

Amalgamation in full globally valued fields

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2017-07-04

A talk which has nothing (much) to do with amalgamation in full
globally valued fields

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The sum formula: a property of function and number fields

$$\forall x \in K^\times \quad \sum_{\omega} m_{\omega} \cdot v_{\omega}(x) = 0, \quad \text{or more generally,} \quad \int_{\Omega} v_{\omega}(x) d\mu(\omega) = 0.$$

Example

- $K = \mathbf{Q}$ and Ω consists of all v_p for prime $p \in \mathbf{Z}$, together with $v_{\infty}(x) = -\log |x|$ (this last valuation is **Archimedean**).
- $K = k(t)$, Ω consists of all v_p for prime $p \in k[t]$, together with $v_{\infty}(f) = -\deg f$.
- Global fields: number fields, function fields in one variable (finite extensions of the previous examples).
- Their algebraic closure. Ω is not discrete, measure essentially unique.
- Function fields of higher-dimensional varieties. Can use intersection numbers to give weight / measure to divisors – measure is not unique.

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Definition

A **globally valued field** is a field K together with a measured family of valuations $v_\omega: K \rightarrow \mathbf{R} \cup \{\infty\}$ such that $\int v_\omega(x) = 0$ for all $x \neq 0$.

This is an inductive elementary class (first order, $\forall\exists$ axioms) in **continuous logic** (and even slightly worse)

Motivation in studying this class

Render certain questions regarding number and function fields, heights, \dots , amenable to Model Theory.

Also, because it is difficult and makes you do interesting maths.

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The class of globally valued fields is elementary, with an $\forall\exists$ theory *GVF*.

Question

- 1 Does *GVF* has a model companion *GVF**?
Is the class of e.c. models elementary?
How does one axiomatise “richness” in GVFs?
Hmmm....
- 2 Is *GVF** a model completion (does it have QE)?
No. No amalgamation over an arbitrary GVF.
- 3 Is *GVF** stable?
No obvious obstruction: the many valuations are coded in an L^1 Banach space, which is stable.

Two approaches for understanding GVFs and their extensions:

Udi A GVF is more or less a function field of a variety, together with some *global* intersection-theoretic data.

Me At each valuation, a GVF is just a valued field, so working *locally* is easier.

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Conjecture (200?)

GVF^* is just GVF plus the **fullness axiom**: $\forall v_\omega \exists x \dots \rightarrow \exists x \forall v_\omega \dots$

This axiom holds in every e.c. GVF , but does it suffice?

Conjecture (201?)

$$\frac{\text{vol}(K[W]_m, \eta^*)}{m \dim K[W]_m} = -\frac{\eta^{\ell+1} \wedge v \mathfrak{C}_W}{(\ell+1) \deg W} + O(m^{-1}) \quad \{\ell = \dim W\}$$

I. This **really cool** estimate holds, and II. it is useful.

Theorem (Spring 2016, assuming the estimate)

Let K be a GVF (non-trivial, alg. clsd, surjective, ...)

Then K is an amalgamation base ("weakly e.c.") if and only if K is full.

Theorem (Autumn 2016)

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And now to something completely different

Enough of that. Instead, let us play my favourite analogy game.

Rules: choose A , solve for X

as $\begin{matrix} \{T, F\} \\ A \end{matrix}$ is to
is to

\mathbb{R}
 X

Classical logic

Continuous logic

\forall, \exists
Equality

\sup, \inf
Distance

Set
Topological spaces

Metric space (complete)
Topometric spaces

Compactness
Stability

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Algebraic geometry

???

Rigid analytic geometry? More algebraic?

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Classical logic
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Continuous logic
Complete metric space
???

Field
Ring = $K[X]/I$
Integral domain

Complete metric field: \mathbf{R} , \mathbf{C} or (K, v)
 $K[X]$ with a valuation

Definition

A **sub-valuation** on a ring A is a function $u: A \rightarrow \mathbf{R} \cup \{\infty\}$ s.t.:

- $u(ab) \geq u(a) + u(b)$ (if $=$, then u is a **valuation**)
- $u(a^n) = nu(a)$
- $u(a + b) \geq \min u(a), u(b)$
- $u(0) = \infty$

A valuation (sub-valuation) $u: A \rightarrow \{0, \infty\} = \{T, F\}$ is just a prime (radical) ideal!

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Theorem (B. '14)

Like *ACF*, the theory *ACMVF* of algebraically closed *metric* valued fields, in continuous logic, has *QE* and is stable.

Definition

Say K is a valued field. A **valuation** on a K -vector space E is a function $u_E: E \rightarrow \mathbb{R} \cup \{\infty\}$ s.t.:

- $u_E(ax) \geq v_K(a) + u_E(x)$
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The **tensor product valuation** on $E \otimes_K F$ is the least valuation such that $(u_E \otimes u_F)(x \otimes y) = u_E(x) + u_F(y)$.

If $K \subseteq L, M$ are algebraically closed, then $L \otimes_K M$ is an integral domain.

Theorem (Poineau '13, B. '15)

If $K \subseteq L, M$ are algebraically closed valued fields, then $v_L \otimes v_M$ (as K -vector spaces) is a valuation on the ring $L \otimes_K M$ (i.e., "prime"). \rightsquigarrow *independence*.

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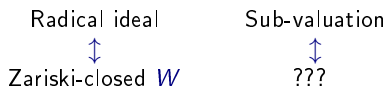
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Definition

Let u be a [homogeneous, almost finitely generated] sub-valuation on $K[X] = K[X_0, \dots, X_n]$.

$$\ker u = \{P : u(P) = \infty\} \quad W = V(\ker u) \subseteq \mathbb{P}^n$$

$$u^*(\xi) = \inf_{\text{homog. } P} v(P(x)) - u(P) \quad - \hat{v}(x) \quad \begin{cases} \xi = [x] \in W \\ \hat{v}(x) = \min_i v(x_i) \end{cases}$$

Such a function $\eta = u^*: W \rightarrow \mathbb{R}$ is a virtual divisor on W .

Theorem ("Valued Nullstellensatz")

Bijection $u \longleftrightarrow (W, u^*)$. *Converse duality:* $\eta = u^* \iff u = \eta^*$.



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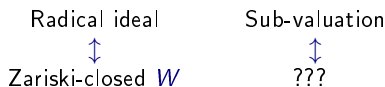
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 \ker u &= \{P : u(P) = \infty\} & W &= V(\ker u) \subseteq \mathbf{P}^n \\
 u^*(\xi) &= \inf_{\text{homog. } P} \frac{v(P(x)) - u(P)}{\deg P} & - \hat{v}(x) & \begin{cases} \xi = [x] \in W \\ \hat{v}(x) = \min_i v(x_i) \end{cases}
 \end{aligned}$$

Such a function $\eta = u^* : W \rightarrow \mathbf{R}$ is a **virtual divisor** on W .

Theorem ("Valued Nullstellensatz")

Bijection $u \longleftrightarrow (W, u^)$. Converse duality: $\eta = u^* \iff u = \eta^*$.*



Definition

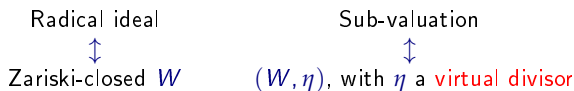
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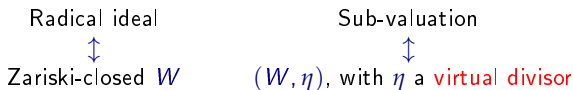
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Lemma

Let $(L, v) \supseteq (K, v)$. Let $f \in L[X]_m$ (homog. of degree m) have no zeros on $\mathbf{P}(K)$. Then

$$\eta(\xi) = \hat{f}(\xi) = v(f(x)) - \hat{v}(x) \quad \left(\xi = [x] \in \mathbf{P}^n(K) \right)$$

is a virtual divisor, and every virtual divisor is a uniform limit of such.

- In other words, a **virtual divisor** is the “echo” of an **actual divisor** from a larger model.
- In model theoretic terms, it is an externally definable (\mathbf{R} -valued) predicate.
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Fact (Resultant)

There exists an irreducible polynomial \mathfrak{C} over \mathbf{Z} which takes the coefficients of $n + 1$ homogeneous polynomials in $n + 1$ unknowns, and vanishes if and only if they have a common zero in \mathbf{P}^n . Let us write

$$f_0 \wedge \cdots \wedge f_n = \mathfrak{C}(f_0, \dots, f_n).$$

Proposition

For $i = 0, \dots, n$ assume that:

- L_i/K is an extension.
- $f_i \in L_i[X]_{d_i}$ (homogeneous of degree d_i).
- $\eta_i = \hat{f}_i$ is a virtual divisor.

Then, in the free amalgam $\text{Frac}(L_0 \otimes_K \cdots \otimes_K L_n)$:

$$\eta_0 \wedge \cdots \wedge \eta_n := \frac{v(f_0 \wedge \cdots \wedge f_n)}{d_0 \cdots d_n} \in \mathbf{R}.$$

depends only on the η_i (i.e., on the traces of the f_i on K).

- Zariski-closed $W \subseteq \mathbf{P}^n \rightsquigarrow$ a pair (W, η) .
 η is a virtual divisor, i.e., the trace on W of a polynomial with external coefficients ($f \in L[X]_d$).
- Assume for simplicity that $W = \mathbf{P}^n$. Then to η we associate

$$\eta^{\wedge n+1} = \eta \wedge \cdots \wedge \eta = \frac{v(\text{resultant of independent copies of } f)}{d^{n+1}} \in \mathbf{R}.$$

- On the other hand, η contains the same information as a sub-valuation $u = \eta^*$ on $K[X]$. How do we recover $\eta^{\wedge n+1}$ from u ?

Definition (“volume = determinant”)

Let E be a valued K -vector space, $\mathbf{x} = (x_0, \dots, x_{k-1})$ a basis.

$$\mathbf{x}^\wedge = x_0 \wedge \cdots \wedge x_{k-1} = \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn} \sigma \cdot x_{\sigma(0)} \otimes \cdots \otimes x_{\sigma(k-1)} \in E^{\otimes k},$$

$$\operatorname{vol}_{\mathbf{x}}(E, u_E) = u_E^{\otimes k}(\mathbf{x}^\wedge).$$

Theorem

Let η be a virtual divisor on \mathbf{P}^n , and η^* the dual valuation on $K[X]$. Then:

$$\frac{\operatorname{vol}_{\mathfrak{M}_m}(K[X]_m, \eta^*)}{m \dim K[X]_m} = -\frac{\eta^{\wedge n+1}}{n+1} + O(m^{-1}),$$

where \mathfrak{M}_m denotes the set of monomials of degree m .

Say $\eta = \hat{f}|_K$, where $f \in L[X]_d$. Take a Morley sequence (many independent copies) $f_i \in L_i[X]_d$.

- $K[X]_m$ admits a basis $\Phi = (\varphi_\zeta)$, where each φ_ζ is **mostly** a product of f_i s

$$\frac{\text{vol}_\Phi(K[X]_m)}{m \dim K[X]_m} = O(m^{-1}).$$

- Change of basis:

$$\Phi^\wedge = \det \Phi \cdot \mathfrak{M}_m^\wedge, \quad \text{vol}_\Phi(E, u_E) = \text{vol}_{\mathfrak{M}_m}(E, u_E) + v(\det \Phi).$$

- By a “generalised Vandermonde matrix identity”:

$$v(\det \Phi) = \binom{m/d}{n+1} v(f_0 \wedge \cdots \wedge f_n) + O(m^n) = \frac{\eta^{\wedge n+1}}{n+1} + O(m^{-1})$$

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Globally: Let K be a GVF. At each valuation v_ω let η_ω be a virtual divisor and η_ω^* the dual sub-valuation on $K[X]$. Integrating both sides:

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(Since K is a GVF, volume is independent of basis)
 Which is the special case ($W = \mathbf{P}^n$) of:

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$$\frac{\text{vol}(K[W]_m, \eta^*)}{m \dim K[W]_m} = -\frac{\eta^{\wedge \ell+1} \wedge v\mathfrak{C}_W}{(\ell+1) \deg W} + O(m^{-1}) \quad \left\{ \ell = \dim W \right.$$

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Definition

Say that a GVF K is **full** if it is (non-trivial, algebraically closed, surjective) and for every valued vector space (E, \mathbf{u}) :

$$(\forall \varepsilon > 0) \quad (\exists x \in E \setminus \{0\}) \quad \int \mathbf{u}(x) > \frac{\text{vol}(E, \mathbf{u})}{\dim E} - \varepsilon.$$

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With earlier hypotheses, there exists $W' \subseteq W$ of dimension $\ell - 1$ such that

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Corollary (Full GVFs are linearly e.c.)

Let K be a full GVF, $E = (E, \mathbf{u})$ a valued vector space. Let $L \supseteq K$ be a larger GVF, $E_L = (E \otimes_K L, \mathbf{u} \otimes \mathbf{v}_L)$. If there exists $x \in E_L$ such that $\int \mathbf{u}_L(x) > 0$ then such x already exists in E .

Corollary

Let K be a full GVF, L_1 and L_2 two GVF extensions. Then both embed over K in some larger GVF M .

Proof.

By the Corollary, on $L_1 \otimes_K L_2$, the valuation $\mathbf{v}_{L_1} \otimes \mathbf{v}_{L_2}$ is sub-global:

$$\forall x \neq 0 \quad \int (\mathbf{v}_{L_1} \otimes \mathbf{v}_{L_2})(x) \leq 0.$$

This we know how to correct to get $= 0$. □

Fullness together with the estimate yields a $g \in K[W]_m$ such that (up to small error):

$$\frac{\eta^*(g)}{m} \geq -\frac{\eta^{\wedge \ell+1} \wedge v\mathfrak{C}_W}{(\ell+1) \deg W}$$

So for all x :

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