PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 45, Fasc. 2 (2025), pp. 227–245
Published online 27.2.2026
doi:10.37190/0208-4147.00270

A NEW PROBABILITY-TYPE INEQUALITY FOR PARTIAL SUMS OF WIDELY ORTHANT DEPENDENT RANDOM VARIABLES AND ITS APPLICATION TO NON-PARAMETRIC REGRESSION MODELS

BY

OMAR ELFAROUK CHERIFI (SIDI BEL ABBES), KHADIDJA ABDELHAK (NAAMA), SAMIR BENAISSA (SIDI BEL ABBES), AND BOUBAKER MECHAB (SIDI BEL ABBES)

Abstract. This paper establishes an exponential inequality for a sequence of widely orthant dependent (WOD) random variables. Leveraging this result, we prove the strong consistency of a non-parametric regression estimator under a general moment condition. We further derive the convergence rate of this estimator, highlighting its alignment with classical results under specific parameter choices. As an application, we examine the nearest neighbor estimator and support our theoretical findings with a numerical study.

2020 Mathematics Subject Classification: Primary 60E15; Secondary 60F15, 62G08.

Key words and phrases: widely orthant dependence, probability inequality, non-parametric regression, strong consistency, convergence rate.

1. INTRODUCTION

Numerous studies have investigated the convergence properties of non-parametric regression estimators, with particular emphasis on both strong and weak consistency. Following the framework established by [10], we consider the model defined, for each $n \geqslant 1$ and $1 \leqslant k \leqslant n$, by

$$Z_{n,k} = R(t_{n,k}) + e_{n,k}, \tag{1.1}$$

where R is an unknown regression function, the design points $t_{n,k}$ belong to a compact subset $S \subset \mathbb{R}^d$ for $d \geqslant 1$, and $e_{n,k}$ are random errors. The corresponding weighted regression estimator of $R(t)$, for any $t \in S$, is

$$R_n(t) = \sum_{k=1}^{n} P_{n,k}(t) Z_{n,k}, \tag{1.2}$$

where $P_{n,k}(t) = P_{n,k}(t, t_{n,1}, t_{n,2}, \ldots, t_{n,n})$ are weight functions.

© Probability and Mathematical Statistics, 2025

A major challenge in this area is the treatment of dependent random errors. We focus on a broad class of dependence structures that includes both positively and negatively dependent random variables, known as widely orthant dependence, introduced in [11]. Let $\{g_u(n),\, n \geqslant 1\}$ and $\{g_l(n),\, n \geqslant 1\}$ be sequences of positive numbers called *dominant coefficients*, and let $\{T_n,\, n \geqslant 1\}$ be a sequence of random variables. The sequence $\{T_n\}$ is said to be *WOD* if, for all $t_k \in \mathbb{R}$ with $1 \leqslant k \leqslant n$, the following two conditions hold:

$$\mathbb{P}\Big(\bigcap_{k=1}^{n} \{T_k > t_k\}\Big) \leqslant g_u(n) \prod_{k=1}^{n} \mathbb{P}(T_k > t_k),$$

and

$$\mathbb{P}\Big(\bigcap_{k=1}^{n} \{T_k \leqslant t_k\}\Big) \leqslant g_l(n) \prod_{k=1}^{n} \mathbb{P}(T_k \leqslant t_k).$$

Several recent works have addressed convergence under widely orthant dependence. For instance, [13, 18] established complete convergence theorems, while [9] studied convergence properties for the maxima of random variables. [16] derived weak laws of large numbers using moment inequalities. For complete moment convergence, see [1, 17], which extend earlier studies by [5, 8] on complete qth moment convergence. Additional contributions include [14] on L^p convergence under R-h-integrability conditions and [15] on Cesàro uniform integrability. In non-parametric regression, consistency results for wavelet and Priestley–Chao estimators were obtained in [2, 3] respectively.

In this paper, we develop fundamental probabilistic tools for analyzing data with dependent structures. First, we present several key lemmas forming the theoretical foundation of our analysis. Next, we establish an exponential inequality for WOD random variables and use it to derive convergence results for the weighted regression estimator (1.2), including the convergence rate for strong consistency. To keep the presentation concise, all the proofs are included in the appendix. Finally, we illustrate the practical utility of these theoretical findings through an application to nearest neighbor estimation, supported by numerical simulations.

2. LEMMAS

This section presents key lemmas that will be used to prove the main results in the next section.

LEMMA 2.1 ([11]). *Let $\{T_n,\, n \geqslant 1\}$ be a sequence of WOD random variables. If $\{\varphi_n,\, n \geqslant 1\}$ is a sequence of increasing (or decreasing) functions, then the sequence $\{\varphi_n(T_n),\, n \geqslant 1\}$ is also WOD.*

LEMMA 2.2 ([6]). *Let $\{T_n,\, n \geqslant 1\}$ be a sequence of WOD random variables. Then for any $r > 0$, the following inequality holds:*

$$\mathbb{E}\Big(\exp\Big\{r \sum_{k=1}^{n} T_k\Big\}\Big) \leqslant g(n) \prod_{k=1}^{n} \mathbb{E}(e^{rT_k}).$$

LEMMA 2.3 ([12]). *Let $\{T_n,\ n \geqslant 1\}$ be a sequence of WOD random variables such that $\mathbb{E}(T_k) = 0$ and $\mathbb{E}(|T_k|^\mu) < \infty$ for $1 \leqslant k \leqslant n$ and some $\mu \geqslant 1$. Then*

$$\mathbb{E}\Big(\Big|\sum_{k=1}^{n} T_k\Big|^\mu\Big) \leqslant (C_{\mu,1} + g(n)C_{\mu,2}) \sum_{k=1}^{n} \mathbb{E}(|T_k|^\mu) \quad \text{for } 1 \leqslant \mu \leqslant 2,$$

and

$$\mathbb{E}\Big(\Big|\sum_{k=1}^{n} T_k\Big|^\mu\Big) \leqslant C_{\mu,1} \sum_{k=1}^{n} \mathbb{E}(|T_k|^\mu) + g(n)C_{\mu,2}\Big(\sum_{k=1}^{n} \mathbb{E}(|T_k|^2)\Big)^{\mu/2} \quad \text{for } \mu \geqslant 2,$$

where $C_{\mu,1}$ and $C_{\mu,2}$ are positive constants.

3. PROBABILITY INEQUALITY FOR WIDELY ORTHANT DEPENDENT RANDOM VARIABLES

In this section, we present an exponential inequality that provides an upper bound, which will be instrumental in proving our main results.

THEOREM 3.1. *Let $\{T_n,\ n \geqslant 1\}$ be a sequence of WOD random variables such that $\mathbb{E}(T_k) = 0$ and $|T_k| \leqslant \Delta$ for every $1 \leqslant k \leqslant n$ and for some constant $\Delta > 0$. Then, for every $1 < \lambda + \pi \leqslant 2$, $1 < \pi < 2$, and $\epsilon > 0$, the following inequality holds:*

$$\mathbb{P}\Big(\Big|\sum_{k=1}^{n} T_k\Big| > \epsilon\Big) \leqslant 2g(n) \exp\Bigg\{-C_{\lambda,\pi}\Bigg(\frac{\epsilon^{\lambda+\pi}}{e\mathbb{E}(\sum_{k=1}^{n} |T_k|^{\lambda+\pi})}\Bigg)^{\frac{1}{\lambda+\pi-1}}\Bigg\}, \tag{3.1}$$

where

$$C_{\lambda,\pi} = (\lambda+\pi)^{-\frac{1}{\lambda+\pi-1}} - (\lambda+\pi)^{-\frac{\lambda+\pi}{\lambda+\pi-1}}.$$

The next result establishes complete convergence for partial sums of WOD random variables.

THEOREM 3.2. *Let $\{T_n,\ n \geqslant 1\}$ be a sequence of WOD random variables such that $\mathbb{E}(T_k) = 0$ and $|T_k| \leqslant \Delta$ for all $1 \leqslant k \leqslant n$ and for some constant $\Delta > 0$. Assume that $\sup_{k\geqslant 1} \mathbb{E}(|T_k|^{\lambda+\pi}) < \infty$ for some λ, π satisfying $1 < \lambda + \pi \leqslant 2$ and $1 < \pi < 2$. If for some constants $d_1, d_2 > 0$,*

$$\sum_{n=1}^{\infty} g(n) \exp\{-d_1 n^{d_2}\} < \infty, \tag{3.2}$$

then $n^{-l}\sum_{k=1}^{n} T_k$ converges completely to zero for every $l > 1/(\lambda+\pi)$.

REMARK 3.1. In Theorem 3.2, if the condition (3.2) is replaced by the stronger summability assumption

$$\sum_{n=1}^{\infty} g(n) n^{-d} < \infty, \tag{3.3}$$

while all the other hypotheses of the theorem remain satisfied, then for any $d > 1$

we obtain the almost sure convergence of $n^{-l}\sum_{k=1}^{n} T_k$, with convergence rate

$$\begin{aligned}\epsilon &= \Big((dC_{\lambda,\pi}^{-1})^{\lambda+\pi-1} e \sup_{k\geqslant 1}\mathbb{E}(|T_k|^{\lambda+\pi})\Big)^{1/(\lambda+\pi)} \\ &\quad \times n^{-l+1/(\lambda+\pi)} \log^{(\lambda+\pi-1)/(\lambda+\pi)} n \\ &= \mathcal{O}(n^{-l+1/(\lambda+\pi)} \log^{(\lambda+\pi-1)/(\lambda+\pi)} n).\end{aligned} \tag{3.4}$$

REMARK 3.2. In Remark 3.1, if $g(n) = 1$ or $g(n) = \omega$, where ω is a positive constant, which corresponds respectively to the negatively orthant dependent and extended negatively orthant dependent cases, we achieve almost sure convergence with the same convergence rate (3.4) without needing to assume condition (3.3).

4. SOME RESULTS ON CONVERGENCE OF THE NON-PARAMETRIC REGRESSION MODEL ESTIMATOR

In this section, we establish the almost sure convergence of the estimator for the non-parametric regression model introduced in (1.2), under certain conditions on the weight function and the dominant coefficients.

We denote by $\mathbb{I}$ indicator functions. The notation $\|t\|$ refers to the Euclidean norm. Throughout this section, the symbol C represents a positive constant that may vary from line to line depending on the context.

The result below establishes the strong consistency of the estimator (1.2). The following assumptions on the weight functions are required for the proof, for any $t \in S$. Let $8/3 < v \leqslant 4$, $\alpha > 0$, $1 < \hat{s} \leqslant 2$, $j \geqslant \hat{s} - 1 - (\hat{s}-1)^2/u$, and $g \geqslant 3(\hat{s}-1)/2$, where $\hat{s}/2 < u < \hat{s}$. Note that the choice of v satisfies the condition $v^{-1} \geqslant (3v-8)/(4v) > 0$.

$(\mathrm{A}_{\mathrm{sc}}^{[1]})$ $|\sum_{k=1}^{n} P_{n,k}(t) - 1| = \mathcal{O}(n^{(8-3v)/(4v)})$,

$(\mathrm{A}_{\mathrm{sc}}^{[2]})$ $\sum_{k=1}^{n} |P_{n,k}(t)| \leqslant C$ for all $n \geqslant 1$,

$(\mathrm{A}_{\mathrm{sc}}^{[3]})$ $\max_{1\leqslant k\leqslant n} |P_{n,k}(t)|^{\hat{s}-1} = \mathcal{O}(n^{-j}\log^{-g} n)$,

$(\mathrm{A}_{\mathrm{sc}}^{[4]})$ $\sum_{k=1}^{n} |P_{n,k}(t)|\,|R(t_{n,k}) - R(t)|\,\mathbb{I}(\|t_{n,k}-t\| > \alpha n^{-1/v}) = \mathcal{O}(n^{(8-3v)/(4v)})$.

THEOREM 4.1. *Let $\{e_n,\ n \geqslant 1\}$ be a sequence of WOD random variables with $\mathbb{E}(e_k) = 0$ for all $1 \leqslant k \leqslant n$ and $\sup_{n\geqslant 1}\mathbb{E}(|e_n|^{\hat{s}}) < \infty$. If the assumptions $(\mathrm{A}_{\mathrm{sc}}^{[1]})$–$(\mathrm{A}_{\mathrm{sc}}^{[4]})$ hold, and*

$$\sum_{n=1}^{\infty} g(n)\exp\{-Cn\log^{a} n\} < \infty \quad \textit{for some } a \geqslant 3/2, \tag{4.1}$$

with $R(t)$ satisfying a local Lipschitz condition around the point t, then for each

$t \in S$, we have

$$\lim_{n\to\infty} R_n(t) = R(t) \quad \textit{almost surely.} \tag{4.2}$$

THEOREM 4.2. *Consider a sequence $\{e_n,\ n \geqslant 1\}$ of WOD random variables such that $\mathbb{E}(e_k) = 0$ for every $1 \leqslant k \leqslant n$. Assume that $\sup_{n\geqslant 1} \mathbb{E}(|e_n|^{2\hat{s}}) < \infty$. If the conditions $(\mathrm{A}_{\mathrm{sc}}^{[1]})$–$(\mathrm{A}_{\mathrm{sc}}^{[4]})$ and (4.1) hold, with $R(t)$ satisfying the local Lipschitz condition around the point t, then, for each $t \in S$, we have*

$$R_n(t) - R(t) = \mathcal{O}(n^{(8-3v)/(4v)}) \quad \textit{almost surely.} \tag{4.3}$$

5. APPLICATION TO THE NEAREST NEIGHBOR ESTIMATOR

In this section, we present an application of our results to the nearest neighbor estimator of $R(t)$. Assume $t_{n,k} = k/n$ for $1 \leqslant k \leqslant n$ where $\lceil t \rceil$ stands for the smallest integer greater than or equal to t. For every $t \in S = [0, 1]$, we reorder the absolute differences $|t_{n,1} - t|, |t_{n,2} - t|, \ldots, |t_{n,n} - t|$ as follows:

$$|t_{\varphi_1(t)} - t| \leqslant |t_{\varphi_2(t)} - t| \leqslant \cdots \leqslant |t_{\varphi_n(t)} - t|.$$

Observe that if $|t_{n,k} - t| = |t_{n,\hat{k}} - t|$, then $|t_{n,k} - t|$ appears before $|t_{n,\hat{k}} - t|$ if $t_{n,k} \leqslant t_{n,\hat{k}}$. The weighted nearest neighbor estimator of $R(t)$ for model (1.1) is defined by

$$R_n^{[\mathrm{nne}]}(t) = \sum_{k=1}^{n} P_{n,k}^{[\mathrm{nne}]}(t) Z_{n,k}, \tag{5.1}$$

where $P_{n,k}^{[\mathrm{nne}]}(t) = 1/\theta_n$ if $|t_{n,k} - t| \leqslant |t_{\varphi_{\theta_n}(t)} - t|$, and 0 otherwise, for every $1 \leqslant \theta_n \leqslant n$.

By applying Theorems 4.1, and 4.2, we obtain the following result.

THEOREM 5.1. *Let $\{e_n,\ n \geqslant 1\}$ be a sequence of WOD random variables with $\mathbb{E}(e_k) = 0$ for all $1 \leqslant k \leqslant n$ and let $\theta_n = \lceil n^{1/2} \rceil$. Assume that condition (4.1) holds and $R(t)$ satisfies a local Lipschitz condition around t. Then, for each $t \in S$, if $\sup_{n\geqslant 1} \mathbb{E}(|e_n|^{\hat{s}}) < \infty$, then*

$$\lim_{n\to\infty} R_n^{[\mathrm{nne}]}(t) = R(t) \quad \textit{almost surely.}$$

Moreover, if $\sup_{n\geqslant 1} \mathbb{E}(|e_n|^{2\hat{s}}) < \infty$, then

$$R_n^{[\mathrm{nne}]}(t) - R(t) = \mathcal{O}(n^{(8-3v)/(4v)}) \quad \textit{almost surely.}$$

5.1. Numerical study. To support the theoretical results, we carried out a simulation study of the nearest neighbor estimator $R_n^{[\mathrm{nne}]}(t)$ for sample sizes $n = 100, 350, 700, 1000$. A collection of regression functions $R(t)$ was examined and

four representative cases were chosen to reflect different patterns of shape and smoothness.

In the simulations, the error terms $\{e_{n,k}\}$ were generated from a multivariate normal distribution with a banded covariance matrix

$$\Sigma = (\sigma_{ij})_{1\leqslant i,j\leqslant n}, \quad \sigma_{ij} = \begin{cases} 1+\theta^2, & i=j, \\ -\theta, & |i-j|=1, \\ 0, & \text{otherwise}, \end{cases}$$

where we set $\theta = 0.7$. For each setting, 200 Monte Carlo replications were performed. The estimator was evaluated at grid points, and the replication yielding the smallest mean squared error (MSE) was selected for visualization. The resulting average MSE values for the regression functions and sample sizes are summarized in Table 1.

TABLE 1. MSE for different functions with varying sample sizes

$R(t)$	t	$n=100$	$n=350$	$n=700$	$n=1000$
			MSE		
$t^4-t^2+0.2$	0.25	$6.27\cdot10^{-4}$	$8.79\cdot10^{-3}$	$2.67\cdot10^{-2}$	$7.38\cdot10^{-3}$
	0.5	$8.75\cdot10^{-4}$	$9.29\cdot10^{-7}$	$1.93\cdot10^{-4}$	$3.58\cdot10^{-3}$
	0.75	$2.65\cdot10^{-2}$	$1.49\cdot10^{-2}$	$4.48\cdot10^{-3}$	$5.22\cdot10^{-3}$
	1	$4.39\cdot10^{-2}$	$3.83\cdot10^{-3}$	$3.07\cdot10^{-3}$	$1.67\cdot10^{-2}$
$\exp\left(-\frac{(t-0.5)^2}{0.1}\right)$	0.25	$3.21\cdot10^{-2}$	$4.78\cdot10^{-5}$	$1.03\cdot10^{-2}$	$6.23\cdot10^{-3}$
	0.5	$8.75\cdot10^{-3}$	$2.31\cdot10^{-2}$	$7.69\cdot10^{-3}$	$1.46\cdot10^{-3}$
	0.75	$6.65\cdot10^{-2}$	$4.63\cdot10^{-3}$	$3.28\cdot10^{-3}$	$1.34\cdot10^{-3}$
	1	$9.13\cdot10^{-2}$	$1.17\cdot10^{-2}$	$1.18\cdot10^{-3}$	$1.71\cdot10^{-4}$
$\cos(2\pi t)$	0.25	$8.26\cdot10^{-2}$	$1.27\cdot10^{-3}$	$3.38\cdot10^{-3}$	$1.74\cdot10^{-3}$
	0.5	$6.42\cdot10^{-3}$	$2.37\cdot10^{-2}$	$1.09\cdot10^{-2}$	$3.90\cdot10^{-3}$
	0.75	$4.24\cdot10^{-2}$	$6.66\cdot10^{-4}$	$3.14\cdot10^{-3}$	$8.00\cdot10^{-4}$
	1	$4.63\cdot10^{-3}$	$2.62\cdot10^{-4}$	$9.14\cdot10^{-4}$	$2.31\cdot10^{-3}$
$t(1-t)\sin(6\pi t)$	0.25	$4.00\cdot10^{-2}$	$1.17\cdot10^{-4}$	$1.24\cdot10^{-2}$	$2.81\cdot10^{-3}$
	0.5	$2.76\cdot10^{-2}$	$2.86\cdot10^{-3}$	$4.19\cdot10^{-4}$	$1.32\cdot10^{-3}$
	0.75	$1.57\cdot10^{-2}$	$3.08\cdot10^{-3}$	$8.38\cdot10^{-3}$	$6.33\cdot10^{-3}$
	1	$5.55\cdot10^{-3}$	$2.84\cdot10^{-2}$	$1.49\cdot10^{-3}$	$2.68\cdot10^{-5}$

As shown in Table 1, the estimator exhibits decreasing MSE as the sample size increases in most cases, confirming the theoretical consistency results.

Figures 1–4 display the estimated curves together with the true regression functions for the four cases, showing how the estimator approaches the true function as n increases, assuming WOD errors. We note that occasional non-monotone behavior of the MSE for finite n, appearing in a minority of the replications, is due to Monte Carlo variability and finite-sample effects under dependence, and does not contradict the asymptotic consistency of the estimator.

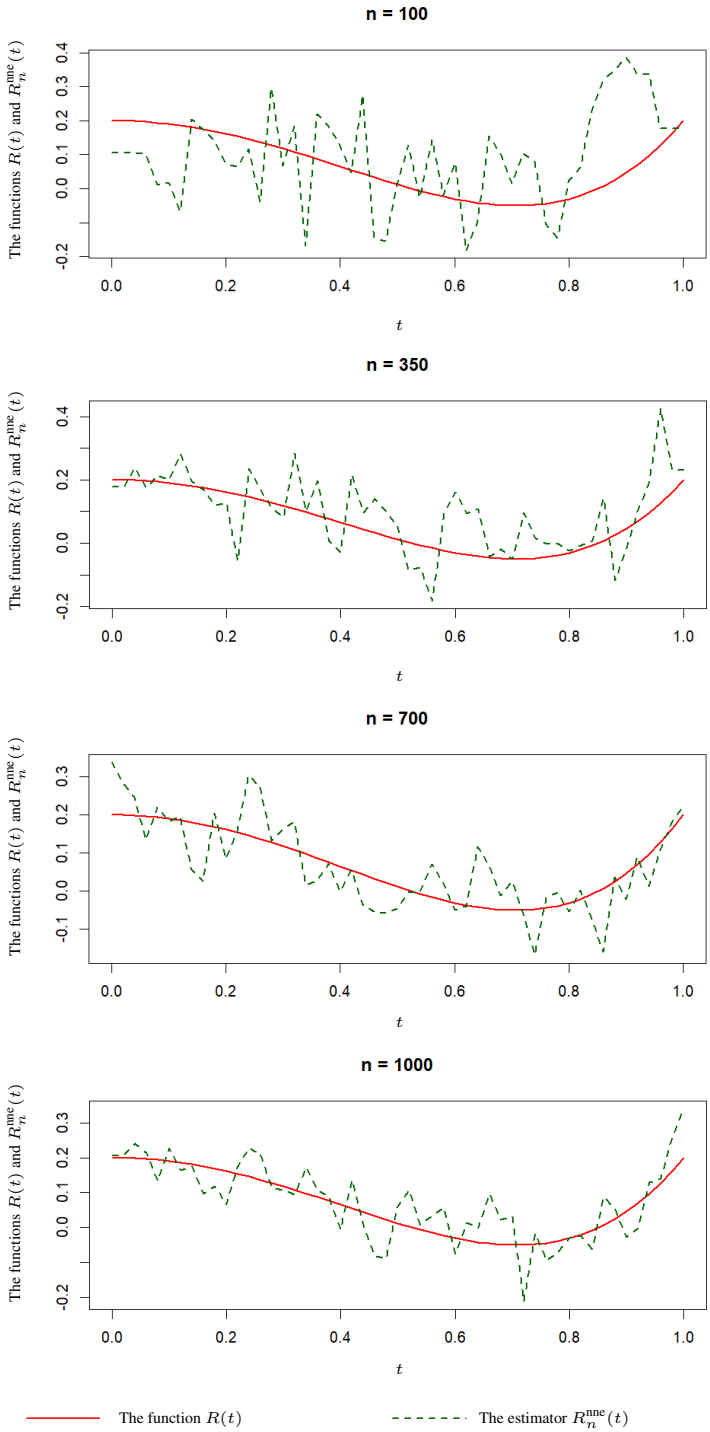


FIGURE 1. Comparison of estimated and true regression functions $(R(t) = t^4 - t^2 + 0.2)$ for sample sizes $n = 100, 350, 700, 1000$

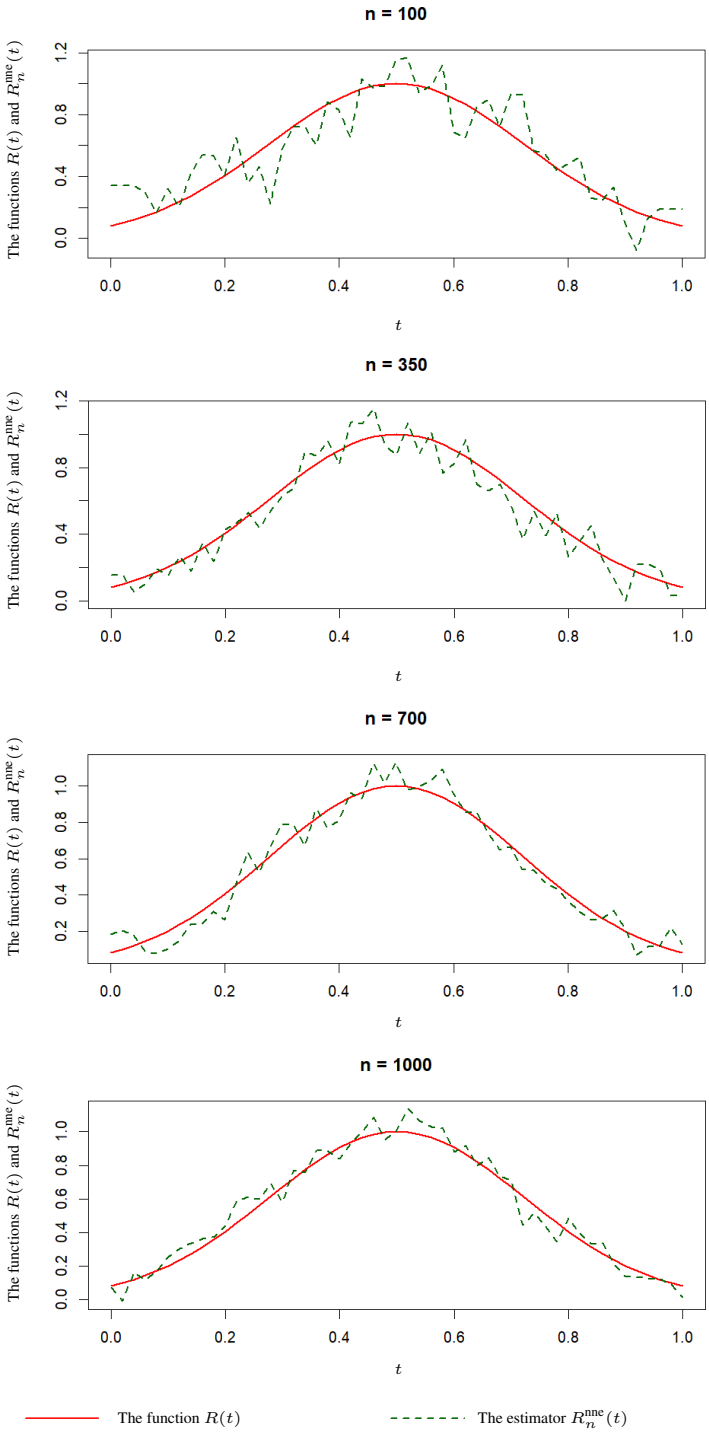


FIGURE 2. Comparison of estimated and true regression functions $(R(t) = \exp(-\frac{(t-0.5)^2}{0.1}))$ for sample sizes $n = 100, 350, 700, 1000$

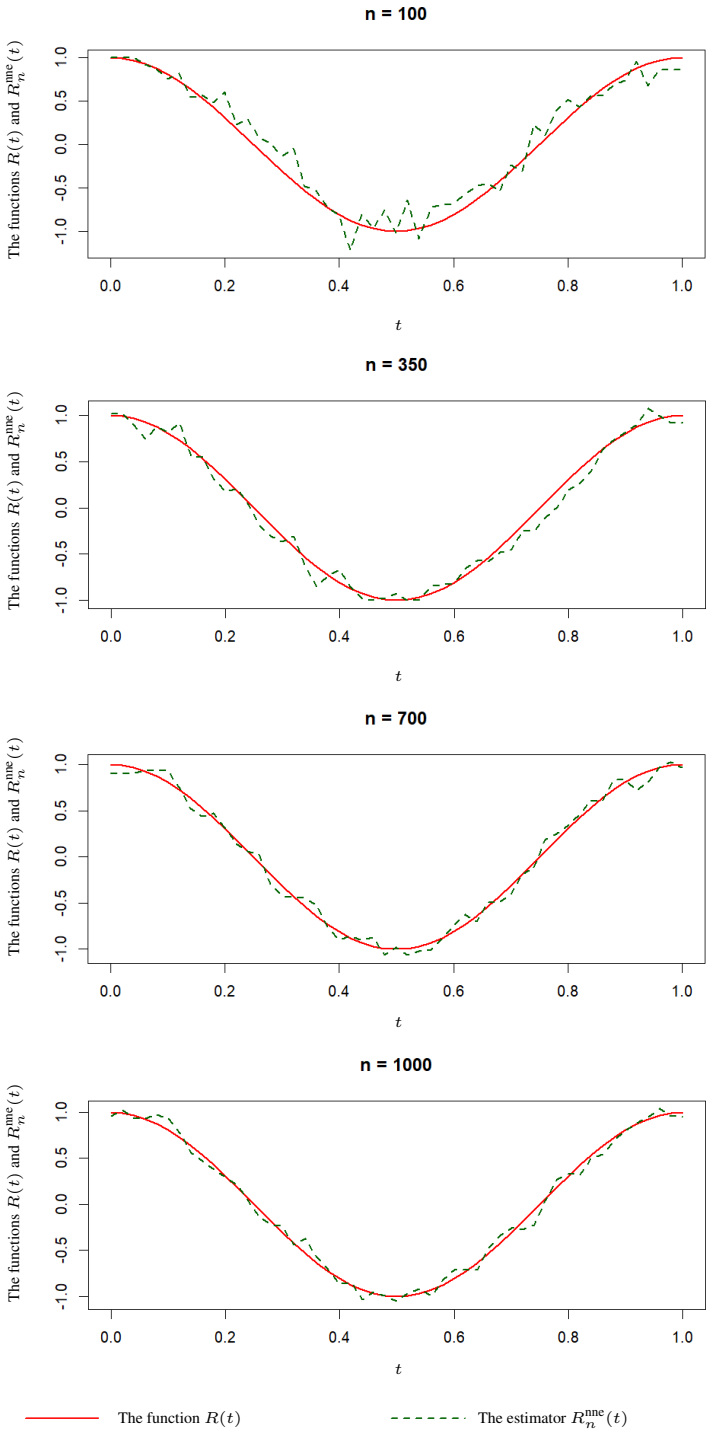


FIGURE 3. Comparison of estimated and true regression functions $(R(t) = \cos(2\pi t))$ for sample sizes $n = 100, 350, 700, 1000$

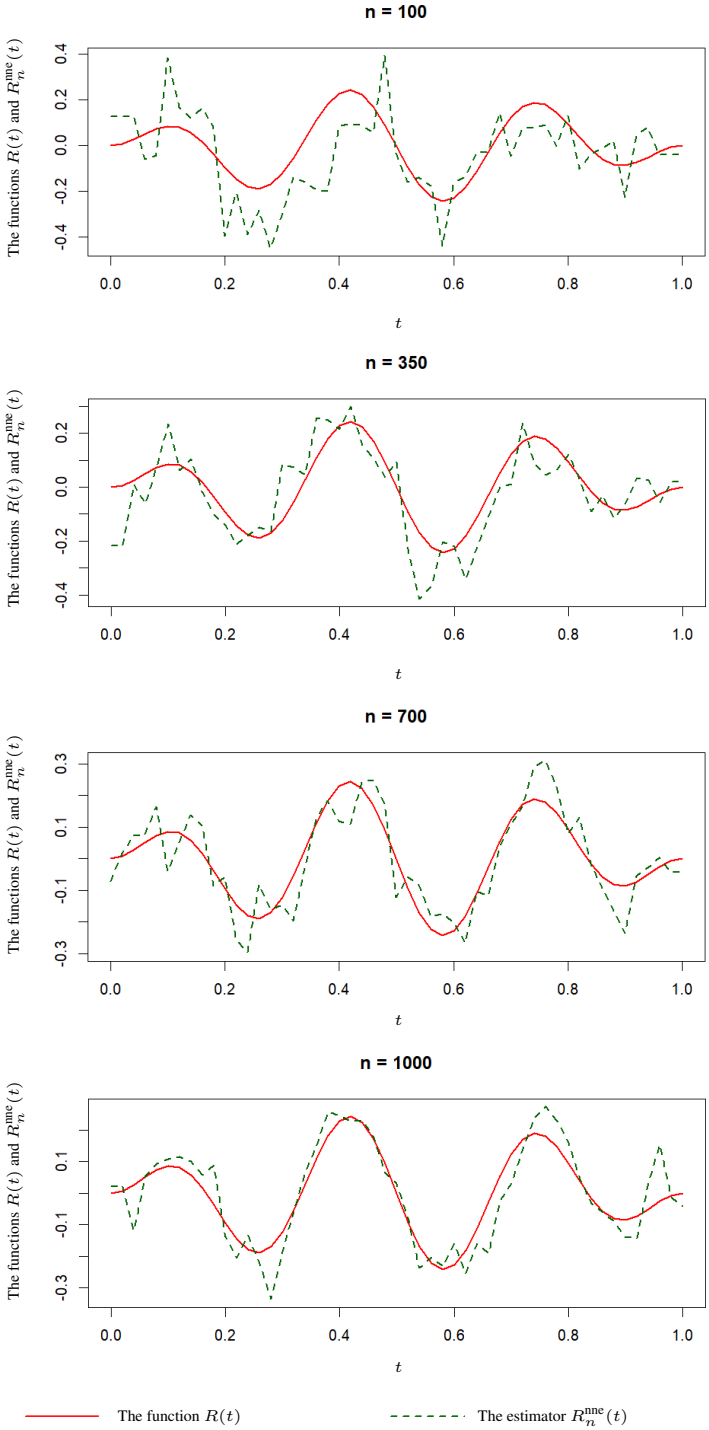


FIGURE 4. Comparison of estimated and true regression functions $(R(t) = t(1-t)\sin(6\pi t))$ for sample sizes $n = 100, 350, 700, 1000$

6. CONCLUSION

This paper extends classical results on non-parametric regression under WOD errors by establishing a new exponential inequality and proving strong consistency under the weaker moment condition $\sup_{n\geqslant 1}\mathbb{E}(|e_n|^{\hat{s}})<\infty$ for $1<\hat{s}\leqslant 2$ (see [7] for the case $\hat{s}=2$). We also derive an $\mathcal{O}(n^{(8-3v)/(4v)})$ convergence rate for $8/3<v\leqslant 4$, which coincides with the classical rate when $v=4$.

The assumption $(\mathrm{A}_{\mathrm{sc}}^{[3]})$ allows for broader applicability through flexible parameters (j,g), adapting to a wider range of moment conditions. In the nearest neighbor estimator, we adopt $\theta_n=n^{1/2}$, consistent with our theoretical framework. This example demonstrates how our general results can be applied to concrete estimators, confirming their validity in practical settings. A numerical study is also provided to illustrate the practical relevance of our results.

7. APPENDIX

7.1. Proof of Theorem 3.1. Since $|T_k|\leqslant\Delta$ for some constant $\Delta>0$, we apply the inequality

$$1+r\leqslant e^r\leqslant 1+r+|r|^{\lambda+\pi}e^{2|r|}\qquad\text{for all } r\in\mathbb{R},$$

which holds for $0<\lambda\leqslant 2-\pi$ and $1<\pi<2$. Then for all $\theta>0$,

$$\begin{aligned}\mathbb{E}(e^{\theta T_k})&\leqslant 1+\mathbb{E}(|\theta T_k|^{\lambda+\pi}e^{2|\theta T_k|})\leqslant 1+\theta^{\lambda+\pi}e^{2\theta\Delta}\mathbb{E}(|T_k|^{\lambda+\pi})\\&\leqslant\exp\{\theta^{\lambda+\pi}e^{2\theta\Delta}\mathbb{E}(|T_k|^{\lambda+\pi})\}.\end{aligned}\tag{7.1}$$

Using Markov's inequality, Lemma 2.2, and inequality (7.1), we obtain

$$\begin{aligned}\mathbb{P}\Big(\sum_{k=1}^n T_k>\epsilon\Big)&\leqslant\mathbb{E}(e^{\theta\sum_{k=1}^n T_k})e^{-\theta\epsilon}\\&\leqslant g(n)\prod_{k=1}^n\mathbb{E}(e^{\theta T_k})e^{-\theta\epsilon}\\&\leqslant g(n)\exp\Big\{\theta^{\lambda+\pi}e^{2\theta\Delta}\sum_{k=1}^n\mathbb{E}(|T_k|^{\lambda+\pi})-\theta\epsilon\Big\}.\end{aligned}\tag{7.2}$$

Now, assume $2\theta\Delta\leqslant 1$ and define

$$\Theta(\theta)=-\theta\epsilon+\theta^{\lambda+\pi}e\mathbb{E}\Big(\sum_{k=1}^n|T_k|^{\lambda+\pi}\Big).$$

Minimizing $\Theta(\theta)$ yields the optimal value

$$\theta=\left(\frac{\epsilon}{(\lambda+\pi)e\mathbb{E}(\sum_{k=1}^n|T_k|^{\lambda+\pi})}\right)^{\frac{1}{\lambda+\pi-1}}.$$

Substituting this into (7.2), we get

$$\mathbb{P}\Big(\sum_{k=1}^{n} T_k > \epsilon\Big) \leqslant g(n)\exp\Bigg\{-C_{\lambda,\pi}\bigg(\frac{\epsilon^{\lambda+\pi}}{e\mathbb{E}(\sum_{k=1}^{n}|T_k|^{\lambda+\pi})}\bigg)^{\frac{1}{\lambda+\pi-1}}\Bigg\}. \tag{7.3}$$

Noting that

$$\mathbb{P}\Big(\Big|\sum_{k=1}^{n} T_k\Big| > \epsilon\Big) = \mathbb{P}\Big(\sum_{k=1}^{n} T_k > \epsilon\Big) + \mathbb{P}\Big(-\sum_{k=1}^{n} T_k > \epsilon\Big),$$

and that, from Lemma 2.1, the sequence $\{-T_k\}$ is also WOD, we similarly obtain

$$\mathbb{P}\Big(\sum_{k=1}^{n} -T_k > \epsilon\Big) \leqslant g(n)\exp\Bigg\{-C_{\lambda,\pi}\bigg(\frac{\epsilon^{\lambda+\pi}}{e\mathbb{E}(\sum_{k=1}^{n}|T_k|^{\lambda+\pi})}\bigg)^{\frac{1}{\lambda+\pi-1}}\Bigg\}. \tag{7.4}$$

The result follows directly from inequalities (7.3) and (7.4).

7.2. Proof of Theorem 3.2. From Theorem 3.1, for any $\epsilon > 0$,

$$\begin{aligned}\mathbb{P}\Big(\Big|\sum_{k=1}^{n} T_k\Big| > n^l\epsilon\Big) &\leqslant 2g(n)\exp\Bigg\{-C_{\lambda,\pi}\bigg(\frac{(n^l\epsilon)^{\lambda+\pi}}{e\mathbb{E}(\sum_{k=1}^{n}|T_k|^{\lambda+\pi})}\bigg)^{\frac{1}{\lambda+\pi-1}}\Bigg\}\\ &\leqslant 2g(n)\exp\{-d_1 n^{d_2}\},\end{aligned} \tag{7.5}$$

where

$$d_1 = \frac{C_{\lambda,\pi}\epsilon^{\frac{\lambda+\pi}{\lambda+\pi-1}}}{(e\sup_{k\geqslant 1}\mathbb{E}(|T_k|^{\lambda+\pi}))^{\frac{1}{\lambda+\pi-1}}} \quad \text{and} \quad d_2 = \frac{l(\lambda+\pi)-1}{\lambda+\pi-1}.$$

Under condition (3.2), the Borel–Cantelli lemma implies that $n^{-l}\sum_{k=1}^{n} T_k$ converges almost surely to 0. This completes the proof.

7.3. Proof of Remark 3.1. If we substitute ϵ from (3.4) into (7.5), then under the assumption (3.3), we obtain complete convergence. A straightforward application of the Borel–Cantelli lemma yields almost sure convergence with convergence rate $\mathcal{O}(n^{-l+1/(\lambda+\pi)}\log^{(\lambda+\pi-1)/(\lambda+\pi)} n)$.

7.4. Proof of Theorem 4.1. Fix $t \in S$. For clarity, we decompose

$$\begin{aligned}|R_n(t) - R(t)| &= |\mathbb{E}(R_n(t)) - R(t) + R_n(t) - \mathbb{E}(R_n(t))|\\ &\leqslant |\mathbb{E}(R_n(t)) - R(t)| + |R_n(t) - \mathbb{E}(R_n(t))| = L_{n,t}^{[1]} + L_{n,t}^{[2]}.\end{aligned}$$

From (1.1) and (1.2), we can write

$$\begin{aligned}L_{n,t}^{[1]} \leqslant{}& \sum_{k=1}^{n} |P_{n,k}(t)|\,|R(t_{n,k}) - R(t)|\,\mathbb{I}(\|t_{n,k} - t\| \leqslant \alpha n^{-1/v})\\ &+ \sum_{k=1}^{n} |P_{n,k}(t)|\,|R(t_{n,k}) - R(t)|\,\mathbb{I}(\|t_{n,k} - t\| > \alpha n^{-1/v})\\ &+ |R(t)|\,\Big|\sum_{k=1}^{n} P_{n,k}(t) - 1\Big|.\end{aligned} \tag{7.6}$$

Under assumptions $(\mathrm{A}_{\mathrm{sc}}^{[1]})$ and $(\mathrm{A}_{\mathrm{sc}}^{[4]})$, we conclude

$$L_{n,t}^{[1]} = \mathcal{O}(n^{(8-3v)/(4v)}). \tag{7.7}$$

The proof follows the same lines as in [4], so it is omitted for brevity. To prove (4.2), it remains to show

$$\lim_{n\to\infty} L_{n,t}^{[2]} = 0 \quad \text{almost surely.} \tag{7.8}$$

Let $e_{n,k} = e_{n,k}^{[1]} + e_{n,k}^{[2]} + e_{n,k}^{[3]}$ and $e_k = e_k^{[1]} + e_k^{[2]} + e_k^{[3]}$ for each $1 \leqslant k \leqslant n$ with $n \geqslant 1$, where

$$\begin{aligned}
e_{n,k}^{[1]} &= k^{1/u}\mathbb{I}(e_{n,k} > k^{1/u}) + e_{n,k}\mathbb{I}(|e_{n,k}| \leqslant k^{1/u}) - k^{1/u}\mathbb{I}(e_{n,k} < -k^{1/u}),\\
e_{n,k}^{[2]} &= (e_{n,k} - k^{1/u})\mathbb{I}(e_{n,k} > k^{1/u}),\\
e_{n,k}^{[3]} &= (e_{n,k} + k^{1/u})\mathbb{I}(e_{n,k} < -k^{1/u}),
\end{aligned}$$

and similarly define $e_k^{[i]}$.

For each fixed k, we assume that $e_{n,k}$ and e_k have the same marginal distribution, which in particular implies that $\mathbb{E}(e_{n,k}) = \mathbb{E}(e_k) = 0$. Then, for the weighted sum we have

$$\begin{aligned}
L_{n,t}^{[2]} &= \sum_{k=1}^{n} P_{n,k}(t)e_{n,k}\\
&= \sum_{k=1}^{n} P_{n,k}(t)[e_{n,k}^{[1]} - \mathbb{E}(e_{n,k}^{[1]}) + e_{n,k}^{[2]} - \mathbb{E}(e_{n,k}^{[2]}) + e_{n,k}^{[3]} - \mathbb{E}(e_{n,k}^{[3]})]\\
&= \sum_{k=1}^{n} (M_{n,k}^{[1]} + M_{n,k}^{[2]} + M_{n,k}^{[3]}),
\end{aligned}$$

where we define $M_{n,k}^{[i]} = P_{n,k}(t)(e_{n,k}^{[i]} - \mathbb{E}(e_{n,k}^{[i]}))$. Under conditions $(\mathrm{A}_{\mathrm{sc}}^{[1]})$–$(\mathrm{A}_{\mathrm{sc}}^{[3]})$, with Jensen's inequality and the inequality $(x+y)^{\hat{s}} \leqslant 2^{\hat{s}-1}(x^{\hat{s}} + y^{\hat{s}})$ for $x, y \in \mathbb{R}$, we have

$$\begin{aligned}
\sum_{k=1}^{n} \mathbb{E}(|M_{n,k}^{[1]}|^{\hat{s}}) &= \sum_{k=1}^{n} \mathbb{E}(|P_{n,k}(t)[e_{n,k}^{[1]} - \mathbb{E}(e_{n,k}^{[1]})]|^{\hat{s}})\\
&\leqslant 2^{\hat{s}} \sum_{k=1}^{n} |P_{n,k}(t)|^{\hat{s}}\mathbb{E}(|e_{n,k}^{[1]}|^{\hat{s}})\\
&\leqslant C \sum_{k=1}^{n} |P_{n,k}(t)|^{\hat{s}}\mathbb{E}(|e_k|^{\hat{s}})\\
&\leqslant C \max_{1\leqslant k\leqslant n} |P_{n,k}(t)|^{\hat{s}-1} \sum_{k=1}^{n} |P_{n,k}(t)|\\
&\leqslant Cn^{-j}\log^{-g}(n).
\end{aligned} \tag{7.9}$$

Applying the inequality in Theorem 3.1 and the upper bound (7.9), we find that for every $\epsilon > 0$ and $8/3 < v \leqslant 4$,

$$\mathbb{P}\Big(\Big|\sum_{k=1}^{n} M_{n,k}^{[1]}\Big| > \epsilon\Big) \leqslant 2g(n)\exp\Big\{-C_{\hat{s}}\Big(\frac{\epsilon^{\hat{s}}}{en^{-j}\log^{-g} n}\Big)^{1/(\hat{s}-1)}\Big\}, \tag{7.10}$$

where $C_{\hat{s}} = \hat{s}^{-1/(\hat{s}-1)} - \hat{s}^{-\hat{s}/(\hat{s}-1)}$. Assume that $\epsilon = n^{(-j+1)/\hat{s}}$ and substitute it into (7.10). So by applying the definition of complete convergence and condition (4.1), we get

$$\sum_{n=1}^{\infty} \mathbb{P}\Big(\Big|\sum_{k=1}^{n} M_{n,k}^{[1]}\Big| > \epsilon\Big) \leqslant \sum_{n=1}^{\infty} 2g(n)\exp\{-Cn\log^{g/(\hat{s}-1)} n\} < \infty. \tag{7.11}$$

By applying the Borel–Cantelli lemma to (7.11), we obtain

$$\lim_{n\to\infty} \sum_{k=1}^{n} M_{n,k}^{[1]} = 0 \quad \text{almost surely.} \tag{7.12}$$

From the condition $\sup_{n\geqslant 1} \mathbb{E}(|e_n|^{\hat{s}}) < \infty$, for every $1 < \hat{s} \leqslant 2$, we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\mathbb{E}(e_{n,k}^{[2]})}{k^{j/(\hat{s}-1)}\log^{g/(\hat{s}-1)-(3v-8)/(4v)}(2k)} &= \sum_{k=1}^{\infty} \frac{\mathbb{E}(e_k^{[2]})}{k^{j/(\hat{s}-1)}\log^{g/(\hat{s}-1)-(3v-8)/(4v)}(2k)} \\
&\leqslant \sum_{k=1}^{\infty} \frac{\mathbb{E}[|e_k|\mathbb{I}(|e_k| > k^{1/u})]}{k^{j/(\hat{s}-1)}\log^{g/(\hat{s}-1)-(3v-8)/(4v)}(2k)} \\
&\leqslant \sum_{k=1}^{\infty} \frac{k^{-(\hat{s}-1)/u}\mathbb{E}(|e_k|^{\hat{s}})}{k^{j/(\hat{s}-1)}\log^{g/(\hat{s}-1)-(3v-8)/(4v)}(2k)} \\
&= \sum_{k=1}^{\infty} \frac{\mathbb{E}(|e_k|^{\hat{s}})}{k^{j/(\hat{s}-1)+(\hat{s}-1)/u}\log^{g/(\hat{s}-1)-(3v-8)/(4v)}(2k)}.
\end{aligned} \tag{7.13}$$

Set $j = \hat{s} - 1$ in (7.13). Then

$$\sum_{k=1}^{\infty} \frac{\mathbb{E}(|e_k|^{\hat{s}})}{k^{1+(\hat{s}-1)/u}\log^{g/(\hat{s}-1)-(3v-8)/(4v)}(2k)} < \infty,$$

which implies

$$\sum_{k=1}^{\infty} \frac{e_{n,k}^{[2]}}{k\log^{g/(\hat{s}-1)-(3v-8)/(4v)}(2k)} < \infty \quad \text{almost surely.} \tag{7.14}$$

Applying Kronecker's lemma to (7.14), we find

$$n^{-1}\log^{(3v-8)/(4v)-g/(\hat{s}-1)}(2n)\sum_{k=1}^{n} e_{n,k}^{[2]} \to 0 \quad \text{almost surely.}$$

By $(\mathrm{A}_{\mathrm{sc}}^{[2]})$–$(\mathrm{A}_{\mathrm{sc}}^{[3]})$, it is easy to see that

$$\begin{aligned}(7.15)\qquad \Big|\sum_{k=1}^{n} P_{n,k}(t)e_{n,k}^{[2]}\Big| &\leqslant \max_{1\leqslant k\leqslant n}|P_{n,k}(t)|\sum_{k=1}^{n} e_{n,k}^{[2]}\\ &\leqslant Cn^{-1}(\log^{-g/(\hat{s}-1)} n)\sum_{k=1}^{n} e_{n,k}^{[2]}\\ &= o(\log^{-(3v-8)/(4v)} n) \quad \text{almost surely.}\end{aligned}$$

We also deduce under the same conditions, and since $n > \log(2n)$, that

$$\begin{aligned}(7.16)\qquad \Big|\sum_{k=1}^{n} P_{n,k}(t)\,\mathbb{E}(e_{n,k}^{[2]})\Big| &= \Big|\sum_{k=1}^{n} P_{n,k}(t)\,\mathbb{E}(e_{k}^{[2]})\Big|\\ &\leqslant \max_{1\leqslant k\leqslant n}|P_{n,k}(t)|\sum_{k=1}^{n}\mathbb{E}[|e_k|\,\mathbb{I}(|e_k|>k^{1/u})]\\ &\leqslant Cn^{-1}\log^{-g/(\hat{s}-1)}(2n)\\ &\quad\times\sum_{k=1}^{n} k^{-(\hat{s}-1)/u}\mathbb{E}[|e_k|^{\hat{s}}\,\mathbb{I}(|e_k|>k^{1/u})]\\ &= \mathcal{O}(\log^{-g/(\hat{s}-1)-(\hat{s}-1)/u}(2n)).\end{aligned}$$

From (7.15), (7.16) and since $g/(\hat{s}-1)+(\hat{s}-1)/u-(3v-8)/(4v)>1$, we have

$$(7.17)\qquad \Big|\sum_{k=1}^{n} M_{n,k}^{[2]}\Big| = o(\log^{-(3v-8)/(4v)}(2n)) \quad \text{almost surely.}$$

Since

$$\begin{aligned}(7.18)\qquad \mathbb{E}(|e_{n,k}^{[3]}|) = \mathbb{E}(|e_k^{[3]}|) &= \mathbb{E}[(-e_k-k^{1/u})\,\mathbb{I}(e_k<-k^{1/u})]\\ &= -\mathbb{E}[e_k\,\mathbb{I}(e_k<-k^{1/u})]-k^{1/u}\,\mathbb{P}(e_k<-k^{1/u})\\ &\leqslant -\mathbb{E}[e_k\,\mathbb{I}(e_k<-k^{1/u})]\\ &\leqslant \mathbb{E}[|e_k|\,\mathbb{I}(|e_k|>k^{1/u})],\end{aligned}$$

the proof for $M_{n,k}^{[3]}$ follows the same steps as that of $M_{n,k}^{[2]}$, where we replace $\mathbb{E}(|e_{n,k}^{[2]}|)$ by $\mathbb{E}(|e_{n,k}^{[3]}|)$ using $(7.18)_3$ in the proof of (7.14), and using $(7.18)_4$ in the proof of (7.16), which yields

$$(7.19)\qquad \Big|\sum_{k=1}^{n} M_{n,k}^{[3]}\Big| = o(\log^{-(3v-8)/(4v)}(2n)) \quad \text{almost surely.}$$

Hence, we conclude that (7.8) has been established based on the results of (7.12), (7.17) and (7.19). This completes the proof.

7.5. Proof of Theorem 4.2. Using the same notation as in the proof of Theorem 4.1, where $\hat{s} < u \leqslant 2\hat{s}$, suppose that $j = (3v-8)\hat{s}/(4v) + \hat{s} - 1$ and $g = 3(\hat{s}-1)/2$. According to (7.7), we only need to prove that

$$L_{n,t}^{[2]} = \mathcal{O}(n^{(8-3v)/(4v)}) \quad \text{almost surely.} \tag{7.20}$$

Applying the inequality from Theorem 3.1 and using the upper bound (7.9), since $\sup_{n\geqslant 1} \mathbb{E}(|e_n|^{2\hat{s}}) < \infty$, we have, for every $\epsilon > 0$ and $8/3 < v \leqslant 4$,

$$\begin{aligned} \mathbb{P}\Big(\Big|\sum_{k=1}^{n} M_{n,k}^{[1]}\Big| > n^{(8-3v)/(4v)}\epsilon\Big) &\leqslant 2g(n)\exp\left\{-C_{\hat{s}}\left(\frac{(\epsilon n^{(8-3v)/(4v)})^{\hat{s}}}{en^{-j}\log^{-g}(2n)}\right)^{1/(\hat{s}-1)}\right\} \\ &\leqslant 2g(n)\exp\{-Cn\log^{g/(\hat{s}-1)}(2n)\}, \end{aligned} \tag{7.21}$$

where $C_{\hat{s}} = \hat{s}^{-1/(\hat{s}-1)} - \hat{s}^{-\hat{s}/(\hat{s}-1)}$. Since $g = 3(\hat{s}-1)/2$ and using the complete convergence condition (4.1) we obtain, by the Borel–Cantelli lemma,

$$\sum_{k=1}^{n} M_{n,k}^{[1]} = \mathcal{O}(n^{(8-3v)/(4v)}) \quad \text{almost surely.} \tag{7.22}$$

To prove the convergence of $\sum_{k=1}^{n} M_{n,k}^{[2]}$, note that since $\sup_{n\geqslant 1} \mathbb{E}(|e_n|^{2\hat{s}}) < \infty$, we can write

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\mathbb{E}(e_{n,k}^{[2]})}{k^{(3v-8)/(4v(\hat{s}-1))+1}\log^{3/2}(2k)} &= \sum_{k=1}^{\infty} \frac{\mathbb{E}(e_k^{[2]})}{k^{(3v-8)/(4v(\hat{s}-1))+1}\log^{3/2}(2k)} \\ &\leqslant \sum_{k=1}^{\infty} \frac{\mathbb{E}[|e_k|\,\mathbb{I}(|e_k| > k^{1/u})]}{k^{(3v-8)/(4v(\hat{s}-1))+1}\log^{3/2}(2k)} \\ &\leqslant \sum_{k=1}^{\infty} \frac{k^{-(2\hat{s}-1)/u}\mathbb{E}(|e_k|^{2\hat{s}})}{k^{(3v-8)/(4v(\hat{s}-1))+1}\log^{3/2}(2k)} \\ &= \sum_{k=1}^{\infty} \frac{\mathbb{E}(|e_k|^{2\hat{s}})}{k^{(3v-8)/(4v(\hat{s}-1))+(2\hat{s}-1)/u+1}\log^{3/2}(2k)} < \infty. \end{aligned} \tag{7.23}$$

This gives

$$\sum_{k=1}^{\infty} \frac{e_{n,k}^{[2]}}{k^{(3v-8)/(4v(\hat{s}-1))+1}\log^{3/2}(2k)} < \infty \quad \text{almost surely.} \tag{7.24}$$

Applying Kronecker's lemma to (7.24), we infer that

$$n^{(8-3v)/(4v(\hat{s}-1))-1}\log^{-3/2}(2n)\sum_{k=1}^{n}e_{n,k}^{[2]}\to 0 \quad \text{almost surely.}$$

By $(\mathrm{A}_{\mathrm{sc}}^{[2]})$–$(\mathrm{A}_{\mathrm{sc}}^{[3]})$, it is easy to see that

$$\begin{aligned}(7.25)\qquad \Big|\sum_{k=1}^{n}P_{n,k}(t)e_{n,k}^{[2]}\Big| &\leqslant \max_{1\leqslant k\leqslant n}|P_{n,k}(t)|\sum_{k=1}^{n}e_{n,k}^{[2]}\\ &\leqslant Cn^{((8-3v)\hat{s})/(4v(\hat{s}-1))-1}\log^{-3/2}(2n)\sum_{k=1}^{n}e_{n,k}^{[2]}\\ &= Cn^{((8-3v)/(4v))(\hat{s}/(\hat{s}-1)-1)-1+(8-3v)/(4v)}\\ &\quad\times\log^{-3/2}(2n)\sum_{k=1}^{n}e_{n,k}^{[2]}\\ &= o(n^{(8-3v)/(4v)}) \quad \text{almost surely.}\end{aligned}$$

Again, since $\hat{s}>\hat{s}-1$ and $n>\log(2n)$, we have

$$\begin{aligned}(7.26)\qquad \Big|\sum_{k=1}^{n}P_{n,k}(t)\mathbb{E}(e_{n,k}^{[2]})\Big| &= \Big|\sum_{k=1}^{n}P_{n,k}(t)\mathbb{E}(e_{k}^{[2]})\Big|\\ &\leqslant \max_{1\leqslant k\leqslant n}|P_{n,k}(t)|\sum_{k=1}^{n}\mathbb{E}[|e_k|\,\mathbb{I}(|e_k|\geqslant k^{1/u})]\\ &\leqslant Cn^{((8-3v)\hat{s})/(4v(\hat{s}-1))-1}\log^{-3/2}(2n)\\ &\quad\times\sum_{k=1}^{n}k^{-(2\hat{s}-1)/u}\mathbb{E}[|e_k|^{2\hat{s}}\mathbb{I}(|e_k|\geqslant k^{1/u})]\\ &\leqslant Cn^{((8-3v)\hat{s})/(4v(\hat{s}-1))}\log^{-3/2-(2\hat{s}-1)/u}(2n)\\ &= \mathcal{O}(n^{(8-3v)/(4v)}\log^{-3/2-(2\hat{s}-1)/u}(2n)).\end{aligned}$$

From (7.25) and (7.26), we have

$$(7.27)\qquad \Big|\sum_{k=1}^{n}M_{n,k}^{[2]}\Big| = o(n^{(8-3v)/(4v)}) \quad \text{almost surely.}$$

The proof for $M_{n,k}^{[3]}$ is similar to that for $M_{n,k}^{[2]}$, replacing $\mathbb{E}(|e_{n,k}^{[2]}|)$ by $\mathbb{E}(|e_{n,k}^{[3]}|)$ in (7.23) and (7.26), noting that $|e_k|=|-e_k|$. Note that

$$\mathbb{E}(|e_{n,k}^{[3]}|)=\mathbb{E}[|e_k|\mathbb{I}(|-e_k|>k^{1/u})]\leqslant k^{-(2\hat{s}-1)/u}\mathbb{E}(|e_k|^{2\hat{s}}).$$

Therefore,

$$(7.28)\qquad \Big|\sum_{k=1}^{n}M_{n,k}^{[3]}\Big| = o(n^{(8-3v)/(4v)}) \quad \text{almost surely.}$$

From (7.22), (7.27), and (7.28), we conclude that

$$L_{n,t}^{[2]} = \mathcal{O}(n^{(8-3v)/(4v)}) \quad \text{almost surely.} \tag{7.29}$$

The result follows by combining (7.7) and (7.29).

7.6. Proof of Theorem 5.1. From the definitions of $\varphi_k(t)$ and $P_{n,k}^{[\text{nne}]}(t)$, we observe that if $P_{n,k}^{[\text{nne}]}(t) > 0$, then for all $\alpha > 0$, $\hat{s}/2 < u < \min\{\hat{s}, 4(\hat{s}-1)\}$ and $t \in [0,1]$,

$$\sum_{k=1}^{n} P_{n,k}^{[\text{nne}]}(t) = \sum_{k=1}^{\theta_n} \frac{1}{\theta_n} = 1, \tag{7.30}$$

$$\max_{1 \leqslant k \leqslant n} |P_{n,k}^{[\text{nne}]}(t)|^{\hat{s}-1} = \frac{1}{\theta_n^{\hat{s}-1}} \leqslant C n^{(1-\hat{s})/2}, \tag{7.31}$$

$$\sum_{k=1}^{n} |P_{n,k}^{[\text{nne}]}(t)|\,|R(t_{n,k}) - R(t)|\,\mathbb{I}(\|t_{n,k} - t\| > \alpha n^{-1/v}) \tag{7.32}$$

$$\leqslant C\alpha^{-2} \sum_{k=1}^{\theta_n} (k/n)^2 \theta_n^{-1} n^{2/v} = \mathcal{O}(n^{(2-v)/v}).$$

Estimates (7.30)–(7.32) show that conditions $(\text{A}_{\text{sc}}^{[1]})$–$(\text{A}_{\text{sc}}^{[4]})$ are satisfied, allowing the application of Theorems 4.1 and 4.2 to prove Theorem 5.1.

REFERENCES

[1] L. Cheng, R. Zhang, Y. Wang and A. Yao, *Complete moment convergence for weighted sums of widely orthant-dependent random variables and its application in nonparametric regression models,* Front. Math. China 16 (2021), 1–20.

[2] L. Ding and C. Ping, *A note on the consistency of wavelet estimators in nonparametric regression model under widely orthant dependent random errors,* Math. Slovaca 69 (2019), 1471–1484.

[3] Q. He, *Consistency of the Priestley–Chao estimator in nonparametric regression model with widely orthant dependent errors,* J. Inequal. Appl. 2019, art. 123, 13 pp.

[4] S. H. Hu, G. M. Pan and Q. B. Gao, *Estimation problems for a regression model with linear process errors,* Appl. Math. J. Chinese Univ. Ser. A 18 (2003), 81–90.

[5] C. Lu, C. Zhuang and J. W. Xue, *Complete f-moment convergence for widely orthant dependent random variables and its application in nonparametric models,* Acta Math. Sinica (Engl. Ser.) 35 (2019), 1917–1936.

[6] D. Lu and J. Wang, *Complete convergence and complete moment convergence for maximal randomly weighted sums of widely orthant-dependent random variables with applications,* Comm. Statist. Theory Methods 50 (2019), 763–791.

[7] A. Shen, *Bernstein-type inequality for widely dependent sequence and its application to nonparametric regression models*, Abstr. Appl. Anal. 2013, art. 862602, 9 pp.

[8] A. Shen and C. Wu, *Complete q-th moment convergence and its statistical applications,* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 114 (2020), art. 35, 25 pp.

[9] A. Shen, X. Li and M. Ning, *Strong convergence properties for weighted sums of WNOD random variables and its applications in nonparametric regression models,* Stochastics 93 (2021), 376–401.

[10] C. J. Stone, *Consistent nonparametric regression,* Ann. Statist. 5 (1977), 595–620.
[11] K. Wang, Y. Wang and Q. Gao, *Uniform asymptotics for the finite-time ruin probability of a dependent risk model with a constant interest rate,* Methodol. Comput. Appl. Probab. 15 (2013), 109–124.
[12] X. Wang, C. Lu, J. Xue and A. Shen, *On complete convergence for widely orthant-dependent random variables and its applications in nonparametric regression models,* Test 23 (2014), 607–629.
[13] X. Wang, Y. Wu and A. Rosalsky, *Complete convergence for arrays of rowwise widely orthant dependent random variables and its applications,* Stochastics 89 (2017), 1228–1252.
[14] Y. Wu, X. Wang, R. Li and C. Deng, *Limiting behaviour for arrays of rowwise widely orthant dependent random variables under conditions of R-h-integrability and its applications,* Stochastics 91 (2019), 916–944.
[15] M. Xi, Y. Yang, L. Zhao and W. Liu, *Some convergence properties for partial sums of widely orthant dependent random variables and their statistical applications,* Statist. Papers 61 (2020), 1663–1684.
[16] C. Yao, X. Li, W. Zhang and Q. He, *Laws of large numbers and complete convergence for WOD random variables and their applications,* Comm. Statist. Simulation Comput. 52 (2023), 175–195.
[17] A. Zhang, Y. Yu and Y. Shen, *Complete moment convergence for weighted sums of widely orthant dependent random variables and its application in nonparametric regression model,* Filomat 36 (2022), 2761–2774.
[18] R. Zhang, Y. Wang, L. Cheng and A. Zhang, *On complete consistency for the weighted estimator of nonparametric regression models,* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 113 (2019), 2319–2333.

Omar Elfarouk Cherifi, Samir Benaissa,
Boubaker Mechab
Statistics and Stochastic Processes Laboratory
Djillali Liabes University
Sidi Bel Abbes, Algeria
E-mail: cherifi.omar74@gmail.com
benaissamir@yahoo.fr
mechaboub@yahoo.fr

Khadidja Abdelhak
Laboratory of Mathematics, Statistics &
Computer Sciences
University Center Salhi Ahmed
Naama, Algeria
E-mail: khadidjamekkaoui14@gmail.com

Received 13.9.2025;
accepted 11.2.2026