each $e \in E$ such that:

- for any $e = vw \in E$, we have $\{\gamma_e(0), \gamma_e(1)\} = \{\varphi(v), \varphi(w)\}$;
- for all $e, f \in E$, if $e \neq f$ then $\gamma_e((0,1)) \cap \gamma_f((0,1)) = \varnothing$; and
- for all $e \in E$, we have $\gamma_e((0,1)) \cap \varphi(V) = \varnothing$.

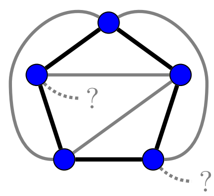
For simplicity, a *drawing of G* means a drawing of G on $\mathbb{R}^2$. We say that G is *planar* if there exists a drawing of G.

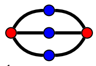
Remark. If a drawing of G exists, it can always be modified to produce a "simple" drawing: for instance, one in which $\gamma_e([0,1])$ is a union of finitely many line segments and circle arcs for any $e \in E(G)$. We will assume that all our drawings are of this "simple" form.

Example 5.1.

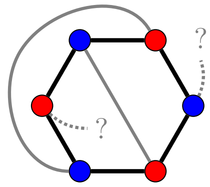
(i) The graph $K_4 =$  is planar, as displayed in the drawing.

However, K_5 is not planar. Indeed, a drawing of a subgraph $C_5 \leq K_5$ separates $\mathbb{R}^2$ into the "inside" and the "outside". In order to complete this to a drawing of K_5 we would need to add five "diagonals"—but in order to avoid intersections of these diagonals, at most two of them can be drawn on each the "inside" and the "outside".



(ii) The graph $K_{2,3} =$  is planar, as displayed in the drawing; similarly, $K_{2,t}$ is planar for any $t \geq 1$.

However, $K_{3,3}$ is not planar. Indeed, a drawing of a subgraph $C_6 \leq K_{3,3}$ separates $\mathbb{R}^2$ into the "inside" and the "outside". In order to complete this to a drawing of $K_{3,3}$ we would need to add three "diagonals"—but in order to avoid intersections of these diagonals, at most one of them can be drawn on each the "inside" and the "outside".



In order to give a criterion for planarity, we introduce the following definition.

Definition (subdivisions)**.** A *subdivision* of a graph G is a graph G' obtained by repeatedly replacing a chosen edge with a path of length 2: that is, removing an edge vw and adding a vertex u together with edges uv and uw.

Since K_5 and $K_{3,3}$ are non-planar, it is clear that any subdivision of K_5 or $K_{3,3}$ is non-planar, as is any graph containing such a subdivision. However, this turns out to be the only obstruction to planarity.

Theorem 5.2 (Kuratowski's Theorem)**.** *A graph G is planar if and only if it contains no subdivisions of K_5 or of $K_{3,3}$ as subgraphs.*

We will postpone the proof of Kuratowski's Theorem for later.

Now recall that a *tree* is a connected graph with at least one vertex and no cycles, and that every tree with ≥ 2 vertices has a *leaf*—a vertex of degree 1 (see Problem 1.5).

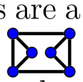
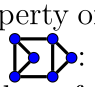
Proposition 5.3. *Every tree is planar.*

Proof. Let T be a tree. We prove the result by induction on $|T|$; if $|T| = 1$, then the result is clear.

Suppose $|T| \geq 2$. Let $v \in T$ be a leaf, so that $N(v) = \{w\}$ for some $w \in T$. Then $T - \{v\}$ is a tree, so by the inductive hypothesis it has a drawing consisting of maps φ and $(\gamma_e \mid e \in E(T) \setminus \{vw\})$. We can pick a neighbourhood of $\varphi(w)$ in the drawing such that in the neighbourhood the drawing only consists of $d_T(w) - 1$ line segments or circle arcs, each being a subpath of $\gamma_{wu}([0, 1])$ for $u \in N_T(w) \setminus \{v\}$. We can then define γ_{wv} to be a path with image a line segment contained in this neighbourhood, thus defining a drawing of T. □

We now turn to the study of drawings themselves.

Definition (faces)**.** A drawing of a graph on a surface X divides X into connected regions. Each such region is called a *face*.

Note that faces are a property of drawings, not of the graphs themselves. For instance, consider drawings  and : the first one has a hexagonal face, but the second one does not. However, the number of faces *is* a property of the graph itself, as the following result shows.

Theorem 5.4 (Euler's Formula)**.** *Let G be a connected planar graph with $|G| = n$ and $e(G) = m$, and suppose there exists a drawing of G with ℓ faces. Then $n - m + \ell = 2$.*

Proof. We prove this by induction on the number of cycles in G. For the base case, note that if G is a tree then $m = n - 1$ (see Problem 1.5) and any drawing of G only has $\ell = 1$ face, so indeed $n - m + \ell = n - (n-1) + 1 = 2$.

Suppose now that G has a cycle, and let $e \in E(G)$ be an edge in this cycle. Then $H := G - \{e\}$ is connected and planar, we have $|H| = n$ and $e(H) = m - 1$, and extending a drawing of H to a drawing of G subdivides one of the faces into two, implying that H can be drawn with $\ell - 1$ faces. By the inductive hypothesis, it then follows that $n - (m-1) + (\ell - 1) = 2$, and therefore $n - m + \ell = 2$, as required. □

Theorem 5.5. *Let G be a planar graph with $|G| = n \geq 3$. Then $e(G) \leq 3n - 6$.*

Proof. Without loss of generality, suppose G is connected (we could add edges to G consistent with a drawing of G if not), and that $n \geq 4$ (the cases $G \cong C_3$ and $G \cong P_2$ can be verified directly). Let $m = e(G)$, and suppose G has a drawing with ℓ faces. By the Euler's Formula (Theorem 5.4), we have $n - m + \ell = 2$. Now each edge belongs to at most 2 faces, and each face has at least 3 edges (that's why we require $n \geq 4$), implying that $\ell \leq \frac{2m}{3}$. Therefore, we have

$$m = 3\left(m - \frac{2m}{3}\right) \leq 3(m - \ell) = 3(n-2) = 3n - 6,$$

as required. □

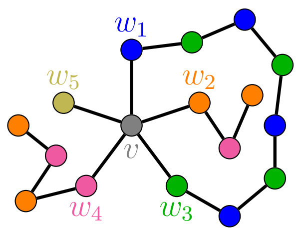
We are now ready to prove that every planar graph has an admissible 5-colouring, as follows.

Theorem 5.6 (Five Colour Theorem)**.** *If G is a planar graph, then $\chi(G) \leq 5$.*

Proof. We need to show that G has an admissible 5-colouring. We do this by induction on $n := |G|$; the case $n \leq 5$ is trivial.

Suppose $n \geq 6$. Since G is planar, by Theorem 5.5 we have $e(G) \leq 3n - 6$, implying that G has average degree $d(G) = \frac{2e(G)}{n} \leq \frac{6n-12}{n} = 6 - \frac{12}{n} < 6$. Therefore, G must have a vertex $v \in G$ with $d(v) \leq 5$. Let $H = G - \{v\}$, and let $c\colon H \to [5]$ be an admissible colouring, which exists by the inductive hypothesis. If $c(N_G(v)) \neq [5]$, then we can extend c to an admissible 5-colouring of G.

Thus, we may assume that $c(N_G(v)) = [5]$. Then $d(v) = 5$, and v has exactly one neighbour of each colour in G. Consider a drawing of G, and let $N_G(v) = \{w_1, \ldots, w_5\}$ so that the labels follow clockwise order around v. Without loss of generality, suppose that $c(w_i) = i$. For each i, consider $V_i := \{u \in V(H) \mid c(u) = i\}$, and for each $i \neq j$, consider the subgraph $H_{ij} := H[V_i \cup V_j]$.



If w_1 and w_3 are in different connected components of H_{13}, then we may define an admissible colouring $c'\colon G \to [5]$ by swapping colours 1 and 3 in the component of H_{13} containing w_1, and set $c'(v) = 1$. Otherwise, w_2 and w_4 are in different faces of the drawing restricted to $G[\{v\} \cup V(H_{13})]$, implying that they must be in different connected components of H_{24} (see the picture)—so again we can define an admissible colouring $c' \colon G \to [5]$ by swapping colours 2 and 4 in the component of H_{24} containing w_2 and setting $c'(v) = 2$. □

In fact, one can do better: it is possible (although very hard) to show that an admissible 4-colouring of a planar graph always exists. This is the best possible result, since K_4 is planar and $\chi(K_4) = 4$.

Theorem 5.7 (Four Colour Theorem; K. Appel and W. Haken, 1976)**.** *If G is a planar graph, then $\chi(G) \leq 4$.* □

5.2 Proof of Kuratowski's Theorem

Here we prove Kuratowski's Theorem (Theorem 5.2). The "only if" direction is easy: see Example 5.1 and the subsequent discussion. We will therefore only need to prove the "if" direction.

We start with a study of 2-connected graphs. In order to do that, we need the following definition.

Definition (ear decomposition)**.** Let G be a graph.

- An *ear* in G is a path $P = v_0 \cdots v_n \leq G$ of length $n \geq 1$ such that $d_G(v_i) = 2$ for $0 < i < n$ and $d_G(v_0), d_G(v_n) \geq 3$.

- We say that G is obtained by *adding an ear* to a subgraph $H \leq G$ if G has an ear $P = v_0 \cdots v_n$ such that $H = G - \{v_1, \ldots, v_{n-1}\}$ if $n \geq 2$ or $H = G - \{v_0v_1\}$ if $n = 1$.

- An *ear decomposition* of G is a sequence of subgraphs $G_0 \leq G_1 \leq \cdots \leq G_k = G$ such that G_0 is a cycle and G_i is obtained by adding an ear to G_{i-1} for $1 \leq i \leq k$.

Theorem 5.8. *Let G be a graph with $|G| \geq 3$. Then G is 2-connected if and only if it has an ear decomposition.*

Proof.

($\Leftarrow$) Since cycles are 2-connected, it is enough to show that a graph G obtained by adding an ear $P = v_0 \cdots v_n$ to a 2-connected graph H must be 2-connected. It is clear that G is connected; let $w \in G$—we aim to show that $G - \{w\}$ is connected. If $w \in H$, then $G - \{w\}$ is obtained from $H - \{w\}$ by either adding the ear P or "gluing" a path $v_0 \cdots v_{n-1}$ or $v_1 \cdots v_n$ at one of its endpoints, and $H - \{w\}$ is connected since H is 2-connected, so $G - \{w\}$ is connected. Otherwise, we have $w = v_i$ for some $0 < i < n$, and $G - \{w\}$ is obtained by "gluing" the paths $v_0 \cdots v_{i-1}$ and $v_{i+1} \cdots v_n$ at their endpoints to the connected graph H, so $G - \{w\}$ is again connected.

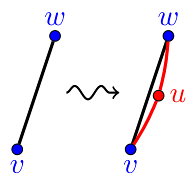
(⇒) Since G is 2-connected and $|G| \geq 3$, it follows that G is not a tree and therefore must contain a cycle. Therefore, it is enough to show (by induction on $e(G)$) that if $H \leq G$ is a subgraph with $H \neq G$ and $|H| \geq 3$, then there exists a graph H' with $H \leq H' \leq G$ obtained by adding an ear to H. Thus, let $e = vw \in E(G)$ such that $v \in H$ and $e \notin E(H)$ (such an edge exists since G is connected and $H \neq G$). Since G is 2-connected and $|H| \geq 2$, we can choose a shortest path $P = w \cdots u$ in $G - \{v\}$ with $u \in H$. Consider the path $Q = v_0 \cdots v_n \leq G$, where $v_0 = v$ and $v_1 \ldots v_n = P$. We then have a subgraph $H' \leq G$ defined by $V(H') = V(H) \cup V(Q)$ and $E(H') = E(H) \cup E(Q)$, and it is easy to see that H' is obtained by adding the ear Q to H. □

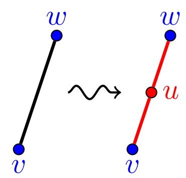
Corollary 5.9. *Let G be a 2-connected graph with $\delta(G) \geq 3$. Then there exists an edge $e \in E(G)$ such that $G - \{e\}$ is still 2-connected.*

Proof. By Theorem 5.8, G has an ear decomposition; on the other hand, G cannot be a cycle since $\delta(G) \geq 3$. Let P be the last ear added in the ear decomposition of G. Since $\delta(G) \geq 3$, the ear P must have length 1, and so $P = vw$ for some edge $e = vw \in E(G)$. Then $G - \{e\}$ still has an ear decomposition, and so is 2-connected. □

We now sketch a proof of Kuratowski's Theorem.

Proof of Theorem 5.2 (sketch). Suppose that there exists a graph G that is non-planar but contains no subdivisions of K_5 or $K_{3,3}$, and choose such a graph with $|G| + e(G)$ as small as possible. For any subgraph $H \leq G$ with $H \neq G$, we know that H still has no subdivisions of K_5 or $K_{3,3}$, implying by the minimality of $|G| + e(G)$ that H is planar. Therefore, it can be verified (see Problem 5.8) that G is 2-connected.





Moreover, we claim that G has no vertices of degree ≤ 2. Indeed, it is clear from non-planarity of G and minimality of $|G| + e(G)$ that G has no vertices of degree 0 or 1 (see the proof of Proposition 5.3). Suppose $N_G(u) = \{v, w\}$ for some $u \in G$, with $v \neq w$. If $v \sim_G w$, then $G - \{u\}$ is planar by the minimality of $|G| + e(G)$, and given a drawing of $G - \{u\}$ we can extend it to a drawing of G by drawing the path vuw "close" to the edge vw (see the top picture), contradicting non-planarity of G. Otherwise, consider a graph obtained by adding the edge vw to the graph $G - \{u\}$: such a graph cannot have a subdivision of $K_{3,3}$ or K_5 and therefore must be planar by the minimality of $|G| + e(G)$, again contradicting non-planarity of G (see the bottom picture). Thus $\delta(G) \geq 3$, as claimed.

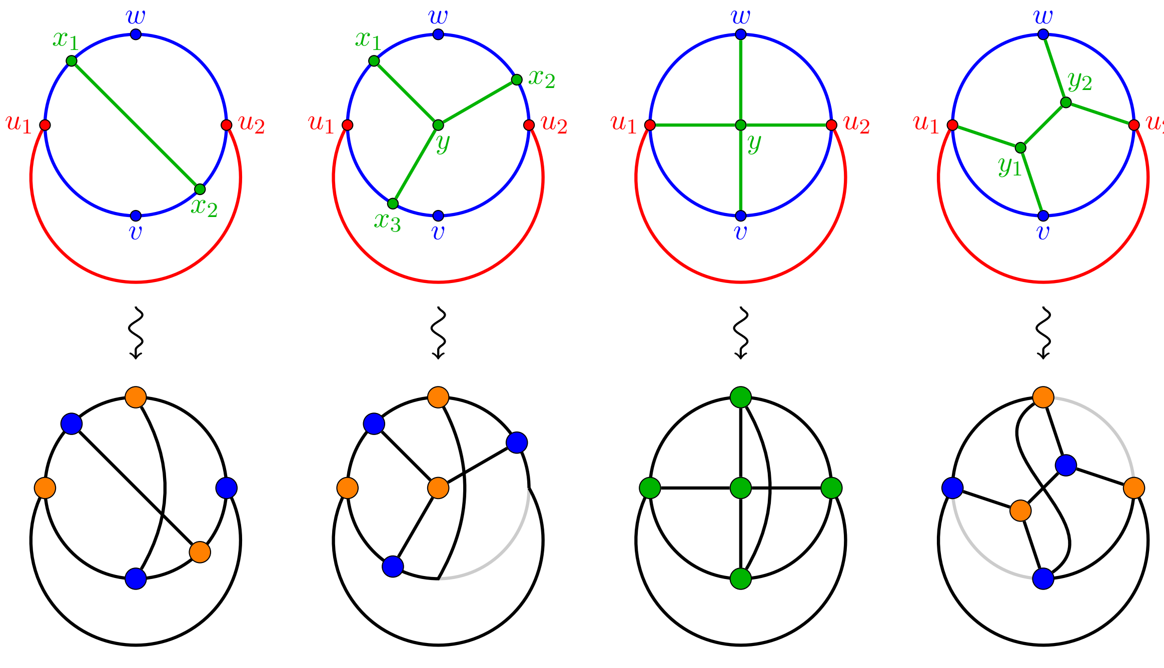
Now by Corollary 5.9, there exists an edge $e = vw \in G$ such that $H := G - \{e\}$ is 2-connected. Moreover, by the minimality of $|G| + e(G)$, we know that H is planar, so we may draw H on the plane. Since H is 2-connected, it follows from Menger's Theorem (Theorem 1.11) that there exists two independent (v, w)-paths $Q_1, Q_2 \leq H$, and therefore a cycle $C \leq H$ containing v and w. Note that a drawing of C separates the plane into two regions, called the *inside* and the *outside* of C, and every edge of $E(H) \setminus E(C)$ is drawn either on the inside or on the outside; without loss of generality, suppose that Q_1 and Q_2 are chosen in such a way that there are as many edges on the inside of C as possible.

Let $H_1, \ldots, H_k$ be the connected components of $H - V(C)$, and for each $1 \leq i \leq k$, let H'_i be the subgraph of H with vertices $V(H'_i) = V(H_i) \cup N_H(V(H_i))$ (so that $V(H'_i) \subseteq V(H_i) \cup V(C)$) and edges $E(H'_i) = \{xy \in E(H) \mid x \in H_i \text{ or } y \in H_i\}$. Each H'_i must contain at least 2 vertices of C, as otherwise the fact that the removal of $V(C) \cap V(H'_i)$ disconnects G would contradict the 2-connectedness of G. Moreover, each H'_i must be drawn either on the inside or the outside of C (we call such an H'_i *interior* or *exterior*, respectively).

For distinct $i, j \in [k]$, we say that H'_i and H'_j *overlap* if every subpath $P \leq C$ containing all vertices in $V(H'_i) \cap V(C)$ also contains some vertex of H'_j that is not an endpoint of P. We can show that no two interior H'_i overlap, and that there exists an interior H'_I containing vertices in both $Q_1 - \{v, w\}$ and $Q_2 - \{v, w\}$ that overlaps some exterior H'_O (see Problem 5.8).

Now since H'_O is exterior, by the maximality of the number of edges on the inside of C we know that H'_O can have at most one vertex in either Q_1 or Q_2, implying that H'_O has exactly two vertices in C, namely, exactly one in each $Q_1 - \{v, w\}$ and $Q_2 - \{v, w\}$. We call these vertices u_1 and u_2, respectively, and let $R \leq H'_O$ be a path from u_1 to u_2.

We can now show that H'_I contains one of the four subgraphs displayed in the top row of Figure 5.1 (see Problem 5.8). In each of these cases, with the addition of the edge vw, we can find a subdivision of $K_{3,3}$ or K_5 in G (see the bottom row of Figure 5.1), contradicting our choice of G. This completes the proof. □



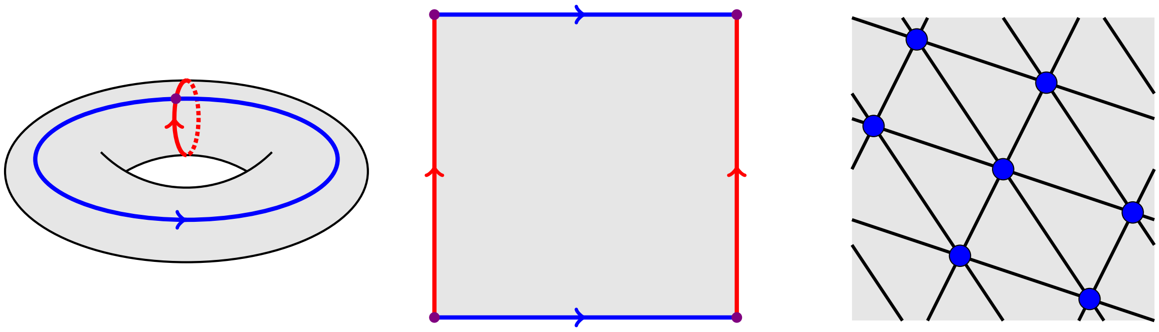
(a) (b) (c) (d)

Figure 5.1: Top: the possible configurations of C (blue), R (red) and a subgraph of H'_I (green) in a drawing of H; in addition to those four, one may get similar configurations by swapping u_1 with u_2 and/or v with w. In (b), one may also possibly have $x_2 = u_2$ and/or $x_3 = v$. Bottom: subdivisions of $K_{3,3}$ (a, b, d) or K_5 (c) that appear in G after adding the edge vw to the displayed configuration.

5.3 Graphs on surfaces

Drawing a graph G on the plane $\mathbb{R}^2$ is equivalent to drawing it on the sphere $\mathbb{S}^2$. Indeed, for any point $x \in \mathbb{S}^2$, the plane is "topologically equivalent" (homeomorphic) to $\mathbb{S}^2 \setminus \{x\}$, so a drawing of G on $\mathbb{R}^2$ gives a drawing on $\mathbb{S}^2$, and a drawing of G on $\mathbb{S}^2$ gives (after removing a point not in the image of the drawing) a drawing on $\mathbb{R}^2$.

Therefore, if a graph G can be drawn on $\mathbb{S}^2$ then $\chi(G) \leq 4$ by Theorem 5.7. What about other surfaces? Consider a torus $\mathbb{T}^2$ (see Figure 5.2a). It can be represented by a square, with each pair of opposite edges identified in an appropriate way (see Figure 5.2b). It turns out that there exists a drawing of K_7 on $\mathbb{T}^2$ (see Figure 5.2c), and we have $\chi(K_7) = 7$.



(a) A torus... (b) ...represented by a square... (c) ...with K_7 drawn on it.

Figure 5.2: A torus represented by a square with K_7 drawn on it.

In fact, closed surfaces have been classified. We first need a (slightly handwavy) definition.

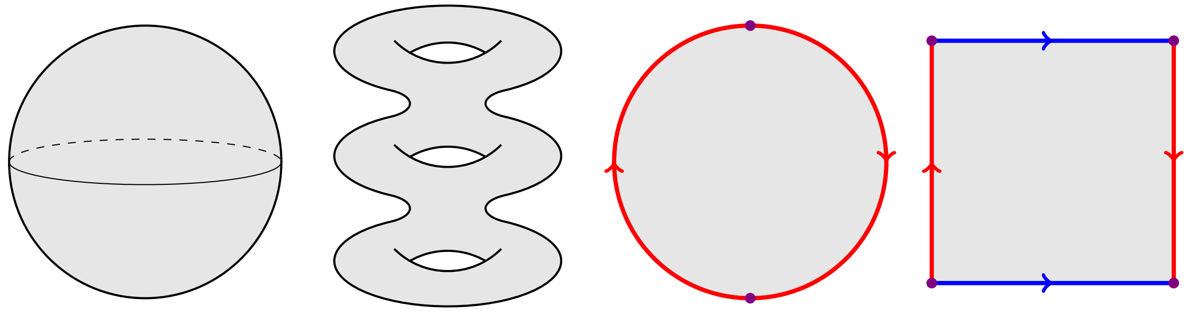
Definition (connected sums)**.** Let X and Y be surfaces without boundary. The *connected sum* of X and Y, denoted $X \# Y$, is a surface obtained by removing open disks from X and Y to get surfaces X' and Y' with boundary, and gluing X' and Y' along their boundary circles. For surfaces $X_1, \dots, X_n$ without boundary, we define $X_1 \# \cdots \# X_n$ inductively, by setting it to be X_1 if $n = 1$ and $(X_1 \# \cdots \# X_{n-1}) \# X_n$ if $n \geq 2$.

Theorem 5.10 (Classification Theorem of Closed Surfaces)**.** *Let X be a closed surface. Then X is homeomorphic to one of*

- *the sphere $\Sigma_0 := \mathbb{S}^2$, also known as the* closed orientable surface of genus 0*;*
- *the connected sum $\Sigma_g := \mathbb{T}^2 \# \cdots \# \mathbb{T}^2$ of $g \geq 1$ copies of the torus $\mathbb{T}^2$, also known as the* closed orientable surface of genus g*; or*
- *the connected sum $N_g := \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$ of $g \geq 1$ copies of the real projective plane $\mathbb{R}P^2$, also known as the* closed non-orientable surface of genus g*.* □

Several examples of these surfaces are displayed in Figure 5.3.

The following result can be viewed as a generalisation of the Euler's Formula (Theorem 5.4) and is given here without a proof.



(a) $\Sigma_0 = \mathbb{S}^2$; (b) Σ_3; (c) $N_1 = \mathbb{R}P^2$; (d) N_2 (Klein bottle).

Figure 5.3: Some closed surfaces.

Theorem 5.11 (Euler–Poincaré Formula)**.** *Let G be a graph with $|G| = n$ and $e(G) = m$, and suppose that there exists a drawing of G on a closed surface X with ℓ faces. Then $n - m + \ell \geq \epsilon(X)$, where $\epsilon(X)$ is the* Euler characteristic *of X, defined as $\epsilon(\Sigma_g) = 2 - 2g$ for $g \geq 0$ and $\epsilon(N_g) = 2 - g$ for $g \geq 1$.* □

Remark. In literature, the Euler characteristic is denoted by the letter χ. Here we use ϵ instead to avoid confusion with chromatic numbers.

We can use Theorem 5.11 to bound chromatic numbers of graphs drawn on surfaces.

Theorem 5.12. *Let X be a closed surface of Euler characteristic ϵ, let G be a graph, and suppose that G can be drawn on X. Then $\chi(G) \leq \left\lfloor \frac{7+\sqrt{49-24\epsilon}}{2} \right\rfloor$.*

Proof. Let $|G| = n$, $e(G) = m$, and suppose G can be drawn on X with ℓ faces. Without loss of generality, suppose that n is as small as possible among all graphs of chromatic number $\geq \chi(G)$ that can be drawn on X—that is, if a graph H with $|H| < n$ can be drawn on X, then $\chi(H) < \chi(G)$. Moreover, we may further suppose that G is connected (if not, we may add edges consistent with a drawing of G on X to make it connected). Since $\left\lfloor \frac{7+\sqrt{49-24\epsilon}}{2} \right\rfloor \geq 4$ for any $\epsilon \leq 2$, we may further assume that $\chi(G) \geq 4$ and therefore $n \geq 4$.

By the Euler–Poincaré Formula (Theorem 5.11), we have $n - m + \ell \geq \epsilon$. On the other hand, as in the proof of Theorem 5.5, we can show that $\ell \leq \frac{2m}{3}$. This implies that $n - \frac{m}{3} \geq \epsilon$ and therefore $m \leq 3(n - \epsilon)$. Thus,

$$\delta(G) \leq d(G) = \frac{2m}{n} \leq \frac{6(n-\epsilon)}{n} = 6 - \frac{6\epsilon}{n}.$$

Now pick a vertex $v \in G$ with $d(v) = \delta(G)$, and let $H = G - \{v\}$. By the assumption on G, we have $\chi(H) < \chi(G)$, and therefore there exists an admissible $(\chi(G)-1)$-colouring of H. In any such colouring, every colour must appear in $N_G(v)$, implying that $\delta(G) = d_G(v) \geq \chi(G) - 1$. In particular, we have $\chi(G) \leq 7 - \frac{6\epsilon}{n}$.

The rest of the argument depends on the value of ϵ:

- If $\epsilon = 2$, then $X = \mathbb{S}^2$ and therefore $\chi(G) \leq 4 = \frac{7+\sqrt{49-24\epsilon}}{2}$ by Theorem 5.7.

- If $\epsilon = 1$, then $\chi(G) \le 7 - \frac{6}{n} < 7$, so $\chi(G) \le 6 = \frac{7+\sqrt{49-24\epsilon}}{2}$.
- If $\epsilon \le 0$, then note that clearly $\chi(G) \le n$. Therefore, we have $\chi(G) \le 7 - \frac{6\epsilon}{\chi(G)}$, implying that $\chi(G)^2 - 7\chi(G) + 6\epsilon \le 0$. Solving this equation gives $\chi(G) \le \frac{7+\sqrt{49-24\epsilon}}{2}$, as required. □

In fact, it can be shown that for every surface $X \not\cong N_2$, the graph K_n can be drawn on X for $n = \left\lfloor \frac{7+\sqrt{49-24\epsilon(X)}}{2} \right\rfloor$, implying that the bound in Theorem 5.12 is optimal. However, we cannot draw K_7 on the Klein bottle N_2, and with a bit more work we can show that if G can be drawn on Klein bottle then actually $\chi(G) \le 6$ (we cannot further decrease the bound as K_6 can be drawn on N_2).

5.4 The chromatic polynomial

If a graph has an admissible x-colouring, one may ask *how many* of such colourings G has. In order to do that, we define a function $p_G\colon \mathbb{N} \to \mathbb{N}$ (with $p_G(x) = 0$ for $x < \chi(G)$) as follows.

Notation. Given a graph G and $x \ge 0$, we write $p_G(x)$ for the number of admissible x-colourings of G.

In order to study the function p_G, we introduce the following terminology.

Definition (edge contraction)**.** Let $G = (V, E)$ be a graph and $e = vw \in E$. *Contraction of the edge* e is an operation resulting in a graph G/e, defined by setting $V(G/e) = (V \setminus e) \sqcup \{u\}$ and $E(G/e) = \{xy \in E \mid x, y \notin e\} \sqcup \{ux \mid x \in N_G(e) \setminus e\}$.

Lemma 5.13. *Let G be a graph and $e \in E(G)$. Then $p_G(x) = p_{G-\{e\}}(x) - p_{G/e}(x)$ for any $x \ge 0$.*

Proof. An admissible x-colouring c of G is precisely an admissible x-colouring of $G - \{e\}$ with $c(v) \ne c(w)$, where $e = vw$. On the other hand, there is a bijection between admissible x-colourings of G/e and admissible x-colourings of $G - \{e\}$ with $c(v) = c(w)$, obtained by sending a colouring $c'\colon V(G/e) \to [x]$ to a colouring $c\colon V(G - \{e\}) \to [x]$ defined by $c(z) = c'(z)$ for $z \in V(G) \setminus \{v, w\}$ and $c(v) = c(w) = c'(u)$, where u is the unique vertex in $V(G/e) \setminus V(G)$. This shows that $p_{G-\{e\}}(x) = p_G(x) + p_{G/e}(x)$, as required. □

Theorem 5.14. *For any graph G, the function p_G is a polynomial of the form*

$$p_G(x) = x^n - mx^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0,$$

where $n = |G|$ and $m = e(G)$.

Proof. We prove the statement by induction on m. If $m = 0$, then clearly $p_G(x) = x^n$.

Suppose that $m \geq 1$, let $e \in E(G)$, and note that $e(G - \{e\}) = e(G/e) = m - 1$. Therefore, it follows by the inductive hypothesis that $p_{G-\{e\}}(x)$ is a polynomial of the form $x^n - (m-1)x^{n-1} + \cdots$, and $p_{G/e}(x)$ is a polynomial of the form $x^{n-1} + \cdots$. By Lemma 5.13, $p_G(x)$ is then a polynomial of the form $x^n - mx^{n-1} + \cdots$, as required. □

Motivated by Theorem 5.14, we call p_G the *chromatic polynomial* of G.

5.5 Edge colourings

We now consider a variation of admissible colourings, as follows.

Definition (admissible edge colourings, edge chromatic number)**.** Let G be a graph.

- For $k \geq 1$, an *admissible k-edge-colouring* of G is a map $c\colon E(G) \to [k]$ such that $c(uv) \neq c(uw)$ whenever $uv, uw \in E(G)$ and $v \neq w$.
- The *edge chromatic number* $\chi'(G)$ of G is the smallest $k \geq 1$ such that G has an admissible k-edge-colouring.

It turns out that it is much easier to bound $\chi'(G)$ than $\chi(G)$. Indeed, note first that clearly $\chi'(G) \geq \Delta(G)$. On the other hand, a greedy algorithm (similar to the one described in Section 4.2 to show that $\chi(G) \leq \Delta(G)+1$) shows that we must have $\chi'(G) \leq 2\Delta(G)-1$. In fact, we can strengthen this bound, as the following result shows.

Theorem 5.15 (Vizing's Theorem)**.** *Let G be a graph. Then $\chi'(G) \leq \Delta(G) + 1$.*

Proof. We use induction on $e(G)$. For $e(G) = 0$, the result is trivial.

Suppose now that $e(G) \geq 1$, and pick an edge $vw_1 \in G$. By the inductive hypothesis, we can find an admissible edge colouring $c\colon G - \{vw_1\} \to [\Delta + 1]$, where $\Delta = \Delta(G)$. Note that since $d(u) < \Delta + 1$ for all $u \in G - \{vw_1\}$, there must be (at least one) colour "missing" at u.

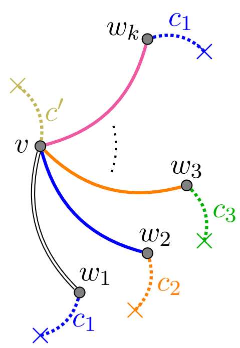
Now define vertices $w_1, w_2, \ldots$ and colours $c_1, c_2, \ldots$ inductively, as follows (note that w_1 has already been defined). For each $k \geq 1$, having already defined w_i and $1 \leq i \leq k$ and c_i for $1 \leq i \leq k-1$, we do the following.

(i) If c_i is "missing" at w_k for some $i < k$, then stop. Otherwise, let c_k be a colour "missing" at w_k.

(ii) If c_k is "missing" at v, then stop. Otherwise, let $w_{k+1} \in N_G(v)$ be such that $c(vw_{k+1}) = c_k$.

By construction we have $c_i \neq c_j$ for $i \neq j$, so since we only have $\Delta + 1$ colours the process must terminate for some $k \leq \Delta + 1$. There are two ways in which this may happen, and in each case we can modify the colouring c and extend it to an admissible $(\Delta + 1)$-edge colouring of G.

(i) Suppose that c_i is "missing" at w_k for some $1 \leq i < k$ (and therefore $k \geq 2$). Suppose, without loss of generality, that $i = 1$: indeed, if that is not the case then we can (re-)colour vw_j with colour c_j for $1 \leq j \leq i-1$ and uncolour the edge vw_i. Let c' be a colour "missing" at v, and consider the subgraph $H \leq G - \{vw_1\}$ consisting of all vertices of G and edges of colours c' and c_1. We have $\Delta(H) \leq 2$, so all connected components of H are paths or cycles. On the other hand, the vertices v, w_1 and w_k all have degree ≤ 1 in H, so they cannot all belong to the same connected component of H.



Let H_v, H_1 and H_k be the connected components of H containing v, w_1 and w_k, respectively. If $H_v \neq H_1$, then we can swap colours c' and c_1 in H_1, and colour vw_1 with colour c'. Otherwise, we have $H_v = H_1 \neq H_k$, so we can swap colours c' and c_1 in H_k, recolour vw_k with colour c', and (re-)colour vw_j with colour c_j for $1 \leq j \leq k-1$.

(ii) Suppose that c_k is "missing" at v for some $k \geq 1$. We can then (re-)colour vw_j with colour c_j for $1 \leq j \leq k$. □

6

Algebraic methods

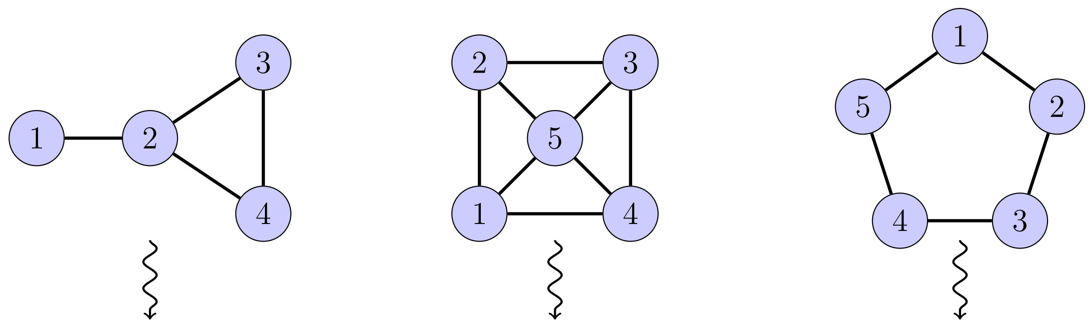
In this chapter we show how methods from linear algebra can be used to study graphs. We will apply these methods to study graphs that are "strongly regular".

6.1 The adjacency matrix

We start with introducing a certain matrix associated to a graph.

Definition (adjacency matrix)**.** Let G be a graph with vertex set $[n]$. The *adjacency matrix* of G is the $n \times n$ matrix A such that $A_{i,j} = 1$ if $i \sim_G j$ and $A_{i,j} = 0$ otherwise.

The adjacency matrix of a graph is symmetric, all its entries are equal to 0 or 1, and all diagonal entries are equal to 0. Conversely, any square matrix satisfying these properties is an adjacency matrix of some graph. Several examples of adjacency matrices are given in Figure 6.1.



$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(a) (b) (c)

Figure 6.1: Some graphs and their adjacency matrices.

Powers of the adjacency matrix have graph-theoretic interpretation, as follows.

Lemma 6.1. *Let G be a graph, A its adjacency matrix, and $m \geq 0$. Then $(A^m)_{i,j}$ is the number of walks of length m in G starting at i and ending at j. In particular, $(A^2)_{i,j} = |N(i) \cap N(j)|$.*

Proof. We prove the statement by induction on m. If $m = 0$ (and therefore $A^m = I_n$) then the result is trivial.

Suppose $m \geq 1$, and let $a_{i,j}(m)$ be the number of walks in G from i to j of length m. Every such walk is of the form $ik \cdots j$ for some $k \in N(i)$ and some walk $k \cdots j$ from k to j of length $m-1$. This implies that $a_{i,j}(m) = \sum_{k \in N(i)} a_{k,j}(m-1)$. By the inductive hypothesis, we have $a_{k,j}(m-1) = (A^{m-1})_{k,j}$, and therefore

$$a_{i,j}(m) = \sum_{k \in N(i)} (A^{m-1})_{k,j} = \sum_{k=1}^{n} A_{i,k}(A^{m-1})_{k,j} = (A \cdot A^{m-1})_{i,j} = (A^m)_{i,j},$$

as required.

In particular, for $m = 2$, a walk from i to j of length 2 is precisely a sequence ikj for some $k \in N(i) \cap N(j)$, implying that $(A^2)_{i,j} = |N(i) \cap N(j)|$. □

We now turn to the study of eigenvalues and eigenvectors of adjacency matrices. As adjacency matrices are real and symmetric, the following well-known result from linear algebra will turn out to be useful.

Theorem 6.2. *Let A be an $n \times n$ symmetric matrix with real entries. Then A is diagonalisable over $\mathbb{R}$, and in particular all its eigenvalues are real. Moreover, eigenvectors of A corresponding to different eigenvalues are orthogonal (with respect to the standard inner product in $\mathbb{R}^n$).* □

We list some properties of the maximal eigenvalue of an adjacency matrix, as follows.

Proposition 6.3. *Let G be a graph with adjacency matrix A. Then $|\lambda| \leq \Delta(G)$ for every eigenvalue λ of A. Moreover, if G is connected, then $\Delta(G)$ is an eigenvalue of G if and only if G is regular, in which case $\Delta(G)$ has multiplicity 1 and eigenvector $(1, 1, \ldots, 1)$.*

Proof. Let $\mathbf{x} \in \mathbb{R}^n$ be an eigenvector of G with eigenvalue λ. Choose $i \in [n]$ such that $|\mathbf{x}_i|$ is as big as possible. After replacing $\mathbf{x}$ with its scalar multiple if necessary, we may assume that $\mathbf{x}_i = 1$, and therefore $|\mathbf{x}_j| \leq 1$ for all $j \in [n]$. We then have

$$|\lambda| = |\lambda \mathbf{x}_i| = |(A\mathbf{x})_i| = \left| \sum_{j \in N(i)} \mathbf{x}_j \right| \leq \sum_{j \in N(i)} |\mathbf{x}_j| \leq \sum_{j \in N(i)} 1 = d(i) \leq \Delta(G), \tag{6.1}$$

proving the first statement.

Suppose now that G is connected. We prove the second statement as follows.

($\Leftarrow$) If G is regular, then every vertex of G has degree $\Delta(G)$, implying that $\Delta(G)$ is an eigenvalue of G with eigenvector $(1, 1, \ldots, 1)$.

($\Rightarrow$) Let $\lambda = \Delta(G)$, and let $\mathbf{x} \in \mathbb{R}^n$ and $i \in [n]$ be as above. Let $\mathcal{J} = \{j \in [n] \mid \mathbf{x}_j = 1\}$, so that $i \in \mathcal{J}$. It is then enough to show that $\mathcal{J} = [n]$ and that $d(j) = \Delta(G)$ for all $j \in \mathcal{J}$, as that will imply that all eigenvectors corresponding to $\Delta(G)$ are scalar multiples of $(1, 1, \ldots, 1)$ and that G is $\Delta(G)$-regular.

Now since $\lambda = \Delta(G)$, all the inequalities in (6.1) must be equalities, implying that $\mathbf{x}_j = 1$ for all $j \in N(i)$ (and therefore $N(i) \subseteq \mathcal{J}$) and that $d(i) = \Delta(G)$. But the same can be said for any other $k \in \mathcal{J}$, implying that $d(k) = \Delta(G)$ for all $k \in \mathcal{J}$, and that $N(k) \subseteq \mathcal{J}$ for all $k \in \mathcal{J}$. As G is connected and $\mathcal{J} \neq \varnothing$, the latter fact implies that $\mathcal{J} = [n]$, as required. □

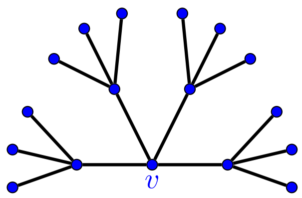
6.2 Moore graphs

We now want to describe graphs with bounded degree and diameter (see the following definition), and with as many vertices as possible.

Definition (diameter)**.** Let G be a connected graph. The *diameter* of G is the smallest integer $D \geq 0$ such that any two vertices of G are endpoints of a path of length $\leq D$.

It is clear that the only graphs of diameter 0 are the graphs with 0 and 1 vertices, and that a connected graph G has diameter 1 if and only if $G \cong K_r$ for some $r \geq 2$. We now concentrate on the study of graphs of diameter 2.

Suppose G is a graph of diameter 2 with maximal degree $\Delta(G) = \Delta$. How many vertices can G have? Fix $v \in G$. Since G has diameter 2, we have $V(G) = \{v\} \cup N(v) \cup N(N(v))$, and therefore $|G| \leq 1 + \Delta + \Delta(\Delta - 1) = \Delta^2 + 1$ (see the picture for the case $\Delta = 4$). This motivates the following definition.



Definition (Moore graphs)**.** Let $\Delta \geq 1$. A connected graph G of diameter 2 with $\Delta(G) = \Delta$ and $|G| = \Delta^2 + 1$ is called a *Moore graph*.

We have the following alternative characterisation of Moore graphs.

Lemma 6.4. *A graph G with $|G| \geq 3$ is a Moore graph if and only if it is Δ-regular for some $\Delta \geq 1$, no two adjacent vertices of G have any common neighbours, and every pair of distinct non-adjacent vertices of G have exactly one common neighbour.*

Proof.

($\Rightarrow$) If G is a Moore graph then for any $v \in G$ we have

(i) $V(G) = \{v\} \sqcup N(v) \sqcup [N(N(v)) \setminus \{v\}]$, and in particular $N(v) \cap N(N(v)) = \varnothing$;

(ii) $|N(v)| = \Delta$; and

(iii) $|N(N(v)) \setminus \{v\}| = \Delta(\Delta - 1)$, and in particular $N(u) \cap N(w) = \{v\}$ for all $u, w \in N(v)$.

Now (ii) implies that G is Δ-regular, and (i) implies that if v and w are adjacent then they have no common neighbours. If u and w are distinct and non-adjacent, then they must have a common neighbour v since G has diameter 2, and (iii) then implies that u and w have exactly one common neighbour.

($\Leftarrow$) The fact that every pair of non-adjacent vertices of G share a neighbour implies that G is connected and has diameter ≤ 2; since $|G| \geq 3$ and no two adjacent vertices of G have common neighbours, G is not complete and so has diameter exactly 2. Now let $v \in G$. Then $|N(v)| = \Delta$ since G is Δ-regular, and no two vertices in $N(v)$ are adjacent since they have a common neighbour, namely v. Moreover, given any two distinct $u, w \in N(v)$, we know that v is the only common neighbour of u and w. This implies that the sets in the collection $\{\{v\}, N(v)\} \cup \{N(w) \setminus \{v\} \mid w \in N(v)\}$ are all pairwise disjoint, and we have $|N(w) \setminus \{v\}| = \Delta - 1$ since G is Δ-regular. Thus $|G| = 1 + \Delta + \Delta(\Delta - 1) = \Delta^2 + 1$, as required. □

We aim to find values of $\Delta \geq 1$ for which a Moore graph G with $\Delta(G) = \Delta$ exists. We will try to build such a graph using Lemma 6.4.

Example 6.5.

(i) For $\Delta = 1$, we have $\Delta^2 + 1 = 2$, and the only connected graph of order 2 is P_1, which is not a Moore graph since it has diameter 1. Therefore, no Moore graphs for $\Delta = 1$ exist.

(ii) For $\Delta = 2$, we start building the graph by picking a vertex v. It must have exactly two neighbours—call them w_1 and w_2—and each of them must have one additional neighbour—so $w_i \sim u_i$ for $i = 1, 2$. We must add an additional edge between u_1 and u_2 (as all the other vertices already have degree 2), and doing so creates a 5-cycle $G = vw_1u_1u_2w_2v \cong C_5$ (see Figure 6.2a). We can check that C_5 indeed has diameter 2, and so is a Moore graph.

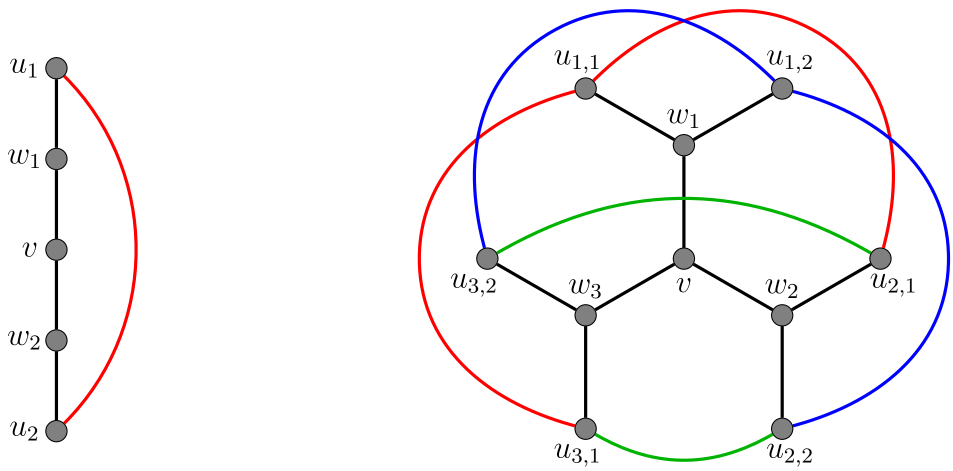
(iii) For $\Delta = 3$, we start by picking a vertex v, its neighbours w_1, w_2 and w_3, and the neighbours of the w_i: we have $u_{i,1} \sim w_i$ and $u_{i,2} \sim w_i$ for $i = 1, 2, 3$, say. We must add edges (six in total) between the $u_{i,j}$ to construct a 3-regular graph that has diameter 2 (see Figure 6.2b).

Now $u_{1,1} \nsim u_{1,2}$ since $u_{1,1}$ and w_1 are adjacent and so cannot have common neighbours. Also, $u_{1,1}$ cannot be adjacent to both $u_{i,1}$ and $u_{i,2}$ for some $i \in \{2, 3\}$, as otherwise w_i and $u_{1,1}$ would be two common neighbours of $u_{i,1}$ and $u_{i,2}$. Since $u_{1,1}$ must have degree 3, it follows that $u_{1,1} \sim u_{2,i}$ and $u_{1,1} \sim u_{3,j}$ for some i and j; without loss of generality (relabelling points if necessary), assume that $u_{1,1} \sim u_{2,1}$ and $u_{1,1} \sim u_{3,1}$ (red edges in the picture).

Next, $u_{1,2}$ cannot be adjacent to $u_{i,1}$ for $i \in \{2, 3\}$, as otherwise $u_{i,1}$ and w_1 would be two common neighbours of $u_{1,1}$ and $u_{1,2}$. Since $d(u_{1,2}) = 3$, we must then have $u_{1,2} \sim u_{2,2}$ and $u_{1,2} \sim u_{3,2}$ (blue edges in the picture). Finally, we must add two more edges between the four points $u_{i,j}$ for $i = 2, 3$ and $j = 1, 2$, and the only way to do this without creating a triangle is to set $u_{2,1} \sim u_{3,2}$ and $u_{2,2} \sim u_{3,1}$ (green edges).

We can check that the resulting graph is 3-regular and has diameter 2, so it is a Moore graph. In fact, this graph is isomorphic to the Petersen graph, which we have already seen earlier (see Problem **??**).

(iv) For $\Delta = 4$, we may try a construction similar to the one in the previous case, but we will eventually get stuck. In fact, there are no 4-regular Moore graphs (more about this later).



(a) The 2-regular Moore graph. (b) The 3-regular Moore graph.

Figure 6.2: The construction of Δ-regular Moore graphs for $\Delta \in \{2, 3\}$.

The description of Moore graphs given by Lemma 6.4 motivates the following definition.

Definition (strongly regular graphs)**.** Let $k, a, b \geq 0$. We say a graph G is (k, a, b)-*strongly regular* if G is k-regular, any two adjacent vertices of G have precisely a common neighbours, and any two non-adjacent vertices of G have precisely b common neighbours.

Therefore, a Moore graph of degree k is precisely a $(k, 0, 1)$-strongly regular graph of order ≥ 3. We use algebraic methods to prove the following result.

Theorem 6.6. *If G is a (k, a, b)-strongly regular graph of order $n \geq 2$, then*

$$\frac{(b-a)(n-1) - 2k}{\sqrt{(a-b)^2 - 4(b-k)}} \in \mathbb{Z}.$$

Proof. Let A be the adjacency matrix of G. Then Lemma 6.1 implies that we have $(A^2)_{i,i} = k$, $(A^2)_{i,j} = a$ if $i \sim j$, and $(A^2)_{i,j} = b$ if $i \neq j$ and $i \nsim j$. Therefore, we have

$$A^2 = kI_n + aA + b(J - I_n - A), \qquad \text{where } J = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

which gives $A^2 + (b-a)A + (b-k)I - bJ = 0$.

Note that since G is k-regular, Proposition 6.3 implies that the matrix A has eigenvector $(1, 1, \ldots, 1)$ with eigenvalue k, and that the eigenvalue k has multiplicity 1. Let

$\mathbf{x} \in \mathbb{R}^n$ be an eigenvector of A with eigenvalue $\lambda \neq k$. Since eigenvectors corresponding to different eigenvalues are orthogonal, we have $J\mathbf{x} = \mathbf{0}$. We then have

$$\mathbf{0} = 0\mathbf{x} = A^2\mathbf{x} + (b-a)A\mathbf{x} + (b-k)\mathbf{x} = (\lambda^2 + (b-a)\lambda + (b-k))\mathbf{x},$$

and as $\mathbf{x} \neq \mathbf{0}$ we then have $\lambda^2 + (b-a)\lambda + (b-k) = 0$. Solving for λ yields $\lambda = \lambda_\pm$, where

$$\lambda_\pm = \frac{1}{2}\left((a-b) \pm \sqrt{(a-b)^2 - 4(b-k)}\right).$$

Thus, the matrix A has at most three eigenvalues: k with multiplicity 1, λ_- with multiplicity m_-, and λ_+ with multiplicity m_+. Since A is an $n \times n$ matrix, we must have $m_- + m_+ + 1 = n$. On the other hand, as all the diagonal entries of A are equal to 0, so is the trace of A, and hence so is the sum of eigenvalues of A (counted with multiplicities)—that is, $m_-\lambda_- + m_+\lambda_+ + k = 0$. This gives the following system of linear equations (with variables m_- and m_+):

$$\begin{cases} m_- + m_+ = n-1, \\ \lambda_- m_- + \lambda_+ m_+ = -k. \end{cases}$$

Solving this system of equations gives $m_\pm = \frac{-k-\lambda_\mp(n-1)}{\pm(\lambda_+ - \lambda_-)}$, that is,

$$\begin{aligned} m_\pm &= \frac{-k - \frac{1}{2}(n-1)\left((a-b) \mp \sqrt{(a-b)^2 - 4(b-k)}\right)}{\pm\sqrt{(a-b)^2 - 4(b-k)}} \\ &= \frac{1}{2}\left((n-1) \pm \frac{(b-a)(n-1) - 2k}{\sqrt{(a-b)^2 - 4(b-k)}}\right). \end{aligned}$$

In particular, we have $m_+ - m_- = \frac{(b-a)(n-1)-2k}{\sqrt{(a-b)^2-4(b-k)}}$. But clearly $m_\pm \in \mathbb{Z}$ and therefore $m_+ - m_- \in \mathbb{Z}$ by construction, which implies the result. □

This allows us to show that there are only finitely many possible Moore graphs, as follows.

Corollary 6.7. *Let G be a Moore graph. Then $\Delta(G) \in \{2, 3, 7, 57\}$.*

Proof. A Moore graph is a $(k, 0, 1)$-strongly regular graph, where $k = \Delta(G)$, so we need to show that $k \in \{2, 3, 7, 57\}$. Substituting $a = 0$, $b = 1$ and $n = k^2 + 1$ into Theorem 6.6 gives $\frac{k^2-2k}{\sqrt{4k-3}} \in \mathbb{Z}$. So either $k^2 - 2k = 0$ and therefore $k = 2$, in which case we are done, or $\sqrt{4k-3}$ is an integer divisor of $k^2 - 2k$.

Suppose $4k - 3 = t^2$, where $t \in \mathbb{N}$ divides $k^2 - 2k$; that is, we have $k^2 - 2k = ut$ for some $u \in \mathbb{Z}$. Substituting $k = \frac{t^2+3}{4}$ yields

$$ut = k(k-2) = \frac{(t^2+3)(t^2-5)}{16} = \frac{t^4 - 2t^2 - 15}{16},$$

rearranging which gives $t(t^3 - 2t - 16u) = 15$. This implies that t divides 15, so $t \in \{1, 3, 5, 15\}$, and hence $k = \frac{t^2+3}{4} \in \{1, 3, 7, 57\}$. It only remains to rule out the case $k = 1$, but this was done in Example 6.5(i). □

Finally, we may ask for which $k \in \{2, 3, 5, 57\}$ a k-regular Moore graph actually exists.

- The 2-regular Moore graph is C_5: see Example 6.5(ii).
- The 3-regular Moore graph is the Petersen graph: see Example 6.5(iii).
- The 7-regular Moore graph of order 50 also exists: it's called the *Hoffman–Singleton graph*.
- It is not known if a 57-regular Moore graph (of order 3250) exists.