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ON THE UNIFORMLY COMPLETE CONSISTENCY OF DENSITY ESTIMATION VIA FREQUENCY POLYGONS UNDER ρ^--MIXING RANDOM FIELDS*

BY

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Abstract. The convergence properties of a density estimator constructed from frequency polygons are investigated under the condition that the samples are drawn from a ρ^--mixing random field. Using the properties and the moment inequalities for ρ^--mixing random fields, the uniformly complete consistency is established for density estimation via frequency polygons for such random fields, and the corresponding convergence rate is derived. The results obtained further improve and extend the relevant research findings of Carbon (2007) and Wang et al. (2021).

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1. INTRODUCTION

Let $d \in \mathbb{N}$. We consider the d-dimensional Euclidean space $\mathbb{R}^d$, furnished with its standard Euclidean norm $\|\cdot\|$ and coordinatewise ordering. We write $\mathbb{N}^d = \{\mathbf{y} = (y_1, \ldots, y_d) \in \mathbb{R}^d : y_i \in \mathbb{N} \text{ for all } i\}$ for the lattice of points with strictly positive integer coordinates. Each lattice point $\mathbf{n} = (n_1, n_2, \ldots, n_d)$ in $\mathbb{N}^d$ is interpreted as a spatial site with positive integer coordinates. We consider a real valued random field $\{X_\mathbf{n},\ \mathbf{n} \in \mathbb{N}^d\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$. A random field $\{X_\mathbf{n},\ \mathbf{n} \in \mathbb{N}^d\}$ is said to be *centered* if $\mathbb{E}X_\mathbf{n} = 0$ for every $\mathbf{n} \in \mathbb{N}^d$. For

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a random variable X with finite pth moment, define $\|X\|_p = (E|X|^p)^{1/p}$. For nonempty and disjoint subsets $B, D \subset \mathbb{N}^d$, define

$$\operatorname{dist}(B, D) = \min\{\|\mathbf{s} - \mathbf{t}\| : \mathbf{s} \in B,\, \mathbf{t} \in D\}.$$

Denote by $\sigma(B)$ the σ-algebra generated by $\{X_{\mathbf{s}},\, \mathbf{s} \in B\}$, and analogously for $\sigma(D)$. Let $\mathscr{C}$ be the class of functions that are coordinatewise increasing.

A random field $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\}$ is said to be *negatively associated* (NA) if, for every pair of disjoint subsets $B, D \subset \mathbb{N}^d$,

$$\operatorname{Cov}\{\phi(X_{\mathbf{s}},\, \mathbf{s} \in B), \psi(X_{\mathbf{t}},\, \mathbf{t} \in D)\} \leqslant 0 \quad \text{for all } \phi, \psi \in \mathscr{C}.$$

The field is called ρ^**-mixing* whenever

$$\rho^*(\tau) = \sup\{\rho(B, D) : B, D \subset \mathbb{N}^d,\, \operatorname{dist}(B, D) \geqslant \tau\} \to 0 \quad \text{as } \tau \to \infty,$$

where

$$\rho(B, D) = \sup\left\{\frac{|E(\phi - E\phi)(\psi - E\psi)|}{\|\phi - E\phi\|_2 \|\psi - E\psi\|_2} : \phi \in L_2(\sigma(B)),\, \psi \in L_2(\sigma(D))\right\}.$$

DEFINITION 1.1. A random field $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\}$ is called ρ^-*-mixing* if

$$\rho^-(\tau) = \sup\{\rho^-(B, D) : B, D \subset \mathbb{N}^d,\, \operatorname{dist}(B, D) \geqslant \tau\} \to 0, \quad \tau \to \infty,$$

where

$$\rho^-(B, D) = 0 \vee \sup\left\{\frac{\operatorname{Cov}\{\phi(X_{\mathbf{s}},\, \mathbf{s} \in B), \psi(X_{\mathbf{t}},\, \mathbf{t} \in D)\}}{\operatorname{Var}\{\phi(X_{\mathbf{s}},\, \mathbf{s} \in B)\} \cdot \operatorname{Var}\{\psi(X_{\mathbf{t}},\, \mathbf{t} \in D)\}} : \phi, \psi \in \mathscr{C}\right\}.$$

Clearly, $\rho^-(\tau) \leqslant \rho^*(\tau)$ for $\tau \geqslant 1$, and the field $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\}$ is NA precisely when $\rho^-(\tau) = 0$. Consequently, ρ^--mixing is strictly weaker than ρ^*-mixing and is interpreted as asymptotic negative association (ANA), also termed the negative quadrant counterpart of ρ^*-mixing.

The introduction of ρ^--mixing random fields is attributed to Zhang and Wang [21], whose foundational contributions included establishing Rosenthal-type inequalities and deriving both complete convergence and almost sure summability rates for the strong law of large numbers. Zhang [20] derived central limit theorems for such random fields under either low moment assumptions or the Lindeberg condition. In the univariate setting ($d = 1$), ρ^--mixing random fields reduce to sequences of ρ^--mixing random variables. This is a well-documented topic in the literature. Notably, Wang and Lu [12] developed inequalities concerning the maximum of partial sums and established weak convergence results. Wang and Zhang [13] explored the law of the iterated logarithm, while Liu and Liu [9] investigated the moment bounds for normalized maxima of partial sums. Yuan and

Wu [19] analyzed the limiting behavior of the maximum of partial sums. Ko [8] derived a Hájek–Rényi-type inequality and proved the strong law of large numbers. More recently, Tang et al. [11] examined the asymptotic normality of wavelet-based estimators in nonparametric regression, and Deng et al. [5] established a central limit theorem under this framework, applying it to the asymptotic analysis of wavelet estimators in regression settings, among other contributions.

This work focuses on density estimation via frequency polygons under the assumption of ρ^--mixing random fields. Next, we provide a detailed exposition of the frequency polygon estimation.

Let $\mathscr{R}_{\mathbf{n}}$ be the rectangular region in $\mathbb{N}^d$ defined as the set of indices $\mathbf{i} = (i_1, \ldots, i_d)$ satisfying $1 \leqslant i_l \leqslant n_l$ for all $l = 1, \ldots, d$. The observations $X_{\mathbf{i}}$ are made on the rectangular region $\mathscr{R}_{\mathbf{n}}$. Define $\hat{\mathbf{n}} = n_1 \ldots n_d$. We write $\mathbf{n} \to \infty$ if $\min\{n_l\} \to \infty$ and there exists a finite constant $C > 0$ such that $|n_l/n_m| < C$ for all $1 \leqslant l, m \leqslant d$.

Consider the sample $\{X_{\mathbf{i}} : \mathbf{i} \in \mathscr{R}_{\mathbf{n}}\}$, which is drawn from a population X with an underlying density function $f(x)$. Divide the real line into consecutive intervals $I_k = [(k-1)h_{\mathbf{n}}, kh_{\mathbf{n}})$ of equal width $h_{\mathbf{n}}$. Let $I_{k_0} = [(k_0-1)h_{\mathbf{n}}, k_0 h_{\mathbf{n}})$ and $I_{k_1} = [k_0 h_{\mathbf{n}}, (k_0+1)h_{\mathbf{n}})$ be two adjacent histogram bins, with $k_1 = k_0 + 1$. We write v_{k_0} and v_{k_1} for the number of samples falling in the bins I_{k_0} and I_{k_1}, respectively. The value of the histogram over bin I_{k_0} is $f_{k_0} = v_{k_0}\hat{\mathbf{n}}^{-1}h_{\mathbf{n}}^{-1}$, while over I_{k_1} it is $f_{k_1} = v_{k_1}\hat{\mathbf{n}}^{-1}h_{\mathbf{n}}^{-1}$. The estimator $f_{\mathbf{n}}(x)$, termed the *frequency polygon*, is defined as

$$f_{\mathbf{n}}(x) = \left(\frac{1}{2} + k_0 - \frac{x}{h_{\mathbf{n}}}\right) f_{k_0} + \left(\frac{1}{2} - k_0 + \frac{x}{h_{\mathbf{n}}}\right) f_{k_1} \quad \text{for } x \in U_{\mathbf{n},k_0},$$

where $U_{\mathbf{n},k_0} = \left[\left(k_0 - \frac{1}{2}\right)h_{\mathbf{n}}, \left(k_0 + \frac{1}{2}\right)h_{\mathbf{n}}\right)$. Define

$$Y_{\mathbf{i},k} = \begin{cases} 1 & \text{if } X_{\mathbf{i}} \in I_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$v_{k_0} = \sum_{\mathbf{i} \in \mathscr{R}_n} Y_{\mathbf{i},k_0} \quad \text{and} \quad v_{k_1} = \sum_{\mathbf{i} \in \mathscr{R}_n} Y_{\mathbf{i},k_1}.$$

The frequency polygon, first rigorously analyzed by Scott [10], offers a compelling alternative to the histogram by delivering not only a faster integrated mean squared error (IMSE) convergence rate that matches that of non-negative kernel estimators but also low computational complexity. This attractive performance profile has sparked considerable interest in its theoretical properties. A significant body of work has focused on random fields. Carbon [2] provided a seminal analysis, deriving the IMSE-optimal bin width and uniform convergence rates for mixing fields. Research then expanded to include asymptotic normality (Carbon et al. [3])

and consistency under different settings (Bensaïd and Dabo-Niang [1]). Parallel developments for random variables began with Carbon et al. [4], who established IMSE and strong consistency for strong mixing processes. Later research extended these results, proving uniform strong consistency for NA, α-mixing, and φ-mixing samples (Xing and Yang [17]; Xing et al. [18]; Kang et al. [7]), uniformly complete consistency for widely orthant dependent data (Wu and Wang [15]), and the Berry–Esseen bound for α-mixing and ρ-mixing cases (Xing et al. [18]; Wu and Wang [16]).

In comparison, the results on frequency polygon estimation under random fields are relatively limited, and even more so for the case of ρ^--mixing random fields. Therefore, this work concentrates on ρ^--mixing random fields, with the goal of establishing the uniformly complete consistency and deriving the corresponding convergence rates for the frequency polygon estimator based on the samples $\{X_{\mathbf{i}},\, \mathbf{i} \in \mathscr{R}_{\mathbf{n}}\}$ from a ρ^--mixing random field.

The organization of the paper is outlined below. Section 2 is devoted to the central theoretical contributions. Section 3 is devoted to the numerical simulation and real data analysis, while the detailed proofs of the main theorems are reserved for Section 4.

We adopt the following notations. Let C be a context-dependent positive constant, not necessarily the same in each instance. The indicator function of an event B is denoted by $I(B)$, and $\lfloor x \rfloor$ represents the floor function, which gives the greatest integer not exceeding x. Consider a sequence of random variables $\{\xi_n,\, n \geqslant 1\}$ and a sequence of positive constants $\{a_n,\, n \geqslant 1\}$. We write $\xi_n = o(a_n)$ completely if $\sum_{n=1}^{\infty} P(|\xi_n|/a_n > \varepsilon) < \infty$ for every $\varepsilon > 0$. Similarly, $|\xi_n| = O(a_n)$ completely is taken to mean that $\sum_{n=1}^{\infty} P(|\xi_n|/a_n > M) < \infty$ for some $M > 0$. Unless otherwise specified, all limits are understood as $\mathbf{n} \to \infty$.

2. MAIN RESULTS

Prior to the derivation of our principal findings, we introduce the following assumptions:

(C1) $h_{\mathbf{n}} \to 0$.

(C2) $\nu_{\mathbf{n}}$ is a sequence of positive constants that tend to zero and

$$\liminf_{\mathbf{n}\to\infty} (\hat{\mathbf{n}}^{\delta} h_{\mathbf{n}} \nu_{\mathbf{n}}) > 0 \quad \text{for some } 0 < \delta < 1/2.$$

(C3) $f(x)$ obeys a Lipschitz condition, i.e. there exists a constant $L > 0$ such that for any x and y in the support of f,

$$|f(x) - f(y)| \leqslant L|x - y|.$$

(C4) The moment $E|X|^{2/T}$ is finite for some $T > 0$.

REMARK 2.1. The regularity conditions presented here are consistent with those adopted in the literature (e.g., Xing and Yang [17]; Wang et al. [14]; Wu and Wang [15]) on the consistency properties of frequency polygon estimators under random sampling.

REMARK 2.2. What needs to be emphasized here is that we can obtain $\hat{\mathbf{n}}^{\delta} h_{\mathbf{n}} \to \infty$ by (C2). From this, $\hat{\mathbf{n}} h_{\mathbf{n}} = \hat{\mathbf{n}}^{\delta} h_{\mathbf{n}} \cdot \hat{\mathbf{n}}^{1-\delta} \to \infty$. Hence, $\hat{\mathbf{n}} h_{\mathbf{n}} \to \infty$ does not require further assumptions.

Underpinned by the preceding assumptions, we now establish our principal results. Suppose G is a compact subset of $\mathbb{R}$.

THEOREM 2.1. *Assume that* (C1)–(C3) *are satisfied. Then*

$$\sup_{x \in G} |f_{\mathbf{n}}(x) - f(x)| = o(\nu_{\mathbf{n}}) + O(h_{\mathbf{n}}) \quad \textit{completely.} \tag{2.1}$$

THEOREM 2.2. *Granted that conditions* (C1)–(C4) *hold, it follows that*

$$\sup_{x \in \mathbb{R}} |f_{\mathbf{n}}(x) - f(x)| = o(\nu_{\mathbf{n}}) + O(h_{\mathbf{n}}) \quad \textit{completely.} \tag{2.2}$$

Substituting $\nu_{\mathbf{n}} = \hat{\mathbf{n}}^{-\delta} h_{\mathbf{n}}^{-1}$ directly yields the corollaries stated below. Let $\Psi_{\mathbf{n}} = \max\left\{h_{\mathbf{n}}, \hat{\mathbf{n}}^{-\delta} h_{\mathbf{n}}^{-1}\right\}$.

COROLLARY 2.1. *Assume that* (C1)–(C3) *are satisfied. Then*

$$\sup_{x \in G} |f_{\mathbf{n}}(x) - f(x)| = O(\Psi_{\mathbf{n}}) \quad \textit{completely.}$$

COROLLARY 2.2. *Granted that conditions* (C1)–(C4) *hold, it follows that*

$$\sup_{x \in \mathbb{R}} |f_{\mathbf{n}}(x) - f(x)| = O(\Psi_{\mathbf{n}}) \quad \textit{completely.}$$

REMARK 2.3. Deviating from the proof approaches of weak and strong consistency of frequency polygon estimation for mixing random field in Carbon [2], the paper employs the properties and moment inequalities of ρ^{-}-mixing random fields to derive complete consistency. The approach not only improves the existing results but also extends them to other types of random fields. Moreover, there have been no prior results concerning frequency polygon estimation under ρ^{-}-mixing random fields in the existing literature.

REMARK 2.4. Setting $d = 1$ yields complete consistency, along with the associated convergence rates, for frequency polygons applied to ρ^{-}-mixing random variables. This research goes a step beyond Wang et al. [14], achieving not only a generalization from random variables to random fields but also a refinement from strong to complete consistency, without requiring stronger assumptions.

3. SIMULATION STUDY

We conduct a comprehensive simulation study in this section. This study is designed to assess the performance of frequency polygons in finite samples, using data generated from or following the structure of ρ^--mixing random fields.

3.1. Numerical simulation

CASE 1. We consider a random field $\{X_{(i,j)}\}$ defined on a two-dimensional rectangular grid $\mathscr{R}_{\mathbf{n}} = \{(i,j) : 1 \leqslant i \leqslant n_1,\ 1 \leqslant j \leqslant n_2\}$, where $n_1 = n_2 \in \{10, 20, 30, 40, 50, 60\}$. To generate a sample from a ρ^--mixing random field, we adopt the following approach. First, all points in the grid $\mathscr{R}_{\mathbf{n}}$ are arranged in lexicographic order, resulting in a sequence of length

$$\hat{\mathbf{n}} = n_1 n_2 = 100, 400, 900, 1600, 2500, 3600.$$

Based on this ordering, we construct an $\hat{\mathbf{n}} \times \hat{\mathbf{n}}$ covariance matrix $\mathbf{\Sigma}$. This matrix is symmetric and banded: its main diagonal elements are all 1, and the first sub-diagonals and super-diagonals have all entries -0.18, with all other entries of the matrix being zero. A multivariate normal random vector $X = (X_1, \ldots, X_{\hat{\mathbf{n}}})' \sim N(\mathbf{0}, \mathbf{\Sigma})$ is generated. The components X_k of X are reassigned to their corresponding grid locations (i,j) via the inverse of the aforementioned lexicographic mapping, thereby obtaining the desired random field observations $\{X_{(i,j)},\ (i,j) \in \mathscr{R}_{\mathbf{n}}\}$. Following Joag-Dev and Proschan [6], the sequence generated by the above procedure, and by extension the mapped random field, possesses the ρ^--mixing property. Furthermore, each marginal distribution $X_{(i,j)}$ follows a standard normal distribution; thus, the true density function $f(x)$ to be estimated is the standard normal density. Let $h_{\mathbf{n}} = \left(\frac{\log \hat{\mathbf{n}}}{\hat{\mathbf{n}}}\right)^{1/3}$. The 200 data points x_i $(i = 1, \ldots, 200)$ are defined over the interval $[-3, 3]$, generated with a constant spacing of 0.03015.

To assess the convergence properties of density estimates obtained through frequency polygons, we perform 1000 independent replications to obtain the corresponding final values of this estimation. These estimates are compared with the true density $f(x)$ in Figures 1–3. It is clearly evident from the results that the frequency polygon estimator performs better and better as $\hat{\mathbf{n}}$ increases. Furthermore, we calculate the mean absolute errors (MAE) for each x_i, and plot the boxplots based on the MAE values for all sample sizes $\hat{\mathbf{n}}$ in Figure 4. The MAE converges to zero as $\hat{\mathbf{n}}$ increases, confirming that the edge frequency polygon estimator is consistent.

CASE 2. We can generate the components X_k of the vector X following the method in Example 1 of Zhang and Wang [21], as described below. Let $\{\xi_k,\ k \geqslant 1\}$, $\{\eta_k,\ k \geqslant 1\}$ and $\{\tau_k, k \geqslant 1\}$ be three independent sequences of independent and identically distributed standard normal random variables.

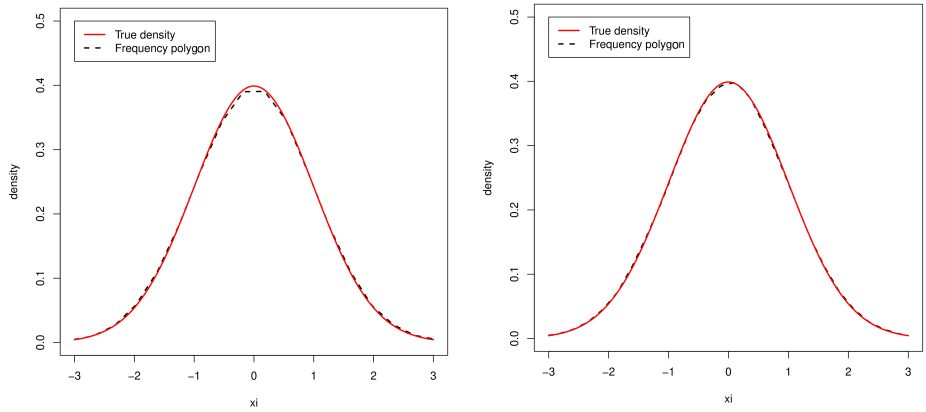


FIGURE 1. Comparative plots of $f_{\mathbf{n}}(x)$ against $f(x)$ for $\hat{\mathbf{n}} = 100$ and $\hat{\mathbf{n}} = 400$

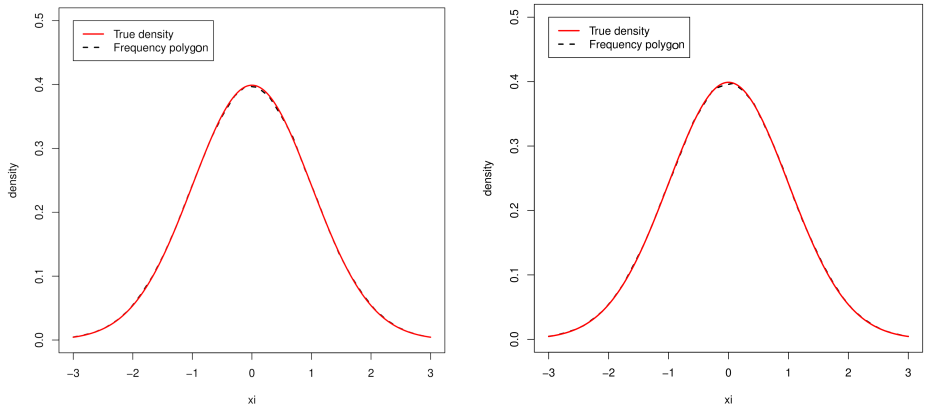


FIGURE 2. Comparative plots of $f_{\mathbf{n}}(x)$ against $f(x)$ for $\hat{\mathbf{n}} = 900$ and $\hat{\mathbf{n}} = 1600$

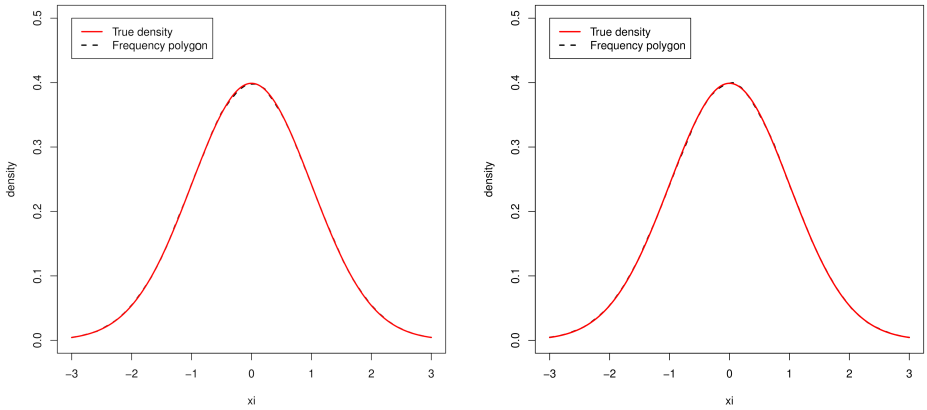


FIGURE 3. Comparative plots of $f_{\mathbf{n}}(x)$ against $f(x)$ for $\hat{\mathbf{n}} = 2500$ and $\hat{\mathbf{n}} = 3600$

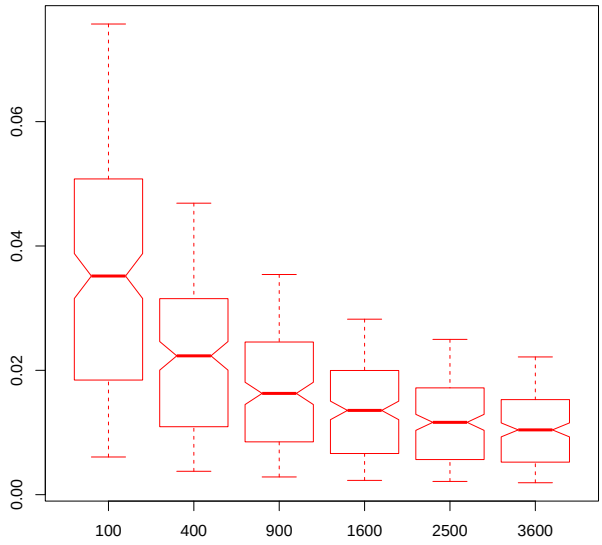


FIGURE 4. The boxplots of the MAE for $\hat{\mathbf{n}} = 100, 400, 900, 1600, 2500, 3600$

Define

$$Y_k = \begin{cases} \xi_m & \text{if } k = 2m-1, \\ -\xi_m & \text{if } k = 2m, \end{cases} \qquad Z_k = \begin{cases} \eta_m & \text{if } k = 2^{2m-1}, \\ -\eta_m & \text{if } k = 2^{2m}, \\ \tau_k & \text{otherwise}, \end{cases}$$

and $X_k = Y_k^2 + Z_k$. Similarly, via the inverse of the aforementioned lexicographic mapping, we can obtain a ρ^--mixing random field $\{X_{(i,j)}, (i,j) \in \mathscr{R}_{\mathbf{n}}\}$, and following the calculations in Wang et al. [14], the marginal density function $f(x)$ of $X_{(i,j)}$ is $\frac{1}{\pi}\int_0^\infty e^{-\frac{s^2+(x-s^2)^2}{2}} ds$. Other settings are the same as those in Case 1. We also compute the estimator for 1000 times to obtain the final values and then compare them with $f(x)$ in Figures 5–7 under different sample sizes. We present boxplots of the MAE values for each sample in Figure 8. It can be seen that Figures 5–8 yield a similar conclusion to those in Case 1.

3.2. Real data analysis. To further illustrate the practical applicability of our theoretical results, we analyze the lead-to-copper ratio from the classic *meuse* spatial dataset. This dataset contains heavy metal concentrations measured at 155 locations in a floodplain of the river Meuse. We first explore the ρ^--mixing assumption by rearranging the spatial observations in lexicographic order based on their coordinates. The autocorrelation function (ACF) and partial autocorrelation function (PACF) plots for the lexicographically ordered lead-to-copper ratio series are presented in Figure 9. The ACF shows a sharp cut-off after the fifth lag, with values remaining insignificant within the confidence bounds thereafter. This pattern is characteristic of a moving average (MA) structure. Crucially, the rapid decay of

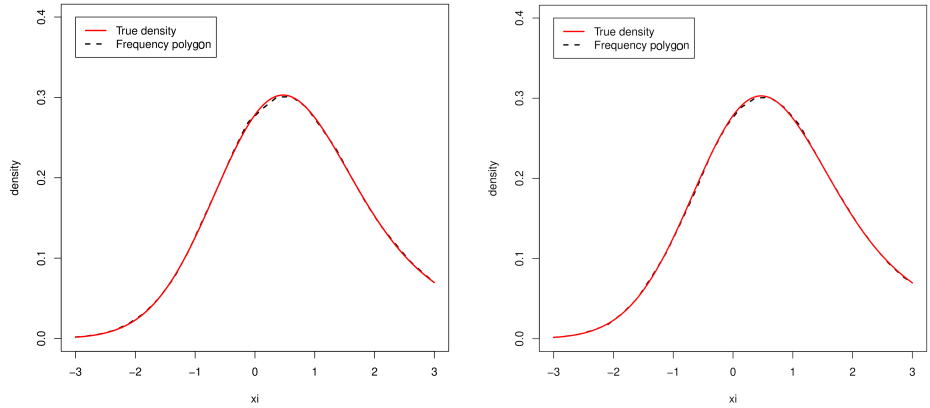


FIGURE 5. Comparative plots of $f_{\mathbf{n}}(x)$ against $f(x)$ for $\hat{\mathbf{n}} = 100$ and $\hat{\mathbf{n}} = 400$

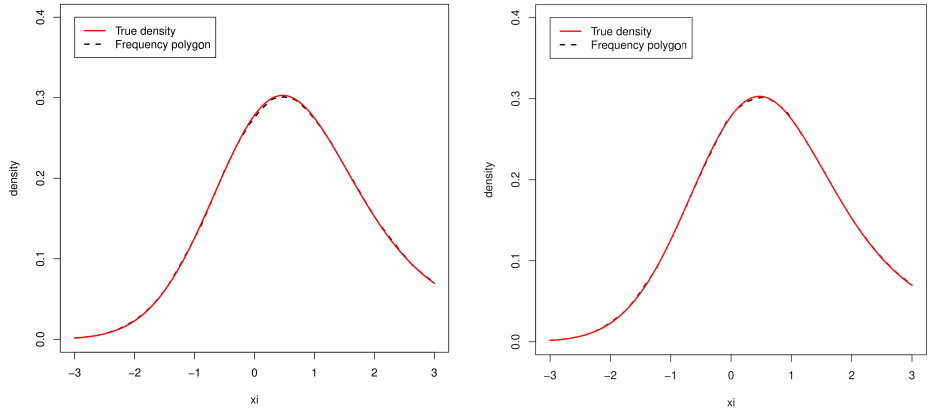


FIGURE 6. Comparative plots of $f_{\mathbf{n}}(x)$ against $f(x)$ for $\hat{\mathbf{n}} = 900$ and $\hat{\mathbf{n}} = 1600$

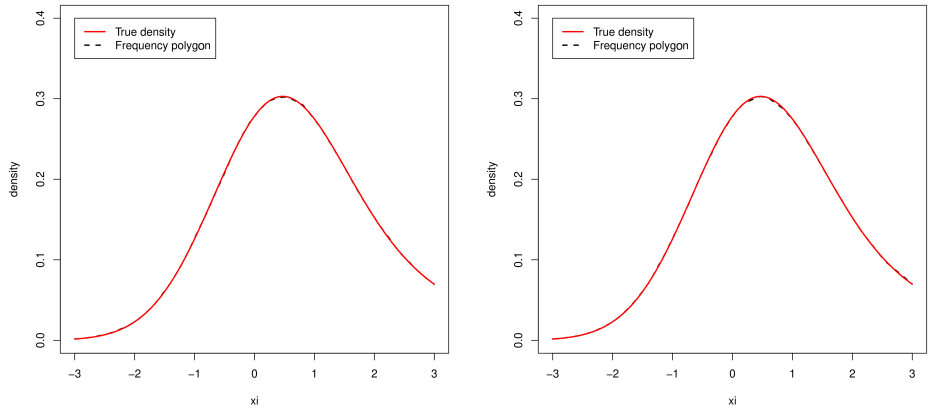


FIGURE 7. Comparative plots of $f_{\mathbf{n}}(x)$ against $f(x)$ for $\hat{\mathbf{n}} = 2500$ and $\hat{\mathbf{n}} = 3600$

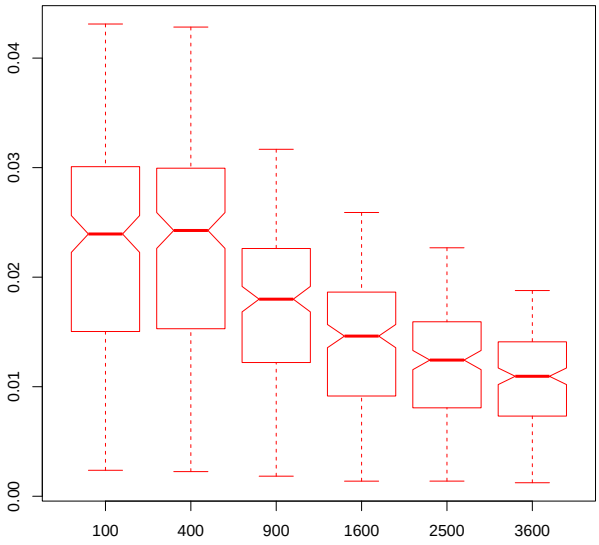


FIGURE 8. The boxplots of the MAE for $\hat{\mathbf{n}} = 100, 400, 900, 1600, 2500, 3600$

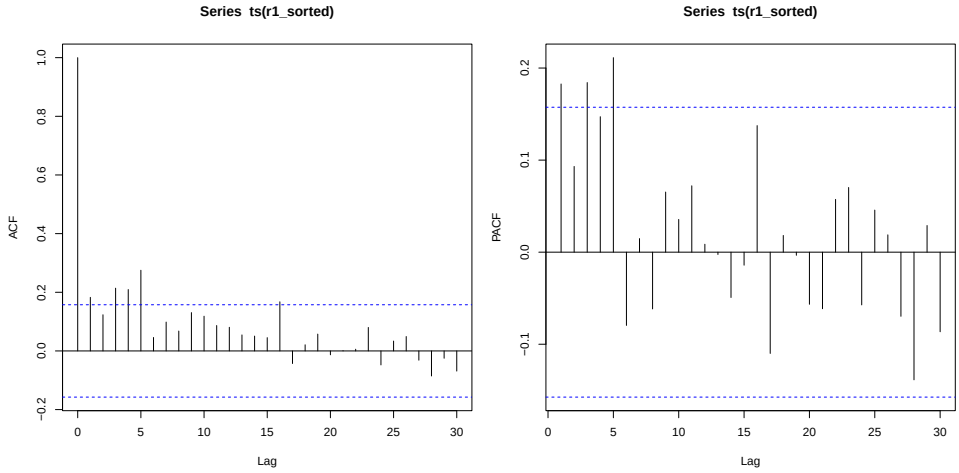


FIGURE 9. ACF and PACF of the lead-to-copper ratio

the ACF signifies that the serial dependence becomes negligible after a few lags. This property of asymptotic uncorrelatedness is a fundamental manifestation of the weak dependence inherent in ρ^--mixing random fields.

Under this ρ^--mixing framework, we apply the frequency polygon estimator with bandwidth $h_{\mathbf{n}} = \left(\frac{\log \hat{\mathbf{n}}}{\hat{\mathbf{n}}}\right)^{1/3}$, where $\hat{\mathbf{n}} = 155$. Figure 10 presents both the histogram and frequency polygon density estimates for the lead-to-copper ratio. The frequency polygon provides a visibly smoother estimate than the histogram while

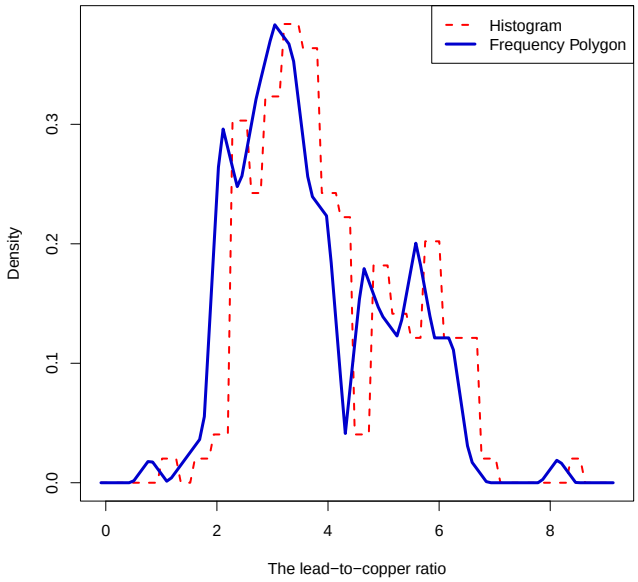


FIGURE 10. The histogram (red line) and the frequency polygon density curve (blue line) of lead-to-copper ratio

preserving the modal structure of the distribution, demonstrating its practical utility. This real data analysis demonstrates that spatial environmental data can exhibit ρ^--mixing properties, and our frequency polygon estimator performs effectively under such dependency structures.

4. PRELIMINARY LEMMAS AND PROOFS OF MAIN RESULTS

4.1. Preliminary lemmas. To pave the way for proving the main theorems, several foundational lemmas that serve as essential building blocks are introduced.

LEMMA 4.1 (cf. Zhang and Wang [21, Property P2]). *A family of increasing functions defined on disjoint subsets of a ρ^--mixing random field $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ inherits the ρ^--mixing property, with its mixing coefficients not exceeding the original $\rho^-(\tau)$.*

The Rosenthal-type inequality developed for ρ^--mixing random fields serves as a foundational component in establishing our principal findings.

LEMMA 4.2 (cf. Zhang and Wang [21, Theorem 2.1]). *Consider a centered ρ^--mixing random field $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ satisfying $\|X_{\mathbf{n}}\|_q < \infty$ for some $q \geqslant 2$ and all $\mathbf{n}$. For any finite set $D \subset \mathbb{N}^d$, there is a constant $B_q > 0$ which relies only on q and $\rho^-(\cdot)$, for which*

$$E\Big|\sum_{\mathbf{n}\in D} X_{\mathbf{n}}\Big|^q \leqslant B_q\Big(\sum_{\mathbf{n}\in D} E|X_{\mathbf{n}}|^q + \Big(\sum_{\mathbf{n}\in D} E|X_{\mathbf{n}}|^2\Big)^{q/2}\Big).$$

LEMMA 4.3. *Assume* $E|X|^{2/T} < \infty$, *and define* $A_{\mathbf{n}} = (-\infty, -\hat{\mathbf{n}}^{\lambda T}) \cup (\hat{\mathbf{n}}^{\lambda T}, +\infty)$ *for some* $\lambda > 1$. *Then*

$$\sup_{x \in A_{\mathbf{n}}} f_{\mathbf{n}}(x) = O(\hat{\mathbf{n}}^{-1}) \quad \textit{completely.}$$

Imposing the additional condition that $f(x)$ *is uniformly continuous leads to the sharper result*

$$\sup_{x \in A_{\mathbf{n}}} f(x) = O(\hat{\mathbf{n}}^{-2\lambda}).$$

Proof. As an immediate consequence of Markov's inequality,

$$\begin{aligned}
\sum_{\mathbf{n}} P\Big(\sup_{x \in A_{\mathbf{n}}} f_{\mathbf{n}}(x) > \hat{\mathbf{n}}^{-1}\Big) &\leqslant \sum_{\mathbf{n}} P\Big(\sup_{x \in A_{\mathbf{n}}} f_{\mathbf{n}}(x) > 0\Big) \\
&\leqslant \sum_{\mathbf{n}} P\Big(f_{\mathbf{n}}(x) > 0, \exists x \in A_{\mathbf{n}}\Big) \leqslant \sum_{\mathbf{n}} P\Big(\bigcup_{\mathbf{i} \in \mathscr{R}_{\mathbf{n}}} (X_{\mathbf{i}} \in B_{\mathbf{n}})\Big) \\
&\leqslant \sum_{\mathbf{n}} \sum_{\mathbf{i} \in \mathscr{R}_{\mathbf{n}}} P(|X_{\mathbf{i}}| > \hat{\mathbf{n}}^{\lambda T}/2) = \sum_{\mathbf{n}} \hat{\mathbf{n}} P(|X| > \hat{\mathbf{n}}^{\lambda T}/2) \\
&= \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} n_1 \cdots n_d P(|X| > (n_1 \cdots n_d)^{\lambda T}/2) \\
&\leqslant C \sum_{n_1=1}^{\infty} n_1^{1-2\lambda} \cdots \sum_{n_d=1}^{\infty} n_d^{1-2\lambda} E|X|^{2/T} < \infty,
\end{aligned}$$

which yields $\sup_{x \in A_{\mathbf{n}}} f_{\mathbf{n}}(x) = O(\hat{\mathbf{n}}^{-1})$ completely, where $B_{\mathbf{n}} = (-\infty, -\hat{\mathbf{n}}^{\lambda T}/2) \cup (\hat{\mathbf{n}}^{\lambda T}/2, +\infty)$.

Combining the uniform continuity of $f(x)$ with the integrability condition $E|X|^{2/T} = \int_{-\infty}^{\infty} |x|^{2/T} f(x)dx < \infty$ yields the asymptotic result that $|x|^{2/T} f(x) \to 0$ as $|x| \to \infty$. As a result, when $\min_{1 \leqslant k \leqslant d} \{n_k\}$ large enough,

$$\sup_{x \in A_{\mathbf{n}}} f(x) \leqslant \sup_{|x| > \hat{\mathbf{n}}^{\lambda T}} \hat{\mathbf{n}}^{-2\lambda} |x|^{2/T} f(x) = O(\hat{\mathbf{n}}^{-2\lambda}).$$

This completes the proof. ∎

4.2. The proof of Theorem 2.1. The uniform deviation $\sup_{x \in G} |f_{\mathbf{n}}(x) - f(x)|$ is bounded by the sum of two nonnegative terms: $\sup_{x \in G} |f_{\mathbf{n}}(x) - Ef_{\mathbf{n}}(x)|$ and $\sup_{x \in G} |Ef_{\mathbf{n}}(x) - f(x)|$. These are denoted by $S_{1\mathbf{n}}$ and $S_{2\mathbf{n}}$ respectively, yielding the upper bound $S_{1\mathbf{n}} + S_{2\mathbf{n}}$. We proceed to study the complete convergence of the two components, $S_{1\mathbf{n}}$ and $S_{2\mathbf{n}}$, individually.

We may restrict our attention to compact sets of the form $G = [-H, H]$ for some $H > 0$, as this assumption simplifies the exposition without diminishing the generality of the results. Let $r_{\mathbf{n}} = \lfloor H/h_{\mathbf{n}} \rfloor + 1$. Since $r_{\mathbf{n}} h_{\mathbf{n}} > H$, we have $[-H, H] \subset \bigcup_{k_0=-r_{\mathbf{n}}}^{r_{\mathbf{n}}} U_{\mathbf{n},k_0}$. Hence, for any $\epsilon > 0$,

$$P(|S_{1\mathbf{n}}| > \epsilon\nu_{\mathbf{n}}) \leqslant P\Big(\max_{-r_{\mathbf{n}}\leqslant k_0\leqslant r_{\mathbf{n}}} \sup_{x\in U_{\mathbf{n},k_0}} |f_{\mathbf{n}}(x) - Ef_{\mathbf{n}}(x)| > \epsilon\nu_{\mathbf{n}}\Big)$$
$$\leqslant \sum_{k_0=-r_{\mathbf{n}}}^{r_{\mathbf{n}}} P\Big(\sup_{x\in U_{\mathbf{n},k_0}} |f_{\mathbf{n}}(x) - Ef_{\mathbf{n}}(x)| > \epsilon\nu_{\mathbf{n}}\Big).$$

Denote $Z_{\mathbf{i},j} = Y_{\mathbf{i},j} - EY_{\mathbf{i},j}$. One may directly verify that

$$f_{\mathbf{n}}(x) - Ef_{\mathbf{n}}(x) = \Big(\frac{1}{2} + k_0 - \frac{x}{h_{\mathbf{n}}}\Big)\hat{\mathbf{n}}^{-1}h_{\mathbf{n}}^{-1}\sum_{\mathbf{i}\in\mathscr{R}_n} Z_{\mathbf{i},k_0}$$
$$+ \Big(\frac{1}{2} - k_0 + \frac{x}{h_{\mathbf{n}}}\Big)\hat{\mathbf{n}}^{-1}h_{\mathbf{n}}^{-1}\sum_{\mathbf{i}\in\mathscr{R}_n} Z_{\mathbf{i},k_0+1}.$$

Thus, by $0 < \frac{1}{2} + k_0 - \frac{x}{h_{\mathbf{n}}} \leqslant 1$ and $0 \leqslant \frac{1}{2} - k_0 + \frac{x}{h_{\mathbf{n}}} < 1$ for any $x \in U_{\mathbf{n},k_0}$, we have

$$P(|S_{1\mathbf{n}}| > \epsilon\nu_{\mathbf{n}}) \leqslant \sum_{k_0=-r_{\mathbf{n}}}^{r_{\mathbf{n}}} P\Big(\Big|\sum_{\mathbf{i}\in\mathscr{R}_n} Z_{\mathbf{i},k_0}\Big| > \frac{\epsilon}{2}\hat{\mathbf{n}}h_{\mathbf{n}}\nu_{\mathbf{n}}\Big)$$
$$+ \sum_{k_0=-r_{\mathbf{n}}}^{r_{\mathbf{n}}} P\Big(\Big|\sum_{\mathbf{i}\in\mathscr{R}_n} Z_{\mathbf{i},k_0+1}\Big| > \frac{\epsilon}{2}\hat{\mathbf{n}}h_{\mathbf{n}}\nu_{\mathbf{n}}\Big). \tag{4.1}$$

The variable

$$Z_{\mathbf{i},k_0} = I((k_0-1)h_{\mathbf{n}} \leqslant X_{\mathbf{i}} < k_0h_{\mathbf{n}}) - EI((k_0-1)h_{\mathbf{n}} \leqslant X_{\mathbf{i}} < k_0h_{\mathbf{n}})$$

can be decomposed as $Z_{\mathbf{i},k_0} = Z^{(1)}_{\mathbf{i},k_0} - Z^{(2)}_{\mathbf{i},k_0}$, with the components defined by $Z^{(1)}_{\mathbf{i},k_0} = I(X_{\mathbf{i}} \geqslant (k_0-1)h_{\mathbf{n}}) - EI(X_{\mathbf{i}} \geqslant (k_0-1)h_{\mathbf{n}})$ and $Z^{(2)}_{\mathbf{i},k_0} = I(X_{\mathbf{i}} \geqslant k_0h_{\mathbf{n}}) - EI(X_{\mathbf{i}} \geqslant k_0h_{\mathbf{n}})$. According to Lemma 4.1, both $\{Z^{(1)}_{\mathbf{i},k_0},\, \mathbf{i}\in\mathbb{N}^d\}$ and $\{Z^{(2)}_{\mathbf{i},k_0},\, \mathbf{i}\in\mathbb{N}^d\}$ are centered ρ^--mixing random fields. Thus, by Lemma 4.2 and $|Z^{(j)}_{\mathbf{i},k_0}| \leqslant 1$ $(j = 1, 2)$, for $q > \frac{2}{1/2-\delta}$ we have

$$\sum_{k_0=-r_{\mathbf{n}}}^{r_{\mathbf{n}}} P\Big(\Big|\sum_{\mathbf{i}\in\mathscr{R}_n} Z_{\mathbf{i},k_0}\Big| > \frac{\epsilon}{2}\hat{\mathbf{n}}h_{\mathbf{n}}\nu_{\mathbf{n}}\Big)$$
$$\leqslant \sum_{k_0=-r_{\mathbf{n}}}^{r_{\mathbf{n}}} P\Big(\Big|\sum_{\mathbf{i}\in\mathscr{R}_n} Z^{(1)}_{\mathbf{i},k_0}\Big| > \frac{\epsilon}{4}\hat{\mathbf{n}}h_{\mathbf{n}}\nu_{\mathbf{n}}\Big) + \sum_{k_0=-r_{\mathbf{n}}}^{r_{\mathbf{n}}} P\Big(\Big|\sum_{\mathbf{i}\in\mathscr{R}_n} Z^{(2)}_{\mathbf{i},k_0}\Big| > \frac{\epsilon}{4}\hat{\mathbf{n}}h_{\mathbf{n}}\nu_{\mathbf{n}}\Big)$$
$$\leqslant C\sum_{k_0=-r_{\mathbf{n}}}^{r_{\mathbf{n}}} (\hat{\mathbf{n}}h_{\mathbf{n}}\nu_{\mathbf{n}})^{-q} E\Big|\sum_{\mathbf{i}\in\mathscr{R}_n} Z^{(1)}_{\mathbf{i},k_0}\Big|^q + C\sum_{k_0=-r_{\mathbf{n}}}^{r_{\mathbf{n}}} (\hat{\mathbf{n}}h_{\mathbf{n}}\nu_{\mathbf{n}})^{-q} E\Big|\sum_{\mathbf{i}\in\mathscr{R}_n} Z^{(2)}_{\mathbf{i},k_0}\Big|^q$$
$$\leqslant C\sum_{k_0=-r_{\mathbf{n}}}^{r_{\mathbf{n}}} (\hat{\mathbf{n}}h_{\mathbf{n}}\nu_{\mathbf{n}})^{-q}\Big(\sum_{\mathbf{i}\in\mathscr{R}_n} E|Z^{(1)}_{\mathbf{i},k_0}|^q + \Big(\sum_{\mathbf{i}\in\mathscr{R}_n} E|Z^{(1)}_{\mathbf{i},k_0}|^2\Big)^{q/2}\Big)$$
$$+ C\sum_{k_0=-r_{\mathbf{n}}}^{r_{\mathbf{n}}} (\hat{\mathbf{n}}h_{\mathbf{n}}\nu_{\mathbf{n}})^{-q}\Big(\sum_{\mathbf{i}\in\mathscr{R}_n} E|Z^{(2)}_{\mathbf{i},k_0}|^q + \Big(\sum_{\mathbf{i}\in\mathscr{R}_n} E|Z^{(2)}_{\mathbf{i},k_0}|^2\Big)^{q/2}\Big)$$
$$\leqslant C\sum_{k_0=-r_{\mathbf{n}}}^{r_{\mathbf{n}}} (\hat{\mathbf{n}}h_{\mathbf{n}}\nu_{\mathbf{n}})^{-q}(\hat{\mathbf{n}} + \hat{\mathbf{n}}^{q/2}) \leqslant C\hat{\mathbf{n}}^{-q/2}h_{\mathbf{n}}^{-q}\nu_{\mathbf{n}}^{-q}(2r_{\mathbf{n}}+1),$$

which together with (C1) and (C2) implies that

$$
\begin{aligned}
(4.2)\qquad &\sum_{\mathbf{n}}\sum_{k_0=-r_{\mathbf{n}}}^{r_{\mathbf{n}}} P\Big(\Big|\sum_{\mathbf{i}\in\mathscr{R}_n} Z_{\mathbf{i},k_0}\Big| > \frac{\epsilon}{2}\hat{\mathbf{n}}h_{\mathbf{n}}\nu_{\mathbf{n}}\Big)\\
&\quad\leqslant C\sum_{\mathbf{n}}\hat{\mathbf{n}}^{-q/2}h_{\mathbf{n}}^{-q}\nu_{\mathbf{n}}^{-q}r_{\mathbf{n}} \leqslant C\sum_{\mathbf{n}}\hat{\mathbf{n}}^{-q/2}h_{\mathbf{n}}^{-1-q}\nu_{\mathbf{n}}^{-q}\\
&\quad\leqslant C\sum_{\mathbf{n}}\hat{\mathbf{n}}^{q\delta-q/2+1}\\
&\quad= C\Big(\sum_{n_1=1}^{\infty} n_1^{q\delta-q/2+1}\Big)\Big(\sum_{n_2=1}^{\infty} n_2^{q\delta-q/2+1}\Big)\cdots\Big(\sum_{n_d=1}^{\infty} n_d^{q\delta-q/2+1}\Big) < \infty.
\end{aligned}
$$

Similarly,

$$
(4.3)\qquad \sum_{\mathbf{n}}\sum_{k_0=-r_{\mathbf{n}}}^{r_{\mathbf{n}}} P\Big(\Big|\sum_{\mathbf{i}\in\mathscr{R}_n} Z_{\mathbf{i},k_0+1}\Big| > \frac{\epsilon}{2}\hat{\mathbf{n}}h_{\mathbf{n}}\nu_{\mathbf{n}}\Big) < \infty.
$$

Hence, by (4.1)–(4.3), we can see that for any $\epsilon > 0$,

$$
(4.4)\qquad \sum_{\mathbf{n}} P(|S_{1\mathbf{n}}| > \epsilon\nu_{\mathbf{n}}) < \infty.
$$

Regarding $S_{2\mathbf{n}}$, it is readily verified that for every $x \in U_{\mathbf{n},k_0}$,

$$
\begin{aligned}
|Ef_{\mathbf{n}}(x) - f(x)| &= \Big|\Big(\frac{1}{2} + k_0 - \frac{x}{h_{\mathbf{n}}}\Big)\hat{\mathbf{n}}^{-1}h_{\mathbf{n}}^{-1}\sum_{\mathbf{i}\in\mathscr{R}_n} EY_{\mathbf{i},k_0}\\
&\quad + \Big(\frac{1}{2} - k_0 + \frac{x}{h_{\mathbf{n}}}\Big)\hat{\mathbf{n}}^{-1}h_{\mathbf{n}}^{-1}\sum_{\mathbf{i}\in\mathscr{R}_n} EY_{\mathbf{i},k_0+1} - f(x)\Big|\\
&= \Big|\Big(\frac{1}{2} + k_0 - \frac{x}{h_{\mathbf{n}}}\Big)h_{\mathbf{n}}^{-1}\int_{(k_0-1)h_{\mathbf{n}}}^{k_0h_{\mathbf{n}}} f(y)\,dy\\
&\quad + \Big(\frac{1}{2} - k_0 + \frac{x}{h_{\mathbf{n}}}\Big)h_{\mathbf{n}}^{-1}\int_{k_0h_{\mathbf{n}}}^{(k_0+1)h_{\mathbf{n}}} f(y)dy - f(x)\Big|\\
&\leqslant \Big(\frac{1}{2} + k_0 - \frac{x}{h_{\mathbf{n}}}\Big)h_{\mathbf{n}}^{-1}\int_{(k_0-1)h_{\mathbf{n}}}^{k_0h_{\mathbf{n}}} |f(y) - f(x)|\,dy\\
&\quad + \Big(\frac{1}{2} - k_0 + \frac{x}{h_{\mathbf{n}}}\Big)h_{\mathbf{n}}^{-1}\int_{k_0h_{\mathbf{n}}}^{(k_0+1)h_{\mathbf{n}}} |f(y) - f(x)|dy\\
&\leqslant h_{\mathbf{n}}^{-1}\int_{(k_0-1)h_{\mathbf{n}}}^{(k_0+1)h_{\mathbf{n}}} |f(y) - f(x)|\,dy.
\end{aligned}
$$

The inequality $|f(y) - f(x)| \leqslant L|y - x|$ holds for all $x, y \in G$ by (C3). Under the condition that $x \in U_{\mathbf{n},k_0}$ and letting y be any point in the segment

$[(k_0-1)h_{\mathbf{n}},(k_0+1)h_{\mathbf{n}}]$, $|y-x| \leqslant 3h_{\mathbf{n}}/2$, we find that when $\min_{1\leqslant k\leqslant d}\{n_k\}$ is sufficiently large,

$$
\begin{aligned}
S_{2\mathbf{n}} &\leqslant \max_{-r_{\mathbf{n}}\leqslant k_0\leqslant r_{\mathbf{n}}} \sup_{x\in U_{\mathbf{n},k_0}} |Ef_{\mathbf{n}}(x)-f(x)| \\
&\leqslant \max_{-r_{\mathbf{n}}\leqslant k_0\leqslant r_{\mathbf{n}}} \sup_{x\in U_{\mathbf{n},k_0}} Lh_{\mathbf{n}}^{-1} \int_{(k_0-1)h_{\mathbf{n}}}^{(k_0+1)h_{\mathbf{n}}} \frac{3}{2}h_{\mathbf{n}}dy \leqslant Ch_{\mathbf{n}},
\end{aligned} \tag{4.5}
$$

that is, $S_{2\mathbf{n}} = O(h_{\mathbf{n}})$. Combining (4.4) and (4.5) directly leads to (2.1). ∎

4.3. The proof of Theorem 2.2. The supremum norm of the difference between $f_{\mathbf{n}}(x)$ and $f(x)$ over $x \in \mathbb{R}$ can be expressed as the sum of $M_{1\mathbf{n}}$, $M_{2\mathbf{n}}$, $M_{3\mathbf{n}}$, where

$$
\begin{aligned}
M_{1\mathbf{n}} &= \sup_{x\in A_{\mathbf{n}}} |f_{\mathbf{n}}(x)-f(x)| \quad (\lambda=2), \\
M_{2\mathbf{n}} &= \sup_{x\in[-\hat{\mathbf{n}}^{2T},\hat{\mathbf{n}}^{2T}]} |f_{\mathbf{n}}(x)-Ef_{\mathbf{n}}(x)|, \\
M_{3\mathbf{n}} &= \sup_{x\in[-\hat{\mathbf{n}}^{2T},\hat{\mathbf{n}}^{2T}]} |Ef_{\mathbf{n}}(x)-f(x)|.
\end{aligned}
$$

It follows directly from the results established in Lemma 4.3 that

$$
M_{1\mathbf{n}} = O(\hat{\mathbf{n}}^{-1}) = o(h_{\mathbf{n}}) \quad \text{completely,}
$$

For $M_{2\mathbf{n}}$, let $G=[-\hat{\mathbf{n}}^{2T},\hat{\mathbf{n}}^{2T}]$, $r_{\mathbf{n}} = \lfloor \hat{\mathbf{n}}^{2T}/h_{\mathbf{n}} \rfloor + 1$. Since $r_{\mathbf{n}}h_{\mathbf{n}} > \hat{\mathbf{n}}^{2T}$, we have $\big[-\hat{\mathbf{n}}^{2T},\hat{\mathbf{n}}^{2T}\big] \subset \bigcup_{k_0=-r_{\mathbf{n}}}^{r_{\mathbf{n}}} U_{\mathbf{n},k_0}$. Similar to the proof of the complete convergence of $S_{1\mathbf{n}}$ in Theorem 2.1, we have, for $q > \frac{2T+2}{1/2-\delta}$,

$$
\begin{aligned}
&\sum_{\mathbf{n}} P(|M_{2\mathbf{n}}| > \epsilon\nu_{\mathbf{n}}) \\
&\quad\leqslant \sum_{\mathbf{n}} P\Big(\max_{-r_{\mathbf{n}}\leqslant k_0\leqslant r_{\mathbf{n}}} \sup_{x\in U_{\mathbf{n},k_0}} |f_{\mathbf{n}}(x)-Ef_{\mathbf{n}}(x)| > \epsilon\nu_{\mathbf{n}}\Big) \\
&\quad\leqslant \sum_{\mathbf{n}} \sum_{k_0=-r_{\mathbf{n}}}^{r_{\mathbf{n}}} P\Big(\sup_{x\in U_{\mathbf{n},k_0}} |f_{\mathbf{n}}(x)-Ef_{\mathbf{n}}(x)| > \epsilon\nu_{\mathbf{n}}\Big) \\
&\quad\leqslant C\sum_{\mathbf{n}} \hat{\mathbf{n}}^{-q/2}h_{\mathbf{n}}^{-q}\nu_{\mathbf{n}}^{-q}(2r_{\mathbf{n}}+1) \leqslant C\sum_{\mathbf{n}} \hat{\mathbf{n}}^{2T-q/2}h_{\mathbf{n}}^{-1-q}\nu_{\mathbf{n}}^{-q} \\
&\quad\leqslant C\sum_{\mathbf{n}} \hat{\mathbf{n}}^{-(1/2-\delta)q+2T+1} \\
&\quad= C\Big(\sum_{n_1=1}^{\infty} n_1^{-(1/2-\delta)q+2T+1}\Big)\cdots\Big(\sum_{n_d=1}^{\infty} n_d^{-(1/2-\delta)q+2T+1}\Big) < \infty,
\end{aligned}
$$

which implies that $M_{2\mathbf{n}} = o(\nu_{\mathbf{n}})$ completely. Following the same reasoning as in the proof for $S_{2\mathbf{n}}$ from Theorem 2.1, we obtain $M_{3\mathbf{n}} = O(h_{\mathbf{n}})$. This, together with the convergence of $M_{1\mathbf{n}}$ and $M_{2\mathbf{n}}$, yields the required result (2.2). ∎

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