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EXACT STRONG LAWS FOR NON-IDENTICALLY-DISTRIBUTED φ-MIXING RANDOM VARIABLES WITH INFINITE MEANS*

BY

WENZHI YANG (HEFEI), ZHENGYU ZHAO, AND ANDRÉ ADLER (DONGGUAN)

Abstract. This paper investigates Exact Strong Laws for non-identically-distributed φ-mixing random variables with infinite means. We obtain the Exact Strong Laws for random variables for which previously only exact weak laws were known. We show how to correctly select the weights and norming sequences in this setting so that the strong law indeed holds. Our results extend those of Adler [Bull. Inst. Math. Acad. Sinica 7 (2012), 417–422] and Nakata [Statist. Probab. Lett. 109 (2016), 124–129], who established Exact Weak Laws for independent Pareto–Zipf distributions, to Exact Strong Laws for φ-mixing Pareto–Zipf distributions. To illustrate these Exact Strong Laws, we provide an algorithm for generating dependent Pareto–Zipf random variables with non-identical distributions and infinite means, and present finite-sample simulations.

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1. INTRODUCTION

The classical laws of large numbers for weighted random variables are studied under the assumption of finite means (see Chow and Teicher [8]). However, when the means are infinite, alternative techniques are required. For example, Adler [2] investigated a weighted Exact Weak Law for independent Pareto–Zipf distributions, where the classical Weak Law fails and the corresponding Strong Law also fails. The work of Adler [2] was extended by Nakata [13]. We return to Adler [2] in search for when a strong law holds. Here we establish the correct weights and norming sequences for a particular class of non-identically-distributed and dependent random variables with infinite means in order to get a strong and not weak law of large numbers.

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First, assume that $\{X_n,\ n \geqslant 1\}$ is a sequence of nonnegative random variables with Pareto–Zipf distributions, i.e.,

$$P\{X_n = 0\} = 1 - \frac{1}{d_n}, \tag{1.1}$$

and for all $x > 0$, the tail probabilities are given by

$$P\{X_n > x\} = \frac{1}{x + d_n}. \tag{1.2}$$

We start with $d_n = (\lg n)^\delta$ for some $\delta \in (0, \infty)$ and in the last section we also consider the case $d_n = n^\delta$ for some $\delta \in (0, 1)$.

On the one hand, by (1.1) and (1.2), $X_1, X_2, \ldots$ are not identically distributed, and each has an infinite mean. On the other hand, when studying the limiting behavior of the sequence $\{X_n,\ n \geqslant 1\}$, a crucial modeling choice lies in its dependence structure. Traditional probability limit theory is typically built upon the strong assumption of independence, which greatly facilitates the derivation of classical results such as the law of large numbers and the central limit theorem. However, in many real-world applications, observed data often exhibit some form of correlation or dependence, making the assumption of complete independence overly idealized and, in many cases, inconsistent with practical scenarios. Thus, this paper investigates Exact Strong Laws for the sequence $\{X_n,\ n \geqslant 1\}$ satisfying (1.1) and (1.2), under the assumption that $\{X_n,\ n \geqslant 1\}$ is a φ-mixing sequence. This framework is of significant practical relevance, particularly in applications such as financial time series analysis.

Let us recall the definition of φ-mixing. Suppose $\{X_n,\ n \geqslant 1\}$ is a sequence of real-valued random variables. Denote $\mathcal{F}_a^b = \sigma(X_i,\ a \leqslant i \leqslant b)$ and define

$$\varphi(n) = \sup_{k \geqslant 1} \sup_{A \in \mathcal{F}_1^k,\, B \in \mathcal{F}_{k+n}^\infty,\, P(A) \neq 0} |P(B|A) - P(B)|.$$

DEFINITION 1.1. If $\varphi(n) \downarrow 0$ as $n \to \infty$, then $\{X_n,\ n \geqslant 1\}$ is said to be a *φ-mixing sequence*.

The φ-mixing condition is an important concept for dependent sequences and has applications in financial time series analysis. We refer to Dobrushin [9], Györfi et al. [10], Lu and Lin [11], Hall and Heyde [15] and the references therein for more properties and applications. There are many works on Exact Weak Laws and Exact Strong Laws for independent and Pareto-type random variables with infinite means: see for example Adler [1, 2, 3, 4], Miao et al. [12], Nakata [13, 14], and Adler and Matuła [5]. Xu et al. [18] and Yang et al. [19] extended these results to dependent settings, including negatively superadditive-dependent (NSD) sequences and weakly dependent $\tilde{\rho}$-mixing sequences [6]. Note that $\tilde{\rho}$-mixing and φ-mixing do not contain each other. For properties and examples of $\tilde{\rho}$-mixing, we refer to Bradley [6, 7].

The remainder of this paper is structured as follows. Section 2 provides some technical lemmas for φ-mixing sequences. In Section 3, we first introduce a norming sequence required to establish Exact Strong Laws, and then present three versions of Exact Strong Laws. Section 4 illustrates supporting simulations, while Section 5 concludes the paper and provides further discussions.

Throughout the paper, we define $\lg x = \ln(\max(e, x))$. We do that to avoid minor issues like dividing by $\ln 1$ and to ensure that probabilities do not exceed 1. Also we let $\lg_2 x = \lg(\lg x)$ (not the logarithm to base 2), and $\lg_m x = \lg(\lg_{m-1} x)$ for all positive integers m. In addition, $e_n \sim f_n$ means $e_n/f_n \to 1$ as $n \to \infty$.

2. PRELIMINARIES

In this section, we give some assumptions and technical lemmas for φ-mixing sequences.

ASSUMPTION 2.1. *For each $n \geqslant 1$, the random variable X_n satisfies* (1.1) *and* (1.2)*, and $\{X_n,\, n \geqslant 1\}$ is a sequence of φ-mixing random variables with $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$.*

LEMMA 2.1 (Maximal inequality, [16, Lemma 1.7]). *Let $\{X_n,\, n \geqslant 1\}$ be a φ-mixing sequence satisfying $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$. For some $r \geqslant 2$, if $EX_n = 0$ and $E\,|X_n|^r < \infty$ for all $n \geqslant 1$, then*

$$E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k} X_i\Big|^r\Big) \leqslant C\Big(\sum_{i=1}^{n} E\,|X_i|^r + \Big(\sum_{i=1}^{n} EX_i^2\Big)^{r/2}\Big),$$

where C is a positive constant depending only on $\varphi(\cdot)$.

COROLLARY 2.1 (Khinchin–Kolmogorov Convergence Theorem). *Let $\{X_n,\, n \geqslant 1\}$ be a sequence of φ-mixing random variables satisfying $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$. If $\sum_{n=1}^{\infty} \mathrm{Var}(X_n) < \infty$, then $\sum_{n=1}^{\infty}(X_n - EX_n)$ converges almost surely.*

Proof. Wu and Jiang [17] used the maximal inequality for $\tilde{\rho}$-mixing to obtain the Khinchin–Kolmogorov Convergence Theorem for $\tilde{\rho}$-mixing sequences. So by Lemma 2.1 together with the proof of [17, Theorem 1], it is easy to obtain Corollary 2.1 for φ-mixing sequences. ∎

LEMMA 2.2 (The Borel–Cantelli Lemma, [11, Theorem 8.2.1]). *Let $\{X_n,\, n \geqslant 1\}$ be a sequence of φ-mixing random variables. Then for all $A_n \in \sigma(X_n)$,*

$$\sum_{n=1}^{\infty} P(A_n) < \infty \iff P(A_n,\ i.o.) = 0,$$

where $i.o.$ means infinitely often.

3. MAIN RESULTS

In order to obtain Exact Strong Laws, the norming sequence should not be selected via the sum of weights, but independent of that sequence. Thus we have to define four sequences $\{a_n,\, n \geqslant 1\}$, $\{b_n,\, n \geqslant 1\}$, $\{c_n,\, n \geqslant 1\}$ and the aforementioned $\{d_n, n \geqslant 1\}$. This leaves us greater freedom in creating an Exact Strong Law. As usual, let

$$a_n = \frac{(\lg n)^\alpha S(n)}{n}$$

where $\alpha > -2$ and $S(n)$ is a very slowly varying function composed of logarithmic factors:

$$S(n) = \prod_{j=2}^{m} (\lg_j n)^{\beta_j}, \tag{3.1}$$

where m is any integer greater than 1, and the $\beta_2, \ldots, \beta_m$ are arbitrary real numbers. Then we select

$$b_n = (\lg n)^{\alpha+2} S(n) \quad \text{and} \quad c_n = b_n/a_n = n(\lg n)^2.$$

We are now ready to establish our strong law of large numbers.

3.1. Exact Strong Law. In this situation we do indeed get a strong law. The proof is quite straightforward, but it was not easy figuring out all four sequences $\{a_n,\, n \geqslant 1\}$, $\{b_n,\, n \geqslant 1\}$, $\{c_n,\, n \geqslant 1\}$, $\{d_n,\, n \geqslant 1\}$ that allowed us to establish this result.

THEOREM 3.1. *Let Assumption* 2.1 *hold with* $d_n = (\lg n)^\delta$ *for some* $\delta > 0$. *Then for all* $\alpha > -2$,

$$\lim_{n\to\infty} \frac{1}{(\lg n)^{\alpha+2} S(n)} \sum_{k=1}^{n} \frac{(\lg k)^\alpha S(k)}{k} X_k = \frac{1}{\alpha+2} \quad \textit{almost surely.}$$

Proof. We define $\{a_n,\, n \geqslant 1\}$, $\{b_n,\, n \geqslant 1\}$ and $\{c_n,\, n \geqslant 1\}$ as above. Let

$$\tilde{X}_k = [X_k I(X_k \leqslant c_k) - E X_k I(X_k \leqslant c_k)], \quad k \geqslant 1.$$

We use the usual partition

$$\begin{aligned} \frac{1}{b_n} \sum_{k=1}^{n} a_k X_k &= \frac{1}{b_n} \sum_{k=1}^{n} a_k \tilde{X}_k + \frac{1}{b_n} \sum_{k=1}^{n} a_k X_k I(X_k > c_k) \\ &\quad + \frac{1}{b_n} \sum_{k=1}^{n} a_k E X_k I(X_k \leqslant c_k) \\ &:= A_1 + A_2 + A_3. \end{aligned} \tag{3.2}$$

First, we consider the term A_1. It is easy to observe that $\{\tilde{X}_n,\ n \geqslant 1\}$ and $\{\frac{1}{c_n}\tilde{X}_n,\ n \geqslant 1\}$ are also mean zero φ-mixing sequences with $\sum_{n=1}^{\infty}\varphi^{1/2}(n) < \infty$. In order to make A_1 go to zero, we use (1.1) and (1.2) and get a bound on the second moment

$$EX_n^2 I(X_n \leqslant c_n) = \int_0^{c_n} \frac{x^2}{(x+d_n)^2}\,dx < \int_0^{c_n} dx = c_n. \tag{3.3}$$

Then

$$\begin{aligned}\sum_{n=1}^{\infty} \mathrm{Var}\left(\frac{\tilde{X}_n}{c_n}\right) &\leqslant \sum_{n=1}^{\infty}\frac{1}{c_n^2}EX_n^2 I(X_n \leqslant c_n)\\ &< \sum_{n=1}^{\infty}\frac{1}{c_n} = \sum_{n=1}^{\infty}\frac{1}{n(\lg n)^2} < \infty.\end{aligned} \tag{3.4}$$

Combining Corollary 2.1 with (3.4), we find that

$$\sum_{n=1}^{\infty}\frac{1}{c_n}\tilde{X}_n \quad \text{converges almost surely.} \tag{3.5}$$

Consequently, it follows from (3.5), $c_n = b_n/a_n$, Kronecker's lemma (see [8]) and $b_n \to \infty$ that

$$A_1 = \frac{1}{b_n}\sum_{k=1}^{n} a_k\tilde{X}_k \to 0 \quad \text{almost surely.} \tag{3.6}$$

Next, we consider A_2. By (1.2),

$$\sum_{n=1}^{\infty} P\{X_n > c_n\} = \sum_{n=1}^{\infty}\frac{1}{c_n+d_n} < \sum_{n=1}^{\infty}\frac{1}{c_n} = \sum_{n=1}^{\infty}\frac{1}{n(\lg n)^2} < \infty. \tag{3.7}$$

Let n_0 be be some positive integer. Applying Lemma 2.2 with (3.7) and $b_n \to \infty$, one can get

$$A_2 = \frac{1}{b_n}\Big[\sum_{k=1}^{n_0} + \sum_{k=n_0}^{n}\Big]a_k X_k I(X_k > c_k) \to 0 \quad \text{almost surely.} \tag{3.8}$$

Last, we consider A_3. It follows from (1.1) and (1.2) that

$$\begin{aligned}EX_n I(X_n \leqslant c_n) &= \int_0^{c_n}\frac{x}{(x+d_n)^2}\,dx = \int_{d_n}^{c_n+d_n}\frac{u-d_n}{u^2}\,du\\ &= \int_{d_n}^{c_n+d_n}\left[\frac{1}{u}-\frac{d_n}{u^2}\right]du = \lg\left(\frac{c_n+d_n}{d_n}\right)+\frac{d_n}{c_n+d_n}-1\\ &= \lg\left(\frac{c_n}{d_n}+1\right)+\frac{1}{\frac{c_n}{d_n}+1}-1\\ &\sim \lg(c_n/d_n) = \lg n + (2-\delta)\lg_2 n \sim \lg n,\end{aligned} \tag{3.9}$$

by using the fact $c_n/d_n = n(\lg n)^{2-\delta}$ for any $\delta > 0$. Using (3.9) and the slow variation of logarithms in (3.1), we establish that

$$
\begin{aligned}
(3.10)\qquad \frac{\sum_{k=1}^{n} a_k E X_k I(X_k \leqslant c_k)}{b_n} &\sim \frac{\sum_{k=1}^{n} \frac{(\lg k)^{\alpha} S(k)}{k} \lg k}{(\lg n)^{\alpha+2} S(n)} \\
&= \frac{\sum_{k=1}^{n} \frac{(\lg k)^{\alpha+1} S(k)}{k}}{(\lg n)^{\alpha+2} S(n)} \sim \frac{\int_1^n \frac{(\lg x)^{\alpha+1} \prod_{j=2}^{m} (\lg_j x)^{\beta_j}}{x}\, dx}{(\lg n)^{\alpha+2} S(n)} \\
&= \frac{\int_{\lg 1}^{\lg n} u^{\alpha+1} \prod_{j=2}^{m} (\lg_{j-1} u)^{\beta_j}\, du}{(\lg n)^{\alpha+2} S(n)} \sim \frac{\frac{(\lg n)^{\alpha+2}}{\alpha+2} \prod_{j=2}^{m} (\lg_{j-1}(\lg n))^{\beta_j}}{(\lg n)^{\alpha+2} S(n)} \\
&= \frac{\frac{(\lg n)^{\alpha+2}}{\alpha+2} S(n)}{(\lg n)^{\alpha+2} S(n)} = \frac{1}{\alpha+2}.
\end{aligned}
$$

Consequently, by (3.2), (3.6), (3.8) and (3.10), the proof is complete. ∎

This proof shows that $S(n)$ can be replaced by any very slowly varying function, such as $S(\lg n)$. A natural question is what happens if $\alpha < -2$. Then the only possible limit would be zero. However, one can obtain a strong law if $\alpha = -2$. But we must select new sequences for our norms and weights.

3.2. When $\alpha = -2$. The sequence $\{d_n,\, n \geqslant 1\}$ naturally stays the same and $\{c_n,\, n \geqslant 1\}$ changes slightly for our new selections of $\{a_n,\, n \geqslant 1\}$ and $\{b_n,\, n \geqslant 1\}$.

THEOREM 3.2. *Let Assumption* 2.1 *hold with $d_n = (\lg n)^{\delta}$ for some $\delta > 0$. Then for all $\beta > -1$,*

$$
\lim_{n\to\infty} \frac{1}{(\lg_2 n)^{\beta+1}} \sum_{k=1}^{n} \frac{(\lg_2 k)^{\beta}}{k(\lg k)^2} X_k = \frac{1}{\beta+1} \quad \textit{almost surely.}
$$

Proof. Similar to the proof of Theorem 3.1, we take

$$
a_n = \frac{(\lg_2 n)^{\beta}}{n(\lg n)^2} \quad \text{and} \quad b_n = (\lg_2 n)^{\beta+1},
$$

thus

$$
c_n = b_n/a_n = n(\lg n)^2 (\lg_2 n).
$$

Obviously,

$$
\sum_{n=1}^{\infty} P\{X_n > c_n\} = \sum_{n=1}^{\infty} \frac{1}{c_n + d_n} < \sum_{n=1}^{\infty} \frac{1}{c_n} = \sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2 (\lg_2 n)} < \infty.
$$

In view of

$$
E X_n^2 I(X_n \leqslant c_n) = \int_0^{c_n} \frac{x^2}{(x+d_n)^2}\, dx < \int_0^{c_n} dx = c_n,
$$

we have

$$\sum_{n=1}^{\infty} \operatorname{Var}\left(\frac{\tilde{X}_n}{c_n}\right) \leqslant \sum_{n=1}^{\infty} \frac{1}{c_n^2} EX_n^2 I(X_n \leqslant c_n) < \sum_{n=1}^{\infty} \frac{1}{c_n} < \infty$$

as well. Combining these with (3.2), (3.6) and (3.8) we see that A_1 and A_2 in our partition (3.2) still converge to zero. As for A_3, by (1.1) and (1.2) we have once again

$$\begin{aligned} EX_n I(X_n \leqslant c_n) &= \int_0^{c_n} \frac{x}{(x+d_n)^2}\, dx \sim \lg(c_n/d_n) \\ &= \lg n + (2-\delta)\lg_2 n + \lg_3 n \sim \lg n, \end{aligned}$$

since $c_n/d_n = n(\lg n)^{2-\delta}(\lg_2 n)$ for any $\delta > 0$. Hence, similar to (3.10), we establish

$$\begin{aligned} \frac{\sum_{k=1}^n a_k EX_k I(X_k \leqslant c_k)}{b_n} &\sim \frac{\sum_{k=1}^n \frac{(\lg_2 k)^\beta}{k(\lg k)^2} \lg k}{(\lg_2 n)^{\beta+1}} = \frac{\sum_{k=1}^n \frac{(\lg_2 k)^\beta}{k \lg k}}{(\lg_2 n)^{\beta+1}} \\ &\sim \frac{\int_1^n \frac{(\lg_2 x)^\beta}{x \lg x}\, dx}{(\lg_2 n)^{\beta+1}} = \frac{\int_{\lg 1}^{\lg n} \frac{(\lg u)^\beta}{u}\, du}{(\lg_2 n)^{\beta+1}} \\ &= \frac{\int_{\lg_2 1}^{\lg_2 n} w^\beta dw}{(\lg_2 n)^{\beta+1}} \sim \frac{\frac{(\lg_2 n)^{\beta+1}}{\beta+1}}{(\lg_2 n)^{\beta+1}} = \frac{1}{\beta+1}, \end{aligned}$$

which completes the proof. ∎

Naturally, the next question is what happens when $\beta = -1$. This is the same as asking what is the smallest real number larger than zero. We can always get tighter and tighter results. We conclude with that result, but we can keep going on, and at this point it should be clear what happens when $\gamma = -1$.

THEOREM 3.3. *Let Assumption* 2.1 *hold with* $d_n = (\lg n)^\delta$ *for some* $\delta > 0$. *Then for all* $\gamma > -1$,

$$\lim_{n\to\infty} \frac{1}{(\lg_3 n)^{\gamma+1}} \sum_{k=1}^n \frac{(\lg_3 k)^\gamma}{k(\lg k)^2(\lg_2 k)} X_k = \frac{1}{\gamma+1} \quad \textit{almost surely.}$$

Proof. Similar to the proof of Theorem 3.2, we take

$$a_n = \frac{(\lg_3 n)^\gamma}{n(\lg n)^2(\lg_2 n)} \quad \text{and} \quad b_n = (\lg_3 n)^{\gamma+1},$$

thus

$$c_n = b_n/a_n = n(\lg n)^2(\lg_2 n)(\lg_3 n).$$

As before,

$$\sum_{n=1}^{\infty} P\{X_n > c_n\} = \sum_{n=1}^{\infty} \frac{1}{c_n + d_n} < \sum_{n=1}^{\infty} \frac{1}{c_n} = \sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2(\lg_2 n)(\lg_3 n)} < \infty$$

and

$$\sum_{n=1}^{\infty} \operatorname{Var}\left(\frac{\tilde{X}_n}{c_n}\right) \leqslant \sum_{n=1}^{\infty} \frac{1}{c_n^2} EX_n^2 I(X_n \leqslant c_n) < \sum_{n=1}^{\infty} \frac{1}{c_n^2} c_n = \sum_{n=1}^{\infty} \frac{1}{c_n} < \infty,$$

using

$$EX_n^2 I(X_n \leqslant c_n) = \int_0^{c_n} \frac{x^2}{(x+d_n)^2}\, dx < \int_0^{c_n} dx = c_n.$$

In addition, the bound on our truncated mean is

$$\begin{aligned} EX_n I(X_n \leqslant c_n) &= \int_0^{c_n} \frac{x}{(x+d_n)^2}\, dx \sim \lg(c_n/d_n) \\ &= \lg n + (2-\delta)\lg_2 n + \lg_3 n + \lg_4 n \sim \lg n, \end{aligned}$$

since $c_n/d_n = n(\lg n)^{2-\delta}(\lg_2 n)(\lg_3 n)$. Hence

$$\begin{aligned} \frac{\sum_{k=1}^n a_k EX_k I(X_k \leqslant c_k)}{b_n} &\sim \frac{\sum_{k=1}^n \frac{(\lg_3 k)^\gamma}{k(\lg k)^2(\lg_2 k)} \lg k}{(\lg_3 n)^{\gamma+1}} = \frac{\sum_{k=1}^n \frac{(\lg_3 k)^\gamma}{k(\lg k)(\lg_2 k)}}{(\lg_3 n)^{\gamma+1}} \\ &\sim \frac{\int_1^n \frac{(\lg_3 x)^\gamma}{x(\lg x)(\lg_2 x)}\, dx}{(\lg_3 n)^{\gamma+1}} \sim \frac{\int_{\lg 1}^{\lg n} \frac{(\lg_2 u)^\gamma}{u(\lg u)}\, du}{(\lg_3 n)^{\gamma+1}} \\ &\sim \frac{\int_{\lg_2 1}^{\lg_2 n} \frac{(\lg w)^\gamma}{w}\, dw}{(\lg_3 n)^{\gamma+1}} \sim \frac{\int_{\lg_3 1}^{\lg_3 n} v^\gamma\, dv}{(\lg_3 n)^{\gamma+1}} \\ &\sim \frac{\frac{(\lg_3 n)^{\gamma+1}}{\gamma+1}}{(\lg_3 n)^{\gamma+1}} = \frac{1}{\gamma+1}. \end{aligned}$$

Combining these with (3.2), (3.6), (3.8) and (3.10) completes the proof. ∎

3.3. Powers of n. In this subsection we use the previous techniques to obtain a far reaching result. In (1.1) and (1.2), Adler [2] considered $d_n = n$ and took weights $a_n = 1/n$, $b_n = \lg n(\lg_2 n)$ to obtain an Exact Weak Law, but this result cannot be extended to a Strong Law. Perhaps other weights should be used to get an Exact Strong Law. In this subsection, we consider $d_n = n^\delta$ for some $0 < \delta < 1$ and get an Exact Strong Law. Set

$$a_n = \frac{(\lg n)^\alpha}{n}, \quad b_n = (\lg n)^{\alpha+2}, \quad c_n = \frac{b_n}{a_n} = n(\lg n)^2.$$

The final theorem is probably the most important in this paper and yet the easiest one to prove.

THEOREM 3.4. *Let Assumption* 2.1 *hold with* $d_n = n^\delta$ *for some* $0 < \delta < 1$. *Then for all* $\alpha > -2$,

$$\lim_{n\to\infty} \frac{1}{(\lg n)^{\alpha+2}} \sum_{k=1}^{n} \frac{(\lg k)^\alpha}{k} X_k = \frac{1-\delta}{\alpha+2} \quad \textit{almost surely.}$$

Proof. Similar to the proof of Theorem 3.3, it is easy to see that

$$\sum_{n=1}^{\infty} P\{X_n > c_n\} = \sum_{n=1}^{\infty} \frac{1}{c_n + d_n} < \sum_{n=1}^{\infty} \frac{1}{c_n} = \sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2} < \infty.$$

Moreover, since

$$EX_n^2 I(X_n \leqslant c_n) = \int_0^{c_n} \frac{x^2}{(x+d_n)^2}\, dx < \int_0^{c_n} dx = c_n,$$

we have

$$\sum_{n=1}^{\infty} \frac{1}{c_n^2} EX_n^2 I(X_n \leqslant c_n) < \sum_{n=1}^{\infty} \frac{1}{c_n^2} c_n = \sum_{n=1}^{\infty} \frac{1}{c_n} < \infty.$$

The only calculation remaining is that of our truncated mean. This term is similar to the truncated means in the previous proofs:

$$\begin{aligned} EX_n I(X_n \leqslant c_n) &= \int_0^{c_n} \frac{x}{(x+d_n)^2}\, dx = \lg\left(\frac{c_n}{d_n} + 1\right) + \frac{1}{\frac{c_n}{d_n}+1} - 1 \\ &\sim \lg(c_n/d_n) = (1-\delta)\lg n + 2\lg_2 n \sim (1-\delta)\lg n, \end{aligned}$$

by using the fact $c_n/d_n = n^{1-\delta}(\lg n)^2$ and $0 < \delta < 1$. Hence our limit is

$$\begin{aligned} \frac{\sum_{k=1}^n a_k EX_k I(X_k \leqslant c_k)}{b_n} &\sim \frac{\sum_{k=1}^n \frac{(\lg k)^\alpha}{k}(1-\delta)\lg k}{(\lg n)^{\alpha+2}} = \frac{(1-\delta)\sum_{k=1}^n \frac{(\lg k)^{\alpha+1}}{k}}{(\lg n)^{\alpha+2}} \\ &\sim \frac{(1-\delta)\frac{(\lg n)^{\alpha+2}}{\alpha+2}}{(\lg n)^{\alpha+2}} = \frac{1-\delta}{\alpha+2}. \end{aligned}$$

Together with (3.2), (3.6), (3.8) and (3.10), this completes the proof. ∎

REMARK 3.1. In view of Theorem 3.1, we may consider the slowly varying function $S(\lg n)$ in Theorem 3.2. For instance, let $a_n = \frac{(\lg_2 n)^\beta S(\lg n)}{n(\lg n)^2}$, $b_n = (\lg_2 n)^{\beta+1} S(\lg n)$, and $c_n = b_n/a_n = n(\lg n)^2(\lg_2 n)$ in the proof of Theorem 3.2. Then, under the conditions of Theorem 3.2, we obtain

$$\lim_{n\to\infty} \frac{1}{(\lg_2 n)^{\beta+1} S(\lg n)} \sum_{k=1}^{n} \frac{(\lg_2 k)^\beta S(\lg k)}{k(\lg k)^2} X_k = \frac{1}{\beta+1} \quad \text{almost surely.}$$

Similarly, for Theorem 3.3 we consider the slowly varying function $S(\lg_2 n)$. Set

$$a_n = \frac{(\lg_3 n)^\gamma S(\lg_2 n)}{n(\lg n)^2(\lg_2 n)}, \quad b_n = (\lg_3 n)^{\gamma+1} S(\lg_2 n),$$
$$c_n = b_n/a_n = n(\lg n)^2(\lg_2 n)(\lg_3 n)$$

in the proof. Then, under the conditions of Theorem 3.3,

$$\lim_{n\to\infty} \frac{1}{(\lg_3 n)^{\gamma+1} S(\lg_2 n)} \sum_{k=1}^{n} \frac{(\lg_3 k)^\gamma S(\lg_2 k)}{k(\lg k)^2(\lg_2 k)} X_k = \frac{1}{\gamma+1} \quad \text{almost surely.}$$

Finally, in the proof of Theorem 3.4, set

$$a_n = \frac{(\lg n)^\alpha S(n)}{n}, \quad b_n = (\lg n)^{\alpha+2} S(n), \quad c_n = b_n/a_n = n(\lg n)^2.$$

Then, under the conditions of Theorem 3.4,

$$\lim_{n\to\infty} \frac{1}{(\lg n)^{\alpha+2} S(n)} \sum_{k=1}^{n} \frac{(\lg k)^\alpha S(k)}{k} X_k = \frac{1-\delta}{\alpha+2} \quad \text{almost surely.}$$

4. SIMULATION WORK

In this section, we report some simulations to illustrate our Exact Strong Laws. First, we consider a random variable X_n with Pareto–Zipf distribution. On the one hand, by (1.1) and (1.2), it is easy to see that

$$P(X_n > x \mid X_n > 0) = \frac{P(X_n > x, X_n > 0)}{P(X_n > 0)} = \frac{d_n}{x + d_n}, \quad x > 0, \tag{4.1}$$

where d_n is a positive constant (for example $d_n = (\lg n)^\delta$ for some $\delta > 0$ or $d_n = n^\delta$ for some $0 < \delta < 1$). Thus, from (4.1), the conditional distribution of X_n given $X_n > 0$ is

$$\begin{aligned} F_{X_n|X_n>0}(x) &= P(X_n \leqslant x \mid X_n > 0) \\ &= 1 - P(X_n > x \mid X_n > 0) = \frac{x}{x + d_n}, \quad x > 0. \end{aligned} \tag{4.2}$$

On the other hand, it is known that if $Y \overset{d}{\sim} \mathrm{Uniform}(0,1)$, then for any given distribution $F(x)$, $x \in \mathbb{R}$, we have $X = F^{-1}(Y) \overset{d}{\sim} F(x)$, where $F^{-1}(u) = \inf\{x : F(x) \geqslant u\}$, $u \in (0,1)$. So we use this method with (4.2) to generate a

random variable X_n satisfying (1.1) and (1.2). In view of the infinite mean, we use the truncated method in the practical simulation. Let $\varepsilon \in (0,1)$ and $M > 0$ be a sufficiently small and a sufficiently large positive constant, respectively. Let $d_n > 0$ be a positive constant defined by (1.1) and (1.2). The algorithm is as follows:

Algorithm 1 Generation of random variable X_n satisfying (1.1) and (1.2)

Parameters: ε, M, and d_n
Generate $U \sim \mathrm{Uniform}(0,1)$
if $U \leqslant 1 - \frac{1}{d_n}$ **then**
    Set $X_n \leftarrow 0$
else
    Generate $V \sim \mathrm{Uniform}(0,1)$
    if $1-\varepsilon \leqslant V \leqslant 1$ **then**
        Set $X_n \leftarrow M$
    else
        Set $X_n \leftarrow \min\big(d_n \frac{V}{1-V}, M\big)$ (since $V = \frac{x}{x+d_n} \Rightarrow x = d_n \frac{V}{1-V}$)
    end if
end if
return X_n

Let $\boldsymbol{Z} = (Z_1, \ldots, Z_n)^\top \overset{d}{\sim} N_n(\mathbf{0}, \boldsymbol{\Sigma})$, where $\mathbf{0}$ is the $n \times 1$ zero vector and $\boldsymbol{\Sigma}$ is an $n \times n$ symmetric positive definite matrix. For example, we take the Toeplitz matrix $\boldsymbol{\Sigma} = (|\rho|^{|i-j|})_{1 \leqslant i,j \leqslant n}$ for some $0 \leqslant |\rho| < 1$. Here, we define $0^0 = 1$. Then we take $U_i = \Phi(Z_i) \overset{d}{\sim} \mathrm{Uniform}(0,1)$, $1 \leqslant i \leqslant n$, where $\Phi(x)$ is the distribution of $N(0,1)$. Thus, we have a dependent vector $\boldsymbol{U} = (U_1, \ldots, U_n)^\top$. For each U_i, we use Algorithm 1 to generate random variables X_i, $1 \leqslant i \leqslant n$. It can be observed that $\{Z_1, \ldots, Z_n\}$, $\{U_1, \ldots, U_n\}$ and $\{X_1, \ldots, X_n\}$ are φ-mixing random variables with $\varphi(i) = O(|\rho|^i)$.

Let n_0 be some positive integer. In the notation of Theorem 3.1,

$$\begin{aligned}
&\frac{1}{(\lg n)^{\alpha+2} S(n)} \sum_{k=1}^{n} \frac{(\lg k)^\alpha S(k)}{k} X_k \\
&= \frac{1}{(\lg n)^{\alpha+2} S(n)} \sum_{k=1}^{n_0} \frac{(\lg k)^\alpha S(k)}{k} X_k + \frac{1}{(\lg n)^{\alpha+2} S(n)} \sum_{k=n_0+1}^{n} \frac{(\lg k)^\alpha S(k)}{k} X_k.
\end{aligned}$$

By Theorem 3.1, the first term goes to zero, while the second goes to $\frac{1}{\alpha+2}$ almost surely. By (1.1) and (1.2), the Pareto–Zipf random variable X_n has an infinite mean. So the first term in finite sample simulation may occasionally exhibit large values, which can obscure the interquartile range in a box plot and complicate interpretation. Therefore, we conduct the simulation on the basis of the second term instead. For example, denote

$$T_{n1} = \frac{1}{(\lg n)^{\alpha+2} S(n)} \sum_{k=n_0+1}^{n} \frac{(\lg k)^\alpha S(k)}{k} X_k - \frac{1}{\alpha+2}, \tag{4.3}$$

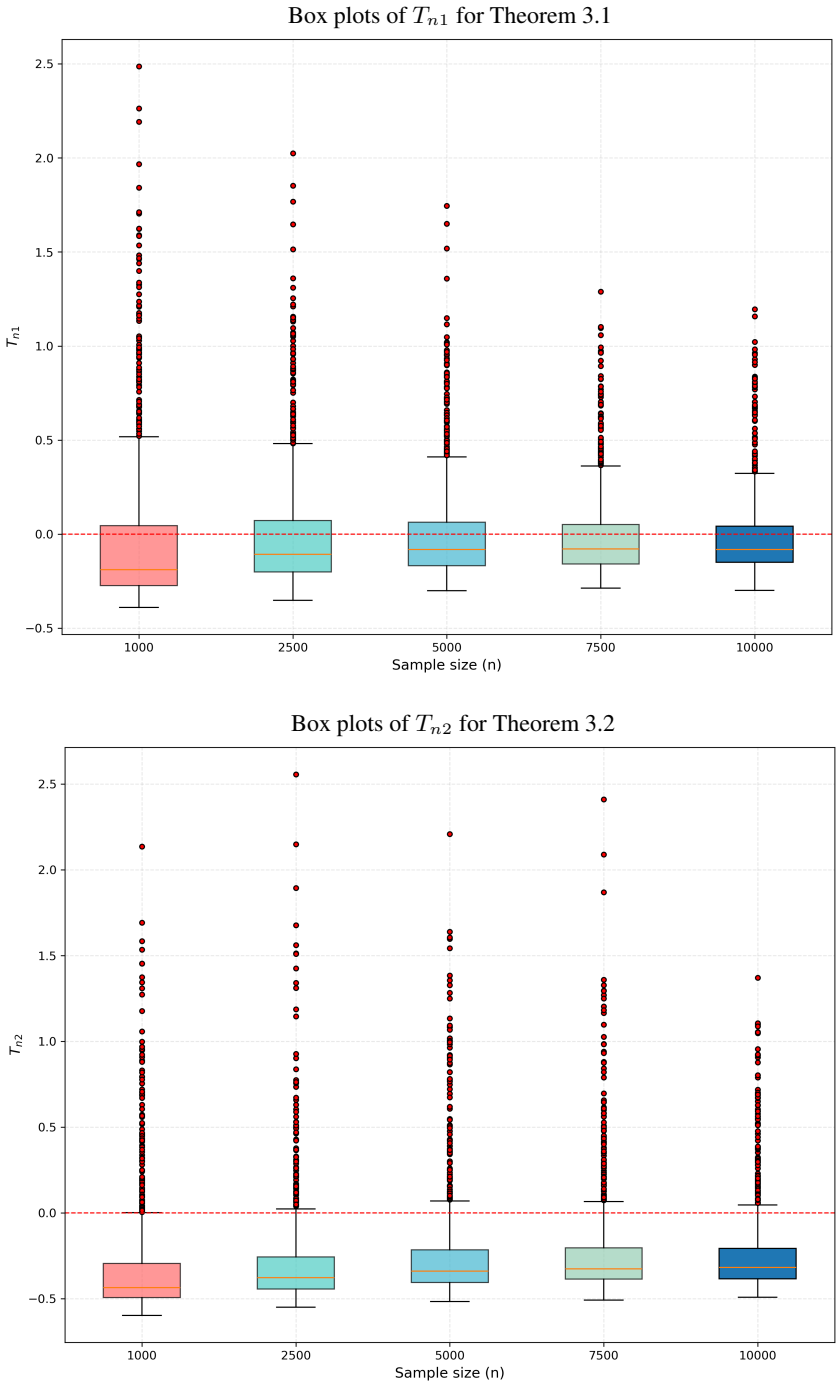


FIGURE 1. Top: the box plot of T_{n1} with parameters $m = 3$, $\beta_1 = 0.5$, $\beta_2 = 0.3$ in (3.1), $d_n = (\lg n)^\delta$, $\delta = 0.5$ and $\alpha = 0$ in (4.3). Bottom: the box plot of T_{n2} with $d_n = (\lg n)^\delta$, $\delta = 0.5$ and $\beta = 0.8$ in (4.4).

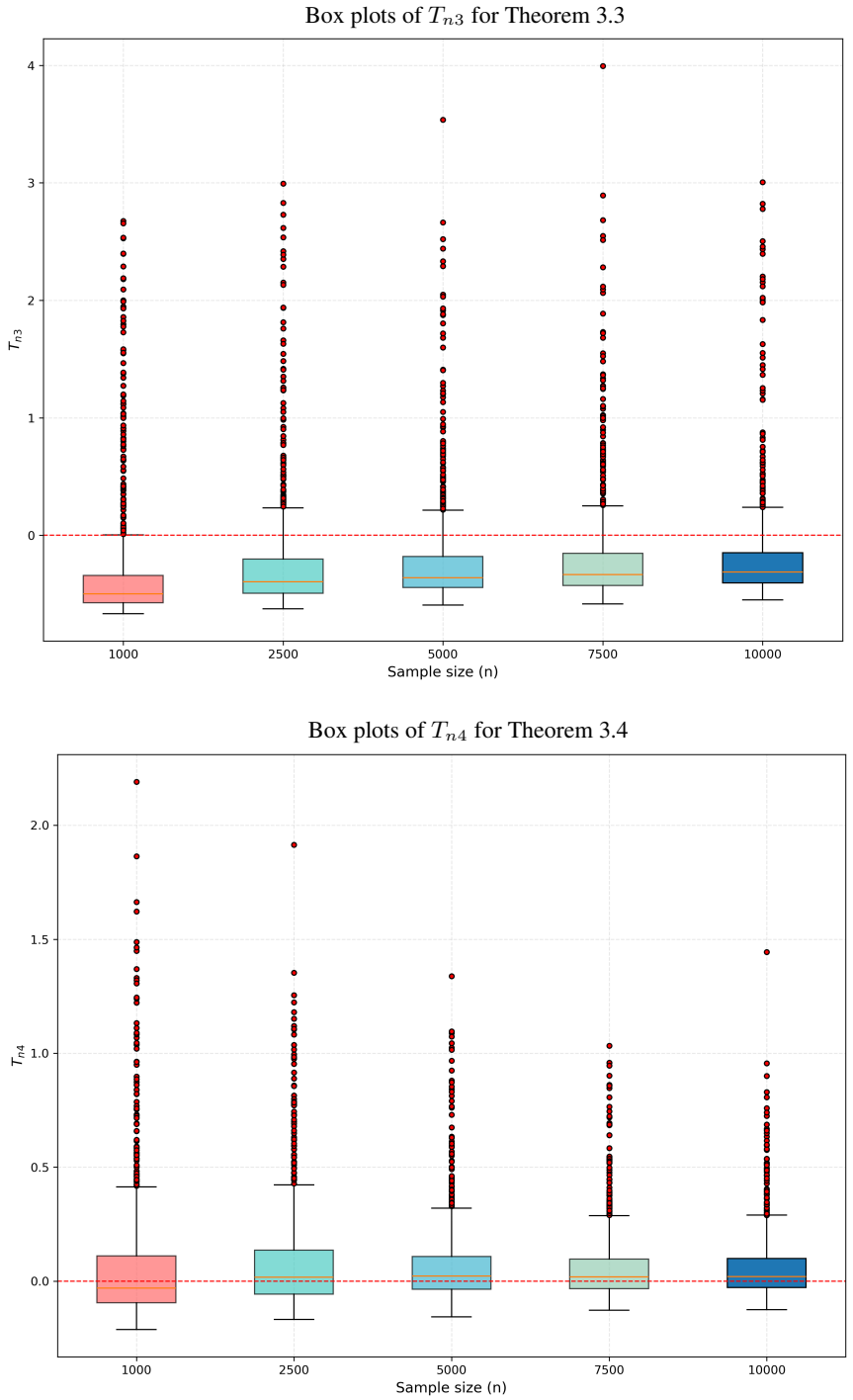


FIGURE 2. Top: the box plot of T_{n3} with $d_n = (\lg n)^{\delta}$, $\delta = 0.5$ and $\gamma = 1$ in (4.5). Bottom: the box plot of T_{n4} with $d_n = n^{\delta}$, $\delta = 0.3$ and $\alpha = 0.5$ in (4.6).

$$T_{n2} = \frac{1}{(\lg_2 n)^{\beta+1}} \sum_{k=n_0+1}^{n} \frac{(\lg_2 k)^{\beta}}{k(\lg k)^2} X_k - \frac{1}{\beta+1}, \tag{4.4}$$

$$T_{n3} = \frac{1}{(\lg_3 n)^{\gamma+1}} \sum_{k=n_0+1}^{n} \frac{(\lg_3 k)^{\gamma}}{k(\lg k)^2(\lg_2 k)} X_k - \frac{1}{\gamma+1}, \tag{4.5}$$

$$T_{n4} = \frac{1}{(\lg n)^{\alpha+2}} \sum_{k=n_0+1}^{n} \frac{(\lg k)^{\alpha}}{k} X_k - \frac{1-\delta}{\alpha+2}, \tag{4.6}$$

where $\alpha > -2$, $\beta > -1$ and $\gamma > -1$ are respectively defined by Theorems 3.1–3.3, while $0 < \delta < 1$ and $\alpha > -2$ are defined by Theorem 3.4. We take $d_n = (\lg n)^{\delta}$ for some $\delta > 0$ in Theorems 3.1–3.3, and $d_n = n^{\delta}$ for some $0 < \delta < 1$ in Theorem 3.4. Specifically, in Theorems 3.1–3.3 we set $\delta = 0.5$, while in Theorem 3.4 we set $\delta = 0.3$. Moreover, we take $\alpha = 0$ in Theorem 3.1 based on a very slowly varying function $S(n)$ with $m = 3$, $\beta_1 = 0.5$ and $\beta_2 = 0.3$ in (3.1); we take $\beta = 0.8$ in Theorem 3.2, $\gamma = 1$ in Theorem 3.3, and $\alpha = 0.5$ in Theorem 3.4. Finally, we set $\varepsilon = 0.001$, $M = 10000$, $n_0 = 100$ and $\rho = 0.5$ in $\mathbf{\Sigma}$. With the sample sizes $n = [1000, 2500, 5000, 7500, 10000]$ and 1000 replications, we obtain the box plots in Figs. 1 and 2 for Theorems 3.1–3.4.

As shown in Fig. 1, the box plots of T_{n1} and T_{n2} converge as the sample size increases. It should be noted that the median values of T_{n1} and T_{n2} are somewhat distant from zero. The reason is that $\frac{1}{(\lg n)^{\alpha+2}S(n)}$ in (4.3) and $\frac{1}{(\lg_2 n)^{\beta+1}}$ in (4.4) are very slowly varying functions of n. In fact, when n takes the values 1000, 2500, 5000, 7500, and 10000, the corresponding values of $\lg n$ are approximately 6.907755, 7.824046, 8.517193, 8.922658 and 9.210340, respectively. Similarly, the corresponding iterated logarithms $\lg_2 n$ are approximately 1.932645, 2.057202, 2.142087, 2.188594 and 2.220327, respectively. The rates of growth are extremely slow. The triple iterated logarithm $\lg_3 n$ grows even more slowly. So we cannot further increase the value of n to provide relevant simulations for finite samples. In Fig. 2, the box plots of T_{n3} and T_{n4} converge as the sample size increases, even though the medians of T_{n3} are still somewhat distant from zero.

5. CONCLUSION AND DISCUSSION

We investigate Exact Strong Laws for a φ-mixing sequence $\{X_n,\, n \geqslant 1\}$ of Pareto–Zipf distributions satisfying (1.1) and (1.2). The random variables $X_1, X_2, \ldots$ are non-identically-distributed with infinite means. We obtain Exact Strong Laws for random variables that previously only had an exact weak law. For example, we show how to correctly select the weights and norming sequences $\{a_n,\, n \geqslant 1\}$, $\{b_n,\, n \geqslant 1\}$, $\{c_n,\, n \geqslant 1\}$ and $\{d_n,\, n \geqslant 1\}$ in this setting so that this strong law does indeed hold. Adler [2] and Nakata [13] obtained Exact Weak Laws for independent Pareto–Zipf distributions. In this paper, we extend these results to the dependent case of φ-mixing sequences and establish Exact Strong Laws. Applying Algorithm 1, we generate the dependent Pareto–Zipf random variables

$X_1, \ldots, X_n$. The empirical results of Exact Strong Laws in Theorems 3.1–3.4 converge as the sample size increases, even though the medians are somewhat distant from zero. The main reason is that $\lg n$, $\lg_2 n$ and $\lg_3 n$ are very slowly varying functions. Therefore, in this paper, we cannot further increase the value of n to provide relevant finite sample simulations. On the other hand, Adler and Matuła [5] investigated the order statistics from a two-sided Pareto distribution, and obtained Exact Weak and Strong Laws for the smallest and largest order statistics. So it is of interest to study Exact Strong Laws of order statistics based on (1.1) and (1.2) in further research.

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Wenzhi Yang, Zhengyu Zhao
School of Big Data and Statistics
Anhui University
Hefei, 230601, P. R. China
E-mail: wzyang@ahu.edu.cn
we24301100@stu.ahu.edu.cn

André Adler (corresponding author)
Guangdong Provincial Key Lab of Mathematical
and Neural Dynamical Systems
Great Bay University
Dongguan, 523000, P. R. China
E-mail: swiss919@gmail.com

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