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MALLIAVIN DIFFERENTIABILITY AND REGULARITY OF DENSITIES IN SEMI-LINEAR STOCHASTIC DELAY EQUATIONS DRIVEN BY WEIGHTED FRACTIONAL BROWNIAN MOTION

BY

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Abstract. The weighted fractional Brownian motion is a limit of occupation time fluctuations of a system with independent particles moving in $\mathbb{R}$ with respect to a symmetric stable Lévy process. It is an extension of fractional Brownian motion. We demonstrate that the law of the solution to multiplicative stochastic delay differential equations driven by weighted fractional Brownian motion is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, and that the density is smooth, using Malliavin calculus. We prove the existence and uniqueness of the solution and provide an explicit expression for it. Additionally, we establish a new generalized maximum inequality in L^p for Wick integrals driven by weighted fractional Brownian motion, which bounds the moments of the solutions and their Malliavin derivatives. This is essential for achieving a smooth density. Due to the existence of Malliavin derivatives of solutions in the upper bound used for deriving the boundedness of moments, we assume some usual additional conditions on the order of derivatives of the coefficients of the equations.

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1. INTRODUCTION

The theory of stochastic differential equations (SDEs) driven by fractional Brownian motion (fBm) is widely studied through different approaches. These models find many applications, including finance, telecommunications, image processing, biology, and turbulence [27, 5, 33, 16, 36]. Additionally, a wide range of systems is commonly described by delayed equations that utilize fBm-driven models. For instance, Dung [26] considered delayed logistic equations driven by fBm for the description of population growth, while neutral stochastic equations used to describe physical systems with delay and memory were considered in [13, 2].

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One of the themes in the study of fractional SDEs is the existence, uniqueness, and regularity of the corresponding solution densities. When $H > \frac{1}{2}$, the existence and uniqueness of the solution have been established by Lyons [24], Zahle [40] and Nualart-Rascanu [30], through Young's integration theory in [7], for the Hurst parameter $H > 1/3$ in [25], and an extension to $H > 1/4$ in [37]. Recently, the existence and positivity of a unique solution have been studied in the context of mean-reverting stochastic volatility models in finance, particularly when the drift is locally Lipschitz and unbounded near the origin [41].

In the presence of delay in SDEs, Wei and Wang [38] addressed the existence and uniqueness of the solution to stochastic functional differential equations with infinite delay. Caraballo et al. [15] studied the existence, uniqueness, and exponential asymptotic behavior of mild solutions to stochastic delay evolution equations driven by a fractional Brownian motion. In 2015, Boudaoui et al. [12] showed the existence of mild solutions to stochastic impulsive evolution equations with time delays using a new fixed-point approach. It is worth mentioning that the diffusion coefficient functions there depend only on the parameter t and not on the state process. Further investigations are discussed in [18], and then in [14] on stochastic delayed differential equations (SDDEs) with fractional Brownian motion when the diffusion functions rely on the past value of the state process with time delay τ (actually, as $B = 0$ in SDDE (1.1)).

Malliavin calculus is a valuable tool to study the regularity of density solutions to SDEs. For more details on this theory, we refer the readers to [29]. The smoothness of the density of solutions to SDEs driven by fractional Brownian motion with Hurst parameter greater than $1/2$ in the one-dimensional case was addressed in [28] by using Doss–Süssman methods, and in [31, 20] under ellipticity assumptions. Rovira and Ferrante [18] established the existence and regularity of the density of SDEs with delay using Young's integration, focusing on an SDE where the diffusion coefficient function depends only on the past values of the solution with time delay, i.e. it depends on $X_{t-\tau}$ for a solution $X(t)$ and time delay τ. In 2012, León and Tindel [23] showed the existence of a C^∞-density for a general class of Young delay equations driven by fBm with Hurst parameter $H > 1/2$. In 2020, Baños et al. [6] presented a new method for constructing solutions of additive SDEs with merely integrable drift coefficients driven by an fBm with $H < 1/2$.

The weighted fractional Brownian motion (wfBm) is a generalization of fBm considered by Bojdecki et al. [9, 10], as a limit of occupation time fluctuations of a system of independent particles moving in $\mathbb{R}$ according to a symmetric α-stable Lévy process, where $0 < \alpha \leqslant 2$, started from an inhomogeneous Poisson configuration with intensity measure $dx/(1 + |x|^\gamma)$ with $0 < \gamma < \alpha$. This process was investigated in [11] for the high-density limit of occupation time fluctuations in the aforementioned particle system, where the initial configuration follows a Poisson measure with finite intensity. In [19], Garzón proved the convergence to a weighted fractional Brownian sheet of rescaled occupation time fluctuations from the aforementioned particle system. Due to memory effects, self-similarity, and long-range

dependence exhibited by various real-world phenomena (such as a wide variety of natural and financial market behaviors), it would be reasonable to explore stochastic models arising from weighted fBm. Several studies have addressed related topics, including the work of Shen et al. [35, 34], Yan et al. [39], and the references therein.

In the case of weighted fractional Brownian motion, to the best of our knowledge, there is no literature examining the existence and smoothness of the density of solutions to linear SDDEs driven by wfBm. Our study begins by establishing the existence and Malliavin differentiability of solutions, followed by analyzing the smoothness of their probability density functions for certain SDDEs driven by wfBm, under standard global Lipschitz conditions on the coefficient functions. We consider the semi-linear SDDE of the form

$$dx(t) = \{Ax(t) + f(x(t-\tau))\}dt + \{Bx(t) + \sigma(x(t-\tau))\}\delta^{a,b}B^{a,b}(t), \quad t \in [0,T], \tag{1.1}$$

$$x(t) = \xi_0(t), \quad t \in [-\tau, 0], \tag{1.2}$$

where A, B are constants, the $\mathbb{R}$-valued functions f and σ satisfy Assumption 1.1 and the stochastic integral is a type of Wick integral related to weighted fBm, reviewed in Section 3.

ASSUMPTION 1.1. Let K_0 be the largest integer such that $K_0\tau < T$, and M be an integer such that $2K_0 + 1 < M$. Also, for some positive integer constant K_1, the functions $f, \sigma : \mathbb{R} \to \mathbb{R}$ are $K_1 + M$-times differentiable, with all derivatives bounded by some constant K_2.

This assumption implies the existence of some constant L such that

$$|f(y)|^2 + |\sigma(y)|^2 \leqslant L(1 + |y|^2).$$

Besides Assumption 1.1, to obtain smoothness of the density, we also assume that the function σ has a lower bound greater than zero, i.e., $|\sigma(x)| > M_0 > 0$ for every $x \in \mathbb{R}$, implying a Hörmander-type condition.

To derive our main results, it is crucial to establish a suitable upper bound inequality for the supremum of the stochastic Wick integral generated by wfBm. This inequality generalizes the L^p-maximal inequality presented in [4] for the stochastic integral involving wfBm (Theorem 4.1), with the upper bound depending on the Malliavin derivative of the integrand process.

Similar to the method used in [18], we will construct the unique solution to SDDE (1.1) step by step within the intervals $[i\tau, (i+1)\tau]$ for each integer $i = 1, \ldots, K_0$, and subsequently in the interval $[K_0\tau, T]$. In fact, we establish an explicit representation for the solution at each step i, which is presented by some variable $Z_i(t)$. The derived expression serves as the foundation for successively

establishing the Malliavin differentiability of the solution, enabling the computation of a fully characterized expression for its Malliavin derivatives at every stage. Our expression arises from the presence of a linear term in the diffusion coefficient of (1.1), the upper bound of the maximal inequality to bound the moments of the solution and its derivatives, and also the application of Itô's formula, which distinguishes our work from that of [18]. Our expression will enable us to achieve upper bounds for the moments of the solution and its Malliavin derivatives recursively.

More precisely, the order of differentiability of the solution is the order of differentiability of the coefficients f and g. According to the expression (5.3) of the solution, to prove the boundedness of the solution at each step i, we need to find an estimate of the process $D^{a,b}Z_i(t)$ which involves Wick integrals of the Malliavin derivative of the solution of (1.1) from the previous step $i-1$, denoted by $D^{a,b}X_{i-1}(\cdot)$ (see (5.5)). Applying a maximal inequality for this integral yields an upper bound that depends on $D^{a,b}D^{a,b}X_{i-1}(\cdot)$. Therefore, to control the moments of the solution X_i, we must ensure the boundedness of the moments of $D^{a,b}D^{a,b}X_{i-1}(\cdot)$ across all steps $i = 1, \dots, K_0$. This requirement extends to higher-order derivatives of the solution at each step. Consequently, to guarantee the boundedness of the moments of the jth derivative of the solution (1.1), where $j = 1, \dots, K_1$, we need to establish the boundedness of the moments of the $j + M - 2i$ higher-order derivatives of the solution X_i, for every $i = 1, \dots, K_0$. This is the reason to assume $M + K_1$-fold differentiability in Assumption 1.1 to ensure the boundedness of the moments of all derivatives of the solution up to order K_1. For more details, we direct the reader to Section 6.

It is important to note that we do not employ the rough-path approach, the Young integral, or Riemann–Stieltjes integral methods as discussed in [23, 31], since our focus is on the concept of Wick integral for the stochastic integral in (1.1).

The article organization is as follows: In Sections 2 and 3, we review fundamental concepts of weighted fractional Brownian motion and Malliavin calculus relevant to this type of stochastic integral. In Section 4, we prove a generalized version of the maximal inequality to achieve our main results. The existence, uniqueness, and Malliavin differentiability of the solution are established in Section 5, along with an explicit formula for the solution, on all intervals affected by time delay. The uniform boundedness of the moments of the solution and its Malliavin derivatives is investigated in Section 6. Finally, we discuss the regularity of the density in Section 7. In the interest of readability, the proofs of several theorems are given in the Appendix.

2. WEIGHTED FRACTIONAL BROWNIAN MOTION

For all parameters a, b with $a > -1$, $|b| < 1$ and $|b| < a + 1$, a *weighted fractional Brownian motion*, denoted by $B^{a,b}$, on the complete probability space (Ω, F, P) is

a mean zero Gaussian process with simple covariance function [35]:

$$R^{a,b}(t,s) = E[B^{a,b}(t)B^{a,b}(s)] = \int_0^{s\wedge t} u^a[(t-u)^b + (s-u)^b]\,du, \qquad s,t \geqslant 0.$$

This process is $\frac{a+b+1}{2}$-self-similar, long-range dependent with Hölder paths. It is neither a semimartingale nor a Markov process if $b \neq 0$, and for some constant $c_{a,b}$ and for any $s,t \geqslant 0$,

$$c_{a,b}(t\vee s)^a|t-s|^{1+b} \leqslant E\big[\big(B^{a,b}(t)-B^{a,b}(s)\big)^2\big] \leqslant K_{a,b}(t\vee s)^a|t-s|^{1+b}, \tag{2.1}$$

where $K_{a,b} = \frac{b}{2\mathbb{B}(a+1,b+1)}$. Some relevant surveys and references can be found in [3, 29, 35].

When $-1 < b < 1$ and $a = 0$, the covariance of the process coincides with the covariance of the fractional Brownian motion with Hurst index $\frac{b+1}{2}$ [8]:

$$E[B^{a,b}(t)B^{a,b}(s)] = \frac{1}{b+1}[t^{b+1} + s^{b+1} - |s-t|^{b+1}].$$

Let H be the Hilbert space defined as the closure of the linear space $\mathcal{E}$ of indicator functions $\{\mathbf{1}_{[0,t]} : t \in [0,T]\}$ with respect to the inner product

$$\langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[0,t]} \rangle_H = R^{a,b}(t,s).$$

In accordance with [35], the space H is $\mathrm{H} = \{u : [0,T] \to \mathbb{R};\ \|u\|_{\mathrm{H}} < \infty\}$, where

$$\begin{aligned} \|u\|_{\mathrm{H}}^2 &= \int_0^T \int_0^T u(s)u(t)\phi(t,s)\,dt\,ds \\ &:= \int_0^T \int_0^T u(s)u(t)b(t\wedge s)^a(t\vee s - t\wedge s)^{b-1}\,dt\,ds. \end{aligned} \tag{2.2}$$

In addition, the map $\mathcal{E} \ni u \mapsto B^{a,b}(u)$ is an isometry from $\mathcal{E}$ to the Gaussian space generated by $B^{a,b}$, and it can be extended to H. So, the Gaussian processes $B^{a,b}(u)$, denoted by $\int_0^T u(s)\,dB^{a,b}(s)$, on H are well-defined, and for all $u_1, u_2 \in H$, we have $\mathbb{E}(B^{a,b}(u_1)B^{a,b}(u_2)) = \langle u_1, u_2\rangle_H$ (see [35]).

We should note that the subspace $|\mathrm{H}|$ of measurable functions u equipped by the norm

$$\|u\|_{|\mathrm{H}|}^2 = \int_0^T \int_0^T |u(s)|\,|u(t)|\phi(t,s)\,dt\,ds < \infty \tag{2.3}$$

is a Banach space and $\mathcal{E}$ is dense in $|\mathrm{H}|$. Moreover, Pipiras and Taqqu [32] have shown that if $a + b < 1$, then

$$L^2([0,T]) \subset L^{2/(a+b+1)} \subset |\mathrm{H}| \subset \mathrm{H}. \tag{2.4}$$

3. PRELIMINARIES ON MALLIAVIN CALCULUS

We briefly recall some Malliavin criteria on the fractional Brownian motion and weighted fractional Brownian motion from [29, 17, 35]. When $b > 0$, we denote by $\mathcal{S}$ the set of smooth functionals of the form

$$F = f\big(B^{a,b}(u_1), \dots, B^{a,b}(u_n)\big),$$

where $f \in C_b^\infty(\mathbb{R}^n)$ (f and all its derivatives are bounded) and $u_i \in \mathrm{H}$, $i = 1, \dots, n$. For every $F \in \mathcal{S}$, define

$$D^{a,b}F = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}\big(B^{a,b}(u_1), \dots, B^{a,b}(u_n)\big)u_i.$$

The derivative operator $D^{a,b}$ is a closable operator from $L^p(\Omega)$ into $L^p(\Omega, \mathrm{H})$ for every $p \geqslant 1$. We denote by $\mathbb{D}^{1,p}$ the closure of $\mathcal{S}$ in the norm

$$\|F\|_{1,p}^p = E|F|^p + E\|D^{a,b}F\|_H^p.$$

For any $k \geqslant 1$, the k-fold iteration of the derivative operator is expressed by $D_{r_1}^{a,b} \cdots D_{r_k}^{a,b}F$. Given a Hilbert space V, the corresponding Sobolev space $\mathbb{D}^{1,p}(V)$ is the domain of the derivative operator of V-valued random variables.

The adjoint of the derivative operator, called $\delta^{a,b}(\varphi)$, is characterized by the duality relationship

$$E[F\delta^{a,b}(\varphi)] = E\langle D^{a,b}F, \varphi\rangle_H \tag{3.1}$$

for any $F \in \mathcal{S}$. For every stochastic process $\varphi \in \mathbb{D}^{1,2}(|\mathrm{H}|)$ the indefinite integral is expressed as

$$\delta^{a,b}(\varphi) = \int_0^T \varphi(s)\delta^{a,b}B^{a,b}(s).$$

As shown in [35, 4], we have

$$\begin{aligned}\int_0^T \varphi(s)\, dB^{a,b}(s) &= \int_0^T \varphi(s)\delta^{a,b}B^{a,b}(s) \\ &\quad + b\int_0^T\int_0^T D_t^{a,b}\varphi(s)(t\wedge s)^a(t\vee s - t\wedge s)^{b-1}\, dt\, ds,\end{aligned} \tag{3.2}$$

providing the second summand is finite, and by [35, p. 399],

$$\begin{aligned}&\mathbb{E}[|\delta^{a,b}(\varphi)|^2] \\ &= \mathbb{E}\|\varphi\|_H^2 + \mathbb{E}\int_{[0,T]^4} D_\xi^{a,b}\varphi(r)D_\eta^{a,b}\varphi(s)\phi(\eta, r)\phi(\xi, s)\, ds\, dr\, d\xi\, d\eta \\ &\leqslant \mathbb{E}\|\varphi\|_{|H|}^2 + \mathbb{E}\int_{[0,T]^4} |D_\xi^{a,b}\varphi(r)|\, |D_\eta^{a,b}\varphi(s)|\, |\phi(\eta, r)|\, |\phi(\xi, s)|\, ds\, dr\, d\xi\, d\eta.\end{aligned} \tag{3.3}$$

For any $p > 1$ we denote by $\mathbb{L}^{1,p}_{(a+b+1)/2}$ the set of processes φ in $\mathbb{D}^{1,2}(|\mathrm{H}|)$ such that

$$\|\varphi\|^p_{\mathbb{L}^{1,p}_{(a+b+1)/2}} = \mathbb{E}\|\varphi\|^p_{L^{2/(a+b+1)}([0,T])} + \mathbb{E}\|D^{a,b}\varphi\|^p_{L^{2/(a+b+1)}([0,T]^2)} < \infty.$$

In view of (2.4) and (3.3), as a consequence of [29, (5.40)], for every $\varphi \in \mathbb{L}^{1,p}_{(a+b+1)/2}$,

$$\begin{aligned}(3.4)\quad &\mathbb{E}[|\delta^{a,b}(\varphi)|^p] \\ &\leqslant C_{(a+b+1)/2,p}(\|\mathbb{E}\varphi\|^p_{L^{2/(a+b+1)}([0,T])} + \mathbb{E}\|D^{a,b}\varphi\|^p_{L^{2/(a+b+1)}([0,T]^2)}).\end{aligned}$$

Let us recall that Itô's formula has been introduced in [17, Theorems 4.5 and 4.2], which is essential in our modification. In fact, the authors have mentioned that their results can be extended to any symmetric and positive definite function $\phi(t,s)$. So Itô's formula therein (Theorem 4.5) and also Theorem 4.2 are also satisfied for wfBm instead of fBm. We rewrite them in the wfBm version here.

Recall $\mathcal{L}(0,T)$ is the set of Malliavin differentiable stochastic processes G such that $\mathbb{E}\|G\|_H + \mathbb{E}\|D^{a,b}G\|_{H\otimes H} < \infty$, and for any sequence of partitions $\pi : 0 = t^n_0 \leqslant \cdots \leqslant t^n_n = T$ of $[0,T]$ such that $|\pi| \to 0$ as $n \to \infty$, and $G^\pi_t = G_{t_i}$ for every $t_i \leqslant t \leqslant t_{i+1}$, $i = 0, \dots, n-1$,

$$\sum_{i=0}^{n} \mathbb{E}\Big\{ \int_{t^n_i}^{t^n_{i+1}} \int_0^T (D^{a,b}_r G^\pi_{t^n_i} - D^{a,b}_r G_s)\phi(r,s)\, dr\, ds \Big\}^2 + \mathbb{E}\{\|G^\pi - G\|_H\} < \infty.$$

For every $G \in \mathcal{L}(0,T)$, define

$$\int_0^T G(s)\delta^{a,b}B^{a,b}_s = L^2(\Omega)\text{-}\lim_{|\pi|\to 0} \sum_{i=0}^{n-1} G^\pi_{t^n_i} \Diamond (B^{a,b}_{t_{i+1}} - B^{a,b}_{t_i}),$$

where $\Diamond$ is the Wick product defined in [29].

PROPOSITION 3.1 ([17, Theorem 4.5]). *Let $\{F_s,\ s \in [0,T]\} \in \mathcal{L}(0,T)$ be a stochastic process such that for some $\alpha_3 > 1 - \frac{b+1}{2}$,*

$$\mathbb{E}(|F_{s_1} - F_{s_2}|^2) \leqslant c_F |s_1 - s_2|^{2\alpha_3},$$

where $|s_1 - s_2| \leqslant \delta$ for some $\delta > 0$ and

$$\lim_{|s_1 - s_2|\to 0} \mathbb{E}\big(|(D^{a,b}_{s_1})^\phi (F_{s_1} - F_{s_2})|^2\big) = 0$$

with $(D^{a,b}_s)^\phi F = \int_0^T \int_0^T D^{a,b}_r F.\phi(r,s)\, dr$. Assume that $\mathbb{E}(\sup_{0\leqslant s\leqslant T} |G_s|) < \infty$ and $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a function in $\mathbb{C}^{1,2}_b(\mathbb{R}_+ \times \mathbb{R})$. If $\eta_t = \xi + \int_0^t G_s\, ds +$

$\int_0^t F_s \delta^{a,b} B_s^{a,b}$ *with* $\xi \in \mathbb{R}$ *for* $t \in [0,T]$ *and* $\frac{\partial f}{\partial x}(s,\eta_s)F_s \in \mathcal{L}(0,T)$*, then for* $t \in [0,T]$,

$$f(t,\eta_t) = f(0,\xi) + \int_0^t \frac{\partial f}{\partial t}(s,\eta_s)\,ds + \sum_{k=1}^n \int_0^t \frac{\partial f}{\partial x}(s,\eta_s)G_s\,ds + \int_0^t \frac{\partial f}{\partial x}(s,\eta_s)F_s\delta^{a,b}B_s^{a,b} + \int_0^t \frac{\partial^2 f}{\partial x^2}(s,\eta_s)F_s(D_s^{a,b})^\phi \eta_s\,ds.$$

PROPOSITION 3.2 ([17, Theorem 4.2]). *If* $\{F_s,\ s \in [0,T]\} \in \mathcal{L}(0,T)$ *and* $\sup_{0\leqslant s\leqslant T}\mathbb{E}(|(D_s^{a,b})^\phi F_s|^2) < \infty$*, then for* $s,t \in [0,T]$,

$$(D_s^{a,b})^\phi \Big\{ \int_0^t F_u \delta^{a,b} B_u^{a,b} \Big\} = \int_0^t (D_s^{a,b})^\phi F_u \delta^{a,b} B_u^{a,b} + \int_0^t F_u \phi(s,u)\,du \quad \textit{a.s.}$$

4. L^p-MAXIMAL INEQUALITY FOR WICK INTEGRAL OF WEIGHTED FBM

In this section, we shall establish an L^p-maximal estimate of the divergence process driven by weighted fractional Brownian motion for every $1 < p_0 < \infty$. To do this, we will apply some upper bound inequality of [1] to a Hardy-type operator $T_{\alpha,\beta}$ of any function $f \in L^{p_0}((a_1,b_1);v)$, i.e. $\int_{a_1}^{a_2} |f(x)|^{p_0} v(x)\,dx < \infty$, defined by

$$T_{\alpha,\beta}f(x) = \int_{a_1}^x \frac{u(s)W^\beta(s)f(s)w(s)\,ds}{(W(x)-W(s))^{1-\alpha}}, \qquad x \in [0,T],$$

where W is a non-negative, strictly increasing, and absolutely continuous function on the interval $I = (a_1,b_1)$ and $\frac{dW(s)}{ds} = w(s)$, and also u is an almost everywhere positive locally integrable function, as well as v is a positive kernel function.

Let $\frac{1}{p_0} + \frac{1}{p'} = 1$ and for every $p_0 \leqslant q < \infty$ denote

$$A_{\alpha,\beta} = \sup_{z\in I} \Big(\int_{a_1}^z u^{p'}(s) W^{p'\beta}(s) w(s)\,ds \Big)^{1/p_0} \Big(\int_z^{b_1} W^{q(\alpha-1)}(s) v(s)\,ds \Big)^{1/q}.$$

In [1] it is shown that for $0 < \alpha < 1$ and $\beta \geqslant 0$ there exists some constant C such that

$$\Big(\int_{a_1}^{b_1} (T_{\alpha,\beta}f(x))^q v(x)\,dx \Big)^{1/q} \leqslant C \Big(\int_{a_1}^{b_1} (f(x))^{p_0} w(x)\,dx \Big)^{1/p_0} \tag{4.1}$$

if and only if $A_{\alpha,\beta} < \infty$. Now, we establish an L^p-maximal estimate for the indefinite integral $\int_0^t \varphi(s)\delta^{a,b}B^{a,b}(s)$.

THEOREM 4.1. *For $a > -1$ and $|b| < 1$ such that $0 < a + b < 1$, let $u = \{u(t),\, t \in [0,T]\}$ be a stochastic process in $\mathbb{L}^{1,p}_{(a+b+1)/2}$. Then for every $p > \frac{4}{a+b+1} \vee \frac{2}{1-a-b}$, there exists some constant C_1 such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \Big| \int_0^t u(s)\delta^{a,b}B^{a,b}_s \Big|^p \Big) \leqslant C_1 T \Big\{ \Big(\int_0^T |\mathbb{E}u(s)|^{\frac{p}{q-1}}\, ds \Big)^{q-1} + \mathbb{E}\Big(\int_0^T \Big(\int_0^T |D^{a,b}_s u(r)|^{p/q}\, dr \Big)^{\frac{q}{q-1}}\, ds \Big)^{q-1} \Big\},$$

where the constant C_1 depends on $p, T, (a+b+1)/2$, and $q = p(a+b+1)/2$.

Proof. The argument is motivated by the proof of [4, Theorem 4]. Put $a_0 = \frac{a+b+1}{2}$, which is a positive number; we will use the fact that

$$c_{a,b} = \int_r^t (t-\theta)^{-a_0}(\theta - r)^{a_0-1}\, d\theta < \infty.$$

For every $F \in \mathbb{D}^{a,b}$, due to (3.1) and then applying Fubini's theorem, we derive

$$\begin{aligned}
&\mathbb{E}\Big(F \int_0^t (t-r)^{-a_0} \Big(\int_0^r u(s)(r-s)^{a_0-1}\delta^{a,b}B^{a,b}(s) \Big)\, dr \Big) \\
&\quad = \int_0^t (t-r)^{-a_0} \mathbb{E}(\langle D^{a,b}F, 1_{\cdot\leqslant r} u(\cdot)(r-\cdot)^{a_0-1}\rangle_H)\, dr \\
&\quad = \int_0^t (t-r)^{-a_0} \mathbb{E}\Big(\int_0^t \int_0^t D^{a,b}_l F 1_{s\leqslant r} u(s)(r-s)^{a_0-1}\phi(l,s)\, dl\, ds \Big)\, dr \\
&\quad = \int_0^t \mathbb{E}\Big(\int_s^t (t-r)^{-a_0} u(s)(r-s)^{a_0-1} \Big(\int_0^t D^{a,b}_l F\phi(l,s)\, dl \Big)\, dr \Big)\, ds \\
&\quad = c_{a,b} \int_0^t \mathbb{E}\Big(u(s) \int_0^t D^{a,b}_l F\phi(l,s)\, dl \Big)\, ds = \mathbb{E}(\langle D^{a,b}F, u(\cdot)c_{a,b}\rangle_H).
\end{aligned}$$

The above equality is valid for every $F \in \mathbb{D}^{a,b}$, therefore comparing it with (3.1) results in

$$\int_0^t u(s)\delta^{a,b}B^{a,b}(s) = c^{-1}_{a,b} \int_0^t (t-r)^{-a_0} \Big(\int_0^r u(s)(r-s)^{a_0-1}\delta^{a,b}B^{a,b}(s) \Big)\, dr.$$

Now, from Hölder's inequality, and $1 - a_0 \frac{p}{p-1} > 0$, there is another constant c such that

$$\Big| \int_0^t u(s)\delta^{a,b}B^{a,b}(s) \Big|^p \leqslant c^{-1} \int_0^t \Big| \int_0^r u(s)(r-s)^{a_0-1}\delta^{a,b}B^{a,b}(s) \Big|^p\, dr.$$

Therefore,

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\Big|\int_0^t u(s)\delta^{a,b}B^{a,b}(s)\Big|^p\Big)\leqslant c^{-1}\mathbb{E}\int_0^T\Big|\int_0^r u(s)(r-s)^{a_0-1}\delta^{a,b}B^{a,b}(s)\Big|^p\,dr.$$

Using now inequality (3.4) we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[0,T]}\Big|\int_0^t u(s)\delta^{a,b}B^{a,b}(s)\Big|^p\Big)\\
&\qquad\leqslant c^{-1}C_{\frac{a+b+1}{2}}\Big\{\int_0^T\Big(\int_0^r(r-s)^{(a_0-1)/a_0}|\mathbb{E}u(s)|^{1/a_0}\,ds\Big)^{pa_0}dr\\
&\qquad\quad+\mathbb{E}\int_0^T\Big(\int_0^r\int_0^T(r-s)^{(a_0-1)/a_0}|D_\theta^{a,b}u(s)|^{1/a_0}\,d\theta\,ds\Big)^{pa_0}dr\Big\}\\
&\qquad=:c^{-1}C_{a_0}(I_1+I_2).
\end{aligned}$$

To bound the terms I_1 and I_2, we apply the Hardy-type operator inequality (4.1) for $q=pa_0$, $p_0=\frac{pa_0}{pa_0-1}$, $\alpha=1+\frac{a_0-1}{a_0}$, $\beta=0$, $v(\cdot)=\frac{1}{T}$ and $W(s)=s$. Under our assumptions, we have $1<p_0<q$ and $0<\alpha<1$ as well as $A_{\alpha,\beta}<\infty$. Consequently,

$$\begin{aligned}
I_1&\leqslant CT\Big(\int_0^T|\mathbb{E}u(r)|^{p_0/a_0}dr\Big)^{q/p_0},\\
I_2&\leqslant C\mathbb{E}\Big(\int_0^T\Big(\int_0^T|D_s^{a,b}u(r)|^{1/a_0}\,ds\Big)^{p_0}dr\Big)^{q/p_0},
\end{aligned}$$

which completes the proof. ∎

REMARK 4.1. According to the proof of this theorem, for any partition $0\leqslant t_1\leqslant\cdots\leqslant t_N=T$ and every $0\leqslant k\leqslant N$, if the weighted function v in the proof of Theorem 4.1 is replaced by $v=\frac{1}{t_{k+1}-t_k}$, then one can infer that there exists some constant which we denote again by C_1 such that

$$\begin{aligned}
\mathbb{E}\Big(\sup_{t\in[t_k,t_{k=1}]}\Big|\int_{t_k}^t u(s)\delta^{a,b}B_s^{a,b}\Big|^p\Big)&\leqslant C_1(t_{k+1}-t_k)\Big\{\Big(\int_{t_k}^{t_{k+1}}|\mathbb{E}u(s)|^{\frac{p}{q-1}}\,ds\Big)^{q-1}\\
&\quad+\mathbb{E}\Big(\int_{t_k}^{t_{k+1}}\Big(\int_{t_k}^{t_{k+1}}|D_s^{a,b}u(r)|^{2/(a+b+1)}\,dr\Big)^{\frac{q}{q-1}}ds\Big)^{q-1}\Big\}.
\end{aligned}$$

5. EXISTENCE AND MALLIAVIN DIFFERENTIABILITY OF THE SOLUTION

Analytically, we first show the existence of the solution of (1.1) step by step in the intervals $[i\tau,(i+1)\tau]$ for any integer $i=1,\dots,K_0$, and then in the interval

$[K_0\tau, T]$. Malliavin's differentiability of the solution in the sense of the stochastic Wick integral will be established recursively.

We assume the following conditions on the function $\xi_0(\cdot)$.

ASSUMPTION 5.1. The measurable function $\xi_0(\cdot)$ is Malliavin differentiable up to order $2M+1$ and for every $p \geqslant 1$, there exists some constant $C_{\xi,p}$ such that

$$\mathbb{E}\Big(\sup_{0\leqslant r\leqslant\tau} |\xi_0(r-\tau)|^p\Big) \leqslant C_{\xi,p} < \infty. \tag{5.1}$$

To show the main results, consider the following linear SDE driven by weighted fractional Brownian motion:

$$d\psi(t) = A\psi(t)dt + B\psi(t)\delta^{a,b}B^{a,b}(t), \quad t \in [0,T], \quad \text{and} \quad \psi(0) = 1. \tag{5.2}$$

We start by showing that (5.2) has a unique solution $\psi(\cdot)$ with an exponential expression, and it has Malliavin derivatives which can be presented as functions of the solution $\psi(\cdot)$.

Following the proof of [21, Theorem 2.5 and Lemma 2.2], thanks to Proposition 3.1, we can show that the solution of (5.2) is

$$\psi(t) = \exp\bigg\{BB^{a,b}(t) - \frac{1}{2}B^2 \int_0^t\int_0^t \phi(s,s')\,ds\,ds'\bigg\}e^{At}\psi(0),$$

and $D_r^{a,b}\psi(t) = \psi(t)B1_{0\leqslant r\leqslant t}$, equivalently $(D_r^{a,b})^\phi\psi(t) = \psi(t)B\int_0^t \phi(r,s)\,ds$. This relation shows that the drift coefficient of (5.2) does not appear in $D_r^{a,b}\psi(t)$ directly, and this leads to the conclusion that the solution to the SDE

$$\begin{aligned} dY(t) &= \Big(-A + 2B^2\int_0^t \phi(s,t)\,ds'\Big)Y(t)dt - BY(t)\delta^{a,b}B^{a,b}(t), \quad t \in [0,T],\\ Y(0) &= 1, \end{aligned}$$

is

$$Y(t) = \exp\bigg\{-BB^{a,b}(t) + \frac{1}{2}B^2 \int_0^t\int_0^t \phi(s,s')\,ds\,ds'\bigg\}e^{-At}Y(0),$$

and $D_r^{a,b}Y(t) = -Y(t)B1_{0\leqslant r\leqslant t}$, which is equivalent to

$$(D_r^{a,b})^\phi Y(t) = -Y(t)B\int_0^t \phi(r,s)\,ds.$$

Since $Y(t).\psi(t) = 1$, we denote $\psi^{-1}(t) := Y(t)$ in the rest of the paper.

THEOREM 5.1. *The solution $\psi(\cdot)$ of the SDE* (5.2) *is unique and has uniformly bounded moments, i.e., there exists some positive constant C_p such that*

$$\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|\psi(t)|^p\Big)\leqslant C_p.$$

Proof. According to the relation $D_r^{a,b}\psi(t)=\psi(t)B1_{0\leqslant r\leqslant t}$ and inequality (3.3) together with the inclusion (2.4) and Gronwall's inequality, the uniqueness of the solution ψ follows. To prove the boundedness of the p-moments of $\psi(\cdot)$, by a classical computation, using Proposition 3.1 and Theorem 4.1, we deduce that there exists some constant c such that

$$\begin{aligned}&\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|\psi(t)|^p\Big)\\&\quad\leqslant c\Big\{\int_0^T|A|^p\mathbb{E}\Big(\sup_{0\leqslant t\leqslant s}|\psi(t)|^p\Big)\,ds+\int_0^T\int_0^T\mathbb{E}\Big(\sup_{0\leqslant t\leqslant s}|D_r^{a,b}\psi(s)|^p\Big)\,ds\,dr\Big\}.\end{aligned}$$

Then, as $D_r^{a,b}\psi(t)=\psi(t)B1_{0\leqslant r\leqslant t}$, by Gronwall's inequality the claim can be obtained. ∎

By a similar argument using $(D_r^{a,b})^\phi\psi^{-1}(t)=-\psi^{-1}(t)B\int_0^t\phi(r,s)\,ds$, one can easily deduce the following proposition.

PROPOSITION 5.1. *For every $p\geqslant 1$, there exists some constant $C_{\psi,p}$ such that*

$$\mathbb{E}\Big(\sup_{0\leqslant r\leqslant T}|\psi^{-1}(r)|^p\Big)+\mathbb{E}\Big(\int_0^T\int_0^T|D_s^{a,b}\psi^{-1}(r)|^p\,dr\,ds\Big)\leqslant C_{\psi,p}<\infty.$$

As a consequence, the stochastic processes $\psi(\cdot)$ and $\psi^{-1}(\cdot)$ satisfy the conditions of Theorem 3.1, by applying Hölder's inequality, (3.4) and the inclusion (2.4).

COROLLARY 5.1. *The stochastic processes $\psi(\cdot)$ and $\psi^{-1}(\cdot)$ are in $\mathbb{L}^{1,p}_{(a+b+1)/2}$ and also in $\mathcal{L}(0,T)$. They satisfy the conditions of Theorem* 3.1 *with $\alpha_3=1$.*

Now, we are ready to construct an expression for the solution of SDE (1.1) step by step by induction, and also obtain its Malliavin derivative. To do this, we define the following processes and derive their differentiability.

For $t\in[0,\tau]$, let $X_0(t-\tau)=X_{-1}(t-\tau)=\xi_0(t-\tau)$. For every $k=0,1,\dots,K_0-1$ and for all $k\tau\leqslant t\leqslant(k+1)\tau$, define

$$X_k(t):=B_\phi\psi(t)Z_k(t)-\Big(\int_{k\tau}^t U_k(s)\,ds\Big)\psi(t),\tag{5.3}$$

where $B_\phi := e^{B^2 \int_0^t \int_0^s \phi(s,u)\,du\,ds}$ and

$$U_k(t) = \int_{k\tau}^{t} \psi^{-1}(u)\sigma(X_{k-1}(u-\tau))\phi(u,t)\,du + B\int_{k\tau}^{t} Z_k(u)\phi(u,t)\,du + \int_0^{k\tau} D_u^{a,b} Z_k(t)\phi(u,t)\,du,$$

$$Z_k(t) := X_{k-1}(k\tau) + \int_{k\tau}^{t} \psi^{-1}(s)\,dM_{k-1}(s)$$

with

$$M_{k-1}(t) = \int_{k\tau}^{t} f(X_{k-1}(s-\tau))\,ds - B\int_{k\tau}^{t} \sigma_\phi(X_{k-1}(s-\tau))\,ds + \int_{k\tau}^{t} \sigma(X_{k-1}(s-\tau))\delta^{a,b}B^{a,b}(t),$$

$$\sigma_\phi(X_k(t-\tau)) = \sigma(X_k(t-\tau))\int_0^t \phi(u,t)\,du.$$

LEMMA 5.1. *Under Assumption* 1.1*, for every* $k = 0, 1, \dots, K_0-1$*, if* $X_{k-1} \in \mathcal{L}[0,T]$*, the process* $Z_k(t)$ *is Malliavin differentiable and*

$$(D_t^{a,b})^\phi Z_k(t) = U_k(t) - BZ_k(t)\int_{k\tau}^{t} \phi(u,t)\,du. \tag{5.4}$$

Proof. Since $X_{k-1} \in \mathcal{L}[0,T]$, we have $\psi^{-1}\sigma(X_{k-1}(.-\tau)) \in \mathcal{L}[0,T]$ and for every $-k\tau \leqslant s < (k+1)\tau$ and $r \geqslant k\tau$ we know $D_r^{a,b}X_{k-1}(s) = 0$, and so

$$D_s^{a,b}\{f(X_{k-1}(s-\tau)) - B\sigma(X_{k-1}(s-\tau))\int_0^s \phi(u,s)\,du\} = 0.$$

Then $Z_k(t)$ is differentiable and for any $k\tau < r \leqslant t$,

$$\begin{aligned} D_r^{a,b}Z_k(t) &= 0 - \int_r^t B\psi^{-1}(s)\Big(f(X_{k-1}(s-\tau)) - B\int_0^s \sigma(X_{k-1}(s-\tau))\phi(u,s)\,du\Big)\,ds \\ &\quad - \int_r^t B\psi^{-1}(s)\sigma(X_{k-1}(s-\tau))\delta^{a,b}B^{a,b}(t) + \psi^{-1}(r)\sigma(X_{k-1}(r-\tau)) \\ &= \psi^{-1}(r)\sigma(X_{k-1}(r-\tau)) - BZ_k(t) + BZ_k(r), \end{aligned}$$

and for any $0 \leqslant r \leqslant k\tau$,

$$
\begin{aligned}
(5.5)\quad & D_r^{a,b} Z_k(t) \\
&= D_r^{a,b} X_{k-1}(t) - BZ_k(t) - BZ_k(k\tau) + \psi^{-1}(r)\sigma(X_{k-1}(r-\tau)) \\
&\quad + \int_{k\tau}^{t} \psi^{-1}(s) f'(X_{k-1}(s-\tau)) D_r^{a,b} X_{k-1}(s-\tau)\, ds \\
&\quad - B \int_{k\tau}^{t} \psi^{-1}(s)\sigma'(X_{k-1}(s-\tau)) D_r^{a,b} X_{k-1}(s-\tau) \int_0^s \phi(u,s)\, du\, ds \\
&\quad + \int_{k\tau}^{t} \psi^{-1}(s)\sigma'(X_{k-1}(s-\tau)) D_r^{a,b} X_{k-1}(s-\tau)\delta^{a,b} B^{a,b}(s).
\end{aligned}
$$

Clearly, it follows that $D_t^{a,b} Z_k(t) = \psi^{-1}(t)\sigma(X_{k-1}(t-\tau))$, which is equivalent to (5.4). ∎

As a consequence, as ψ is differentiable, the process U_k and hence $X_k(t)$ are also Malliavin differentiable. Now, apply Itô's formula (3.1) for $X_k(t)$ and substitute $(D_t^{a,b})^{\phi}\psi(t)$ and $(D_t^{a,b})^{\phi} Z_k(t)$ to obtain

$$
\begin{aligned}
& dX_k(t) \\
&= B_\phi \Big\{ B^2 \Big(\int_0^t \phi(t,u)\, du \Big) \psi(t) Z_k(t)\, dt + A\psi(t) Z_k(t)\, dt + B\psi(t) Z_k(t) \delta^{a,b} B^{a,b}(t) \\
&\quad + \psi(t)\psi^{-1}(t)\, dM_{k-1}(t) + (D_t^{a,b})^{\phi}\psi(t).\psi^{-1}(t)\sigma(X_{k-1}(t-\tau))\, dt \\
&\quad + B\psi(t)(D_t^{a,b})^{\phi} Z_k(t)\, dt \Big\} \\
&\quad - \psi(t) U_k(t) dt - A\psi(t) \Big(\int_{k\tau}^t U_k(s)\, ds \Big) dt - B\psi(t) \Big(\int_{k\tau}^t U_k(s)\, ds \Big) \delta^{a,b} B^{a,b}(t) \\
&= \{AX_k(t) + f((X_{k-1}(t-\tau))\} dt + \{BX_k(t) + \sigma(X_{k-1}(t-\tau))\} \delta^{a,b} B^{a,b}(t).
\end{aligned}
$$

Thus SDE (1.1) has the solution $B_\phi \psi(t) Z_k(t) - (\int_{k\tau}^t U_k(u)\, du)\psi(t)$, i.e., for every $k\tau \leqslant t \leqslant (k+1)\tau$,

$$
\begin{aligned}
X_k(t) &= X_{k-1}(k\tau) + \int_{k\tau}^{t} \{AX_k(s) + f(X_{k-1}(s-\tau))\}\, ds \\
&\quad + \int_{k\tau}^{t} \{BX_k(s) + \sigma(X_{k-1}(s-\tau))\} \delta^{a,b} B^{a,b}(t).
\end{aligned}
$$

THEOREM 5.2. *Under Assumption* 1.1 *and the first part of Assumption* 5.1, *SDE* (1.1) *admits a unique solution on* $[k\tau, (k+1)\tau]$ *for all* $k = 0, \ldots, K_0 - 1$, *which is also Malliavin differentiable.*

Proof. Assume there are two Malliavin differentiable solutions X_k^1 and X_k^2 to (1.1). Then

$$X_k^1(t) - X_k^2(t) = \int_0^t A(X_k^1(s) - X_k^2(s))\,ds + B\int_0^t (X_k^1(s) - X_k^2(s))\delta^{a,b}B^{a,b}(t)$$

and as in finding the derivative of $\psi(t)$, we obtain $D_r^{a,b}(X_k^1(t) - X_k^2(t)) = B(X_k^1(t) - X_k^2(t))1_{0\leqslant r\leqslant t\leqslant\tau}$. From (3.4) and (2.4), we derive

$$\begin{aligned}
&\mathbb{E}\big(|X_k^1(t) - X_k^2(t)|^2\big)\\
&\qquad\leqslant T\int_{k\tau}^t |A|^2\mathbb{E}(|X_k^1(s) - X_k^2(s)|^2)\,ds\\
&\qquad\quad + B^2\mathbb{E}\Big|\int_{k\tau}^t (X_k^1(s) - X_k^2(s))\delta^{a,b}B^{a,b}(t)\Big|^2\\
&\qquad\leqslant T\int_0^t (|A|^2 + B^2C_{\frac{a+b+1}{2},2})\mathbb{E}(|X_k^1(s) - X_k^2(s)|^2)\,ds\\
&\qquad\quad + TB^2C_{\frac{a+b+1}{2},2}\mathbb{E}\Big(\int_{k\tau}^T\int_0^s |D_r^{a,b}(X_k^1(s) - X_k^2(s))|^2\,dr\,ds\Big)\\
&\qquad\leqslant T\int_{k\tau}^t (|A|^2 + (B^2 + B^4)C_{\frac{a+b+1}{2},2})\mathbb{E}(|X_k^1(s) - X_k^2(s)|^2)\,ds.
\end{aligned}$$

Now, Gronwall's inequality completes the proof of the uniqueness of the solution.

Next, Proposition 3.2 implies that for every $t \in [k\tau, (k+1)\tau]$ this solution has also a Malliavin derivative satisfying

$$\begin{aligned}
(5.6)\quad D_r^{a,b}X_k(t) &= D_r^{a,b}X_{k-1}(k\tau) + BX_k(r) + \sigma(X_{k-1}(r-\tau))1_{k\tau\leqslant r\leqslant t-\tau}\\
&+ \int_{k\tau\vee r}^t \big(AD_r^{a,b}X_k(s) + f'(X_{k-1}(s-\tau))D_r^{a,b}X_{k-1}(s-\tau)1_{r\leqslant s-\tau}\big)\,ds\\
&+ \int_{k\tau\vee r}^t \big(BD_r^{a,b}X_k(s) + \sigma'(X_{k-1}(s-\tau))D_r^{a,b}X_{k-1}(s-\tau)1_{r\leqslant s-\tau}\big)\delta^{a,b}B^{a,b}(s)
\end{aligned}$$

for every $0 \leqslant r \leqslant t - \tau$ and also

$$\begin{aligned}
(5.7)\quad &D_r^{a,b}X_k(t)\\
&= BX_k(r) + \int_{k\tau\vee r}^t AD_r^{a,b}X_k(s)\,ds + \int_{k\tau\vee r}^t BD_r^{a,b}X_k(s)\delta^{a,b}B^{a,b}(s)
\end{aligned}$$

for every $t - \tau < r \leqslant t$. ∎

Performing the same step for $k = K_0$ we deduce that for every $t \in [K_0\tau, T]$, (1.1) has a unique solution $B_\phi\psi(t)Z_{K_0}(t) - (\int_{k\tau}^{t} U_{K_0}(u)\,du)\psi(t)$ for every $K_0\tau \leqslant t \leqslant T$ with Malliavin derivative satisfying (5.6) and (5.7). In this sense, we can find an expression for a unique solution to (1.1) in the following theorem.

THEOREM 5.3. *Under Assumption* 1.1 *and the first part of Assumption* 5.1*, SDE* (1.1) *admits a unique solution* $x(t)$ *on* $[0, T]$ *given by*

$$x(t) = \sum_{k=0}^{K_0-1} X_k(t)1_{k\tau\leqslant t\leqslant(k+1)\tau} + X_{K_0}(t)1_{K_0\tau\leqslant t\leqslant T}, \tag{5.8}$$

which is also Malliavin differentiable up to order $K_1 + M$.

Proof. Obviously, for every $0 \leqslant r \leqslant t - \tau$, $D^{a,b}x(t)$ should satisfy, for $t \in [0, T]$,

$$\begin{aligned} D_r^{a,b}x(t) &= D_r^{a,b}\xi_0(0) + Bx(r) + \sigma(x(r-\tau))1_{k\tau\leqslant r\leqslant t-\tau} \\ &\quad + \int_r^t \{AD_r^{a,b}x(s) + f'(x(s-\tau))D_r^{a,b}x(s-\tau)1_{0\leqslant r\leqslant s-\tau}\}\,ds \\ &\quad + \int_r^t (BD_r^{a,b}x(s) + \sigma'(x(s-\tau))D_r^{a,b}x(s-\tau)1_{0\leqslant r\leqslant s-\tau})\delta^{a,b}B^{a,b}(s), \end{aligned} \tag{5.9}$$

and for every $t - \tau < r \leqslant t$,

$$\begin{aligned} &D_r^{a,b}x(t) \\ &\quad = Bx(r) + \int_r^t AD_r^{a,b}x(s)\,ds + \int_r^t BD_r^{a,b}x(s)\delta^{a,b}B^{a,b}(s), \quad t \in [0, T]. \end{aligned} \tag{5.10}$$

Now, one can easily continue the above constructions for (5.9) instead of (1.1) to derive Malliavin differentiability of $D_r^{a,b}x(t)$. Finally, we repeat this procedure up to order $K_1 + M$, the order of differentiability of the functions f and g, to deduce the assertion. ∎

6. BOUNDS OF MOMENTS OF THE SOLUTION AND ITS DERIVATIVES

Construction of the solution of (1.1) in the previous section allows us to show that this solution and its Malliavin derivatives have uniformly bounded p-moments for every $p \geqslant 1$.

THEOREM 6.1. *For every* $p \geqslant 2$*, under Assumptions* 1.1 *and* 5.1*, there exists some positive constant* $C_{0,p}$ *such that for every* $1 \leqslant l \leqslant K_1 + M$*,*

$$\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|X_0(t)|^p\Big) + \mathbb{E}\Big(\sup_{0\leqslant r_1,\dots,r_l\leqslant t\leqslant T}|D_{r_l}^{a,b}\cdots D_{r_1}^{a,b}X_0(t)|^p\Big) \leqslant C_{0,p}. \tag{6.1}$$

Proof. First, we note that according to the definition of $X_0(t)$, in connection with Theorem 5.1 and Proposition 5.1, to show the boundedness of moments of X_0, it is sufficient to derive the uniform boundedness of moments of $Z_0(\cdot)$, which will be done in Appendix A for every Z_k in general. To show the second part of (6.1), we note that from (5.7), $D_r^{a,b}X_0(t) = BX_0(r)\psi(t)\psi^{-1}(r)$. Therefore, Theorem 5.1 and Proposition 5.1 yield

$$\mathbb{E}\Big(\sup_{0\leqslant t\leqslant\tau}|D_r^{a,b}X_0(t)|^p\Big) < \infty.$$

In addition, for $0 \leqslant r_2 \leqslant r_1 \leqslant t$,

$$\begin{aligned} D_{r_2}^{a,b}D_{r_1}^{a,b}X_0(t) &= BD_{r_2}^{a,b}X_0(r_1)\psi(t)\psi^{-1}(r_1) + BD_{r_2}^{a,b}\psi(t)X_0(r_1)\psi^{-1}(r_1) \\ &\quad + BD_{r_2}^{a,b}\psi^{-1}(r_1)X_0(r_1)\psi(t) \\ &= BD_{r_2}^{a,b}X_0(r_1)\psi(t)\psi^{-1}(r_1) + B^2\psi(t)1_{0\leqslant r_2\leqslant r_1}X_0(r_1)\psi^{-1}(r_1) \\ &\quad - B^2\psi^{-1}(r_1)1_{0\leqslant r_2\leqslant r_1}X_0(r_1)\psi(t), \end{aligned}$$

which is a function of ψ, ψ^{-1}, X_0 and $D_{r_2}^{a,b}X_0(r_1)$. Take higher derivatives to show that the l-order derivative of $X_0(t)$ depends only on the m-order derivatives of $X_0(t)$ for $0 \leqslant m \leqslant l-1$, i.e., for every $0 \leqslant l \leqslant K_1 + M$, there exists some function F_l such that

$$\begin{aligned} &D_{r_l}^{a,b}\cdots D_{r_1}^{a,b}X_0(t) \\ &\quad = F_l(\psi(t), \psi^{-1}(r_1), X_0(r_1), D_{r_2}^{a,b}X_0(r_1), \ldots, D_{r_l}^{a,b}\cdots D_{r_2}^{a,b}X_0(r_1)). \end{aligned}$$

Hence, induction on l, the derivative order, completes the proof. ∎

To prove the boundedness of the solutions X_i, $i = 1, \ldots, K_0$, and their derivatives, we need their boundedness in the previous step $i-1$. In other words, as we mentioned in the Introduction, the upper bound in the maximal inequality depends on the higher derivative of the process. Accordingly, expression (5.8) together with equations (5.9) and (5.10) were imposed to establish the boundedness of the solution X_{i-1} and all its derivatives, and thereby to deduce the corresponding boundedness of X_i. For this reason, for every $0 \leqslant k \leqslant K_0$, define $\mathcal{M}_k$ to be the set of $(K_1 + M - 2k)$-times Malliavin differentiable processes G_t such that all their derivatives have bounded moments, i.e., $G_t \in \mathcal{M}_k$ if and only if for every $p \geqslant 2$ there exists some constant $C_{k,p}$ such that for every $1 \leqslant l \leqslant K_1 + M - 2k$,

$$\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|G_t|^p\Big) + \mathbb{E}\Big(\sup_{0\leqslant r_1,\ldots,r_l\leqslant t\leqslant T}|D_{r_l}^{a,b}\cdots D_{r_1}^{a,b}G_t|^p\Big) \leqslant C_{k,p}.$$

We will show that for every $0 \leqslant k \leqslant K_0$, if $X_{k-1}(t) \in \mathcal{M}_{k-1}$, then $X_k(t) \in \mathcal{M}_k$. Thus, since X_0 is $K_1 + M$-times differentiable and $2K_0 + 1 < M$, the process $x(t)$ has uniformly bounded Malliavin derivatives up to order K_1.

THEOREM 6.2. *For every $0 \leqslant k \leqslant K_0$, if $X_{k-1}(t) \in \mathcal{M}_{k-1}$, then $X_k(t)$ has uniformly bounded moments.*

We prove this theorem in Appendix A. To this end, from Hölder's inequality, Theorem 5.1 and Proposition 5.1, it is sufficient to derive uniform bounds of moments of $Z_k(\cdot)$ and $U_k(\cdot)$ in terms of the bound of moments of the derivatives of $X_{k-1}(t)$ up to order $K_1 + M - 2k + 2$.

Now, we are ready to prove some upper bounds for the moments of higher derivatives of the solution to (1.1).

THEOREM 6.3. *Under Assumptions* 1.1 *and* 5.1*, if $X_{k-1} \in \mathcal{M}_{k-1}$, then for every $p \geqslant 2$ and $1 \leqslant l \leqslant K_1 + M - 2k$ there exists some positive constant $C'_{l,p}$ such that the l-th Malliavin derivative of the solution to SDE* (1.1) *satisfies*

$$\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant u_1,\dots,u_l\leqslant t}|D^{a,b}_{u_l}\cdots D^{a,b}_{u_1}X_k(t)|^p\Big) \leqslant C'_{l,p}.$$

We prove this theorem in Appendix C. It is crucial that in the proof, the order of derivatives for all terms on the right-hand side of the generated equation is at most the order $K_1 + M - 2k + 2$ of $X_{k-1}(\cdot)$.

We gather the results of Theorems 6.2 and 6.3 in the following theorem.

THEOREM 6.4. *For every $p \geqslant 2$, under Assumptions* 1.1 *and* 5.1*, there exists some positive constant $C_{x,p}$ such that for every $1 \leqslant l \leqslant K_1$,*

$$\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|x(t)|^p\Big) + \mathbb{E}\Big(\sup_{0\leqslant r_1,\dots,r_l\leqslant t\leqslant T}|D^{a,b}_{r_l}\cdots D^{a,b}_{r_1}x(t)|^p\Big) \leqslant C_{x,p}. \tag{6.2}$$

7. REGULARITY OF THE DENSITY

To achieve the regularity of the density of the solution to (1.1), we assume that the functions f and σ are infinitely differentiable with bounded derivatives; in fact, $K_1 = \infty$ in Assumption 1.1. Also, we consider the following hypothesis.

HYPOTHESIS H. There exists some constant $M_0 > 0$ such that $|\sigma(x)| > M_0$ for all x.

THEOREM 7.1. *Under Hypothesis H and assumption $K_1 = \infty$, for every $t \in [0,T]$ the solution of the SDDE* (1.1) *has an infinitely differentiable density with respect to Lebesgue's measure on $\mathbb{R}$.*

Proof. Fix $t \in [0,T]$. Theorem 6.4 guarantees the infinitely Malliavin differentiable solution $x(t)$ as $K_1 = \infty$. To prove the second part, we have to show that $\mathbb{E}(\int_0^T |D_u x(t)|^2\,du) < \infty$. By using Malliavin's criterion, it is sufficient to check that for every $p \geqslant 1$ there exists some $\epsilon_0 > 0$ such that for all $\epsilon_1 \leqslant \epsilon_0$,

$$P\Big(\int_0^T |D_u x(t)|^2\,du < \epsilon_1\Big) < \epsilon_1^p.$$

To this end, we estimate

$$\begin{aligned}
&P\Big(\int_0^T |D_u x(t)|^2\,du < \epsilon_1\Big)\\
&\quad\leqslant P\Big(\int_{t-\epsilon_1^\theta}^t |\psi(t)|^2|\psi(u)|^{-2}|Bx(u)+\sigma(x(u-\tau))1_{r>\tau}|^2\,du < \epsilon_1\Big)\\
&\quad\leqslant P\Big(A_0, \inf_{0\leqslant t\leqslant T}\psi(r) > \delta, \sup_{0\leqslant t\leqslant T}\psi(r) < \delta^{-1}\Big)\\
&\qquad + P\Big(\inf_{0\leqslant t\leqslant T}\psi(r)\leqslant\delta\Big) + P\Big(\sup_{0\leqslant t\leqslant T}\psi(r)\geqslant\delta^{-1}\Big)\\
&\quad=: P_{1,\epsilon_1} + P_{2,\epsilon_1} + P_{3,\epsilon_1}
\end{aligned}$$

with

$$A_0 = \Big\{\int_{t-\epsilon_1^\theta}^t |\psi(t)|^2|\psi(u)|^{-2}|Bx(u)+\sigma(x(u-\tau))1_{r>\tau}|^2\,du < \epsilon_1\Big\}.$$

Let $\delta = \epsilon_1^{1/8}$ and apply Chebyshev's inequality to show that $P_{3,\epsilon_1} \leqslant \epsilon_1^p$. Indeed,

$$P_{3,\epsilon_1} \leqslant \delta^{8p}\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|\psi(t)|^{8p}\Big) \leqslant \delta^{8p}C_{8p}.$$

From Proposition 5.1, the infimum part is handled by a simple modification of this argument for $\psi^{-1}(\cdot)$ and therefore $P_{2,\epsilon_1} \leqslant \epsilon_1^p$. On the other hand,

$$\begin{aligned}
P_{1,\epsilon_1} &\leqslant P\Big(\int_{t-\epsilon_1^\theta}^t |Bx(u)+\sigma(x(u-\tau))1_{r>\tau}|^2\,du < \epsilon_1\delta^{-4}\Big)\\
&\leqslant P\Big(\int_{t-\epsilon_1^\theta}^t |Bx(u)+\sigma(x(u-\tau))1_{r>\tau}|^2\,du < \epsilon_1\delta^{-4}, \int_{t-\epsilon_1^\theta}^t |Bx(u)|\,du \leqslant \epsilon_1^\gamma\Big)\\
&\quad + P\Big(\int_{t-\epsilon_1^\theta}^t |Bx(u)|\,du > \epsilon_1^\gamma\Big).
\end{aligned}$$

Since $|\sigma(y)| > 0$ for all y, when $\theta < 1/2$, one can easily see that the first summand in the above inequality vanishes. By using Chebyshev's inequality and applying (6.2) we get, for every $q > 1$,

$$P\Big(\int_{t-\epsilon_1^\theta}^t |Bx(u)| > \epsilon_1^\gamma\Big) \leqslant \frac{1}{\epsilon_1^{\gamma q}}\mathbb{E}\Big(\sup_{t-\epsilon_1^\theta\leqslant u\leqslant t}|Bx(u)|^q\Big) \leqslant B^q C_{0,q}\epsilon_1^{(\theta-\gamma)q}.$$

So, choosing $0 < \gamma < \theta < 1/2$ yields the assertion. ∎

8. CONCLUSION

In this article, we give an explicit expression to prove the existence and uniqueness of the solution to a stochastic differential equation with delay driven by weighted

fractional Brownian motion. We present the solution step by step as we proceed with a time delay. This solution is Malliavin differentiable of higher order m, the differentiability of the coefficients, and its Malliavin derivatives of order less than $m - M$ have uniformly bounded moments, where $\frac{1}{2}M$ is the number of time steps of the delay needed to reach T. The solution has an infinitely differentiable density with respect to Lebesgue's measure on $\mathbb{R}$ for $d \geqslant 1$. Our result has many applications, for example in numerical methods. For instance, one can easily check that the Euler approximation process of (1.1), in the continuous version, has Malliavin derivatives with uniformly bounded moments. Also, one can see that the law of the Euler approximation process is smooth, which is essential to show weak convergence of the approximate process to the true solution.

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APPENDIX A

Proof of Theorem 6.2. For ease of notation we define, for any function N_J,

$$J_{k,p}(N_J) := \mathbb{E}\Big(\int_{k\tau}^{(k+1)\tau}\Big(\int_{k\tau}^{(k+1)\tau}|N_J|^{p/q}\,ds\Big)^{\frac{q}{q-1}}\,dr\Big)^{q-1},$$

$$\psi_{k,k+1,p} = \mathbb{E}\Big(\sup_{k\tau\leqslant s\leqslant(k+1)\tau}|\psi^{-1}(s)|^p\Big),$$

$$\psi_{\sup,k,k+1,p} = \mathbb{E}\Big(\sup_{k\tau\leqslant s\leqslant(k+1)\tau}|\psi^{-1}(s)|^p\Big),$$

$$\psi_{D,k,k+1,p} = \mathbb{E}\Big(\int_{k\tau}^{(k+1)\tau}\int_{k\tau}^{(k+1)\tau}|D_s^{a,b}\psi^{-1}(r)\sigma(X_{k-1}(r-\tau))|^p\,dr\,ds\Big).$$

From Hölder's inequality, Theorem 5.1 and Proposition 5.1, it is sufficient to derive uniform bounds of moments of $Z_k(\cdot)$ and $U_k(\cdot)$ in terms of the bounds of moments of the derivatives of $X_{k-1}(t)$ up to order $K_1 + M - 2k + 2$. To do this, from the definition of X_{k-1}, Remark 4.1, and the fact $(z+y)^p \leqslant 2^p(z^p + y^p)$ one derives

$$\begin{aligned}(\text{A.1})\quad &\mathbb{E}\Big(\sup_{k\tau\leqslant t\leqslant(k+1)\tau}|Z_k(t)|^p\Big)\\ &\leqslant 2^p\Big\{\mathbb{E}(|X_{k-1}(k\tau)|^p) + \mathbb{E}\Big(\sup_{0\leqslant t\leqslant\tau}|\psi^{-1}(s)|^p\Big|\int_{k\tau}^{(k+1)\tau}f_\phi(X_{k-1}(s-\tau))\,ds\Big|^p\Big)\\ &\quad + C_1\tau\Big\{\Big(\int_{k\tau}^{(k+1)\tau}|\mathbb{E}(\psi^{-1}(s)\sigma(\xi_0(s-\tau)))|^{\frac{p}{q-1}}\,ds\Big)^{q-1} + J_{k,p}(F^\sigma)\Big\}\Big\}\end{aligned}$$

$$
\begin{aligned}
&\leqslant 2^p\Big\{\mathbb{E}(|X_{k-1}(k\tau)|^p) + 2^{p-1}L^p\tau^{2p}\int_{k\tau}^{(k+1)\tau}(1+\mathbb{E}(|X_{k-1}(s-\tau)|^{2p}))\,ds\\
&\quad+\frac{1}{2}\psi_{\sup,k,k+1,2p}\\
&\quad+C_1\tau^2\int_{k\tau}^{(k+1)\tau}|\mathbb{E}(\psi^{-1}(s)\sigma(X_{k-1}(s-\tau)))|^p ds + C_1\tau J_{k,p}(F^\sigma)\Big\}\\
&\leqslant 2^p\Big\{\mathbb{E}(|X_{k-1}(k\tau)|^p) + 2^{p-1}L^p\tau^{2p}\int_{k\tau}^{(k+1)\tau}(1+\mathbb{E}(|X_{k-1}(s-\tau)|^{p}))\,ds\\
&\quad+\frac{1}{2}\psi_{\sup,k,k+1,2p}+\frac{1}{2}C_1\tau^2\psi_{k,k+1,2p}\\
&\quad+\frac{1}{2}C_1\tau^2L^p2^p\int_{k\tau}^{(k+1)\tau}\mathbb{E}(1+|X_{k-1}(s-\tau)|^{2p})\,ds + C_1\tau J_{k,p}(F^\sigma)\Big\},
\end{aligned}
$$

where

$$
\begin{aligned}
f_\phi(X_{k-1}(s-\tau)) &:= f(X_{k-1}(s-\tau)) - B\int_{k\tau}^{t}\sigma_\phi(X_{k-1}(s-\tau))\,ds,\\
F^\sigma &:= D_s^{a,b}(\psi^{-1}(r)\sigma(X_{k-1}(r-\tau))).
\end{aligned}
$$

The fact that $D_s\sigma(X_{k-1}(r-\tau))=0$, for all $s\geqslant k\tau$, and Jensen's inequality imply

$$
\begin{aligned}
&J_{k,p}(F^\sigma)\\
&\quad\leqslant L^p\tau^2\Big(1+\mathbb{E}\Big(\sup_{k\tau\leqslant r\leqslant(k+1)\tau}|X_{k-1}(r-\tau)|^p\Big)\Big)+\tau^2\psi_{D,k,k+1,2p}.
\end{aligned}
\tag{A.2}
$$

On the other hand, to bound U_k, from Hölder's inequality, Assumption 1.1 and Proposition 5.1 we derive

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{k\tau\leqslant s\leqslant t}|U_k(s)|^p\Big)\\
&\qquad\leqslant 3^{p-1}\Big\{\frac{1}{2}t^p\psi_{k,t,2p}+(2L)^{p/2}\frac{1}{2}\int_{k\tau}^{t}\big(1+\mathbb{E}(|X_{k-1}(s-\tau)|^p)\big)\,ds\\
&\qquad\quad+B^p\mathbb{E}\Big(\sup_{k\tau\leqslant u\leqslant t}|Z_k(u)|^{2p}\Big)+\frac{3}{2}B^p\mathbb{E}\Big(\int_{k\tau}^{t}|\phi(u,s)|^{2p}\,ds\Big)\\
&\qquad\quad+\mathbb{E}\Big(\sup_{k\tau\leqslant s\leqslant t}\Big|\int_0^{k\tau}D_u^{a,b}Z_k(s)\phi(s,u)\,du\Big|^p\Big)\Big\}
\end{aligned}
\tag{A.3}
$$

$$\leqslant 3^{p-1}\Big\{\frac{1}{2}t^p\psi_{k,t,2p}+(2L)^{\frac{p}{2}}\frac{1}{2}\int_{k\tau}^{t}\big(1+\mathbb{E}(|X_{k-1}(s-\tau)|^p)\big)\,ds$$
$$+B^p\mathbb{E}\Big(\sup_{k\tau\leqslant u\leqslant t}|Z_k(u)|^{2p}\Big)+\frac{3}{2}(B^p+k\tau)\mathbb{E}\Big(\int_{k\tau}^{t}|\phi(u,s)|^{2p}\,ds\Big)$$
$$+\frac{k\tau}{2}\mathbb{E}\Big(\sup_{k\tau\leqslant s\leqslant t}\sup_{0\leqslant u\leqslant k\tau}|D_u^{a,b}Z_k(s)|^{2p}\Big)\Big\}.$$

Also, we employ (5.5) and Remark 4.1 to get

$$\text{(A.4)}\quad \mathbb{E}\Big(\sup_{k\tau\leqslant s\leqslant t}\sup_{0\leqslant u\leqslant k\tau}|D_u^{a,b}Z_k(s)|^{2p}\Big)$$
$$\leqslant 7^p\Big\{\mathbb{E}\Big(\sup_{0\leqslant u\leqslant k\tau}D_u^{a,b}X_{k-1}(k\tau)\Big)+2B^p\mathbb{E}\Big(\sup_{k\tau\leqslant s\leqslant t}|Z_k(s)|^{2p}\Big)$$
$$+\mathbb{E}\Big(\sup_{k\tau\leqslant s\leqslant t}|\psi^{-1}(u)|^{2p}(1+|X_{k-1}(u-\tau)|^2)^p\Big)$$
$$+t^{2p}K_2^{2p}\psi_{k,t,4p}+t^{2p}K_2^{2p}\mathbb{E}\Big(\int_{k\tau}^{t}|D_u^{a,b}X_{k-1}(s-\tau)|^{4p}\,ds\Big)$$
$$+C_1t^{4p+1}L^{4p}\int_{k\tau}^{t}\big|\mathbb{E}\big(\psi^{-1}(s)D_u^{a,b}X_{k-1}(s-\tau)\,ds\big)\big|^{2p}+C_1\tau J_{k,2p}(F^{k,\sigma})\Big\},$$

where

$$F^{k,\sigma}:=D_s^{a,b}\big(\psi^{-1}(r)\sigma'(X_{k-1}(r-\tau))D_u^{a,b}X_{k-1}(r-\tau)\big).$$

Here, we note that if we replace p with $2p$ in the definition of $J_{k,p}$, then q should also be $2q$. Then from Assumption 1.1,

$$\text{(A.5)}\quad J_{k,2p}(F^{k,\sigma})$$
$$\leqslant 3^{p-1}t^2K_2^{2p}\psi_{\sup,k,k+1,4p}$$
$$+3^{p-1}t^2K_2^{2p}\mathbb{E}\Big(\int_{k\tau}^{t}\int_{k\tau}^{t}|D_s^{a,b}X_{k-1}(r-\tau)D_u^{a,b}X_{k-1}(r-\tau)|^{4p}\,dr\,ds\Big)$$
$$+3^{p-1}t^2K_2^{2p}\Big[\psi_{\sup,k,k+1,4p}+\mathbb{E}\Big(\int_{k\tau}^{t}\int_{k\tau}^{t}|D_s^{a,b}D_u^{a,b}X_{k-1}(r-\tau)|^{8p}\,dr\,ds\Big)\Big]$$
$$+3^{p-1}t^2K_2^{2p}\Big[\psi_{D,k,t,4p}+\mathbb{E}\Big(\int_{k\tau}^{t}\int_{k\tau}^{t}|D_s^{a,b}X_{k-1}(r-\tau)|^{8p}\,dr\,ds\Big)\Big].$$

Now, substitute (A.5) and (A.4) in (A.3) and also substitute (A.2) in (A.1). Hence, Assumption 5.1, Proposition 5.1 and the condition $X_{k-1}\in\mathcal{M}_{k-1}$ yield the boundedness of moments of $X_k(\cdot)$ in $[k\tau,(k+1)\tau]$, and finally, by repeating this computation for every $K_0\tau\leqslant t\leqslant T$ the claim can be obtained.

APPENDIX B

Some expressions for higher derivatives and their upper bounds. In this part, we derive a recursive formula for the higher-order Malliavin derivatives of the solution $x(\cdot)$, obtained by continuing to differentiate (5.6) or (5.9). Our expression deals with the case $0 \leqslant r_1, \dots, r_l \leqslant t-\tau$; the other cases have the same computation. We note that our modification will be useful in proving Theorem 6.3.

First, let us define the processes $H_{r_l \cdots r_1}, G_{r_l \cdots r_1}(s), F_{r_l \cdots r_1}(s)$ for every $2 \leqslant l \leqslant 2M$ and $k = 1, \dots, K_0$ as follows for simplicity:

$$
\begin{aligned}
&1_{(1,\dots,l)} := 1_{r_l > (k-1)\tau} \prod_{i=1}^{l-1} 1_{r_i+\tau<r_l}, \quad 1_{(2,\dots,l)} := 1_{r_1>(k-1)\tau} \prod_{i=2}^{l} 1_{r_i+\tau<r_1},\\
&1_{(1,\dots,j)} = 1_{r_j>(k-1)\tau} \prod_{i=1, i\neq j}^{l-1} 1_{r_i+\tau<r_j}, \quad 2 \leqslant j \leqslant l-1,\\
&D^{a,b}_{r_l,\dots,r_1} := D^{a,b}_{r_l} \cdots D^{a,b}_{r_1}, \qquad D^{a,b}_{r_l,r_1,-r_j} := D^{a,b}_{r_l} \cdots D^{a,b}_{r_{j+1}} D^{a,b}_{r_{j-1}} \cdots D^{a,b}_{r_1},\\
&F_{r_l \cdots r_1}(X_{k-1}, s) := D^{a,b}_{r_l} \cdots D^{a,b}_{r_1}\{f(X_{k-1}(s-\tau))\},\\
&G_{r_l \cdots r_1}(X_{k-1}, s) := D^{a,b}_{r_l} \cdots D^{a,b}_{r_1}\{\sigma(X_{k-1}(s-\tau))\},\\
&H_{r_l \cdots r_1}(k) := BD^{a,b}_{r_l,\dots,r_2} X_k(r_1) + D^{a,b}_{r_l,\dots,r_2}\big(\sigma(X_{k-1}(r_1-\tau))\big)1_{(2,\dots,l)}\\
&\qquad\qquad + BD^{a,b}_{r_{l-1},\dots,r_1} X_k(r_l) + D^{a,b}_{r_{l-1},\dots,r_1}\big(\sigma(X_{k-1}(r_l-\tau))\big)1_{(1,\dots,l)}\\
&\qquad\qquad + B\sum_{j=2}^{l-1} D^{a,b}_{r_l,r_1,-r_j} X_k(r_j)\\
&\qquad\qquad + \sum_{j=2}^{l-1} D^{a,b}_{r_l,r_1,-r_j}\big(\sigma(X_{k-1}(r_j-\tau))\big)1_{(1,\dots,j)}.
\end{aligned}
$$

Then the lth derivative for every $k\tau \leqslant t \leqslant (k+1)\tau$ with $k = 1, \dots, K_0$ and also every $0 \leqslant r_1, \dots, r_l \leqslant t-\tau$ satisfies

$$
\begin{aligned}
\text{(B.1)}\quad & D^{a,b}_{r_l,\dots,r_1} X_k(t) = D^{a,b}_{r_l,\dots,r_1} X_{k-1}(k\tau) + H_{r_l \cdots r_1}(k)\\
&+ \int_{k\tau \vee r_1 \vee \cdots \vee r_l}^{t} \Big\{ AD^{a,b}_{r_l,\dots,r_1} X_k(s) + F_{r_l \cdots r_1}(X_{k-1}, s) \prod_{i=1}^{l} 1_{r_i+\tau<s} \Big\}\, ds\\
&+ \int_{k\tau \vee r_1 \vee \cdots \vee r_l}^{t} \Big\{ BD^{a,b}_{r_l,\dots,r_1} X_k(s) + G_{r_l \cdots r_1}(X_{k-1}, s) \prod_{i=1}^{l} 1_{r_i+\tau<s} \Big\} \delta^{a,b} B^{a,b}(s),
\end{aligned}
$$

and for every $K_0\tau \leqslant t \leqslant T$ it also satisfies the same equation with $K_0\tau \vee r_1 \vee \cdots \vee r_l$ instead of $k_{r,\tau} := k\tau \vee r_1 \vee \cdots \vee r_l$. Note that for every $0 \leqslant r_1, \dots, r_l \leqslant t-\tau$ we have

(B.2) $$D^{a,b}_{r_l,\dots,r_1}x(t) = \sum_{k=0}^{K_0-1} D^{a,b}_{r_l,\dots,r_1}X_k(t)1_{k\tau\leqslant t\leqslant(k+1)\tau} + D^{a,b}_{r_l,\dots,r_1}X_{K_0}(t)1_{K_0\tau\leqslant t\leqslant T},$$

where

$$\begin{aligned}
D^{a,b}_{r_l,\dots,r_1}X_k(t) &= B_\phi\psi(t)Z_{k,r_1,\dots,r_l}(t) - \Big(\int_{k_{r,\tau}}^{t} U_{k,r_1,\dots,r_l}(u)\,du\Big)\psi(t),\\
Z_{k,r_1,\dots,r_l}(t) &:= D^{a,b}_{r_l,\dots,r_1}X_{k-1}(k\tau) + H_{r_l\cdots r_1}(k)\\
&\quad + \int_{k_{r,\tau}}^{t} \psi^{-1}(s)\,dM_{k-1,r_1,\dots,r_l}(s),\\
M_{k,r_1,\dots,r_l}(t) &:= \int_{k_{r,\tau}}^{t} F_{r_l\cdots r_1}(X_k,s)\,ds + \int_{k_{r,\tau}}^{t} G_{r_l\cdots r_1}(X_k,s)\delta^{a,b}B^{a,b}(t)\\
&\quad - B\int_{k_{r,\tau}}^{t}\int_{0}^{s} G_{r_l\cdots r_1}(X_k,s)\phi(u,s)\,du\,ds,\\
U_{k,r_1,\dots,r_l}(t) &:= \int_{k_{r,\tau}}^{t} \psi^{-1}(u)G_{r_l\cdots r_1}(X_{k-1},u)\phi(u,t)\,du\\
&\quad + B\int_{k_{r,\tau}}^{t} Z_{k,r_1,\dots,r_l}(u)\phi(u,t)\,du\\
&\quad + \int_{k_{r,\tau}}^{t}\int_{0}^{k\tau} D^{a,b}_{r_l,\dots,r_1}Z_{k,r_1,\dots,r_l}(s)\phi(u,s)\,du\,ds.
\end{aligned}$$

From the definition of $F_{r_l\cdots r_1}(X_k,t)$ and $G_{r_l\cdots r_1}(X_k,t)$, we understand that these processes depend on higher derivatives of f and σ, respectively, and Malliavin derivatives of X_{k-1} up to order l. Then

$$\begin{aligned}
&|F_{r_l\cdots r_1}(X_{k-1},t)|^p\\
&\quad = \mathcal{P}_1(f'(X_{k-1}),\dots,f^{(l)}(X_{k-1}),D^{a,b}_{r_1}X_{k-1},\dots,D^{a,b}_{r_l,\dots,r_1}X_{k-1})(t),\\
&|G_{r_l\cdots r_1}(X_{k-1},t)|^p\\
&\quad = \mathcal{P}_2(\sigma'(X_{k-1}),\dots,\sigma^{(l)}(X_{k-1}),D^{a,b}_{r_1}X_{k-1},\dots,D^{a,b}_{r_l,\dots,r_1}X_{k-1})(t),
\end{aligned}$$

where $\mathcal{P}_1$ and $\mathcal{P}_2$ are polynomial functions. From Assumption 1.1 we infer that for all $p\geqslant 2$, $0\leqslant r_1,\dots,r_l\leqslant T$ and $0\leqslant t_1\leqslant t_2\leqslant T$,

(B.3) $$\begin{aligned}
&\mathbb{E}\Big(\sup_{t_1\leqslant t\leqslant t_2}|F_{r_l\cdots r_1}(X_k,t)|^p\Big) + \mathbb{E}\Big(\sup_{t_1\leqslant t\leqslant t_2}|G_{r_l\cdots r_1}(X_k,t)|^p\Big)\\
&\qquad\leqslant 2^{lp}K_2\sum_{j=2}^{l}\mathbb{E}\Big(\sup_{t_1\leqslant t\leqslant t_2}|D^{a,b}_{r_j,\dots,r_1}X_{k-1}(t-\tau)|^{q'_j}\Big),
\end{aligned}$$

where the integers q'_j satisfy $1\leqslant q'_j\leqslant p$. In addition, when $l\leqslant K_1+M-2k$, for

all $0 \leqslant r, u \leqslant t-\tau$,

$$\text{(B.4)}\quad \mathbb{E}\Big(\sup_{t_1\leqslant t\leqslant t_2}|D_r^{a,b}G_{r_l\cdots r_1}(X_k,t)|^p\Big) \leqslant 2^{lp}K_2\sum_{j=1}^{l+1}\mathbb{E}\Big(\sup_{t_1\leqslant t\leqslant t_2}|D^{a,b}_{r_j,\ldots,r_1}X_{k-1}(t-\tau)|^{q_{j,G}}\Big),$$

and

$$\text{(B.5)}\quad \mathbb{E}\Big(\sup_{t_1\leqslant t\leqslant t_2}|D_u^{a,b}D_r^{a,b}G_{r_l\cdots r_1}(X_k,t)|^p\Big) \leqslant 2^{lp}K_2\sum_{j=1}^{l+2}\mathbb{E}\Big(\sup_{t_1\leqslant t\leqslant t_2}|D^{a,b}_{r_j,\ldots,r_1}X_{k-1}(t-\tau)|^{q_{j,G}}\Big),$$

where the integers $q_{j,G}$ satisfy $1 \leqslant q_{j,G} \leqslant p$. Define the vectors

$$\begin{aligned}
V_i &:= \big(\sigma^{(i)}(X_{k-1}(r_1-\tau)),\ldots,\sigma^{(i)}(X_{k-1}(r_l-\tau))\big),\\
W_{j,1} &:= \big(D^{a,b}_{r_j,\ldots,r_2}X_k(r_1),\ldots,D^{a,b}_{r_j,\ldots,r_2}X_k(r_l)\big),\\
W_{j,2} &:= \big(D^{a,b}_{r_j,\ldots,r_2}X_{k-1}(r_1-\tau),\ldots,D^{a,b}_{r_j,\ldots,r_2}X_{k-1}(r_l-\tau)\big),\\
w_{j,j'} &:= \big(D^{a,b}_{r_{j'},r_1,-r_j}X_{k-1}(r_1-\tau),\ldots,D^{a,b}_{r_{j'}}\cdots D^{a,b}_{r_{j'},r_1,-r_j}X_{k-1}(r_l-\tau)\big),\\
w^{(1)}_{j,j'} &:= \big(D^{a,b}_{r_{j'},r_1,-r_j}X_k(r_1-\tau),\ldots,D^{a,b}_{r_{j'},r_1,-r_j}X_k(r_l-\tau)\big),\\
v_{j''} &:= \big(D^{a,b}_{r_{j''},\ldots,r_1}X_{k-1}(r_1-\tau),\ldots,D^{a,b}_{r_{j''},\ldots,r_1}X_{k-1}(r_l-\tau)\big),\\
v^{(1)}_{j''} &:= \big(D^{a,b}_{r_{j''},\ldots,r_1}X_k(r_1-\tau),\ldots,D^{a,b}_{r_{j''},\ldots,r_1}X_k(r_l-\tau)\big),
\end{aligned}$$

for all $1 \leqslant i \leqslant l$, $2 \leqslant j \leqslant l$, $3 \leqslant j' \leqslant l$ and $1 \leqslant j'' \leqslant l-1$. Therefore $|H_{r_l\cdots r_1}|^p$ would be a polynomial composition of the elements of these vectors, i.e., there exists some polynomial $\mathcal{P}_3$ such that

$$\begin{aligned}
|H_{r_l\cdots r_1}|^p = \mathcal{P}_3(&V_1,\ldots,V_l,W_{2,1},W_{2,2},\ldots,\\
&W_{l,1},W_{l,2},w_{2,3},w^{(1)}_{2,3},\ldots,w_{l,l},w^{(1)}_{l,l},v_1,v^{(1)}_1,\ldots,v_{l-1},v^{(1)}_{l-1}).
\end{aligned}$$

We again conclude from Assumption 1.1 that for every $p \geqslant 2$ there exists some constant c_{p,H_1} such that

$$\begin{aligned}
\text{(B.6)}\quad \mathbb{E}(|H_{r_l\cdots r_1}|^p) &\leqslant c_{p,H_1}\Big(\sum_{i=1}^{l}\mathbb{E}(|V_i|^{q_i})+\sum_{j=1}^{l-1}\mathbb{E}(|v_{j''}|^{q_{j''}}+|v^{(1)}_{j''}|^{q_{j''}})\\
&\quad+\sum_{j'=3}^{l}\sum_{j=2}^{l}\{\mathbb{E}(|W_{j,1}|^{q_{j,1}})+\mathbb{E}(|W_{j,2}|^{q_{j,2}})\}\\
&\quad+\sum_{j'=3}^{l}\sum_{j=2}^{l}\{\mathbb{E}(|w_{j,j'}|^{q_{j,j'}}+|w^{(1)}_{j,j'}|^{q_{j,j'}})\}\Big)\\
&=: AA,
\end{aligned}$$

where all integers $q_j, q_{j,1}, q_{j,2}, q_{j,j'}$ and $q_{j''}$ are between 1 and p.

It is worth mentioning that the order of differentiability in the components of the vectors $V_i, W_{j,1}, W_{j,2}, w_{j,j'}, w^{(1)}_{j,j'}$, and $v_{j'}, v^{(1)}_{j'}$ is at most $l-1$. This fact helps us to find some upper bounds for the moments of $D^{a,b}_{u_l}\cdots D^{a,b}_{u_1}x(\cdot)$ that will be demonstrated in Theorem 6.3.

APPENDIX C

Proof of Theorem 6.3. We first make the following computation for every $k = 0,\dots,K_0$ and $1 \leqslant l \leqslant K_1+M-2k$. Assume $\sup_{0\leqslant u_1,\dots,u_l\leqslant t}|D^{a,b}_{u_l,\dots,u_1}X_{k-1}(t)|^p$ is achieved at $u_1=r_1,\dots,u_l=r_1$, where we omitted the dependence of r_is' on t for simplicity. Let

$$F^k_0 := D^{a,b}_r(\psi^{-1}(s)G_{r_l\cdots r_1}(X_{k-1},s)),$$
$$F^{k,\sigma}_0 := D^{a,b}_s(\psi^{-1}(r)D^{a,b}_u G_{r_l\cdots r_1}(X_{k-1},r)).$$

From (B.4) and (B.3),

(C.1) $J_{k,p}(F^k_0)$

$$\begin{aligned}
&\leqslant 2^{p-1}\Big\{\tau^4\psi_{\sup,0.T,2p}+\tau^2\mathbb{E}\Big(\int_{k\tau}^{(k+1)\tau}\int_{k\tau}^{(k+1)\tau}|D^{a,b}_r G_{r_l\cdots r_1}(X_{k-1},s)|^{2p}\,dr\,ds\Big)\\
&\quad+\tau^2\psi_{D,k,k+1,2p}\Big\}+2^{p-1}\tau^2\mathbb{E}\Big(\int_{k\tau}^{(k+1)\tau}\int_{k\tau}^{(k+1)\tau}|G_{r_l\cdots r_1}(X_{k-1},s)|^{2p}\,dr\,ds\Big)\\
&\leqslant 2^{p-1}\Big\{\tau^4\psi_{\sup,0.T,2p}+\tau^2\psi_{D,k,k+1,2p}\\
&\quad+\tau^4 2^{2lp}K_2\sum_{j=2}^{l}\mathbb{E}\Big(\sup_{t_1\leqslant t\leqslant t_2}|D^{a,b}_{r_j,\dots,r_1}X_{k-1}(t-\tau)|^{2q'_j}\Big)\\
&\quad+2\tau^4 2^{2lp}K_2^2\sum_{j=1}^{l+1}\mathbb{E}\Big(\sup_{k\tau\leqslant s\leqslant(k+1)\tau}|D^{a,b}_{r_j,\dots,r_1}X_{k-1}(s-\tau)|^{2q_{j,G}}\Big)\Big\},
\end{aligned}$$

and

(C.2) $J_{k,2p}(F^{k,\sigma}_0)$

$$\begin{aligned}
&\leqslant 3^{p-1}t^2\mathbb{E}\Big(\int_{k_{r,\tau}}^{t}\int_{k_{r,\tau}}^{t}|\psi^{-1}(r)D^{a,b}_s D^{a,b}_u G_{r_l\cdots r_1}(X_{k-1},r)|^{2p}\,dr\,ds\Big)\\
&\quad+3^{p-1}t^2\mathbb{E}\Big(\int_{k_{r,\tau}}^{t}\int_{k_{r,\tau}}^{t}|D^{a,b}_s\psi^{-1}(r)D^{a,b}_u G_{r_l\cdots r_1}(X_{k-1},r)|^{2p}\,dr\,ds\Big)\\
&\leqslant 3^{p-1}t^2\Big\{\psi_{\sup,k_{r,\tau},k+1,4p}+\mathbb{E}\Big(\int_{k_{r,\tau}}^{t}\int_{k_{r,\tau}}^{t}|D^{a,b}_s D^{a,b}_u G_{r_l\cdots r_1}(X_{k-1},r)|^{4p}\,dr\,ds\Big)\Big\}\\
&\quad+3^{p-1}t^2K_2^{2p}\Big\{\psi_{D,k_{r,\tau},t,4p}+\mathbb{E}\Big(\int_{k_{r,\tau}}^{t}\int_{k_{r,\tau}}^{t}|D^{a,b}_u G_{r_l\cdots r_1}(X_{k-1},r)|^{4p}\,dr\,ds\Big)\Big\}.
\end{aligned}$$

According to Theorem 4.1,

$$
\begin{aligned}
\text{(C.3)}\quad I_1 &:= \mathbb{E}\Big(\sup_{k\tau\leqslant t\leqslant(k+1)\tau}|Z_{k,r_1,\dots,r_l}(t)|^p\Big)\\
&\leqslant 3^p\bigg\{2^p\mathbb{E}(|D^{a,b}_{r_l,\dots,r_1}X_{k-1}(k\tau)|^p)+\frac{1}{2}\psi_{k,k+1,2p}\\
&\quad+\frac{1}{2}\mathbb{E}\Big(\int_{k\tau}^{(k+1)\tau}|F_{r_l\cdots r_1}(X_{k-1},s)|^{2p}\,ds\Big)+2^p\mathbb{E}(|H_{r_l\cdots r_1}(k)|^p)\\
&\quad+\mathbb{E}\Big(\sup_{k\tau\leqslant t\leqslant(k+1)\tau}\Big|\int_{k\tau}^{t}\psi^{-1}(s)G_{r_l\cdots r_1}(X_{k-1},s)\delta^{a,b}B^{a,b}(s)\Big|^p\Big)\bigg\}\\
&\leqslant 3^p\bigg\{2^p\mathbb{E}(|D^{a,b}_{r_l,\dots,r_1}X_{k-1}(k\tau)|^p)+2^p\mathbb{E}(|H_{r_l\cdots r_1}(k)|^p)+\frac{1}{2}(1+C_1\tau)\psi_{k,k+1,2p}\\
&\quad+\frac{1}{2}\tau\mathbb{E}\Big(\sup_{k\tau\leqslant s\leqslant(k+1)\tau}|F_{r_l\cdots r_1}(X_{k-1},s)|^{2p}\Big)\\
&\quad+\frac{1}{2}C_1\tau\int_{k\tau}^{(k+1)\tau}\mathbb{E}(|\,G_{r_l\cdots r_1}(X_{k-1},s)|^{2p})+C_1\tau J_{k,p}(F_0^k)\bigg\}.
\end{aligned}
$$

Now, substitute (B.6), (B.3) and (C.1) in (C.3) to obtain

(C.4)

$$
\begin{aligned}
I_1\leqslant 3^p\bigg\{&2^p\mathbb{E}(|D^{a,b}_{r_l,\dots,r_1}X_{k-1}(k\tau)|^p)+2^pAA+(1+C_1\tau)\psi_{\sup,0,T,2p}\\
&+2^{2lp}\frac{1}{2}(1+C_1\tau)\tau K_2^2\sum_{j=2}^{l}\mathbb{E}\Big(\sup_{k\tau\leqslant t\leqslant(k+1)\tau}|D^{a,b}_{r_j,\dots,r_2}X_{k-1}(t)|^{q_j}\Big)\\
&+2^pC_1\tau^5\psi_{\sup,0,T,2p}+2^pC_1\tau^3\psi_{D,k,k+1,2p}\\
&+2^{p+1}C_1\tau^3 2^{2lp}K_2^2\sum_{j=2}^{l+1}\mathbb{E}\Big(\sup_{k\tau\leqslant s\leqslant(k+1)\tau}|D^{a,b}_{r_j,\dots,r_1}X_{k-1}(s-\tau)|^{2q_{j,G}}\Big)\bigg\}<\infty.
\end{aligned}
$$

On the other hand, to bound $U_{k,r_1,\dots,r_l}$, from Hölder's inequality, Assumption 1.1 and Proposition 5.1 and the same computation as (A.3) we derive

$$
\begin{aligned}
\text{(C.5)}\quad I_2 &:= \mathbb{E}\Big(\sup_{k\tau\leqslant s\leqslant t}|U_{k,r_1,\dots,r_l}(s)|^p\Big)\\
&\leqslant 3^{p-1}\bigg\{\frac{1}{2}t^p\psi_{k_{r,\tau},t,2p}+\int_{k_{r,\tau}}^{t}\mathbb{E}(|G_{r_l\cdots r_1}(X_{k-1},s)|^p)\,ds\\
&\quad+B^p\mathbb{E}\Big(\sup_{k\tau\leqslant u\leqslant t}|Z_{k,r_1,\dots,r_l}(u)|^{2p}\Big)+\frac{3}{2}(B^p+k\tau)\mathbb{E}\Big(\int_{k_{r,\tau}}^{t}|\phi(u,s)|^{2p}\,ds\Big)\\
&\quad+\frac{k\tau}{2}\mathbb{E}\Big(\sup_{k\tau\leqslant s\leqslant t}\sup_{0\leqslant u\leqslant k\tau}|D^{a,b}_u Z_{k,r_1,\dots,r_l}(s)|^{2p}\Big)\bigg\}.
\end{aligned}
$$

Also, arguing in a way similar to (5.5) and applying Remark 4.1 results in

$$
\begin{aligned}
(\mathrm{C.6}) \quad & \mathbb{E}\Big(\sup_{k\tau\leqslant s\leqslant t}\sup_{0\leqslant u\leqslant k\tau}|D_u^{a,b}Z_{k,r_1,\dots,r_l}(s)|^{2p}\Big)\\
&\leqslant 8^p\Big\{\mathbb{E}\Big(\sup_{0\leqslant u\leqslant k\tau}D_u^{a,b}D^{a,b}_{r_l,\dots,r_1}X_{k-1}(k\tau)\Big)+2B^p\mathbb{E}\Big(\sup_{k\tau\leqslant s\leqslant t}|Z_{k,r_1,\dots,r_l}(s)|^{2p}\Big)\\
&\quad+\mathbb{E}\Big(\sup_{0\leqslant u\leqslant k\tau}D_u^{a,b}H_{r_l\cdots r_1}(k)\Big)\\
&\quad+\mathbb{E}\Big(\sup_{k\tau\leqslant s\leqslant t}\sup_{0\leqslant u\leqslant k\tau}|\psi^{-1}(u)|^{2p}G_{r_l\cdots r_1}(X_{k-1},u)^p\Big)\\
&\quad+t^{2p}\psi_{k_{r,\tau},t,4p}+t^{2p}\mathbb{E}\Big(\int_{k_{r,\tau}}^{t}|D_u^{a,b}(F_{r_l\cdots r_1}(X_{k-1},s)+G_{r_l\cdots r_1}(X_{k-1},s))|^{4p}\,ds\Big)\\
&\quad+C_1t^{4p+1}\int_{k_{r,\tau}}^{t}|\mathbb{E}(\psi^{-1}(s)D_u^{a,b}G_{r_l\cdots r_1}(X_{k-1},s))|^{2p}\,ds+C_1\tau J_{k,2p}(F_0^{k,\sigma})\Big\}.
\end{aligned}
$$

Substitute (C.2) into (C.6) and then into (C.5) and apply (B.6), (B.4), (B.5) and (B.3) to find some upper bound in terms of the derivatives of X_{k-1} and ψ^{-1}. Here, it is crucial that in this computation the order of derivatives for all terms on the right-hand side of the generated equation is at most the order $K_1+M-2k+2$ of $X_{k-1}(\cdot)$. So, applying the condition $X_{k-1}\in\mathcal{M}_{k-1}$ and employing Proposition 5.1 in (C.4) and (C.5) we find that I_1 and then I_2 are bounded, which completes the proof.

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