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ASYMPTOTIC NORMALITY FOR ESTIMATORS IN HETEROSCEDASTIC SEMIPARAMETRIC EV MODELS WITH α-MIXING ERRORS*

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Abstract. In this paper, the heteroscedastic semiparametric errors-in-variables (EV) model, $y_i = \xi_i\beta + g(t_i) + \epsilon_i, x_i = \xi_i + \mu_i$, is considered, where $\epsilon_i = \sigma_i e_i, \sigma_i^2 = f(u_i)$, β is an unknown parameter to be estimated, and $g(\cdot)$ and $f(\cdot)$ are unknown functions to be estimated. Under the assumption of α-mixing random errors, we establish the asymptotic normality for estimators of β, $g(\cdot)$ and $f(\cdot)$ in both homoscedastic and heteroscedastic cases. Numerical simulations are conducted to examine the finite-sample performance of the proposed estimators.

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1. INTRODUCTION

Consider the following heteroscedastic semiparametric errors-in-variables (EV) model:

$$\begin{cases} y_i = \xi_i\beta + g(t_i) + \epsilon_i, \\ x_i = \xi_i + \mu_i, \end{cases} \tag{1.1}$$

where $1 \leqslant i \leqslant n, \epsilon_i = \sigma_i e_i, \sigma_i^2 = f(u_i), (t_i, u_i)$ are known design points, ξ_i denote the unobservable latent variables, y_i are the response variables which can be observed, x_i are observed with measurement errors μ_i and $E\mu_i = 0$, while e_i

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are random errors with $Ee_i = 0$ and $Ee_i^2 = 1$, β is an unknown parameter to be estimated, and $g(\cdot)$ and $f(\cdot)$ are unknown functions defined on a compact set $A \subset \mathbb{R}$. In this paper, u_i is treated as a fixed design point; it may be chosen as a deterministic function of t_i or of the sampling scheme, but it is assumed not to depend on the unobservable variables ξ_i or on the random errors e_i and μ_i.

In recent years, semiparametric EV models have attracted considerable attention. For example, Cui and Li [3] obtained estimators of the parameter β, the variance σ_i^2 and the smooth function $g(\cdot)$ by using the nearest neighbor generalized least squares method; Wang [21] obtained estimators of β and $g(\cdot)$ based on the least squares criterion and proved their strong consistency under independence assumption; Zhou et al. [32] proposed a new estimation method by taking the measurement errors into account and developed a variable selection procedure for the parametric component; Zhang et al. [28] established the asymptotic normality for estimators in model (1.1) under stationary strong mixing and independence assumption; Zhang and Wang [29] investigated the rates of strong consistency for estimators in model (1.1) with negatively associated (NA) samples.

More recently, semiparametric and errors-in-variables models have been further studied from different perspectives. Lin et al. [17] investigated partially linear single index models in the presence of measurement error. Yan et al. [22] studied empirical likelihood methods for partially linear errors-in-variables models with longitudinal data. Zhao et al. [24] considered variable selection for longitudinal varying coefficient errors-in-variables models. Zhang et al. [25] discussed weak consistency for estimators in a semiparametric regression model based on negatively associated random errors. He et al. [10] established complete f-moment convergence for arrays of random variables and applied the results to semiparametric and EV regression models. Zhang et al. [30] studied symmetrical convergence rates and asymptotic properties of estimators in a semiparametric errors-in-variables model with strong mixing errors and missing responses. Emami and Khademnoe [6] considered shrinkage and pretest Liu estimators in semiparametric linear measurement error models, while Huang et al. [11] developed effective estimation methods for nonlinear errors-in-variables models.

Moreover, EV models have been widely applied in economics, biology and forestry. Many covariates in practical research, such as blood pressure, urinary sodium chloride level and pollutant exposure, are inevitably contaminated by measurement errors (see Liang et al. [13]). However, the independence assumption on the errors ϵ_i in model (1.1) is not always appropriate in applications, especially for sequentially collected economic, biomedical and environmental data, which often exhibit evident dependence in the errors. Although many important results have been obtained for semiparametric EV models, relatively few works have systematically investigated the asymptotic normality of estimators for heteroscedastic semiparametric EV models with both measurement errors and α-mixing random errors. In this paper, we consider the asymptotic normality for estimators of β, $g(\cdot)$ and $f(\cdot)$ under α-mixing random errors.

Compared with the aforementioned works, the present paper focuses on the heteroscedastic semiparametric EV model in which both the regression function $g(\cdot)$ and the variance function $f(\cdot)$ are unknown and the random errors are allowed to be α-mixing. This setting brings additional difficulties, since the measurement errors, the unknown heteroscedastic variance function and the dependence structure are intertwined in the estimation procedure. To overcome these difficulties, we combine least squares and weighted least squares estimation with nonparametric smoothing techniques, and employ moment inequalities for α-mixing sequences together with Bernstein's big-block and small-block arguments.

Since α-mixing is the weakest among the most frequently used mixing conditions, we will consider the asymptotic normality for the estimators of the unknown parameter β and the unknown functions $g(\cdot)$ and $f(\cdot)$ in (1.1) with strong mixing random errors. In Section 3, we will discuss it in three cases: (a) the errors are homoscedastic, ξ_i can be observed exactly and $\xi_i = x_i$, corresponding to Theorems 3.1(i) and 3.2(i); (b) the errors are heteroscedastic and $\sigma_i^2 = f(u_i)$ are known, corresponding to Theorems 3.1(ii) and 3.2(ii); (c) the errors are heteroscedastic and σ_i^2 are unknown, corresponding to Theorems 3.3–3.5.

Now let us recall the concept of α-mixing random variables.

DEFINITION 1.1. For a sequence $\{X_n,\, n \geqslant 1\}$ of random variables or random vectors, the *α-mixing coefficients* $\alpha(n)$ are defined as

$$\alpha(n) := \sup_{k \geqslant 1} \sup \{|P(AB) - P(A)P(B)| : A \in \mathcal{F}_{n+k}^{\infty},\, B \in \mathcal{F}_1^k\},$$

where $\mathcal{F}_n^m = \sigma(X_i : n \leqslant i \leqslant m)$ denotes the σ-algebra generated by $X_n, \ldots, X_m$ with $n \leqslant m$. Then the sequence $\{X_n,\, n \geqslant 1\}$ is said to be *α-mixing* or *strong mixing* if $\alpha(n) \to 0$ as $n \to \infty$.

α-mixing was introduced by Rosenblatt [18]; it is weaker than most of the other mixing conditions. The diagram below illustrates the relationships between the mixing conditions:

$$\psi\text{-mixing} \Longrightarrow \varphi\text{-mixing} \Longrightarrow \beta\text{-mixing}/\rho\text{-mixing} \Longrightarrow \alpha\text{-mixing}.$$

One can refer to Bradley [2] for instance. Furthermore, it is well known that $\alpha(n) \leqslant \frac{1}{4}\rho(n) \leqslant \frac{1}{2}\varphi^{1/2}(n)$. In addition, independent sequences and m-dependent sequences are also strong mixing. There are several examples of α-mixing sequences, including many time series models, which have many applications in practice. For example, (i) the well known stationary ARMA-sequences fulfill the α-mixing condition but generally fail the φ-mixing condition; (ii) under very mild assumptions, linear autoregressive and more generally bilinear time series models are strong mixing with mixing coefficients decaying exponentially, i.e., $\alpha(k) = O(\rho^k)$, for some $0 < \rho < 1$; (iii) if $\{X_n\}$ is a stationary ARCH(1) model, $X_n = (\beta + \lambda X_{n-1}^2)^{1/2} Z_n, n \in \mathbb{Z}, Z_n \overset{\text{i.i.d.}}{\sim} N(0,1)$, where β and λ are positive parameters, then $\{X_n\}$ is strong mixing with geometric rate (see Davis and Mikosch

[4] for instance). For more examples of strong mixing time series, one can refer to Doukhan [5] and Fan and Yao [7] for instance.

Noting that α-mixing sequences have many important applications, it is of practical significance to study the asymptotic normality of estimators in heteroscedastic semiparametric EV model (1.1) with α-mixing random errors. This article provides an innovation and promotion of the model of Zhang et al. [28], and it has two main points: firstly, we extend the independent errors to α-mixing errors; secondly, we consider the asymptotic normality of estimates versus real values rather than estimates versus their expectations, which is more reasonable.

The article is organized as follows. Estimators of β, $g(\cdot)$ and $f(\cdot)$ are listed in Section 2. Some assumptions and main results are given in Section 3. The numerical simulations are listed in Section 4. The proofs of the main results and the preliminary lemmas are provided in Sections 5 and 6, respectively.

Throughout the paper, $a_n = O(b_n)$ denotes that there exists a positive constant C such that $|a_n| \leqslant C|b_n|$, while $a_n = o(b_n)$ means $a_n/b_n \to 0$ as $n \to \infty$. Further, $\xrightarrow{D}$ denotes convergence in distribution. Let $C, C_1, C_2, \ldots$ denote positive constants whose values may vary at each occurrence. All limits are taken as the sample size n tends to ∞, unless otherwise specified.

2. ESTIMATORS

In this section, we will present estimators of β, $g(\cdot)$ and $f(\cdot)$. One can refer to Zhang et al. [28] for more details.

Firstly, we assume that the errors are homoscedastic and $\xi_i = x_i$ can be observed exactly. We can apply the least squares estimation (LSE) to estimate the unknown parameter β. Liang et al. [13] improved the LSE method on the basis of the usual partially linear model and employed the following estimator of β:

$$\hat{\beta}_L = \Big[\sum_{i=1}^{n} \tilde{x}_i^2 - n\sigma_\mu^2\Big]^{-1} \sum_{i=1}^{n} \tilde{x}_i \tilde{y}_i, \tag{2.1}$$

where $\tilde{x}_i = x_i - \sum_{j=1}^{n} W_{nj}(t_i)x_j$, $\tilde{y}_i = y_i - \sum_{j=1}^{n} W_{nj}(t_i)y_j$, $\sigma_\mu^2 = E\mu_i^2$ and the $W_{nj}(\cdot)$ are nonnegative weight functions such that $\sum_{j=1}^{n} W_{nj}(\cdot) = 1$.

Secondly, we consider the heteroscedastic case and discuss two subcases based on whether $f(\cdot)$ is known. If $\sigma_i^2 = f(u_i)$ are known, then $\hat{\beta}_L$ is modified to the following weighted least squares estimator (WLSE):

$$\hat{\beta}_{W_1} = \Big[\sum_{i=1}^{n} \sigma_i^{-2}(\tilde{x}_i^2 - \sigma_\mu^2)\Big]^{-1} \sum_{i=1}^{n} \sigma_i^{-2} \tilde{x}_i \tilde{y}_i. \tag{2.2}$$

However, the $\sigma_i^2 = f(u_i)$ are unknown in principle and must be estimated. In this case, we have $E[y_i - \xi_i\beta - g(t_i)]^2 = f(u_i)$ from $Ee_i^2 = 1$. Therefore, an

estimator of $f(u_i)$ can be defined by

$$\hat{f}_n(u_i) = \sum_{j=1}^{n} \widehat{W}_{nj}(u_i)(\tilde{y}_j - \tilde{x}_j\hat{\beta}_L)^2 - \sigma_\mu^2\hat{\beta}_L^2, \tag{2.3}$$

where the $\widehat{W}_{nj}(\cdot)$ are also nonnegative weight functions such that $\sum_{j=1}^{n}\widehat{W}_{nj}(\cdot) = 1$.

For convenience, we assume that $\min_{1\leqslant i\leqslant n}\hat{f}_n(u_i) > 0$. Then we use $\hat{\sigma}_{ni}^2 = \hat{f}_n(u_i)$ to define a nonparametric estimator of σ_i^2. As a consequence, when the errors ϵ_i are heteroscedastic and the σ_i^2 are unknown, the WLSE of β is defined by

$$\hat{\beta}_{W_2} = \Big[\sum_{i=1}^{n}\hat{\sigma}_{ni}^{-2}(\tilde{x}_i^2 - \sigma_\mu^2)\Big]^{-1}\sum_{i=1}^{n}\hat{\sigma}_{ni}^{-2}\tilde{x}_i\tilde{y}_i. \tag{2.4}$$

Using $\hat{\beta}_L$, $\hat{\beta}_{W_1}$, $\hat{\beta}_{W_2}$, we define the following three estimators for $g(\cdot)$:

$$\hat{g}_L(t) = \sum_{j=1}^{n} W_{nj}(t)(y_j - x_j\hat{\beta}_L), \tag{2.5}$$

$$\hat{g}_{W_1}(t) = \sum_{j=1}^{n} W_{nj}(t)(y_j - x_j\hat{\beta}_{W_1}), \tag{2.6}$$

$$\hat{g}_{W_2}(t) = \sum_{j=1}^{n} W_{nj}(t)(y_j - x_j\hat{\beta}_{W_2}). \tag{2.7}$$

The aim of this work is to establish the asymptotic normality for the above estimators of $\beta, g(\cdot)$ and $f(\cdot)$, which will be presented in Section 3.

3. ASSUMPTIONS AND MAIN RESULTS

3.1. Basic assumptions. In this subsection, we list some assumptions as follows.

$(\boldsymbol{H_1})$ Let $\{(e_i, \mu_i),\ i \geqslant 1\}$ be a sequence of identically distributed α-mixing random vectors with $Ee_i = 0$, $E\mu_i = 0$ and mixing coefficients $\alpha(n)$. The sequence $\{e_i,\ i \geqslant 1\}$ is independent of $\{\mu_i,\ i \geqslant 1\}$. Assume that $Ee_i^2 = 1$, $E\mu_i^2 = \sigma_\mu^2 > 0$, $E|e_1|^{4+2\delta} < \infty$ and $E|\mu_1|^{4+2\delta} < \infty$ for some $\delta > 0$.

$(\boldsymbol{H_2})$ There exists a function $h(\cdot)$ defined on $[0,1]$ such that $\xi_i = h(t_i) + \eta_i$ and

(i) $\lim\limits_{n\to\infty} n^{-1}\sum\limits_{i=1}^{n}\eta_i^2 = \sigma_\eta^2\ (0 < \sigma_\eta^2 < \infty)$;

(ii) $\limsup\limits_{n\to\infty}(\sqrt{n}\log n)^{-1}\max\limits_{1\leqslant m\leqslant n}\Big|\sum\limits_{i=1}^{m}\eta_{k_i}\Big| < \infty$ for all permutations $(k_1, \ldots, k_n)$ of $(1, \ldots, n)$.

$(\boldsymbol{H_3})$ (i) $0 < m_0 \leqslant \min_{1\leqslant i\leqslant n} f(u_i) \leqslant \max_{1\leqslant i\leqslant n} f(u_i) \leqslant M_0 < \infty$, where m_0 and M_0 are two positive constants;

(ii) $g(\cdot)$, $f(\cdot)$ and $h(\cdot)$ satisfy the first-order Lipschitz condition.

$(\boldsymbol{H_4})$ The weight functions $W_{nj}(\cdot)$ satisfy the following conditions:

(i) $\max\limits_{1\leqslant j\leqslant n}\sum\limits_{i=1}^{n} W_{nj}(t_i) = O(1)$;

(ii) $\max\limits_{1\leqslant i,j\leqslant n} W_{nj}(t_i) = O(n^{-s}\log^{-1} n)$ for some $1/2 \leqslant s < 1$;

(iii) $\max\limits_{1\leqslant i\leqslant n}\sum\limits_{j=1}^{n} |W_{nj}(t_i)|I(|t_i - t_j| > a) = o(n^{-1/4})$ for any $a > 0$.

$(\boldsymbol{H_5})$ The weight functions $W_{nj}(\cdot)$ satisfy the following conditions:

(i) $\sup\limits_{0\leqslant t\leqslant 1}\max\limits_{1\leqslant j\leqslant n} W_{nj}(t) = O(n^{-s}\log^{-1} n)$ for some $1/2 \leqslant s < 1$;

(ii) $\sup\limits_{0\leqslant t\leqslant 1}\sum\limits_{j=1}^{n} |W_{nj}(t)|I(|t - t_j| > a) = O(n^{-1/2})$ for any $a > 0$.

$(\boldsymbol{H_5'})$ The weight functions $\widehat{W}_{nj}(\cdot)$ satisfy all conditions in $(\boldsymbol{H_5})$ with t, t_j and W_{nj} replaced by u, u_j and $\widehat{W}_{nj}$, respectively.

$(\boldsymbol{H_6})$ There exist positive integers $p := p(n)$ and $q := q(n)$ such that

(i) $p + q \leqslant n$, $p \to \infty$, $q \to \infty$, $qp^{-1} \to 0$, $p^2 = O(nq)$ and $np^{-1}\alpha(q) \to 0$;

(ii) $(qp^{-1})^{1/2}\max_{1\leqslant i\leqslant n}|\eta_i| = o(1)$.

REMARK 3.1. Assumptions $(\boldsymbol{H_2})$–$(\boldsymbol{H_5})$ are standard regularity conditions and have been used commonly in the literature. Assumption $(\boldsymbol{H_2})$ is assumed in Härdle et al. [9] and Liang and Wang [15]. Assumptions $(\boldsymbol{H_3})$–$(\boldsymbol{H_5})$ are assumed in Liang and Jing [14] and Zhou and Hu [32]. Assumption $(\boldsymbol{H_6})$ is used to employ Bernstein's big-block and small-block technique to prove asymptotic normality for an α-mixing sequence and can be satisfied easily, which can be seen in Zhang and Liang [27] for instance.

REMARK 3.2. Remark 3 in Gao et al. [8] and Remark 2.3 in Baek and Liang [1] mentioned that the following two weight functions satisfy assumption $(\boldsymbol{H_4})$ under some mild conditions:

$$W_{nj}^{(1)}(t) = \frac{1}{b_n}\int_{s_{j-1}}^{s_j} K\left(\frac{t-s}{b_n}\right) ds,$$

$$W_{nj}^{(2)}(t) = K\left(\frac{t-t_j}{b_n}\right)\left[\sum_{j=1}^{n} K\left(\frac{t-t_j}{b_n}\right)\right]^{-1},$$

where $s_j = (t_j + t_{j-1})/2$, $j = 1, \ldots, n-1$, $s_0 = 0$, $s_n = 1$, $K(\cdot)$ is the Parzen–Rosenblatt kernel function, and b_n is a bandwidth parameter.

3.2. Main results. Now, we introduce some notations which will be used in the theorems below:

$$\tilde{\xi}_i = \xi_i - \sum_{j=1}^{n} W_{nj}(t_i)\xi_j, \quad S_n^2 = \sum_{i=1}^{n} \tilde{\xi}_i^2, \quad I_n^2 = \sum_{i=1}^{n} \sigma_i^{-2}\tilde{\xi}_i^2,$$

$$\Gamma_n^2(t) = \operatorname{Var}\Big[\sum_{i=1}^{n} W_{ni}(t)(\epsilon_i - \mu_i\beta)\Big],$$

$$\Sigma_n^2 = \operatorname{Var}\Big[\sum_{i=1}^{n} (\tilde{\xi}_i + \mu_i)(\epsilon_i - \mu_i\beta)\Big],$$

$$\Sigma_{1n}^2 = \operatorname{Var}\Big[\sum_{i=1}^{n} \sigma_i^{-2}(\tilde{\xi}_i + \mu_i)(\epsilon_i - \mu_i\beta)\Big],$$

$$\Delta_n^2(u) = \sum_{i=1}^{n} \widehat{W}_{ni}^2(u) \operatorname{Var}\left[(\epsilon_i - \mu_i\beta)^2\right],$$

$$\Delta_n'^2(u) = \sum_{i=1}^{n} \widehat{W}_{ni}^2(u) \operatorname{Var}\left[(\epsilon_i - \mu_i\hat{\beta}_L)^2\right].$$

When $f(\cdot)$ is known, we have the following results.

THEOREM 3.1. *Suppose that assumptions* $(\boldsymbol{H_1})$–$(\boldsymbol{H_4})$ *and* $(\boldsymbol{H_6})$ *hold and* $\alpha(n) = O(n^{-\gamma})$ *for some*

$$\gamma > \max\left\{\frac{(2+\delta)(1+\delta)}{\delta}, \frac{3 - s + \frac{1}{2+\delta}}{s - \frac{1}{2+\delta}}\right\}.$$

Assume that $\max_{1\leqslant j\leqslant n} |\eta_j| = O(n^{1/2}\log^{-1} n)$. *Then the following hold:*

(3.1) (i) *if* $\Sigma_n^2 \geqslant Cn$, *then* $S_n^2(\hat{\beta}_L - \beta)/\Sigma_n \xrightarrow{D} N(0,1)$;

(3.2) (ii) *if* $\Sigma_{1n}^2 \geqslant Cn$, *then* $I_n^2(\hat{\beta}_{W_1} - \beta)/\Sigma_{1n} \xrightarrow{D} N(0,1)$.

THEOREM 3.2. *Suppose that assumptions* $(\boldsymbol{H_1})$–$(\boldsymbol{H_3})$, $(\boldsymbol{H_5})$, $(\boldsymbol{H_6})$ *hold and* $\alpha(n) = O(n^{-\gamma})$ *for some*

$$\gamma > \max\left\{\frac{(2+\delta)(1+\delta)}{\delta}, \frac{3 - s + \frac{1}{2+\delta}}{s - \frac{1}{2+\delta}}\right\}.$$

Assume that $\max_{1\leqslant j\leqslant n} |\eta_j| = O(1)$. *For each* $t \in [0,1]$, *if* $n\Gamma_n^2(t) \to \infty$, *then*

(3.3) (i) $\Gamma_n^{-1}(t)(\hat{g}_L(t) - g(t)) \xrightarrow{D} N(0,1)$;

(3.4) (ii) $\Gamma_n^{-1}(t)(\hat{g}_{W_1}(t) - g(t)) \xrightarrow{D} N(0,1)$.

When $f(\cdot)$ is unknown, we obtain the following results:

THEOREM 3.3. *Suppose that assumptions* $(\boldsymbol{H_1})$–$(\boldsymbol{H_4})$, $(\boldsymbol{H_5'})$, $(\boldsymbol{H_6})$ *hold and* $\alpha(n) = O(n^{-\gamma})$ *for some*

$$\gamma > \max\left\{\frac{(2+\delta)(1+\delta)}{\delta},\ \frac{3-s+\frac{1}{2+\delta}}{s-\frac{1}{2+\delta}}\right\}.$$

Assume that $\inf_i \mathrm{Var}[(\epsilon_i - \mu_i\beta)^2] \geqslant C > 0$ *and* $\max_{1\leqslant j\leqslant n}|\eta_j| = O(n^{1/4})$. *For each* $u \in [0,1]$, *if* $n\sum_{i=1}^n \widehat{W}_{ni}^2(u) \to \infty$, *then*

$$\Delta_n^{-1}(u)(\hat{f}_n(u) - f(u)) \xrightarrow{D} N(0,1). \tag{3.5}$$

THEOREM 3.4. *Suppose that assumptions* $(\boldsymbol{H_1})$–$(\boldsymbol{H_4})$, $(\boldsymbol{H_5'})$, $(\boldsymbol{H_6})$ *hold for some* $\frac{1}{4} + \frac{1}{2+\delta} < s < 1$ *and* $\alpha(n) = O(n^{-\gamma})$ *for some*

$$\gamma > \max\left\{\frac{(2+\delta)(1+\delta)}{\delta},\ \frac{3-s+\frac{1}{4}+\frac{1}{2+\delta}}{s-\frac{1}{4}-\frac{1}{2+\delta}}\right\}.$$

Assume that $\max_{1\leqslant j\leqslant n}|\eta_j| = O(n^{1/2}\log^{-1} n)$. *If* $\Sigma_{1n}^2 \geqslant Cn$, *then*

$$I_n^2(\hat{\beta}_{W_2} - \beta)/\Sigma_{1n} \xrightarrow{D} N(0,1). \tag{3.6}$$

THEOREM 3.5. *Suppose that assumptions* $(\boldsymbol{H_1})$–$(\boldsymbol{H_3})$, $(\boldsymbol{H_5})$, $(\boldsymbol{H_5'})$, *and* $(\boldsymbol{H_6})$ *hold for some* $\frac{1}{4} + \frac{1}{2+\delta} < s < 1$, *and* $\alpha(n) = O(n^{-\gamma})$ *for some*

$$\gamma > \max\left\{\frac{(2+\delta)(1+\delta)}{\delta},\ \frac{3-s+\frac{1}{4}+\frac{1}{2+\delta}}{s-\frac{1}{4}-\frac{1}{2+\delta}}\right\}.$$

Assume that $\inf_i \mathrm{Var}[(\epsilon_i - \mu_i\beta)^2] \geqslant C > 0$ *and* $\max_{1\leqslant j\leqslant n}|\eta_j| = O(1)$.

If $n\sum_{i=1}^n \widehat{W}_{ni}^2(u) \to \infty$ *for each* $u \in [0,1]$, *and* $n\Gamma_n^2(t) \to \infty$ *for each* $t \in [0,1]$, *then*

$$\Gamma_n^{-1}(t)(\hat{g}_{W_2}(t) - g(t)) \xrightarrow{D} N(0,1). \tag{3.7}$$

REMARK 3.3. Zhang et al. [28] also derived the asymptotic normality for the estimators in heteroscedastic semiparametric EV models. Comparing with Zhang et al. [28], we can come to the following conclusions:

(a) In Zhang et al. [28], $\{e_i,\ i \geqslant 1\}$ is assumed to be a sequence of stationary α-mixing random variables and $\{\mu_i,\ i \geqslant 1\}$ is assumed to be a sequence of independent random variables when $f(\cdot)$ is known. When $f(\cdot)$ is unknown, $\{e_i,\ i \geqslant 1\}$ also has to be an independent sequence. In this paper, we assume $\{(e_i, \mu_i),\ i \geqslant 1\}$

to be identically distributed α-mixing random vectors, which is a more general setting.

(b) In Theorems 3.2, 3.4 and 3.5 of Zhang et al. [28], the asymptotic normality for $\hat{g}_L(t) - E\hat{g}_L(t)$, $\hat{g}_{W_1}(t) - E\hat{g}_{W_1}(t)$, $\hat{g}_{W_2}(t) - E\hat{g}_{W_2}(t)$ and $\hat{f}_n(u) - E\hat{f}_n(u)$ was obtained, respectively. By contrast, we derive the asymptotic normality for $\hat{g}_L(t) - g(t)$, $\hat{g}_{W_1}(t) - g(t)$, $\hat{g}_{W_2}(t) - g(t)$ and $\hat{f}_n(u) - f(u)$, which is more natural and reasonable for statistical inference.

REMARK 3.4. The convergence rates of the nonparametric estimators can be read directly from the normalizing factors in Theorems 3.2, 3.3 and 3.5. More precisely for each fixed $t \in [0,1]$ and $u \in [0,1]$,

$$\hat{g}_L(t) - g(t) = O_p(\Gamma_n(t)), \quad \hat{g}_{W_1}(t) - g(t) = O_p(\Gamma_n(t)),$$
$$\hat{g}_{W_2}(t) - g(t) = O_p(\Gamma_n(t)), \quad \hat{f}_n(u) - f(u) = O_p(\Delta_n(u)).$$

These rates are generally slower than the root-n rate of parametric estimators, since they depend on the smoothing weights and the local information around t or u. For example, when the errors are independent and standard local smoothing weights with bandwidth b_n are used, one typically has

$$\sum_{i=1}^{n} W_{ni}^2(t) = O((nb_n)^{-1}), \qquad \sum_{i=1}^{n} \widehat{W}_{ni}^2(u) = O((nb_n)^{-1}),$$

and hence the stochastic orders reduce to the usual nonparametric order $O_p((nb_n)^{-1/2})$, up to the bias terms controlled by the regularity assumptions.

4. SIMULATION

In this section, we will carry out a numerical simulation to study the finite sample performance for the LSE and WLSE of β and $g(\cdot)$. Take two AR(1) sequences

$$e_i = \rho e_{i-1} + \nu_i, \qquad \mu_i = \rho \mu_{i-1} + \nu_i',$$

for each $1 \leqslant i \leqslant n$, with $e_0 = 0$, $\mu_0 = 0$ and $\rho = 0.2$. For the Q-Q plots and fitting curves, we use normal innovations

$$\nu_i, \nu_i' \overset{\text{i.i.d.}}{\sim} N(0,1).$$

To further examine the robustness of the proposed estimators, we also consider standardized Student's t_5 innovations,

$$\nu_i, \nu_i' \overset{\text{i.i.d.}}{\sim} t_5/\sqrt{5/3},$$

so that the innovations have mean zero and variance 1 but heavier tails than the normal distribution. It is easy to check that $\{e_n,\, n \geqslant 1\}$ and $\{\mu_n,\, n \geqslant 1\}$ are

both zero mean α-mixing random variables. One can refer to Zhang et al. [28], for instance. The observations are generated from the following model:

$$y_i = 2\xi_i + \sin(2\pi t_i) + \sigma_i e_i, \quad x_i = \xi_i + \mu_i, \quad i = 1, \ldots, n,$$

where $\sigma_i^2 = f(u_i) = [2 + \cos(\pi u_i)]^2$, $u_i = (i-1)/n$, $t_i = (i-0.5)/n$, $\xi_i = 2t_i^2 + \eta_i$ with $\eta_i \overset{\text{i.i.d.}}{\sim} U(-1,1)$ for $1 \leqslant i \leqslant n$. In order to verify the asymptotic properties of the estimators, we choose the nearest neighbor weights for $W_{nj}(t)$. Let $A = [0,1]$. For any $t \in A$, we rewrite $|t_1 - t|, \ldots, |t_n - t|$ as follows:

$$|t_{R_1(t)} - t| \leqslant |t_{R_2(t)} - t| \leqslant \cdots \leqslant |t_{R_n(t)} - t|;$$

if $|t_j - t| = |t_k - t|$, then we set $j < k$ in the ordering. Let $k_n = \lfloor n^{0.7} \rfloor$ and define the nearest neighbor weight functions as follows:

$$W_{nj}(t) = \begin{cases} 1/k_n & \text{if } |t_j - t| \leqslant |t_{R_{k_n}(t)} - t|, \\ 0 & \text{otherwise.} \end{cases}$$

The weights for $\widehat{W}_{nj}(t)$ are chosen in the same way.

We compare the Q-Q plots of the estimators for β and the fitting curves of the estimators for $g(\cdot)$ based on different sample sizes by R software. For the Q-Q plots and fitting curves, the sample sizes are $n = 100, 300, 500, 1000$, respectively, and the nearest-neighbor number is chosen to be $k_n = \lfloor n^{0.7} \rfloor$.

To further examine the robustness of the proposed method, we also conduct additional simulations under standardized Student's t_5 innovations and different choices of the nearest-neighbor number. Specifically, we consider

$$k_n = \lfloor n^a \rfloor, \quad a = 0.7, 0.8.$$

The corresponding numerical results for $n = 500$ are summarized in Table 1.

Figures 1–12 depict the Q-Q plots for $\hat{\beta}_L$, $\hat{\beta}_{W_1}$ and $\hat{\beta}_{W_2}$ when $n = 100, 300, 500, 1000$, respectively. The Q-Q plots show that the finite sample distributions of $\hat{\beta}_L$, $\hat{\beta}_{W_1}$ and $\hat{\beta}_{W_2}$ are close to the normal distribution, especially when the sample size becomes large. Figures 13–24 reveal that the fitting curves of the estimators $\hat{g}_L(t)$, $\hat{g}_{W_1}(t)$ and $\hat{g}_{W_2}(t)$ gradually approximate the true function $g(t)$ with the increasing sample size n.

Table 1 reports the simulation results under normal and standardized t_5 innovations for different choices of $k_n = \lfloor n^a \rfloor$, where $a = 0.7, 0.8$. Under normal innovations, the three estimators are close to the true value $\beta = 2$, and the weighted estimators generally have smaller standard deviations than the unweighted estimator. Under standardized t_5 innovations, the estimators still remain reasonably stable, although the standard deviations and RMSEs are slightly larger than those under normal innovations. This is consistent with the heavier-tailed nature of the t_5 distribution. Moreover, the results for $a = 0.7$ and $a = 0.8$ are close to each other,

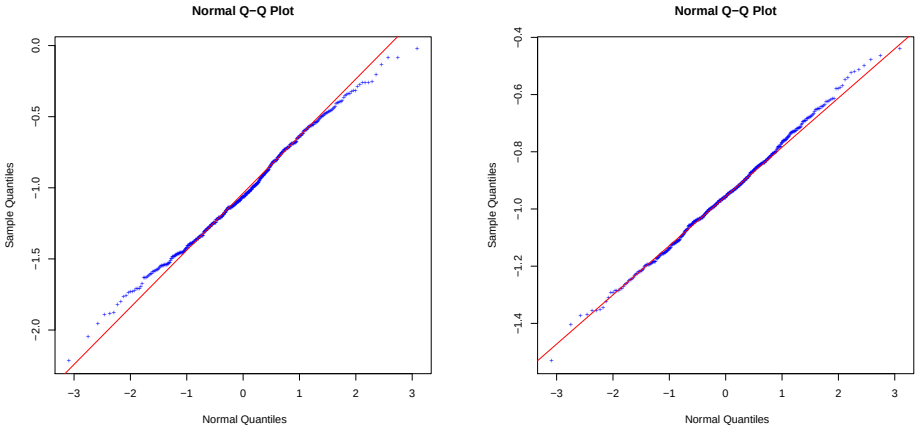


FIGURE 1. Q-Q plots of $\hat{\beta}_L$ for sample sizes $n = 100$ (left) and $n = 300$ (right)

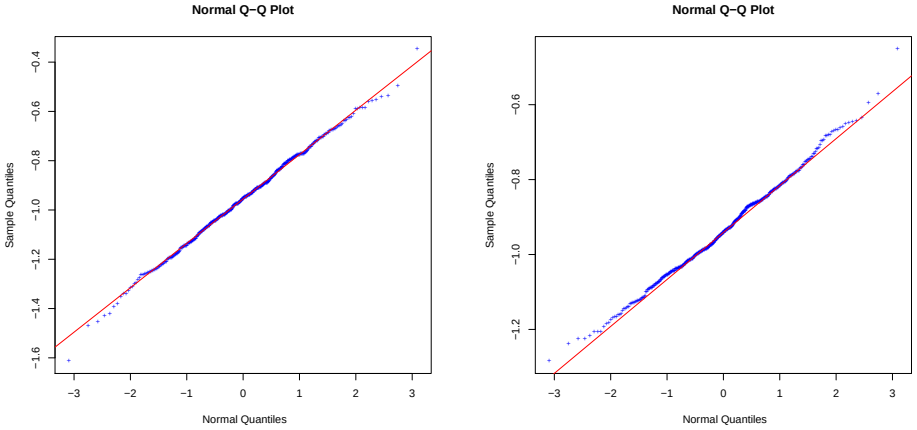


FIGURE 2. Q-Q plots of $\hat{\beta}_L$ for sample sizes $n = 500$ (left) and $n = 1000$ (right)

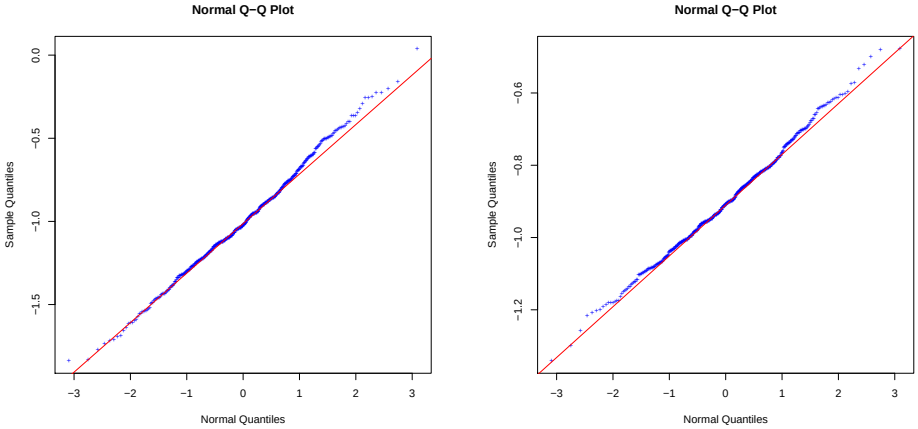


FIGURE 3. Q-Q plots of $\hat{\beta}_{W_1}$ for sample sizes $n = 100$ (left) and $n = 300$ (right)

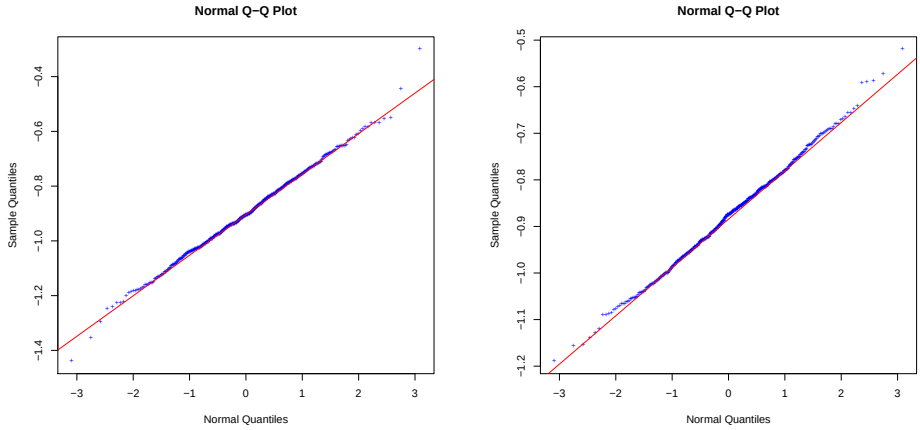


FIGURE 4. Q-Q plots of $\hat{\beta}_{W_1}$ for sample sizes $n = 500$ (left) and $n = 1000$ (right)

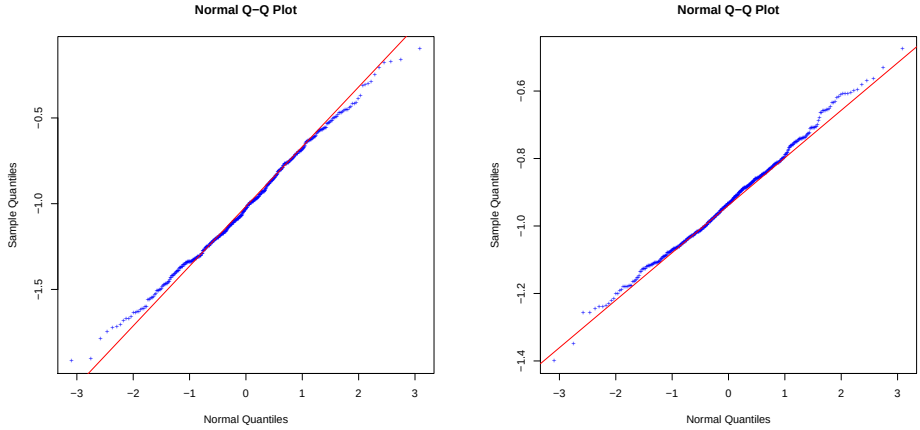


FIGURE 5. Q-Q plots of $\hat{\beta}_{W_2}$ for sample sizes $n = 100$ (left) and $n = 300$ (right)

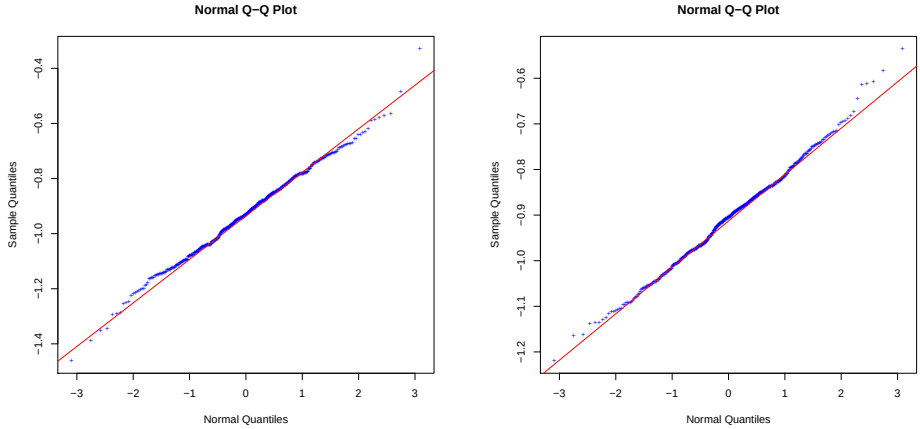


FIGURE 6. Q-Q plots of $\hat{\beta}_{W_2}$ for sample sizes $n = 500$ (left) and $n = 1000$ (right)

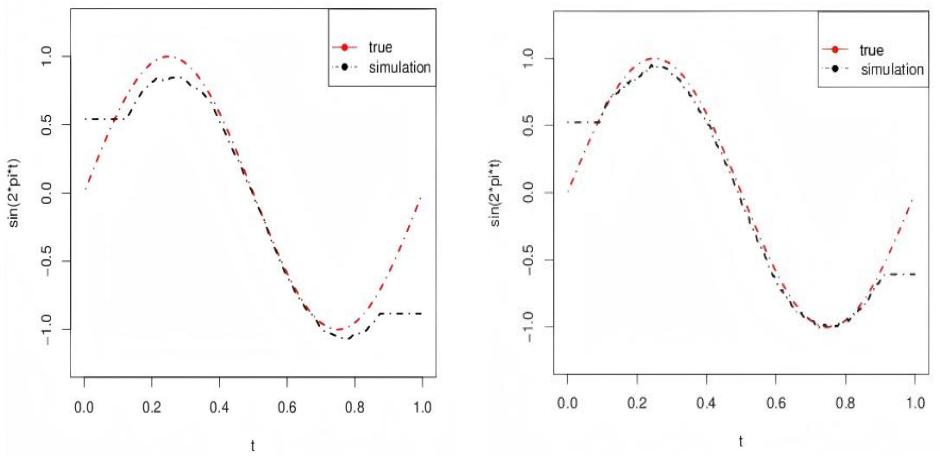


FIGURE 7. Fitting curves of $\hat{g}_L(t)$ for sample sizes $n = 100$ (left) and $n = 300$ (right)

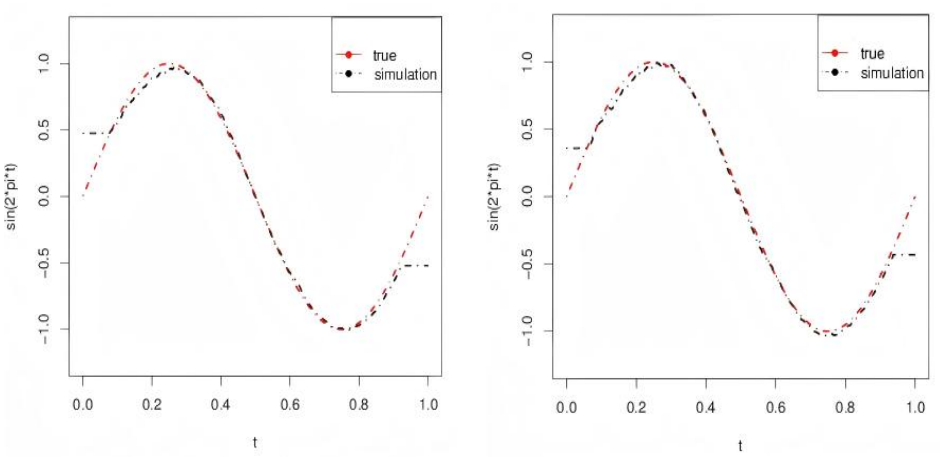


FIGURE 8. Fitting curves of $\hat{g}_L(t)$ for sample sizes $n = 500$ (left) and $n = 1000$ (right)

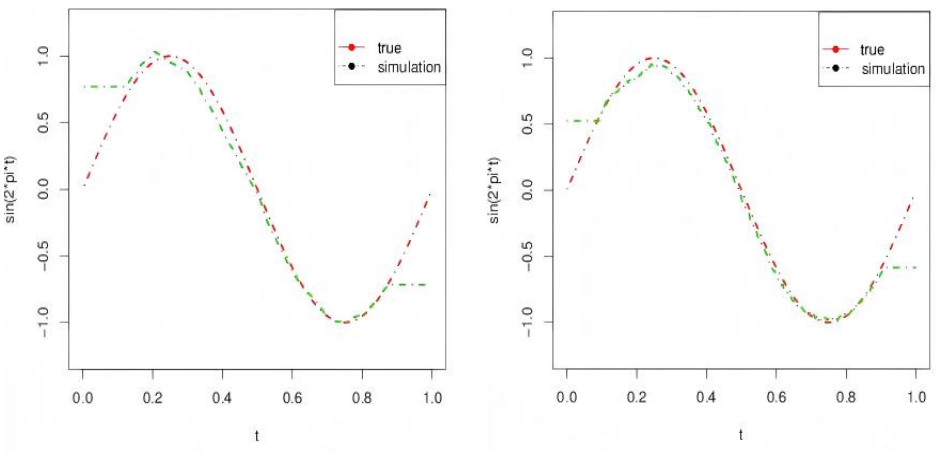


FIGURE 9. Fitting curves of $\hat{g}_{W_1}(t)$ for sample sizes $n = 100$ (left) and $n = 300$ (right)

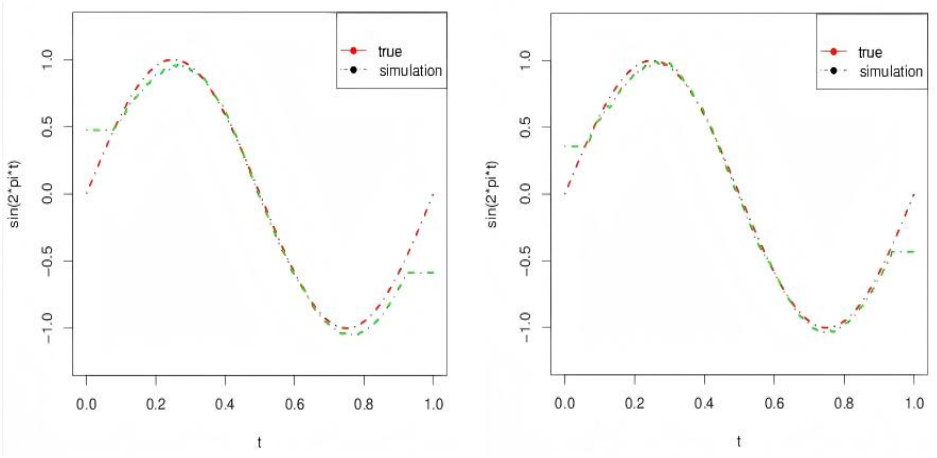


FIGURE 10. Fitting curves of $\hat{g}_{W_1}(t)$ for sample sizes $n = 500$ (left) and $n = 1000$ (right)

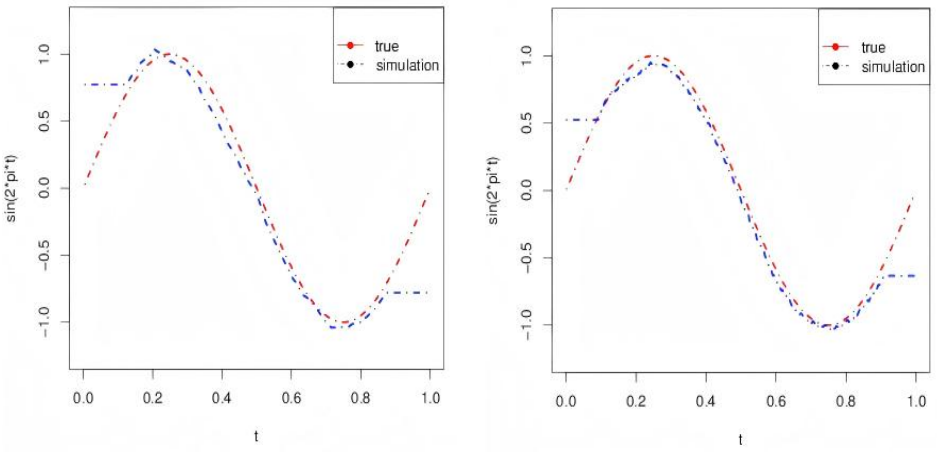


FIGURE 11. Fitting curves of $\hat{g}_{W_2}(t)$ for sample sizes $n = 100$ (left) and $n = 300$ (right)

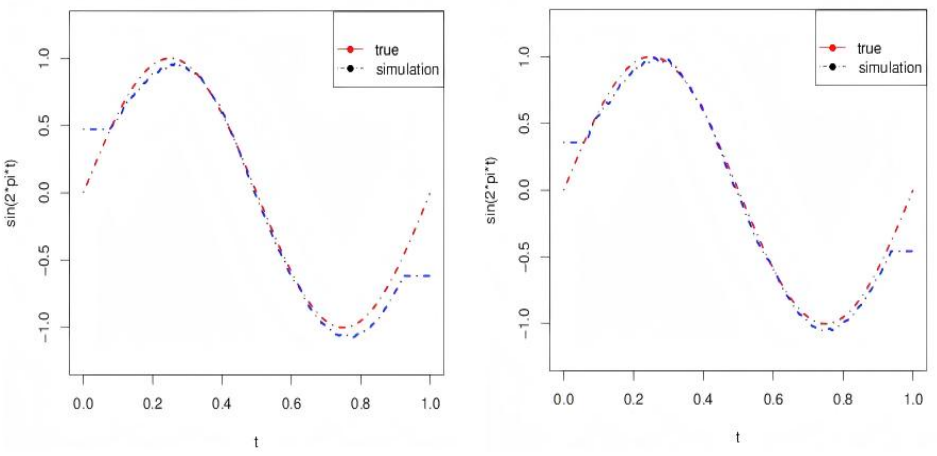


FIGURE 12. Fitting curves of $\hat{g}_{W_2}(t)$ for sample sizes $n = 500$ (left) and $n = 1000$ (right)

TABLE 1. Robustness and sensitivity analysis under normal and standardized t_5 innovations when $n = 500$.

Innovation	a	k_n	Estimator	Mean	Bias	SD	MAE	RMSE
Normal	0.7	77	$\hat{\beta}_L$	2.0293	0.0293	0.2201	0.1784	0.2220
Normal	0.7	77	$\hat{\beta}_{W_1}$	2.0501	0.0501	0.1773	0.1462	0.1841
Normal	0.7	77	$\hat{\beta}_{W_2}$	2.0700	0.0700	0.1851	0.1564	0.1978
Normal	0.8	144	$\hat{\beta}_L$	2.0631	0.0631	0.2073	0.1715	0.2166
Normal	0.8	144	$\hat{\beta}_{W_1}$	2.1174	0.1174	0.1689	0.1665	0.2057
Normal	0.8	144	$\hat{\beta}_{W_2}$	2.1092	0.1092	0.1712	0.1638	0.2030
t_5	0.7	77	$\hat{\beta}_L$	2.0316	0.0316	0.2113	0.1702	0.2136
t_5	0.7	77	$\hat{\beta}_{W_1}$	2.0471	0.0471	0.1886	0.1525	0.1943
t_5	0.7	77	$\hat{\beta}_{W_2}$	2.0923	0.0923	0.4367	0.1771	0.4461
t_5	0.8	144	$\hat{\beta}_L$	2.0698	0.0698	0.2031	0.1728	0.2146
t_5	0.8	144	$\hat{\beta}_{W_1}$	2.1308	0.1308	0.1835	0.1813	0.2252
t_5	0.8	144	$\hat{\beta}_{W_2}$	2.1234	0.1234	0.1843	0.1773	0.2217

indicating that the proposed estimators are not very sensitive to moderate changes in the nearest-neighbor number. Overall, these simulation results are consistent with the theoretical findings obtained in this paper and provide additional evidence for the robustness of the proposed method.

5. SOME IMPORTANT LEMMAS

To prove the main results of the paper, we need the following useful lemmas. In this section, let $\{e_i,\, i \geqslant 1\}$ be a sequence of α-mixing random variables with mixing coefficients $\{\alpha(n),\, n \geqslant 1\}$.

The first one is the Kolmogorov-type strong law of large numbers for α-mixing random variables.

LEMMA 5.1. *Let* $Ee_i = 0$ *and* $\sup_i E|e_i|^r < \infty$ *for some* $r > 1$. *Suppose that* $\alpha(n) = O(\log^{-\gamma} n)$ *for some* $\gamma > r/(r-1)$. *Then* $n^{-1}\sum_{i=1}^{n} e_i = o(1)\ a.s.$

The second one is the Rosenthal-type maximum inequality and the Rosenthal-type inequality for α-mixing random variables.

LEMMA 5.2.

(i) *Let* $r > 2$, $\delta > 0$, $Ee_i = 0$ *and* $E|e_i|^{r+\delta} < \infty$. *Suppose that* $\alpha(n) = O(n^{-\gamma})$ *for some* $\gamma > r(r+\delta)/(2\delta)$. *Then, for any* $\varepsilon > 0$, *there exists a positive constant* $C = C(\varepsilon, r, \delta, \gamma)$ *such that*

$$E\Big(\max_{1\leqslant m\leqslant n}\Big|\sum_{i=1}^{m} e_i\Big|^r\Big) \leqslant C\Big\{n^{\varepsilon}\sum_{i=1}^{n} E|e_i|^r + \Big(\sum_{i=1}^{n}\|e_i\|_{r+\delta}^2\Big)^{r/2}\Big\}.$$

(ii) *If* $Ee_i = 0$ *and* $E|e_i|^{2+\delta} < \infty$ *for some* $\delta > 0$, *then*

$$E\Big(\sum_{i=1}^{n} e_i\Big)^2 \leqslant \Big[1 + 16\sum_{l=1}^{n} \alpha^{\delta/(2+\delta)}(l)\Big] \sum_{i=1}^{n} |e_i|^2_{2+\delta}.$$

LEMMA 5.3. *Let* $V_1, \dots, V_m$ *be* α*-mixing random variables measurable with respect to the* σ*-algebras* $\mathcal{F}_{i_1}^{j_1}, \dots, \mathcal{F}_{i_m}^{j_m}$, *respectively, with* $1 \leqslant i_1 \leqslant j_1 < \cdots < j_m \leqslant n, i_{l+1} - j_l \geqslant \omega \geqslant 1$ *and* $|V_j| \leqslant 1$ *for* $l, j = 1, \dots, m$. *Then*

$$\Big|E\Big(\prod_{j=1}^{m} V_j\Big) - \prod_{j=1}^{m} EV_j\Big| \leqslant 16(m-1)\alpha(\omega),$$

where $\mathcal{F}_a^b = \sigma\{V_i,\ a \leqslant i \leqslant b\}$ *and* $\alpha(\omega)$ *is the mixing coefficient.*

The next one is the exponential probability inequality for α-mixing random variables.

LEMMA 5.4. *Assume that* $\{e_i,\ i \geqslant 1\}$ *is a sequence of* α*-mixing random variables with* $Ee_i = 0$. *Let* $|e_i| \leqslant S < \infty$ *a.s.*, $i = 1, \dots, n$, *where* S *is a positive constant. Set* $D_m = \max_{1 \leqslant j \leqslant 2m} \operatorname{Var}(\sum_{i=1}^{j} e_i)$. *Then, for* $n, m \in \mathbb{N}$ *with* $0 < m \leqslant n/2$ *and for any* $\varepsilon > 0$,

$$P\Big(\Big|\sum_{i=1}^{n} e_i\Big| > \varepsilon\Big) \leqslant 4\exp\Big\{-\frac{\varepsilon^2}{16}\Big(nm^{-1}D_m + \frac{1}{3}\varepsilon S m\Big)^{-1}\Big\} + 32\frac{S}{\varepsilon} n\alpha(m).$$

By the exponential probability inequality, we can get the strong law of large numbers for weighted sums of α-mixing random variables.

LEMMA 5.5. *Let* $\{e_i,\ i \geqslant 1\}$ *be a sequence of identically distributed* α*-mixing random variables with* $Ee_1 = 0$ *and* $E|e_1|^p < \infty$ *for some* $p > 2$, *and* $\alpha(n) = O(n^{-\gamma})$ *for some* $\gamma > 0$.

(i) *Suppose that* $\{a_{nij},\ 1 \leqslant i, j \leqslant n,\ n \geqslant 1\}$ *is an array of real numbers satisfying*

$$\max_{1 \leqslant i,j \leqslant n} |a_{nij}| = O(n^{-s}\log^{-1} n) \quad \textit{for some } \frac{1}{2} \leqslant s \leqslant \frac{1}{p} + 1$$

and

$$\max_{1 \leqslant j \leqslant n} \Big|\sum_{i=1}^{n} a_{nij}\Big| = O(1).$$

If $\gamma > \max\big\{\big(3 - s + \frac{1}{p}\big)/\big(s - \frac{1}{p}\big), p/(p-2)\big\}$, *then*

$$\max_{1 \leqslant j \leqslant n} \Big|\sum_{i=1}^{n} a_{nij} e_i\Big| = o(1) \quad \textit{a.s.}$$

(ii) *Let* $0 < \lambda < \min\{s, 1 - 1/p\}$. *Suppose that* $\{a_{nij},\ 1 \leqslant i, j \leqslant n, n \geqslant 1\}$ *is an array of real numbers satisfying*

$$\max_{1\leqslant i,j\leqslant n} |a_{nij}| = O(n^{-s}\log^{-1} n) \quad \textit{for some } \frac{1}{2} \leqslant s - \lambda \leqslant \frac{1}{p} + 1$$

and

$$\max_{1\leqslant j\leqslant n} \Big|\sum_{i=1}^{n} a_{nij}\Big| = O(1).$$

If $\gamma > \max\big\{\big(3 - s + \lambda + \frac{1}{p}\big)/\big(s - \lambda - \frac{1}{p}\big), p/(p-2)\big\}$, *then*

$$\max_{1\leqslant j\leqslant n} \Big|\sum_{i=1}^{n} a_{nij}e_i\Big| = o(n^{-\lambda}) \quad \textit{a.s.}$$

Proof. The proof of (i) is similar to that of (ii), so we omit it here. We just prove (ii) and use the same method as that in the proof of Lemma 5.7 in Zhang and Liang [27].

Denote $\omega_{ni} = e_i I(|e_i| \leqslant n^{1/p})$ and $\omega'_{ni} = e_i I(|e_i| > n^{1/p})$ for $1 \leqslant i \leqslant n$. Then

$$\max_{1\leqslant j\leqslant n} \Big|\sum_{i=1}^{n} a_{nij}e_i\Big| \leqslant \max_{1\leqslant j\leqslant n} \Big|\sum_{i=1}^{n} a_{nij}\omega_{ni}\Big| + \max_{1\leqslant j\leqslant n} \Big|\sum_{i=1}^{n} a_{nij}\omega'_{ni}\Big| \doteq H_{1n} + H_{2n}.$$

We first prove $H_{2n} = o(n^{-\lambda})$ a.s. It follows from $E|e_1|^p < \infty$ that

$$\sum_{i=1}^{\infty} E|e_i|I(|e_i| > i^{1/p}) < \infty,$$

which yields $\sum_{i=1}^{\infty} |e_i|I(|e_i| > i^{1/p}) < \infty\ a.s.$ Thus,

$$\begin{aligned} n^{\lambda}H_{2n} &\leqslant n^{\lambda} \max_{1\leqslant i,j\leqslant n} |a_{nij}| \sum_{i=1}^{n} |e_i|I(|e_i| > n^{1/p}) \\ &\leqslant Cn^{\lambda-s}\log^{-1} n \sum_{i=1}^{n} |e_i|I(|e_i| > i^{1/p}) \xrightarrow{\text{a.s.}} 0, \end{aligned} \tag{5.1}$$

which implies that $H_{2n} = o(n^{-\lambda})$ a.s.

Next, we will deal with H_{1n}. It follows from $Ee_i = 0$ and $E|e_1|^p < \infty$ that

$$\begin{aligned} n^{\lambda} \max_{1\leqslant j\leqslant n} \Big|\sum_{i=1}^{n} a_{nij}E\omega_{ni}\Big| &\leqslant n^{\lambda} \max_{1\leqslant j\leqslant n} \Big|\sum_{i=1}^{n} a_{nij}\Big| E|e_1|I(|e_1| > n^{1/p}) \\ &\leqslant Cn^{\lambda+1/p-1}E|e_1|^p \xrightarrow{\text{a.s.}} 0. \end{aligned} \tag{5.2}$$

To prove the desired result, it suffices to show that for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{n} a_{nij}(\omega_{ni} - E\omega_{ni})\Big| > \varepsilon n^{-\lambda}\Big) < \infty. \tag{5.3}$$

Set $\zeta_{nij} = n^{s-1/p}a_{nij}(\omega_{ni} - E\omega_{ni})$. Then we get

$$\max_{1\leqslant i,j\leqslant n}|\zeta_{nij}| \leqslant C\log^{-1} n < \infty$$

and

$$\begin{aligned} D_m &\doteq \max_{1\leqslant j\leqslant n}\max_{1\leqslant k\leqslant 2m}\mathrm{Var}\Big(\sum_{i=1}^{k}\zeta_{nij}\Big) \\ &\leqslant C\max_{1\leqslant j\leqslant n}\sum_{i=1}^{2m}\|\zeta_{nij}\|_p^2 \leqslant Cmn^{-2/p}\log^{-2} n \end{aligned}$$

by Lemma 5.2(ii). Hence, applying Lemma 5.4 with $m = \lfloor n^{s-\lambda-1/p}\log^{-1} n\rfloor$, we infer from $\frac{1}{2} \leqslant s - \lambda \leqslant \frac{1}{p} + 1$ and

$$\gamma > \max\left\{\left(3 - s + \lambda + \frac{1}{p}\right)\Big/\left(s - \lambda - \frac{1}{p}\right),\ \frac{p}{p-2}\right\}$$

that

$$\begin{aligned} &\sum_{n=1}^{\infty} P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{n} a_{nij}(\omega_{ni} - E\omega_{ni})\Big| > \varepsilon n^{-\lambda}\Big) \\ &\leqslant \sum_{n=1}^{\infty}\sum_{j=1}^{n} P\Big(\Big|\sum_{i=1}^{n}\zeta_{nij}\Big| > \varepsilon n^{s-\lambda-1/p}\Big) \\ &\leqslant C\sum_{n=1}^{\infty} n\Bigg[\exp\Bigg(-\frac{C_1\varepsilon^2 n^{2s-2\lambda-2/p}}{n^{1-2/p}\log^{-2} n + \varepsilon m n^{s-\lambda-1/p}\log^{-1} n}\Bigg) \\ &\qquad\qquad + \frac{C_2 n\alpha(m)\log^{-1} n}{\varepsilon n^{s-\lambda-1/p}}\Bigg] \\ &\leqslant C\sum_{n=1}^{\infty} n\left[\exp\left(-\frac{C_3\varepsilon^2 n^{2s-2\lambda-2/p}}{\varepsilon n^{2s-2\lambda-2/p}\log^{-2} n}\right) + \frac{C_2 n^{1-s+\lambda+1/p}\log^{-1} n}{n^{\gamma(s-\lambda-1/p)}\log^{-\gamma} n}\right] \\ &\leqslant C\sum_{n=1}^{\infty} n^{1-C_3\varepsilon\log n} + C\sum_{n=1}^{\infty} n^{\lambda+1/p-s+2-\gamma(s-\lambda-1/p)}\log^{\gamma-1} n < \infty, \end{aligned} \tag{5.4}$$

which, together with the Borel–Cantelli lemma and (6.2), yields $H_{1n} = o(n^{-\lambda})$ a.s. ∎

LEMMA 5.6. (i) *Assumptions* $(\boldsymbol{H_2})$*–*$(\boldsymbol{H_4})$ *imply that*

$$n^{-1}\sum_{i=1}^{n}\tilde{\xi}_i^2 \to \sigma_\eta^2, \qquad \sum_{i=1}^{n}|\tilde{\xi}_i|/\sum_{i=1}^{n}\tilde{\xi}_i^2 \leqslant C.$$

(ii) *Assumptions* $(\boldsymbol{H_2})$–$(\boldsymbol{H_4})$ *imply that*

$$C_1 \leqslant n^{-1}\sum_{i=1}^{n}\sigma_i^{-2}\tilde{\xi}_i^2 \leqslant C_2, \quad I_n^{-2}\sum_{i=1}^{n}|\sigma_i^{-2}\tilde{\xi}_i| \leqslant C.$$

(iii) *Let* $\tilde{A}_i = A(t_i) - \sum_{j=1}^{n} W_{nj}(t_i)A(t_j)$, *where* $A = g$ *or* h. *Then* $(\boldsymbol{H_3})$(ii) *and* $(\boldsymbol{H_4})$(iii) *imply that* $\max_{1\leqslant i\leqslant n}|\tilde{A}_i| = o(n^{-1/4})$.

(iv) *Let* $\tilde{A}(t) = A(t) - \sum_{j=1}^{n} W_{nj}(t)A_j$, *where* $A = g$ *or* h. *Then* $(\boldsymbol{H_3})$(ii) *and* $(\boldsymbol{H_5})$(ii) *imply that* $\sup_{0\leqslant t\leqslant 1}|\tilde{A}(t)| = O(n^{-1/2})$. *By* $(\boldsymbol{H_3})$(ii) *and* $(\boldsymbol{H_5'})$ *we have* $\sup_{0\leqslant u\leqslant 1}|\tilde{f}(u)| = O(n^{-1/2})$, *where* $\tilde{f}(u) = f(u) - \sum_{j=1}^{n}\widehat{W}_{nj}(u)f(u_j)$.

(v) *Assumptions* $(\boldsymbol{H_2})$(ii), $(\boldsymbol{H_3})$(ii), $(\boldsymbol{H_4})$ *and* $(\boldsymbol{H_6})$ *imply that*

$$\left(qp^{-1}\right)^{1/2}\max_{1\leqslant i\leqslant n}|\tilde{\xi}_i| = o(1).$$

The final lemma is essential for establishing Theorem 3.4. We can get it from the proof of Theorem 3.3 easily.

LEMMA 5.7. *Suppose that assumptions* $(\boldsymbol{H_1})$–$(\boldsymbol{H_4})$, $(\boldsymbol{H_5'})$, $(\boldsymbol{H_6})$(i) *hold and* $\alpha(n) = O(n^{-\gamma})$ *for some* $\gamma > \max\left\{(2+\delta)(1+\delta)/\delta, \left(3 - s + \frac{1}{2+\delta}\right)/\left(s - \frac{1}{2+\delta}\right)\right\}$. *Assume that* $\max_{1\leqslant j\leqslant n}|\eta_j| = O(n^{1/2})$. *Then*

$$\max_{1\leqslant j\leqslant n}|\hat{f}_n(u_j) - f(u_j)| = o_p(n^{-1/4}). \tag{5.5}$$

Proof. Since $\max_{1\leqslant j\leqslant n}|\eta_j| = O(n^{1/2})$, it is easy to see that $\max_{1\leqslant j\leqslant n}|\tilde{\xi}_j| = O(n^{1/2})$. According to the proof of Theorem 3.3, we can easily get $F_{1n}'' = o_p(n^{-1/4})$, $F_{1n}''' = o_p(n^{-1/4})$ and $F_{ln} = o_p(n^{-1/4}), l = 2, \ldots, 12$. To prove (5.5), we only need to verify $F_{1n}' = o_p(n^{-1/4})$. This is obtained immediately by Lemma 5.5(ii) with $\lambda = 1/4$. ∎

6. PROOFS OF THE RESULTS

With the lemmas above at hand, we can now present the proofs of the main results.

6.1. Proof of Theorem 3.1. We only need to prove (3.1), since the proof of (3.2) is completely analogous and $\sigma_i^2 = f(u_i)$ is bounded by M_0 in view of assumption $(\boldsymbol{H_3})$. In this case, the errors are homoscedastic and $\xi_i = x_i$ can be observed exactly. So without loss of generality, we assume that $\sigma_i^2 = f(u_i) = \sigma^2$ is a

constant. It follows from (2.1) that

$$
\begin{aligned}
(6.1)\quad & \hat{\beta}_L - \beta \\
&= S_{1n}^{-2}\Big[\sum_{i=1}^{n}\tilde{x}_i\tilde{g}_i + n^{-1}\sum_{i=1}^{n}\tilde{x}_i(\tilde{\epsilon}_i - \tilde{\mu}_i\beta) + n\sigma_\mu^2\beta\Big] \\
&= S_{1n}^{-2}\Big[\sum_{i=1}^{n}\big[(\tilde{\xi}_i + \tilde{\mu}_i)(\tilde{\epsilon}_i - \tilde{\mu}_i\beta) + \sigma_\mu^2\beta\big] + \sum_{i=1}^{n}\tilde{\xi}_i\tilde{g}_i + \sum_{i=1}^{n}\tilde{\mu}_i\tilde{g}_i\Big] \\
&= S_{1n}^{-2}\Big[\sum_{i=1}^{n}\big[(\tilde{\xi}_i + \mu_i)(\epsilon_i - \mu_i\beta) + \sigma_\mu^2\beta\big] + \sum_{i=1}^{n}\tilde{\xi}_i\tilde{g}_i + \sum_{i=1}^{n}\tilde{\mu}_i\tilde{g}_i \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n}W_{nj}(t_i)\tilde{\xi}_i\mu_j\beta - \sum_{i=1}^{n}\sum_{j=1}^{n}W_{nj}(t_i)\tilde{\xi}_i\epsilon_j \\
&\quad - \sum_{i=1}^{n}\sum_{j=1}^{n}W_{nj}(t_i)\epsilon_i\mu_j - \sum_{i=1}^{n}\sum_{j=1}^{n}W_{nj}(t_i)\mu_i\epsilon_j + 2\sum_{i=1}^{n}\sum_{j=1}^{n}W_{nj}(t_i)\mu_i\mu_j\beta \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}W_{nj}(t_i)W_{nk}(t_i)\mu_j\epsilon_k - \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}W_{nj}(t_i)W_{nk}(t_i)\mu_j\mu_k\beta\Big] \\
&\doteq S_{1n}^{-2}\sum_{l=1}^{10}A_{ln}.
\end{aligned}
$$

Set $\omega_i = (\tilde{\xi}_i + \mu_i)(\epsilon_i - \mu_i\beta) + \sigma_\mu^2\beta$ and $Z_{ni} = \omega_i/\Sigma_n$. Since $\{(e_i, \mu_i),\ i \geqslant 1\}$ is a sequence of identically distributed α-mixing random vectors and the sequence $\{e_i,\ i \geqslant 1\}$ is independent of $\{\mu_i,\ i \geqslant 1\}$, follows that $\{\omega_i,\ i \geqslant 1\}$ is still a sequence of α-mixing random variables. We employ Bernstein's big-block and small-block procedure. Put $k = \lfloor n/(p+q)\rfloor$ and let

$$
y_{nm} = \sum_{i=k_m}^{k_m+p-1}Z_{ni}, \quad y'_{nm} = \sum_{i=l_m}^{l_m+q-1}Z_{ni}, \quad y'_{nk+1} = \sum_{i=k(p+q)+1}^{n}Z_{ni},
$$

where $k_m = (m-1)(p+q)+1, l_m = (m-1)(p+q)+p+1, m = 1, \ldots, k$. Then

$$
\frac{A_{1n}}{\Sigma_n} = \sum_{i=1}^{n}Z_{ni} = \sum_{m=1}^{k}y_{nm} + \sum_{m=1}^{k}y'_{nm} + y'_{nk+1} \doteq T'_n + T''_n + T'''_n.
$$

Thus, to prove $S_n^2(\hat{\beta}_L - \beta)/\Sigma_n \xrightarrow{D} N(0,1)$, we only need to show that

$$
\begin{aligned}
&S_{1n}^2/S_n^2 - 1 \xrightarrow{\text{a.s.}} 0, \quad E(T''_n)^2 + E(T'''_n)^2 \to 0, \\
&T'_n \xrightarrow{D} N(0,1), \quad \frac{A_{ln}}{\Sigma_n} \xrightarrow{P} 0 \quad \text{for } l = 2, \ldots, 10.
\end{aligned}
$$

Firstly, we prove

$$
(6.2)\qquad S_{1n}^2/S_n^2 - 1 \xrightarrow{\text{a.s.}} 0.
$$

Actually, it is easy to check that

$$
\begin{aligned}
(6.3)\quad & S_n^{-2}|S_{1n}^2 - S_n^2| \\
&= S_n^{-2}\Big|\sum_{i=1}^n (\tilde{x}_i^2 - \sigma_\mu^2) - \sum_{i=1}^n \tilde{\xi}_i^2\Big| = S_n^{-2}\Big|\sum_{i=1}^n (2\tilde{\xi}_i\tilde{\mu}_i + \tilde{\mu}_i^2 - \sigma_\mu^2)\Big| \\
&\leqslant S_n^{-2}\Big|\sum_{i=1}^n (\tilde{\mu}_i^2 - \sigma_\mu^2)\Big| + 2S_n^{-2}\Big|\sum_{i=1}^n \tilde{\xi}_i\mu_i\Big| + 2S_n^{-2}\Big|\sum_{i=1}^n \tilde{\xi}_i\Big(\sum_{j=1}^n W_{nj}(t_i)\mu_j\Big)\Big| \\
&\doteq S_{2n} + 2S_{3n} + 2S_{4n}.
\end{aligned}
$$

It suffices to prove that $S_{kn} = o(1)$ a.s. for $k = 2, 3, 4$.

Note that $\{\mu_i^2 - E\mu_i^2,\ i \geqslant 1\}$ and $\{|\mu_i| - E|\mu_i|,\ i \geqslant 1\}$ are both α-mixing random variables with mean zero. Applying Lemma 5.1 with $E|\mu_1^2|^{\frac{2+\delta}{2}} = E|\mu_1|^{2+\delta} < \infty$ yields

$$
n^{-1}\sum_{i=1}^n \mu_i^2 \xrightarrow{\text{a.s.}} \sigma_\mu^2 < \infty \quad \text{and} \quad n^{-1}\sum_{i=1}^n |\mu_i| \xrightarrow{\text{a.s.}} E|\mu_1| < \infty.
$$

For S_{2n}, by Lemmas 5.1 and 5.5(i) we have

$$
\begin{aligned}
(6.4)\quad S_{2n} &\leqslant Cn^{-1}\Big|\sum_{i=1}^n \Big[\Big(\mu_i - \sum_{j=1}^n W_{nj}(t_i)\mu_j\Big)^2 - \sigma_\mu^2\Big]\Big| \\
&= C\Big|n^{-1}\sum_{i=1}^n \mu_i^2 - \sigma_\mu^2 \\
&\quad + n^{-1}\sum_{i=1}^n\Big(\sum_{j=1}^n W_{nj}(t_i)\mu_j\Big)^2 - 2n^{-1}\sum_{i=1}^n\Big(\mu_i\sum_{j=1}^n W_{nj}(t_i)\mu_j\Big)\Big| \\
&\leqslant C\Big|n^{-1}\sum_{i=1}^n \mu_i^2 - \sigma_\mu^2\Big| + C\Big(\max_{1\leqslant i\leqslant n}\Big|\sum_{j=1}^n W_{nj}(t_i)\mu_j\Big|\Big)^2 \\
&\quad + C\max_{1\leqslant i\leqslant n}\Big|\sum_{j=1}^n W_{nj}(t_i)\mu_j\Big| n^{-1}\sum_{i=1}^n |\mu_i| \\
&\xrightarrow{\text{a.s.}} 0,
\end{aligned}
$$

which means that $S_{2n} = o(1)$ a.s.

For S_{3n}, by $\max_{1\leqslant i\leqslant n}|\eta_i| = O(n^{1/2}\log^{-1} n)$ and Lemma 5.6(i), we get

$$
\max_{1\leqslant i\leqslant n}|n^{-1}\tilde{\xi}_i| = O(n^{-1/2}\log^{-1} n), \qquad \sum_{i=1}^n |n^{-1}\tilde{\xi}_i| = O(1).
$$

It follows from $E|\mu_1|^{2+\delta} < \infty$ and Lemma 5.5(i) that

$$
(6.5)\qquad S_{3n} \leqslant C\Big|n^{-1}\sum_{i=1}^n \tilde{\xi}_i\mu_i\Big| \xrightarrow{\text{a.s.}} 0.
$$

For S_{4n}, it follows from Lemmas 5.5(i) and 5.6(i) that

$$S_{4n} \leqslant Cn^{-1}\Big|\sum_{i=1}^{n}\tilde{\xi}_i\Big(\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\Big)\Big| \leqslant C\max_{1\leqslant i\leqslant n}\Big|\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\Big|n^{-1}\sum_{i=1}^{n}|\tilde{\xi}_i| \xrightarrow{\text{a.s.}} 0 \tag{6.6}$$

From (6.3)–(6.6), we can get $S_n^{-2}|S_{1n}^2 - S_n^2| = o(1)$ a.s., which yields (6.2) immediately.

Secondly, we need to prove

$$E(T_n'')^2 + E(T_n''')^2 \to 0. \tag{6.7}$$

It is readily seen that $qp^{-1}\max_{1\leqslant i\leqslant n}|\tilde{\xi}_i|^2 = o(1)$ by Lemma 5.6(v). In addition, by $(\boldsymbol{H_1})$, $(\boldsymbol{H_6})$, Lemma 5.2(ii) and $\sum_n^2 \geqslant Cn, \gamma > (2+\delta)/\delta$, we get

$$\begin{aligned}
&E(T_n'')^2 + E(T_n''')^2 \\
&= E\Big(\sum_{m=1}^{k}\sum_{i=l_m}^{l_m+q-1}Z_{ni}\Big)^2 + E\Big(\sum_{i=k(p+q)+1}^{n}Z_{ni}\Big)^2 \\
&\leqslant \frac{C}{n}E\Big[\sum_{m=1}^{k}\sum_{i=l_m}^{l_m+q-1}(\tilde{\xi}_i\epsilon_i - \tilde{\xi}_i\mu_i\beta + \mu_i\epsilon_i - (\mu_i^2-\sigma_\mu^2)\beta)\Big]^2 \\
&\quad + \frac{C}{n}E\Big[\sum_{i=k(p+q)+1}^{n}(\tilde{\xi}_i\epsilon_i - \tilde{\xi}_i\mu_i\beta + \mu_i\epsilon_i - (\mu_i^2-\sigma_\mu^2)\beta)\Big]^2 \\
&\leqslant \frac{Ckq}{n}\max_{1\leqslant i\leqslant n}|\tilde{\xi}_i|^2 + \frac{C_1}{n}E\Big(\sum_{m=1}^{k}\sum_{i=l_m}^{l_m+q-1}\mu_i\epsilon_i\Big)^2 \\
&\quad + \frac{C_2}{n}\sum_{m=1}^{k}\sum_{i=l_m}^{l_m+q-1}(E|\mu_i^2-\sigma_\mu^2|^{2+\delta})^{\frac{2}{2+\delta}} + \frac{Cp}{n}\max_{1\leqslant i\leqslant n}|\tilde{\xi}_i|^2 \\
&\quad + \frac{C_3}{n}E\Big(\sum_{i=k(p+q)+1}^{n}\mu_i\epsilon_i\Big)^2 + \frac{C_4}{n}\sum_{i=k(p+q)+1}^{n}(E|\mu_i^2-\sigma_\mu^2|^{2+\delta})^{\frac{2}{2+\delta}} \\
&\leqslant \frac{Cq}{p}\max_{1\leqslant i\leqslant n}|\tilde{\xi}_i|^2 + \frac{C_5q}{p} + \frac{Cp}{n}\max_{1\leqslant i\leqslant n}|\tilde{\xi}_i|^2 + \frac{C_6p}{n} = o(1),
\end{aligned} \tag{6.8}$$

which implies that (6.7) holds.

Thirdly, we need to prove

$$\frac{A_{ln}}{\Sigma_n} \xrightarrow{P} 0 \quad \text{for } l = 2,\dots,10. \tag{6.9}$$

For A_{2n}/Σ_n, by $(\boldsymbol{H_2})$(ii), $(\boldsymbol{H_4})$(i), Lemma 5.6(iii), Abel's inequality and $\Sigma_n^2 \geqslant Cn$, we have

$$
\begin{aligned}
(6.10)\qquad \Big|\frac{A_{2n}}{\Sigma_n}\Big| &\leqslant \frac{C}{\sqrt{n}}\Big|\sum_{i=1}^n \tilde{\xi}_i\tilde{g}_i\Big| \\
&\leqslant \frac{C}{\sqrt{n}}\Big\{\Big|\sum_{i=1}^n \tilde{h}_i\tilde{g}_i\Big| + \Big|\sum_{i=1}^n \eta_i\tilde{g}_i\Big| + \Big|\sum_{i=1}^n\Big(\sum_{j=1}^n W_{nj}(t_i)\eta_j\Big)\tilde{g}_i\Big|\Big\} \\
&\leqslant \frac{C}{\sqrt{n}}\Big\{n\max_{1\leqslant i\leqslant n}|\tilde{h}_i|\max_{1\leqslant i\leqslant n}|\tilde{g}_i| + \max_{1\leqslant i\leqslant n}|\tilde{g}_i|\max_{1\leqslant m\leqslant n}\Big|\sum_{i=1}^m \eta_{k_i}\Big| \\
&\quad + \max_{1\leqslant i\leqslant n}|\tilde{g}_i|\max_{1\leqslant m'\leqslant n}\Big|\sum_{i=1}^{m'}\Big(\sum_{j=1}^n W_{nj}(t_{k_i})\eta_j\Big)\Big|\Big\} \\
&\leqslant \frac{C}{\sqrt{n}}\Big\{n\max_{1\leqslant i\leqslant n}|\tilde{h}_i|\max_{1\leqslant i\leqslant n}|\tilde{g}_i| + \max_{1\leqslant i\leqslant n}|\tilde{g}_i|\max_{1\leqslant m\leqslant n}\Big|\sum_{i=1}^m \eta_{k_i}\Big| \\
&\quad + \max_{1\leqslant i\leqslant n}|\tilde{g}_i|\max_{1\leqslant j\leqslant n}\sum_{i=1}^n W_{nj}(t_{k_i})\max_{1\leqslant m\leqslant n}\Big|\sum_{j=1}^m \eta_{k_j}\Big|\Big\} \\
&= o(1) + o(n^{-1/4}\log n),
\end{aligned}
$$

which yields $A_{2n}/\Sigma_n \to 0$.

For A_{3n}/Σ_n, since $\gamma > (2+\delta)/\delta$, we can deduce by $(\boldsymbol{H_1})$, $(\boldsymbol{H_4})$(i), Lemmas 5.2(ii) and 5.6(iii) that

$$
\begin{aligned}
(6.11)\qquad & E\Big(\frac{A_{3n}}{\Sigma_n}\Big)^2 \\
&\leqslant \frac{C}{n}\Big[E\Big(\sum_{i=1}^n \tilde{g}_i\mu_i\Big)^2 + E\Big(\sum_{i=1}^n \tilde{g}_i\sum_{j=1}^n W_{nj}(t_i)\mu_j\Big)^2\Big] \\
&\leqslant \frac{C}{n}\Big[\sum_{i=1}^n \tilde{g}_i^2(E|\mu_i|^{2+\delta})^{\frac{2}{2+\delta}} + \sum_{j=1}^n\Big(\sum_{i=1}^n W_{nj}(t_i)\tilde{g}_i\Big)^2(E|\mu_j|^{2+\delta})^{\frac{2}{2+\delta}}\Big] \\
&\leqslant \frac{C}{n}\Big[\sum_{i=1}^n \tilde{g}_i^2 + \sum_{j=1}^n\Big(\sum_{i=1}^n W_{nj}(t_i)\tilde{g}_i\Big)^2\Big] \\
&\leqslant C\max_{1\leqslant i\leqslant n}\tilde{g}_i^2 + C\max_{1\leqslant i\leqslant n}\tilde{g}_i^2\max_{1\leqslant j\leqslant n}\Big(\sum_{i=1}^n W_{nj}(t_i)\Big)^2 = o(n^{-1/2}),
\end{aligned}
$$

which yields $A_{3n}/\Sigma_n \xrightarrow{P} 0$ by Chebyshev's inequality.

For A_{4n}/Σ_n, note that

$$
\begin{aligned}
(6.12)\qquad A_{4n} &= \sum_{i=1}^n \tilde{h}_i\Big(\sum_{j=1}^n W_{nj}(t_i)\mu_j\beta\Big) + \sum_{i=1}^n \eta_i\Big(\sum_{j=1}^n W_{nj}(t_i)\mu_j\beta\Big) \\
&\quad - \sum_{i=1}^n\sum_{s=1}^n W_{ns}(t_i)\eta_s\Big(\sum_{j=1}^n W_{nj}(t_i)\mu_j\beta\Big) \\
&\doteq A_1 + A_2 + A_3,
\end{aligned}
$$

By $(\boldsymbol{H_2})$(ii), $(\boldsymbol{H_4})$(i, ii), Lemmas 5.2(ii), 5.5(ii), 5.6(iii), Abel's inequality and $\Sigma_n^2 \geqslant Cn$, we get

$$\begin{aligned}
E\Big(\frac{A_1}{\Sigma_n}\Big)^2 &\leqslant \frac{C}{n}\sum_{j=1}^{n}\Big[\sum_{i=1}^{n} W_{nj}(t_i)\tilde{h}_i\Big]^2 \\
&\leqslant C\max_{1\leqslant i\leqslant n}|\tilde{h}_i|^2 \max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{n} W_{nj}(t_i)\Big|^2 = o(n^{-1/2}), \\
E\Big(\frac{A_2}{\Sigma_n}\Big)^2 &\leqslant \frac{C}{n}\sum_{j=1}^{n}\Big[\sum_{i=1}^{n} W_{nj}(t_i)\eta_i\Big]^2 \\
&\leqslant C\max_{1\leqslant m\leqslant n}\Big|\sum_{i=1}^{m}\eta_{k_i}\Big|^2 \max_{1\leqslant i,j\leqslant n} W_{nj}^2(t_i) = o(1).
\end{aligned}$$

and

$$\begin{aligned}
(6.13)\quad & E\Big(\frac{A_3}{\Sigma_n}\Big)^2 \\
&\leqslant Cn^{-1}E\Big|\sum_{i=1}^{n}\Big(\sum_{s=1}^{n} W_{ns}(t_i)\eta_s\Big)\Big(\sum_{j=1}^{n} W_{nj}(t_i)\mu_j\beta\Big)\Big|^2 \\
&= Cn^{-1}E\Big|\sum_{j=1}^{n}\Big(\sum_{i=1}^{n}\Big(\sum_{s=1}^{n} W_{ns}(t_i)\eta_s\Big)W_{nj}(t_i)\Big)\mu_j\beta\Big|^2 \\
&\leqslant Cn^{-1}\sum_{j=1}^{n}\Big(\sum_{i=1}^{n}\Big(\sum_{s=1}^{n} W_{ns}(t_i)\eta_s\Big)W_{nj}(t_i)\Big)^2 \\
&\leqslant Cn^{-1}\sum_{j=1}^{n}\Big(\max_{1\leqslant i\leqslant n} W_{nj}(t_i)\max_{1\leqslant m'\leqslant n}\Big|\sum_{i=1}^{m'}\Big(\sum_{s=1}^{n} W_{ns}(t_{k_i})\eta_s\Big)\Big|\Big)^2 \\
&= Cn^{-1}\sum_{j=1}^{n}\Big(\max_{1\leqslant i\leqslant n} W_{nj}(t_i)\max_{1\leqslant m'\leqslant n}\Big|\sum_{s=1}^{n}\Big(\sum_{i=1}^{m'} W_{ns}(t_{k_i})\Big)\eta_s\Big|\Big)^2 \\
&\leqslant Cn^{-1}\sum_{j=1}^{n}\Big(\max_{1\leqslant i\leqslant n} W_{nj}(t_i)\cdot\max_{1\leqslant m'\leqslant n}\max_{1\leqslant m\leqslant n}\Big|\sum_{s=1}^{m}\eta_{k_s}\Big| \\
&\quad\cdot\max_{1\leqslant s\leqslant n}\sum_{i=1}^{m'} W_{ns}(t_{k_i})\Big)^2 \\
&\leqslant C\Big(\max_{1\leqslant i,j\leqslant n} W_{nj}(t_i)\max_{1\leqslant m\leqslant n}\Big|\sum_{s=1}^{m}\eta_{k_s}\Big|\max_{1\leqslant s\leqslant n}\sum_{i=1}^{n} W_{ns}(t_i)\Big)^2 = o(1)
\end{aligned}$$

It follows that $A_{4n}/\Sigma_n \xrightarrow{P} 0$. Similarly, by Lemma 5.5(ii), we can get $A_{5n}/\Sigma_n \xrightarrow{P} 0$.

For A_{6n}/Σ_n, by $(\boldsymbol{H_1})$, Lemma 5.2(i, ii), it is easy to see that for $\gamma > (2+\delta)(1+\delta)/\delta$ and any $\epsilon < s(2+\delta)/2$,

$$
\begin{aligned}
(6.14)\qquad E\Big(\frac{A_{6n}}{\Sigma_n}\Big)^2 &\leqslant \frac{C}{n}E\Big[\sum_{i=1}^n\sum_{j=1}^n W_{nj}(t_i)\epsilon_i\mu_j\Big]^2\\
&\leqslant \frac{C}{n}\sum_{i=1}^n\Big\|\sum_{j=1}^n W_{nj}(t_i)\mu_j\Big\|_{2+\delta}^2\cdot\|\epsilon_i\|_{2+\delta}^2\\
&\leqslant \frac{C}{n}\sum_{i=1}^n\Big[n^{\epsilon}\sum_{j=1}^n W_{nj}^{2+\delta}(t_i)+\Big(\sum_{j=1}^n W_{nj}^2(t_i)\Big)^{\frac{2+\delta}{2}}\Big]^{\frac{2}{2+\delta}}\\
&= o(n^{2\epsilon/(2+\delta)-s}\log^{-1}n)+o(n^{-s}\log^{-1}n),
\end{aligned}
$$

which implies $A_{6n}/\Sigma_n \xrightarrow{P} 0$. The proofs for $A_{7n}/\Sigma_n \xrightarrow{P} 0$ and $A_{9n}/\Sigma_n \xrightarrow{P} 0$ are analogous.

For A_{8n}/Σ_n, by $E|\mu_1|^{4+2\delta}<\infty$, $(\boldsymbol{H_4})$(i) and $\Sigma_n^2 \geqslant Cn$ we have

$$
\begin{aligned}
(6.15)\quad & E\Big(\frac{A_{8n}}{\Sigma_n}\Big)^2\\
&\leqslant \frac{C}{n}\sum_{i_1=1}^n\sum_{i_2=1}^n\sum_{j_1=1}^n\sum_{j_2=1}^n W_{nj_1}(t_{i1})W_{nj_2}(t_{i2})E(\mu_{i_1}\mu_{i_2}\mu_{j_1}\mu_{j_2})\\
&\leqslant \frac{C}{n}\Big[\sum_{i=1}^n\sum_{j=1}^n W_{nj}^2(t_i)E(\mu_i^2\mu_j^2)+\sum_{i_1=1}^n\sum_{i_2=1}^n W_{ni_1}(t_{i_1})W_{ni_2}(t_{i_2})E(\mu_{i_1}^2\mu_{i_2}^2)\Big]\\
&\leqslant \frac{C}{n}\Big[\max_{1\leqslant j\leqslant n}\sum_{i=1}^n W_{nj}(t_i)\cdot\sum_{j=1}^n\max_{1\leqslant i\leqslant n}W_{nj}(t_i)+\sum_{i_1=1}^n W_{ni_1}(t_{i_1})\cdot\sum_{i_2=1}^n W_{ni_2}(t_{i_2})\Big]\\
&= O(n^{-1}),
\end{aligned}
$$

which implies $A_{8n}/\Sigma_n \xrightarrow{P} 0$. Similarly, we get $A_{10n}/\Sigma_n \xrightarrow{P} 0$.

At last, we prove that

$$
(6.16)\qquad T_n' \xrightarrow{D} N(0,1).
$$

By Lindeberg's condition, it suffices to show that

$$
(6.17)\qquad \mathrm{Var}(T_n')\to 1,
$$

$$
(6.18)\qquad \Big|E\exp\Big(it\sum_{m=1}^k y_{nm}\Big)-\prod_{m=1}^k E\exp(ity_{nm})\Big|\to 0,
$$

$$
(6.19)\qquad A_n(\omega)\doteq\sum_{m=1}^k Ey_{nm}^2 I(|y_{nm}|>\omega)\to 0 \quad \text{for any } \omega>0.
$$

It is easy to see that $E(T_n'+T_n''+T_n''')^2 = E(A_{1n}/\Sigma_n)^2 = 1$ and

$$
\begin{aligned}
E(T_n'+T_n''+T_n''')^2 &= E(T_n')^2+E(T_n'')^2+E(T_n''')^2\\
&\quad +2E(T_n'T_n'')+2E(T_n'T_n''')+2E(T_n''T_n''').
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
(6.20)\quad |E(T_n')^2 - 1| &\leqslant E(T_n'')^2 + E(T_n''')^2 + 2\sqrt{E(T_n')^2 E(T_n'')^2} \\
&\quad + 2\sqrt{E(T_n')^2 E(T_n''')^2} + 2\sqrt{E(T_n'')^2 E(T_n''')^2} \\
&= o(1),
\end{aligned}
$$

which means that (6.17) holds. For (6.18), by Lemma 5.3 and $(\boldsymbol{H_6})$(i), we have

$$
\begin{aligned}
(6.21)\quad &\Big|E\exp\Big(it\sum_{m=1}^{k} y_{nm}\Big) - \prod_{m=1}^{k} E\exp(ity_{nm})\Big| \\
&\qquad \leqslant C(k-1)\alpha(q) \leqslant Cnp^{-1}\alpha(q) \to 0,
\end{aligned}
$$

which implies (6.18).

Finally, we only need to prove (6.19). Note that $\gamma > (2+\delta)/\delta$. Set $r = 2(1+\tau), \nu = \delta - 2\tau$, where $0 < \tau < [\gamma\delta - (2+\delta)]/(2\gamma + 2 + \delta) < \delta/2$. Obviously, $r + \nu = 2 + \delta$ and

$$
\frac{2+\delta}{\delta} < \frac{(2+\delta)(1+\tau)}{\delta - 2\tau} = \frac{r(r+\nu)}{2\nu} < \gamma.
$$

By Lemmas 5.2(i), 5.6(i, v) and $\Sigma_n^2 \geqslant Cn$, for $0 < \varepsilon < \tau$, we get

$$
\begin{aligned}
A_n(\omega) &\leqslant C\sum_{m=1}^{k} E|y_{nm}|^r = C\sum_{m=1}^{k} E\Big|\sum_{i=k_m}^{k_m+p-1} Z_{ni}\Big|^r \\
&\leqslant Cn^{-r/2}\sum_{m=1}^{k}\Big[p^{\varepsilon}\sum_{i=k_m}^{k_m+p-1} E|\omega_i|^r + \Big(\sum_{i=k_m}^{k_m+p-1}\|\omega_i\|_{r+\delta}^2\Big)^{r/2}\Big] \\
&\leqslant Cn^{-(1+\tau)}\Big\{p^{\varepsilon}\sum_{m=1}^{k}\sum_{i=k_m}^{k_m+p-1}(|\tilde{\xi}_i|^{2(1+\tau)} + O(1)) \\
&\quad + \sum_{m=1}^{k}\Big[\sum_{i=k_m}^{k_m+p-1}(\tilde{\xi}_i^2 + O(1))\Big]^{1+\tau}\Big\} \\
&\leqslant Cn^{-(1+\tau)}\Big\{p^{\varepsilon}\sum_{m=1}^{k}\sum_{i=k_m}^{k_m+p-1}\tilde{\xi}_i^2|\tilde{\xi}_i|^{2\tau} + np^{\epsilon} \\
&\quad + \sum_{m=1}^{k}\Big[\Big(\sum_{i=k_m}^{k_m+p-1}\tilde{\xi}_i^2\Big)\Big(\sum_{i=k_m}^{k_m+p-1}\tilde{\xi}_i^2\Big)^{\tau}\Big] + np^{\tau}\Big\} \\
&\leqslant Cn^{-(1+\tau)}\Big\{np^{\varepsilon}\Big(\max_{1\leqslant i\leqslant n}|\tilde{\xi}_i|\Big)^{2\tau} + np^{\epsilon} + np^{\tau}\Big(\max_{1\leqslant i\leqslant n}|\tilde{\xi}_i|\Big)^{2\tau} + np^{\tau}\Big\} \\
&= C\Big(\frac{p^2}{nq}\Big)^{\tau}\Big(\frac{q}{p}\max_{1\leqslant i\leqslant n}|\tilde{\xi}_i|^2\Big)^{\tau} + O\Big(\frac{p^{\tau}}{n^{\tau}}\Big) = o\Big(\frac{p^{2\tau}}{n^{\tau}q^{\tau}}\Big) + o(1) \\
&= o(1). \ \blacksquare
\end{aligned}
$$

6.2. Proof of Theorem 3.2. We only present the proof of the first equation, as the second one is completely analogous. In this case, the errors are homoscedastic. So without loss of generality, we assume that $\sigma_i^2 = f(u_i) = \sigma^2$ is a constant. It can be easily seen that

$$\begin{aligned}(6.22)\quad \hat{g}_L(t) - g(t) &= \sum_{j=1}^{n} W_{nj}(t)(y_j - x_j\hat{\beta}_L) - g(t)\\ &= \sum_{j=1}^{n} W_{nj}(t)\big(\xi_j\beta + g(t_j) + \epsilon_j - (\xi_j + \mu_j)\hat{\beta}_L\big) - g(t)\\ &= \sum_{j=1}^{n} W_{nj}(t)(\epsilon_j - \mu_j\beta) + \sum_{j=1}^{n} W_{nj}(t)\xi_j(\beta - \hat{\beta}_L)\\ &\quad + \sum_{j=1}^{n} W_{nj}(t)\mu_j(\beta - \hat{\beta}_L) - \tilde{g}(t)\\ &\doteq G_{1n} + G_{2n} + G_{3n} + G_{4n}\end{aligned}$$

Similar to the proof of A_{1n}, we can get $G_{1n}(t)/\Gamma_n(t) \to N(0,1)$ by using Bernstein's big-block and small-block procedure. Moreover, noting that $n\Gamma_n^2(t) \to \infty$ as $n \to \infty$, we only need to prove $G_{kn} = O_p(n^{-1/2})$, $k = 2, 3, 4$.

Before, we first verify $(\beta - \hat{\beta}_L)^2 = O_p(n^{-1})$. It follows from (6.1) that

$$(6.23)\qquad E[S_{1n}^2(\beta - \hat{\beta}_L)]^2 = E\Big[\sum_{k=1}^{10} A_{kn}\Big]^2 \leqslant C\sum_{k=1}^{10} EA_{kn}^2.$$

For EA_{1n}^2, by Lemmas 5.2(ii) and 5.6(i), $(\boldsymbol{H_1})$ and $(\boldsymbol{H_3})$(i), we derive

$$\begin{aligned}(6.24)\quad &\sup_n n^{-1}EA_{1n}^2\\ &\leqslant \sup_n Cn^{-1}\Big\{E\Big(\sum_{i=1}^{n}\tilde{\xi}_i\epsilon_i\Big)^2 + E\Big(\sum_{i=1}^{n}\tilde{\xi}_i\mu_i\beta\Big)^2 + E\Big(\sum_{i=1}^{n}\epsilon_i\mu_i\Big)^2\\ &\quad + \sum_{i=1}^{n}\Big(E|\mu_i^2 - \sigma_\mu^2|^{2+\delta}\Big)^{\frac{2}{2+\delta}}\beta^2\Big\}\\ &\leqslant \sup_n Cn^{-1}\Big[C_1\sum_{i=1}^{n}\tilde{\xi}_i^2\sigma_i^2 + C_2\sum_{i=1}^{n}\tilde{\xi}_i^2\beta^2 + C_3\sum_{i=1}^{n}\sigma_i^2 + C_4 n\Big] < \infty.\end{aligned}$$

By (6.10)–(6.15), one can achieve that $\sup_n n^{-1}EA_{kn}^2 < \infty$ for $k = 2, \ldots, 10$ immediately. By (6.2), we know $S_{1n}^2 \to S_n^2$ a.s. Hence, by Lemma 5.6(i) we derive

$$(6.25)\qquad (\beta - \hat{\beta}_L)^2 = O_p(n^{-1}), \quad |\beta - \hat{\beta}_L| = O_p(n^{-1/2}).$$

For G_{2n}, by $(\boldsymbol{H_3})$(ii), $(\boldsymbol{H_5})$ and $\max_{1\leqslant j\leqslant n}|\eta_j| = O(1)$, we have

$$
\begin{aligned}
|G_{2n}| &= |\beta - \hat{\beta}_L| \cdot \Big|\sum_{j=1}^{n} W_{nj}(t)\xi_j\Big| \\
&\leqslant |\beta - \hat{\beta}_L| \cdot \Big[\sum_{j=1}^{n} W_{nj}(t)|h(t_j)| + \sum_{j=1}^{n} W_{nj}(t)|\eta_j|\Big] \\
&= O_p(n^{-1/2}).
\end{aligned} \tag{6.26}
$$

For G_{3n}, it follows from $(\boldsymbol{H_1})$, $(\boldsymbol{H_5})$(i) and Lemma 5.5(i) that

$$
|G_{3n}| \leqslant |\beta - \hat{\beta}_L| \cdot \Big|\sum_{j=1}^{n} W_{nj}(t)\mu_j\Big| = O_p(n^{-1/2}). \tag{6.27}
$$

For G_{4n}, it is obvious that $G_{4n} = O_p(n^{-1/2})$ by Lemma 5.6(iv). ∎

6.3. Proof of Theorem 3.3. It is easy to check that

$$
\begin{aligned}
&\hat{f}_n(u) - f(u) = \sum_{j=1}^{n} \widehat{W}_{nj}(u)(\tilde{y}_j - \tilde{x}_j\hat{\beta}_L)^2 - \sigma_\mu^2\hat{\beta}_L^2 - f(u) \qquad (6.28)\\
&= \sum_{j=1}^{n} \widehat{W}_{nj}(u)(\tilde{\xi}_j\beta + \tilde{g}(t_j) + \tilde{\epsilon}_j - (\tilde{\xi}_j + \tilde{\mu}_j)\hat{\beta}_L)^2 - \sigma_\mu^2\hat{\beta}_L^2 - f(u) \\
&= \sum_{j=1}^{n} \widehat{W}_{nj}(u)[(\tilde{\epsilon}_j - \tilde{\mu}_j\beta) + \tilde{\mu}_j(\beta - \hat{\beta}_L) + \tilde{\xi}_j(\beta - \hat{\beta}_L) + \tilde{g}(t_j)]^2 - \sigma_\mu^2\hat{\beta}_L^2 - f(u) \\
&= \sum_{j=1}^{n} \widehat{W}_{nj}(u)[(\tilde{\epsilon}_j - \tilde{\mu}_j\beta)^2 - (\sigma_j^2 + \sigma_\mu^2\beta^2)] + \sum_{j=1}^{n} \widehat{W}_{nj}(u)(\tilde{\mu}_j^2 - \sigma_\mu^2)(\beta - \hat{\beta}_L)^2 \\
&\quad + \sum_{j=1}^{n} \widehat{W}_{nj}(u)\tilde{\xi}_j^2(\beta - \hat{\beta}_L)^2 + \sum_{j=1}^{n} \widehat{W}_{nj}(u)\tilde{g}^2(t_j) + 2\sigma_\mu^2\beta(\beta - \hat{\beta}_L) - \tilde{f}(u) \\
&\quad + 2\sum_{j=1}^{n} \widehat{W}_{nj}(u)\tilde{\mu}_j(\tilde{\epsilon}_j - \tilde{\mu}_j\beta)(\beta - \hat{\beta}_L) + 2\sum_{j=1}^{n} \widehat{W}_{nj}(u)\tilde{\xi}_j(\tilde{\epsilon}_j - \tilde{\mu}_j\beta)(\beta - \hat{\beta}_L) \\
&\quad + 2\sum_{j=1}^{n} \widehat{W}_{nj}(u)\tilde{g}(t_j)(\tilde{\epsilon}_j - \tilde{\mu}_j\beta) + 2\sum_{j=1}^{n} \widehat{W}_{nj}(u)\tilde{\xi}_j\tilde{\mu}_j(\beta - \hat{\beta}_L)^2 \\
&\quad + 2\sum_{j=1}^{n} \widehat{W}_{nj}(u)\tilde{\mu}_j(\beta - \hat{\beta}_L)\tilde{g}(t_j) + 2\sum_{j=1}^{n} \widehat{W}_{nj}(u)\tilde{\xi}_j(\beta - \hat{\beta}_L)\tilde{g}(t_j) \\
&\doteq \sum_{l=1}^{12} F_{ln}.
\end{aligned}
$$

For F_{1n}, we obtain

$$\begin{aligned}
F_{1n} &= \sum_{j=1}^{n} \widehat{W}_{nj}(u)\Big\{\Big(\epsilon_j - \sum_{i=1}^{n} W_{ni}(t_j)\epsilon_i\Big) - \Big(\mu_j - \sum_{i=1}^{n} W_{ni}(t_j)\mu_i\Big)\beta\Big\}^2 \\
&\qquad\qquad - (\sigma_j^2 + \sigma_\mu^2\beta^2)\Big\} \\
&= \sum_{j=1}^{n} \widehat{W}_{nj}(u)\Big\{(\epsilon_j - \mu_j\beta) - \sum_{i=1}^{n} W_{ni}(t_j)(\epsilon_i - \mu_i\beta)\Big\}^2 - (\sigma_j^2 + \sigma_\mu^2\beta^2)\Big\} \\
&= \sum_{j=1}^{n} \widehat{W}_{nj}(u)\big[(\epsilon_j - \mu_j\beta)^2 - (\sigma_j^2 + \sigma_\mu^2\beta^2)\big] \\
&\quad + \sum_{j=1}^{n} \widehat{W}_{nj}(u)\Big[\sum_{i=1}^{n} W_{ni}(t_j)(\epsilon_i - \mu_i\beta)\Big]^2 \\
&\quad - 2\sum_{j=1}^{n} \widehat{W}_{nj}(u)(\epsilon_j - \mu_j\beta)\sum_{i=1}^{n} W_{ni}(t_j)(\epsilon_i - \mu_i\beta) \\
&\doteq F_{1n}' + F_{1n}'' - 2F_{1n}'''.
\end{aligned}$$

The proof of this theorem is analogous to that of Theorem 3.1. We only need to prove that $F_{1n}'/\Delta_n(u) \xrightarrow{D} N(0,1)$ $F_{1n}''/\Delta_n(u) \xrightarrow{P} 0$, $F_{1n}'''/\Delta_n(u) \xrightarrow{P} 0$ and $F_{ln}/\Delta_n(u) \xrightarrow{P} 0$ for $l = 2, \ldots, 12$.

STEP 1. Set $Y_j = (\epsilon_j - \mu_j\beta)^2 - (\sigma_j^2 + \sigma_\mu^2\beta^2)$. Notice that $\{Y_j,\, j \geqslant 1\}$ is still a sequence of zero mean α-mixing random variables and $\sup_{j\geqslant 1} E|Y_j|^{2+\delta} < \infty$. Similar to the proof of A_{1n} in Theorem 3.1, one can verify that $F_{1n}'/\Delta_n(u) \xrightarrow{D} N(0,1)$.

STEP 2. Since $n\sum_{j=1}^{n} \widehat{W}_{nj}^2(u) \to \infty$ and $\inf_{1\leqslant j\leqslant n} \mathrm{Var}[(\epsilon_j - \mu_j\beta)^2] \geqslant C > 0$, it is obvious that $n\Delta_n^2(u) \to \infty$. Therefore, we just need to prove $F_{1n}'' = O_p(n^{-1/2})$, $F_{1n}''' = O_p(n^{-1/2})$ and $F_{ln} = O_p(n^{-1/2})$ for $l = 2, \ldots, 12$.

For F_{1n}'' and F_{1n}''', we find by $(\boldsymbol{H_4})$, $(\boldsymbol{H_5'})$ and Lemma 5.2(ii) that

$$\begin{aligned}
EF_{1n}'' &= \sum_{j=1}^{n} \widehat{W}_{nj}(u) E\Big[\sum_{i=1}^{n} W_{ni}(t_j)(\epsilon_i - \mu_i\beta)\Big]^2 \\
&\leqslant C\sum_{j=1}^{n} \widehat{W}_{nj}(u)\sum_{i=1}^{n} W_{ni}^2(t_j)\|\epsilon_i - \mu_i\beta\|_{2+\delta}^2 \\
&\leqslant C\sum_{j=1}^{n} \widehat{W}_{nj}(u)\sum_{i=1}^{n} W_{ni}^2(t_j)(|\sigma_i|^{2+\delta}|Ee_1|^{2+\delta} + |\beta|^{2+\delta}|E\mu_1|^{2+\delta})^{\frac{2}{2+\delta}} \\
&\leqslant C\sum_{j=1}^{n} \widehat{W}_{nj}(u)\sum_{i=1}^{n} W_{ni}^2(t_j) = O(n^{-1/2})
\end{aligned}$$

and

$$\begin{aligned} EF_{1n}'''^2 &= \sum_{j_1,j_2=1}^{n} \sum_{i_1,i_2=1}^{n} \widehat{W}_{nj_1}(u)\widehat{W}_{nj_2}(u)W_{ni_1}(t_{j_1})W_{ni_2}(t_{j_2}) \\ &\quad \cdot E[(\epsilon_{j_1} - \mu_{j_1}\beta)(\epsilon_{j_2} - \mu_{j_2}\beta)(\epsilon_{i_1} - \mu_{i_1}\beta)(\epsilon_{i_2} - \mu_{i_2}\beta)] \\ &\leqslant C \sum_{j=1}^{n} \sum_{i=1}^{n} \widehat{W}_{nj}^2(u)W_{ni}^2(t_j)E[(\epsilon_j - \mu_j\beta)^2(\epsilon_i - \mu_i\beta)^2] \\ &\quad + C \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \widehat{W}_{nj_1}(u)\widehat{W}_{nj_2}(u)W_{nj_1}(t_{j_1})W_{nj_2}(t_{j_2}) \\ &\quad \cdot E[(\epsilon_{j_1} - \mu_{j_1}\beta)^2(\epsilon_{j_2} - \mu_{j_2}\beta)^2] \\ &= O(n^{-1}), \end{aligned}$$

which implies that $F_{1n}'' = O_p(n^{-1/2})$ and $F_{1n}''' = O_p(n^{-1/2})$.

Next, we show that the remaining terms are asymptotically negligible. It follows from the proof of Theorem 3.2 that

$$|\hat{\beta}_L - \beta| = O_p(n^{-1/2}).$$

Together with Assumptions $(\boldsymbol{H_3})$–$(\boldsymbol{H_5'})$ and Lemmas 5.5–5.6, this yields

$$\max_{1\leqslant j\leqslant n} |\tilde{g}(t_j)| = O(n^{-1/2}), \qquad \sup_{0\leqslant u\leqslant 1} |\tilde{f}(u)| = O(n^{-1/2}),$$

and

$$\max_{1\leqslant j\leqslant n} |\tilde{\xi}_j| = O(n^{1/4}).$$

Moreover, for $X_j = \epsilon_j$ or μ_j,

$$\sum_{j=1}^{n} \widehat{W}_{nj}(u)|X_j| = O_p(1),$$

which follows from Lemma 5.5 and Assumption $(\boldsymbol{H_5'})$.

Using these facts, the terms containing $(\hat{\beta}_L - \beta)^2$, $\tilde{g}^2(t_j)$ or $\tilde{f}(u)$ satisfy

$$\begin{aligned} &F_{2n} = O_p(n^{-1}), \quad F_{3n} = O_p(n^{-1/2}), \quad F_{4n} = O(n^{-1/2}), \\ &F_{5n} = O_p(n^{-1/2}), \quad F_{6n} = O(n^{-1/2}). \end{aligned}$$

For the cross-product terms, by Cauchy's inequality and Lemma 5.2,

$$F_{7n} = O_p(n^{-1/2}), \quad F_{8n} = O_p(n^{-1/2}), \quad F_{9n} = O_p(n^{-1/2}),$$

and similarly

$$F_{10n} = O_p(n^{-1/2}), \quad F_{11n} = O_p(n^{-1/2}), \quad F_{12n} = O_p(n^{-1/2}).$$

Therefore,

$$F_{ln} = O_p(n^{-1/2}), \quad l = 2, \dots, 12.$$

Since $n\Delta_n^2(u) \to \infty$, we have

$$\frac{F_{ln}}{\Delta_n(u)} \xrightarrow{P} 0, \quad l = 2, \dots, 12.$$

Combining this with the asymptotic normality of $F'_{1n}/\Delta_n(u)$ and the negligibility of $F''_{1n}/\Delta_n(u)$ and $F'''_{1n}/\Delta_n(u)$ completes the proof. ∎

6.4. Proof of Theorem 3.4. In order to prove (3.6), we only need to prove $\sqrt{n}(\hat{\beta}_{W_2} - \hat{\beta}_{W_1}) = o_p(1)$, since we have proved $I_n^2(\hat{\beta}_{W_1} - \beta)/\Sigma_{1n} \xrightarrow{D} N(0,1)$ and $\Sigma_{1n}^2 \geqslant Cn$. Note that

$$\begin{aligned}
(6.29)\quad & \hat{\beta}_{W_1} - \hat{\beta}_{W_2} \\
&= \frac{1}{I_{1n}^2} \sum_{i=1}^n \sigma_i^{-2} \tilde{x}_i (\tilde{x}_i\beta - \tilde{\mu}_i\beta + \tilde{g}_i + \tilde{\epsilon}_i) \\
&\quad - \frac{1}{I_{2n}^2} \sum_{i=1}^n \hat{\sigma}_{ni}^{-2} \tilde{x}_i (\tilde{x}_i\beta - \tilde{\mu}_i\beta + \tilde{g}_i + \tilde{\epsilon}_i) \\
&= \left(\frac{1}{I_{1n}^2} - \frac{1}{I_{2n}^2} \right) \sum_{i=1}^n \sigma_i^{-2} \big[(\tilde{\xi}_i + \tilde{\mu}_i)(\tilde{g}_i + \tilde{\epsilon}_i - \tilde{\mu}_i\beta) + \sigma_\mu^2 \beta \big] \\
&\quad + \frac{1}{I_{2n}^2} \sum_{i=1}^n (\sigma_i^{-2} - \hat{\sigma}_{ni}^{-2}) \big[(\tilde{\xi}_i + \tilde{\mu}_i)(\tilde{g}_i + \tilde{\epsilon}_i - \tilde{\mu}_i\beta) + \sigma_\mu^2 \beta \big] \\
&= \frac{I_{2n}^2 - I_{1n}^2}{I_{1n}^2 I_{2n}^2} \sum_{i=1}^n \sigma_i^{-2} \big[\tilde{\xi}_i\tilde{g}_i + \tilde{\xi}_i\tilde{\epsilon}_i - \tilde{\xi}_i\tilde{\mu}_i\beta + \tilde{\mu}_i\tilde{g}_i + \tilde{\mu}_i\tilde{\epsilon}_i - (\tilde{\mu}_i^2 - \sigma_\mu^2)\beta \big] \\
&\quad + \frac{1}{I_{2n}^2} \sum_{i=1}^n (\sigma_i^{-2} - \hat{\sigma}_{ni}^{-2}) \big[\tilde{\xi}_i\tilde{g}_i + \tilde{\xi}_i\tilde{\epsilon}_i - \tilde{\xi}_i\tilde{\mu}_i\beta + \tilde{\mu}_i\tilde{g}_i + \tilde{\mu}_i\tilde{\epsilon}_i - (\tilde{\mu}_i^2 - \sigma_\mu^2)\beta \big] \\
&\doteq \Delta.
\end{aligned}$$

We just need to verify that $\Delta = o_p(n^{-1/2})$.

First, we prove that

$$(6.30)\qquad I_{1n}^2/I_n^2 \xrightarrow{\text{a.s.}} 1.$$

It can be easily deduced from $(\boldsymbol{H_3})$(i), (6.5) and Lemma 5.5(ii) that

$$\begin{aligned}
(6.31)\quad & I_n^{-2}|I_{1n}^2 - I_n^2| \\
&= I_n^{-2} \Big| \sum_{i=1}^n \sigma_i^{-2} (\tilde{x}_i^2 - \sigma_\mu^2 - \tilde{\xi}_i^2) \Big| \leqslant Cn^{-1} \Big| \sum_{i=1}^n (2\tilde{\xi}_i\tilde{\mu}_i + \mu_i^2 - \sigma_\mu^2) \Big| \\
&\leqslant Cn^{-1} \Big\{ \Big| \sum_{i=1}^n (\mu_i^2 - \sigma_\mu^2) \Big| + 2 \Big| \sum_{i=1}^n \tilde{\xi}_i\mu_i \Big| + 2 \sum_{i=1}^n |\tilde{\xi}_i| \Big| \sum_{j=1}^n W_{nj}(t_i)\mu_j \Big| \Big\} \xrightarrow{\text{a.s.}} 0,
\end{aligned}$$

which implies (6.30). By Lemma 5.6(ii) and (6.30), we have

$$C_1 n \leqslant I_{1n}^2 \leqslant C_2 n \quad \text{a.s.} \tag{6.32}$$

Next, we will show that

$$\frac{I_{2n}^2}{I_{1n}^2} \xrightarrow{\text{a.s.}} 1. \tag{6.33}$$

It follows from Theorem 3.3 and $(\boldsymbol{H_3})$(i) that for all n large enough,

$$0 < m_1 \leqslant \min_{1\leqslant i\leqslant n} \hat{f}(u_i) \leqslant \max_{1\leqslant i\leqslant n} \hat{f}(u_i) \leqslant M_1 < \infty,$$

which together with Lemma 5.7 yields

$$\max_{1\leqslant i\leqslant n} |\hat{\sigma}_{ni}^{-2} - \sigma_i^{-2}| = \max_{1\leqslant i\leqslant n} \frac{|\hat{f}_n(u_i) - f(u_i)|}{\hat{f}_n(u_i) f(u_i)} = o_p(n^{-1/4}), \tag{6.34}$$

where m_1 and M_1 are two positive constants.

Thus, by (6.5), (6.34) and Lemmas 5.1, 5.5(ii) with $\lambda = 1/4$ and 5.6(i), we can get

$$\begin{aligned}
I_{1n}^{-2}|I_{1n}^2 - I_{2n}^2| &\leqslant Cn^{-1}|I_{1n}^2 - I_{2n}^2| \\
&\leqslant C \max_{1\leqslant i\leqslant n} \Big|\hat{\sigma}_{ni}^{-2} - \sigma_i^{-2}\Big| n^{-1} \Big|\sum_{i=1}^n (\tilde{x}_i^2 - \sigma_\mu^2)\Big| \\
&= C \max_{1\leqslant i\leqslant n} \frac{|\hat{f}_n(u_i) - f(u_i)|}{\hat{f}_n(u_i) f(u_i)} n^{-1} \Big|\sum_{i=1}^n (\tilde{x}_i^2 - \sigma_\mu^2)\Big| \\
&\leqslant C \max_{1\leqslant i\leqslant n} \frac{|\hat{f}_n(u_i) - f(u_i)|}{\hat{f}_n(u_i) f(u_i)} n^{-1} \Big\{ \sum_{i=1}^n \tilde{\xi}_i^2 + \Big|\sum_{i=1}^n (\mu_i^2 - \sigma_\mu^2)\Big| \\
&\quad + \sum_{i=1}^n \Big(\sum_{j=1}^n W_{nj}(t_i)\mu_j\Big)^2 + 2\Big|\sum_{i=1}^n \tilde{\xi}_i \mu_i\Big| + 2\sum_{i=1}^n |\tilde{\xi}_i| \Big|\sum_{j=1}^n W_{nj}(t_i)\mu_j\Big| \\
&\quad + 2\sum_{i=1}^n |\mu_i| \Big|\sum_{j=1}^n W_{nj}(t_i)\mu_j\Big| \Big\} \\
&= o_p(n^{-1/4}),
\end{aligned} \tag{6.35}$$

which implies (6.33). Thus, it follows from (6.32) and (6.33) that

$$C_3 n \leqslant I_{2n}^2 \leqslant C_4 n \quad \text{a.s.} \tag{6.36}$$

Combining this with (6.29), (6.34) and (6.36), we obtain

$$
\begin{aligned}
(6.37)\qquad |\Delta| &= o_p(n^{-1/4})\Big\{\Big|n^{-1}\sum_{i=1}^{n}\tilde{\xi}_i\tilde{g}_i\Big| + \Big|n^{-1}\sum_{i=1}^{n}\tilde{\xi}_i\tilde{\epsilon}_i\Big| + \Big|n^{-1}\sum_{i=1}^{n}\tilde{\xi}_i\tilde{\mu}_i\beta\Big| \\
&\quad + \Big|n^{-1}\sum_{i=1}^{n}\tilde{\mu}_i\tilde{g}_i\Big| + \Big|n^{-1}\sum_{i=1}^{n}\tilde{\mu}_i\tilde{\epsilon}_i\Big| + \Big|n^{-1}\sum_{i=1}^{n}(\tilde{\mu}_i^2-\sigma_\mu^2)\beta\Big|\Big\} \\
&\doteq o_p(n^{-1/4})\sum_{l=1}^{6}B_{ln}.
\end{aligned}
$$

Hence, in order to prove $\Delta = o_p(n^{-1/2})$, we only need to show that $B_{ln} = O(n^{-1/4})$ a.s. for $l = 1, \ldots, 6$.

By Lemma 5.6(i, iii), it is easy to see that $B_{1n} = o(n^{-1/4})$.

For B_{2n}, by Lemmas 5.1, 5.5(ii) with $\lambda = 1/4$, we have

$$
(6.38)\qquad B_{2n} \leqslant \Big|n^{-1}\sum_{i=1}^{n}\tilde{\xi}_i\epsilon_i\Big| + n^{-1}\sum_{i=1}^{n}|\tilde{\xi}_i|\Big|\sum_{j=1}^{n}W_{nj}(t_i)\sigma_j e_j\Big| = o(n^{-1/4}) \quad \text{a.s.}
$$

Similarly, we can prove $B_{3n} = o(n^{-1/4})$ a.s.

It follows from Lemmas 5.1, 5.5(i) and 5.6(iii) that

$$
B_{4n} \leqslant \max_{1\leqslant i\leqslant n}|\tilde{g}_i| n^{-1}\sum_{i=1}^{n}\Big(|\mu_i| + \Big|\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\Big|\Big) = o(n^{-1/4}) \quad \text{a.s.}
$$

By Lemmas 5.1 and 5.5(ii) with $\lambda = 1/4$, we can get

$$
\begin{aligned}
B_{5n} &= \Big|n^{-1}\sum_{i=1}^{n}\Big(\mu_i - \sum_{j=1}^{n}W_{nj}(t_i)\mu_j\Big)\cdot\Big(\epsilon_i - \sum_{j=1}^{n}W_{nj}(t_i)\epsilon_j\Big)\Big| \\
&= \Big|n^{-1}\sum_{i=1}^{n}\mu_i\epsilon_i - n^{-1}\sum_{i=1}^{n}\mu_i\sum_{j=1}^{n}W_{nj}(t_i)\epsilon_j - n^{-1}\sum_{i=1}^{n}\epsilon_i\sum_{j=1}^{n}W_{nj}(t_i)\mu_j \\
&\quad + n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\sum_{j=1}^{n}W_{nj}(t_i)\epsilon_j\Big| \\
&\leqslant \Big|n^{-1}\sum_{i=1}^{n}\mu_i\sigma_i e_i\Big| + \max_{1\leqslant i\leqslant n}\Big|\sum_{j=1}^{n}W_{nj}(t_i)\sigma_j e_j\Big| n^{-1}\sum_{i=1}^{n}|\mu_i| \\
&\quad + \max_{1\leqslant i\leqslant n}|\sigma_i|\max_{1\leqslant i\leqslant n}\Big|\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\Big| n^{-1}\sum_{i=1}^{n}|e_i| \\
&\quad + \Big(\max_{1\leqslant i\leqslant n}\Big|\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\Big|\Big)\Big(\max_{1\leqslant i\leqslant n}\Big|\sum_{j=1}^{n}W_{nj}(t_i)\sigma_j e_j\Big|\Big) \\
&= o(n^{-1/4}) \quad \text{a.s.}
\end{aligned}
$$

and

$$\begin{aligned}
B_{6n} &= |\beta|\Big|n^{-1}\sum_{i=1}^{n}(\mu_i^2-\sigma_\mu^2)+n^{-1}\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\Big)^2 \\
&\quad -2n^{-1}\sum_{i=1}^{n}\mu_i\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\Big| \\
&\leqslant C_1\Big|n^{-1}\sum_{i=1}^{n}(\mu_i^2-\sigma_\mu^2)\Big|+C_2\Big(\max_{1\leqslant i\leqslant n}\Big|\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\Big|\Big)^2 \\
&\quad +C_3\max_{1\leqslant i\leqslant n}\Big|\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\Big|n^{-1}\sum_{i=1}^{n}|\mu_i| \\
&= o(n^{-1/4}) \quad \text{a.s.} \ \blacksquare
\end{aligned}$$

6.5. Proof of Theorem 3.5. Note that

$$\begin{aligned}
(6.39)\quad \hat{g}_{W_2}(t)-g(t) &= \sum_{j=1}^{n}W_{nj}(t)(y_j-x_j\hat{\beta}_{W_2})-g(t) \\
&= \sum_{j=1}^{n}W_{nj}(t)(\xi_j\beta+g(t_j)+\sigma_j e_j-(\xi_j+\mu_j)\hat{\beta}_{W_2})-g(t) \\
&= \sum_{j=1}^{n}W_{nj}(t)\Big[\xi_j(\beta-\hat{\beta}_{W_2})+\sigma_j e_j-\mu_j\hat{\beta}_{W_2}\Big]+\sum_{j=1}^{n}W_{nj}(t)g(t_j)-g(t) \\
&= \sum_{j=1}^{n}W_{nj}(t)(\sigma_j e_j-\mu_j\beta)+\sum_{j=1}^{n}W_{nj}(t)\xi_j(\beta-\hat{\beta}_{W_2}) \\
&\quad +\sum_{j=1}^{n}W_{nj}(t)\mu_j(\beta-\hat{\beta}_{W_2})-\Big(g(t)-\sum_{j=1}^{n}W_{nj}(t)g(t_j)\Big) \\
&\doteq C_{1n}+C_{2n}+C_{3n}+C_{4n}.
\end{aligned}$$

Similar to the proof of G_{1n}, we can get $C_{1n}(t)/\Gamma_n(t)\to N(0,1)$ by using Bernstein's big-block and small-block procedure. In addition, we have $n\Gamma_n^2(t)\to\infty$ as $n\to\infty$. From the proof of Theorem 3.4, it suffices to verify that $(\hat{\beta}_{W_2}-\beta)^2=O_p(n^{-1})$.

Similar to the proofs of $G_{kn}=O_p(n^{-1/2})$ for $k=2,3,4$ in Theorem 3.2, we can get $C_{jn}=O_p(n^{-1/2})$ for $j=2,3,4$ immediately. ∎

REFERENCES

[1] J. Baek and H. Liang, *Asymptotics of estimators in semiparametric model under NA samples*, J. Statist. Planning Inference 136 (2006), 3362–3382.

[2] R. C. Bradley, *Basic properties of strong mixing conditions. A survey and some open questions*, Probab. Surv. 2 (2005), 107–144.

[3] H. J. Cui and R. C. Li, *On parameter estimation for semi-linear errors-in-variables models*, J. Multivar. Anal. 64 (1998), 1–24.
[4] R. A. Davis and T. Mikosch, *The sample autocorrelations of heavy-tailed processes with applications to ARCH*, Ann. Statist. 26 (1998), 2049–2080.
[5] P. Doukhan, *Mixing Properties and Examples*, Lecture Notes in Statist. 85, Springer, Berlin, 1994.
[6] H. Emami and O. Khademnoe, *Shrinkage and pretest Liu estimators in semiparametric linear measurement error models*, Statist. Papers 66 (2025), art. 49, 35 pp.
[7] J. Q. Fan and Q. W. Yao, *Nonlinear Time Series: Nonparametric and Parametric Methods*, Springer, 2006.
[8] J. T. Gao, X. R. Chen and L. C. Zhao, *Asymptotic normality of a class of estimators in partial linear models*, Acta Math. Sinica 37 (1994), 256–268.
[9] W. Härdle, H. Liang and J. T. Gao, *Partial Linear Models*, Physica-Verlag, 2000.
[10] Y. P. He, Y. Shen, R. Wang and X. J. Wang, *Complete f-moment convergence for arrays of random variables and its applications in semiparametric and EV regression models*, Probab. Math. Statist. 44 (2024), 87–118.
[11] Z. Huang, S. Meng and Z. Ye, *Effective estimation of nonlinear errors-in-variables models*, Comm. Statist. Simul. Comput. 54 (2025), 2345–2363.
[12] X. C. Li and J. Zhang, *Calibration and goodness-of-fit testing for standard power distributions under multiplicative distortion errors: methodological frameworks and statistical inference*, Comm. Statist. Simul. Comput. (2025), 1–26.
[13] H. Liang, W. Härdle and R. J. Carroll, *Estimation in a semiparametric partially linear errors-in-variables model*, Ann. Statist. 27 (1999), 1519–1535.
[14] H. Y. Liang and B. Y. Jing, *Asymptotic normality in partial linear models based on dependent errors*, J. Statist. Planning Inference 139 (2009), 1357–1371.
[15] H. Y. Liang and X. Z. Wang, *Convergence rate of wavelet estimator in semiparametric models with dependent $MA(\infty)$ error process*, Chin. J. Appl. Probab. 26 (2010), 35–46.
[16] E. Liebscher, *Estimation of the density and the regression function under mixing conditions*, Statist. Risk Model. 19 (2001), 9–26.
[17] H. Lin, J. Shi, T. Tong and R. Zhang, *Partially linear single-index model in the presence of measurement error*, J. Systems Sci. Complex. 35 (2022), 2361–2380.
[18] M. Rosenblatt, *A central limit theorem and a strong mixing condition*, Proc. Nat. Acad. Sci. USA 42 (1956), 43–47.
[19] Q. M. Shao, *Complete convergence for α-mixing sequences*, Statist. Probab. Lett. 16 (1993), 279–287.
[20] V. A. Volkonskii and Y. A. Rozanov, *Some limit theorems for random functions*, Theory Probab. Appl. 4 (1959), 178–197.
[21] Q. H. Wang, *Estimation of partial linear errors-in-variables models with validation data*, J. Multivar. Anal. 69 (1999), 30–64.
[22] L. Yan, X. Tan and X. Chen, *Empirical likelihood for partially linear errors-in-variables models with longitudinal data*, Acta Math. Appl. Sinica English Ser. 38 (2022), 664–683.
[23] S. C. Yang, *Maximal moment inequality for partial sums of strong mixing sequences and application*, Acta Math. Sinica English Ser. 23 (2007), 1013–1024.
[24] M. Zhao, Y. Gao and Y. Cui, *Variable selection for longitudinal varying coefficient errors-in-variables models*, Comm. Statist. Theory Methods 51 (2022), 3713–3738.
[25] L. Zhang, W. Yu and X. Wang, *Weak consistency for the estimators in a semiparametric regression model based on negatively associated random errors*, J. Statist. Comput. Simul. 92 (2022), 19–42.
[26] J. J. Zhang and H. Y. Liang, *Berry–Esseen type bounds in heteroscedastic semi-parametric model*, J. Statist. Planning Inference 141 (2011), 3447–3462.
[27] J. J. Zhang and H. Y. Liang, *Asymptotic normality of estimators in heteroscedastic semiparametric model with strong mixing errors*, Comm. Statist. Theory Methods 41 (2012), 2172–2201.

[28] J. J. Zhang, H. Y. Liang and A. Amei, *Asymptotic normality of estimators in heteroscedastic errors-in-variables model*, Adv. Statist. Anal. 98 (2014), 165–195.
[29] J. J. Zhang and T. Wang, *Strong consistency rate of estimators in heteroscedastic errors-in-variables model for negative association samples*, Filomat 32 (2018), 4639–4654.
[30] J. J. Zhang, H. Q. Yan and T. T. Hu, *Symmetrical convergence rates and asymptotic properties of estimators in a semi-parametric errors-in-variables model with strong mixing errors and missing responses*, Symmetry 16 (2024), 1544.
[31] X. C. Zhou and S. H. Hu, *Moment consistency of estimators in semi-parametric regression model under NA samples*, Pure Appl. Math. 26 (2010), 262–269.
[32] H. B. Zhou, J. H. You and B. Zhou, *Statistical inference for fixed-effects partially linear regression models with errors in variables*, Statist. Papers 51 (2010), 629–650.

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