Amalgamation in full globally valued fields

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2017-07-04

A talk which has nothing (much) to do with amalgamation in full globally valued fields

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The sum formula: a property of function and number fields

$$\forall x \in K^\times \qquad \sum_\omega m_\omega \cdot v_\omega(x) = 0, \quad \text{or more generally,} \quad \int_\Omega v_\omega(x)\,d\mu(\omega) = 0.$$

Example

- $K = \mathbf{Q}$ and Ω consists of all v_p for prime $p \in \mathbf{Z}$, together with $v_\infty(x) = -\log|x|$ (this last valuation is Archimedean).
- $K = k(t)$, Ω consists of all v_p for prime $p \in k[t]$, together with $v_\infty(f) = -\deg f$.
- Global fields: number fields, function fields in one variable (finite extensions of the previous examples).
- Their algebraic closure. Ω is not discrete, measure essentially unique.
- Function fields of higher-dimensional varieties. Can use intersection numbers to give weight / measure to divisors – measure is not unique.

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$\forall x \in K^\times \qquad \sum_\omega m_\omega \cdot v_\omega(x) = 0,$ or more generally, $\int_\Omega v_\omega(x) d\mu(\omega) = 0.$

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Definition

A globally valued field is a field K together with a measured family of valuations $v_\omega : K \to \mathbf{R} \cup \{\infty\}$ such that $\int v_\omega(x) = 0$ for all $x \neq 0$.

This is an inductive elementary class (first order, $\forall\exists$ axioms) in continuous logic (and even slightly worse)

Motivation in studying this class

Render certain questions regarding number and function fields, heights, ..., amenable to Model Theory.

Also, because it is difficult and makes you do interesting maths.

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The class of globally valued fields is elementary, with an $\forall\exists$ theory GVF.

Question

1. Does GVF has a model companion GVF^*?
   Is the class of e.c. models elementary?
   How does one axiomatise "richness" in GVFs?
   Hmmm....
2. Is GVF^* a model completion (does it have QE)?
   No. No amalgamation over an arbitrary GVF.
3. Is GVF^* stable?
   No obvious obstruction: the many valuations are coded in an L^1 Banach space, which is stable.

Two approaches for understanding GVFs and their extensions:

Udi A GVF is more or less a function field of a variety, together with some global intersection-theoretic data.

Me At each valuation, a GVF is just a valued field, so working locally is easier.

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Conjecture (200?)

GVF^* is just GVF plus the fullness axiom: $\forall v_\omega \exists x \ldots \to \exists x \forall v_\omega \ldots$

This axiom holds in every e.c. GVF, but does it suffice?

Conjecture (201?)

$$\frac{\mathrm{vol}(K[W]_m, \eta^*)}{m \dim K[W]_m} = -\frac{\eta^{\wedge \ell+1} \wedge v\mathfrak{C}_W}{(\ell+1)\deg W} + O(m^{-1}) \qquad \{\ell = \dim W$$

I. This really cool estimate holds, and II. it is useful.

Theorem (Spring 2016, assuming the estimate)

Let K be a GVF (non-trivial, alg. clsd, surjective, ...)
Then K is an amalgamation base ("weakly e.c.") if and only if K is full.

Theorem (Autumn 2016)

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Enough of that. Instead, let us play my favourite analogy game.

Rules: choose A, solve for X

	$\{T, F\}$	is to	$\mathbf{R}$
as	A	is to	X

Classical logic	Continuous logic
$\forall$, $\exists$ Equality	sup, inf Distance
Set Topological spaces	Metric space (complete) Topometric spaces
Compactness Stability	Compactness Stability
Algebraic geometry	??? Rigid analytic geometry? More algebraic?

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Set	Complete metric space
Algebraic geometry	???
Field	Complete metric field: $\mathbf{R}$, $\mathbf{C}$ or (K, v)
Ring $= K[X]/I$	
Integral domain	$K[X]$ with a valuation

Definition

A sub-valuation on a ring A is a function $u\colon A \to \mathbf{R} \cup \{\infty\}$ s.t.:

- $u(ab) \geq u(a) + u(b)$ (if $=$, then u is a valuation)
- $u(a^n) = nu(a)$
- $u(a+b) \geq \min u(a), u(b)$
- $u(0) = \infty$

A valuation (sub-valuation) $u\colon A \to \{0, \infty\} = \{T, F\}$ is just a prime (radical) ideal!

Classical logic	Continuous logic
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Theorem (B. '14)

Like ACF, the theory ACMVF of algebraically closed metric valued fields, in continuous logic, has QE and is stable.

Definition

Say K is a valued field. A valuation on a K-vector space E is a function $u_E : E \to \mathbf{R} \cup \{\infty\}$ s.t.:

- $u_E(ax) \geq v_K(a) + u_E(x)$
- $u_E(a + b) \geq \min u_E(a), u_E(b)$

The tensor product valuation on $E \otimes_K F$ is the least valuation such that $(u_E \otimes u_F)(x \otimes y) = u_E(x) + u_F(y)$.

If $K \subseteq L, M$ are algebraically closed, then $L \otimes_K M$ is an integral domain.

Theorem (Poineau '13, B. '15)

If $K \subseteq L, M$ are algebraically closed valued fields, then $v_L \otimes v_M$ (as K-vector spaces) is a valuation on the ring $L \otimes_K M$ (i.e., "prime"). $\rightsquigarrow$ independence.

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Radical ideal	Sub-valuation
$\updownarrow$	$\updownarrow$
Zariski-closed W	???

Definition

Let u be a [homogeneous, almost finitely generated] sub-valuation on $K[X] = K[X_0, \dots, X_n]$.

$$\ker u = \{P : u(P) = \infty\} \qquad W = V(\ker u) \subseteq \mathbf{P}^n$$

$$u^*(\xi) \quad = \quad \inf_{\text{homog. } P} v(P(x)) - u(P) \quad - \hat{v}(x) \qquad \begin{cases} \xi = [x] \in W \\ \hat{v}(x) = \min_i v(x_i) \end{cases}$$

Such a function $\eta = u^* : W \to \mathbf{R}$ is a virtual divisor on W.

Theorem ("Valued Nullstellensatz")

Bijection $u \longleftrightarrow (W, u^*)$. Converse duality: $\eta = u^* \iff u = \eta^*$.

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Theorem ("Valued Nullstellensatz")

Bijection $u \longleftrightarrow (W, u^*)$. Converse duality: $\eta = u^* \iff u = \eta^*$.

Radical ideal $\updownarrow$ Zariski-closed W

Sub-valuation $\updownarrow$ (W, η), with η a virtual divisor

Definition

Let u be a [homogeneous, almost finitely generated] sub-valuation on $K[X] = K[X_0, \ldots, X_n]$.

$$\ker u = \{P : u(P) = \infty\} \qquad W = V(\ker u) \subseteq \mathbf{P}^n$$

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But why "virtual divisor"?

Lemma

Let $(L, v) \supseteq (K, v)$. *Let* $f \in L[X]_m$ *(homog. of degree* m*) have no zeros on* $\mathbf{P}(K)$. *Then*

$$\eta(\xi) = \hat{f}(\xi) = v(f(x)) - \hat{v}(x) \qquad \left(\xi = [x] \in \mathbf{P}^n(K) \right)$$

is a virtual divisor, and every virtual divisor is a uniform limit of such.

- In other words, a virtual divisor is the "echo" of an actual divisor from a larger model.
- In model theoretic terms, it is an externally definable ($\mathbf{R}$-valued) predicate.
- By stability, a virtual divisor is a definable predicate.

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Fact (Resultant)

There exists an irreducible polynomial $\mathfrak{C}$ *over* $\mathbf{Z}$ *which takes the coefficients of* $n+1$ *homogeneous polynomials in* $n+1$ *unknowns, and vanishes if and only if they have a common zero in* $\mathbf{P}^n$*. Let us write*

$$f_0 \wedge \cdots \wedge f_n = \mathfrak{C}(f_0, \ldots, f_n).$$

Proposition

For $i = 0, \ldots, n$ *assume that:*

- L_i / K *is an extension.*
- $f_i \in L_i[X]_{d_i}$ *(homogeneous of degree* d_i*).*
- $\eta_i = \hat{f}_i$ *is a virtual divisor.*

Then, in the free amalgam $\mathrm{Frac}(L_0 \otimes_K \cdots \otimes_K L_n)$*:*

$$\eta_0 \wedge \cdots \wedge \eta_n := \frac{v(f_0 \wedge \cdots \wedge f_n)}{d_0 \cdots d_n} \in \mathbf{R}.$$

depends only on the η_i *(i.e., on the traces of the* f_i *on* K*).*

- Zariski-closed $W \subseteq \mathbf{P}^n \rightsquigarrow$ a pair (W, η).
  η is a virtual divisor, i.e., the trace on W of a polynomial with external coefficients ($f \in L[X]_d$).
- Assume for simplicity that $W = \mathbf{P}^n$. Then to η we associate

$$\eta^{\wedge n+1} = \eta \wedge \cdots \wedge \eta = \frac{v(\text{resultant of independent copies of } f)}{d^{n+1}} \in \mathbf{R}.$$

- On the other hand, η contains the same information as a sub-valuation $u = \eta^*$ on $K[X]$. How do we recover $\eta^{\wedge n+1}$ from u?

Definition (“volume = determinant”)

Let E be a valued K-vector space, $\mathbf{x} = (x_0, \dots, x_{k-1})$ a basis.

$$\mathbf{x}^\wedge = x_0 \wedge \cdots \wedge x_{k-1} = \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn} \sigma \cdot x_{\sigma(0)} \otimes \cdots \otimes x_{\sigma(k-1)} \in E^{\otimes k},$$

$$\operatorname{vol}_{\mathbf{x}}(E, u_E) = u_E^{\otimes k}(\mathbf{x}^\wedge).$$

Theorem

Let η be a virtual divisor on $\mathbf{P}^n$, and η^* the dual valuation on $K[X]$. Then:

$$\frac{\operatorname{vol}_{\mathfrak{M}_m}\left(K[X]_m, \eta^*\right)}{m \dim K[X]_m} = -\frac{\eta^{\wedge n+1}}{n+1} + O(m^{-1}),$$

where $\mathfrak{M}_m$ denotes the set of monomials of degree m.

Say $\eta = \hat{f}\restriction_K$, where $f \in L[X]_d$. Take a Morley sequence (many independent copies) $f_i \in L_i[X]_d$.

- $K[X]_m$ admits a basis $\Phi = (\varphi_\xi)$, where each φ_ξ is mostly a product of f_is

$$\frac{\mathrm{vol}_\Phi\left(K[X]_m\right)}{m \dim K[X]_m} = O(m^{-1}).$$

- Change of basis:

$$\Phi^\wedge = \det \Phi \cdot \mathfrak{M}_m^\wedge, \qquad \mathrm{vol}_\Phi(E, u_E) = \mathrm{vol}_{\mathfrak{M}_m}(E, u_E) + v(\det \Phi).$$

- By a "generalised Vandermonde matrix identity":

$$v(\det \Phi) = \binom{m/d}{n+1} v(f_0 \wedge \cdots \wedge f_n) + O(m^n) = \frac{\eta^{\wedge n+1}}{n+1} + O(m^{-1})$$

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Globally: Let K be a GVF. At each valuation v_ω let η_ω be a virtual divisor and η^*_ω the dual sub-valuation on $K[X]$. Integrating both sides:

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Which is the special case ($W = \mathbf{P}^n$) of:

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$$\frac{\mathrm{vol}\left(K[W]_m,\eta^*\right)}{m\dim K[W]_m} = -\frac{\eta^{\wedge \ell+1}\wedge v\mathfrak{C}_W}{(\ell+1)\deg W} + O(m^{-1}) \quad \left\{\ell = \dim W\right.$$

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Definition

Say that a GVF K is full if it is (non-trivial, algebraically closed, surjective) and for every valued vector space $(E, \mathbf{u})$:

$$\Big(\forall \varepsilon > 0\Big) \quad \Big(\exists x \in E \smallsetminus \{0\}\Big) \quad \int \mathbf{u}(x) > \frac{\mathrm{vol}(E, \mathbf{u})}{\dim E} - \varepsilon.$$

Corollary (K full GVF, $\boldsymbol{\eta} = (\boldsymbol{\eta}_\omega : \omega \in \Omega)$ virtual divisors on W)

With earlier hypotheses, there exists $W' \subseteq W$ of dimension $\ell - 1$ such that

$$\frac{\boldsymbol{\eta}^{\wedge \ell+1} \wedge v\mathfrak{C}_W}{(\ell+1)\deg W} \leq \frac{\boldsymbol{\eta}^{\wedge \ell} \wedge v\mathfrak{C}_{W'}}{\ell \deg W'} + \varepsilon \leq \cdots \leq \int \eta(\xi) + \varepsilon', \qquad \xi \in W$$

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Corollary (Full GVFs are linearly e.c.)

Let K be a full GVF, $E = (E, \mathbf{u})$ a valued vector space. Let $L \supseteq K$ be a larger GVF, $E_L = (E \otimes_K L, \mathbf{u} \otimes \mathbf{v}_L)$. If there exists $x \in E_L$ such that $\int \mathbf{u}_L(x) > 0$ then such x already exists in E.

Corollary

Let K be a full GVF, L_1 and L_2 two GVF extensions. Then both embed over K in some larger GVF M.

Proof.

By the Corollary, on $L_1 \otimes_K L_2$, the valuation $\mathbf{v}_{L_1} \otimes \mathbf{v}_{L_2}$ is sub-global:

$$\forall x \neq 0 \qquad \int (\mathbf{v}_{L_1} \otimes \mathbf{v}_{L_2})(x) \leq 0.$$

This we know how to correct to get $= 0$. □

Fullness together with the estimate yields a $g \in K[W]_m$ such that (up to small error):

$$\frac{\eta^*(g)}{m} \geq -\frac{\eta^{\wedge \ell+1} \wedge v\mathfrak{C}_W}{(\ell+1)\deg W}$$

So for all x:

$$\frac{v \circ g(x)}{m} - \eta([x]) - \hat{v}(x) \geq -\frac{\eta^{\wedge \ell+1} \wedge v\mathfrak{C}_W}{(\ell+1)\deg W}$$

This holds in particular for "$x \in \eta^{\wedge \ell} \wedge v\mathfrak{C}_W$":

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Thank you