PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 45, Fasc. 2 (2025), pp. 183–190
Published online 2.12.2025
doi:10.37190/0208-4147.00268

ON THE MEAN EXIT TIME FROM A BALL FOR A SYMMETRIC STABLE PROCESS

BY

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Abstract. Getoor (1961) calculated the mean exit time from a ball for the standard isotropic α-stable process in $\mathbb{R}^d$ starting from the interior of the ball. The purpose of this note is to show that, up to a multiplicative constant, the same formula is valid for any symmetric α-stable process.

2020 Mathematics Subject Classification: Primary 60G52; Secondary 60J75.

Key words and phrases: stable process, exit time.

1. INTRODUCTION

Let X_t be a d-dimensional symmetric α-stable process of index $\alpha \in (0,2)$, $d \in \mathbb{N}$. Let $\mathbb{E}^x = \mathbb{E}(\,\cdot\,|\,X_0 = x)$ and let $B_r = \{z \in \mathbb{R}^d : |z| < r\}$. In this note we deal with the mean exit time of the process X from B_r, that is, we study $\mathbb{E}^x \tau_r^X$, where

$$\tau_r^X = \inf\{t \geqslant 0 : |X_t| \geqslant r\}, \quad r > 0.$$

Getoor [1] studied the mean exit time from a ball in the case of the standard isotropic stable process $\tilde{X}$ and he calculated that

$$\mathbb{E}^x \tau_r^{\tilde{X}} = c(\alpha, d)(r^2 - |x|^2)_+^{\alpha/2},$$

where $c(\alpha,d) = \Gamma(d/2)[2^\alpha \Gamma(1+\alpha/2)\Gamma((d+\alpha)/2)]^{-1}$. It is a bit surprising that the above formula holds (up to a multiplicative constant) for any symmetric stable process. The main result of this note is the following theorem.

THEOREM 1.1. *Let ν be the Lévy measure of a symmetric α-stable process X_t. Then*

$$\mathbb{E}^x \tau_r^X = \frac{\kappa_\alpha}{\nu(B_1^c)}(r^2 - |x|^2)_+^{\alpha/2}, \quad x \in \mathbb{R}^d,$$

where $\kappa_\alpha = \frac{1}{\Gamma(1-\alpha/2)\Gamma(1+\alpha/2)}$.

* M. Ryznar was supported in part by the National Science Centre, Poland, grant no. 2023/49/B/ST1/03964.

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Our approach relies on the one-dimensional version of the already mentioned result obtained by Getoor [1]. Actually we apply the fact that the function $\mathbb{R} \ni u \mapsto (r^2 - |u|^2)_+^{\alpha/2}$ is in the domain of the generator of the L^1 semigroup generated by the one-dimensional standard α-stable process and the value of the generator is a negative constant in the interval $(-r, r)$ (see [1, Theorem 5.2]). Next, using this result, we easily show that the value of a generator of any d-dimensional symmetric stable process evaluated (pointwise) for the function $w(x) = (r^2 - |x|^2)_+^{\alpha/2}$, $x \in \mathbb{R}^d$, is equal to a negative constant if $|x| < r$. This allows us to use an argument based on Dynkin's lemma applied to some sufficiently smooth approximation of w.

As a matter of fact, we are able to prove a slightly more general version of Theorem 1.1. Let $\mathcal{A} = \{A(x) : x \in \mathbb{R}^d\}$ be a family of $d \times d$ invertible matrices such that $x \mapsto A(x)$ and $x \mapsto A^{-1}(x)$ are continuous maps (in the space of matrices we use the operator norm). Consider the stochastic differential equation

$$dY_t = A(Y_{t-})dX_t. \tag{1.1}$$

THEOREM 1.2. *Let ν be the Lévy measure of a symmetric α-stable process X_t and let Y_t be a solution to* (1.1)*. Then*

$$\frac{\kappa_\alpha}{A_{\max}\nu(B_1^c)}(r^2 - |x|^2)_+^{\alpha/2} \leqslant \mathbb{E}^x \tau_r^Y \leqslant \frac{\kappa_\alpha}{A_{\min}\nu(B_1^c)}(r^2 - |x|^2)_+^{\alpha/2}, \quad x \in \mathbb{R}^d,$$

where $A_{\max} = \max_{\{|x|\leqslant r,\, |z|=1\}} |A(x)z|^\alpha$ *and* $A_{\min} = \min_{\{|x|\leqslant r,\, |z|=1\}} |A(x)z|^\alpha$.

Throughout this note we denote by $C(\alpha)$ a positive constant depending on α whose value may change from line to line. For two vectors $x, y \in \mathbb{R}^d$ we denote by $x \cdot y$ their standard scalar product, and for a $d \times d$ matrix A, we denote by $|A|$ its operator norm.

2. MEAN EXIT TIME CALCULATIONS

Let $\mathcal{S} = \{z \in \mathbb{R}^d : |z| = 1\}$ be the standard unit sphere in $\mathbb{R}^d$. Let μ be a finite positive measure on $\mathcal{S}$. Let $\mathcal{A} = \{A(x) : x \in \mathbb{R}^d\}$ be a family of $d \times d$ matrices. For a given $0 < \alpha < 2$ let ν be the Lévy measure of an α-stable symmetric process in $\mathbb{R}^d$. We consider the operator defined by

$$\begin{aligned} \mathcal{K}_\nu^{\mathcal{A}} f(x) &= \text{p.v.} \int_{\mathbb{R}^d} [f(x + A(x)y) - f(x)]\, \nu(dy) \\ &= \int_{\mathbb{R}^d} [f(x + A(x)y) - f(x) - \nabla f(x) \cdot A(x) y \mathbf{1}_{|y|\leqslant 1}]\, \nu(dy) \end{aligned}$$

for any Borel function $f : \mathbb{R}^d \to \mathbb{R}$ and any $x \in \mathbb{R}^d$ such that the right-hand side exists. It is well known that the measure ν has the factorization

$$\nu(dy) = \frac{dw}{|w|^{1+\alpha}}\mu(dz), \quad y = wz,\ w \in \mathbb{R},\ z \in \mathcal{S},$$

where μ is a finite positive and symmetric measure on $\mathcal{S}$ so we can write

$$\mathcal{K}_\nu^{\mathcal{A}} f(x) = \text{p.v.} \int_{\mathcal{S}\times\mathbb{R}} [f(x + A(x)zw) - f(x)] \frac{dw}{|w|^{1+\alpha}} \mu(dz).$$

If a bounded f is locally C^2 at x, that is, it has second order partial derivatives continuous in some neighborhood of x, then $\mathcal{K}_\nu^{\mathcal{A}}$ can be written as an iterated integral .

$$\mathcal{K}_\nu^{\mathcal{A}} f(x) = \int_{\mathcal{S}} \int_{\mathbb{R}} [f(x + A(x)zw) - f(x)] \frac{dw}{|w|^{1+\alpha}} \mu(dz),$$

where the inner integral is a principal value integral.

From now on we assume that the solution to (1.1) exists for any initial value $x \in \mathbb{R}^d$. Then by an application of Ito's formula it is not difficult to show that the operator $\mathcal{K}_\nu^{\mathcal{A}}$ is the generator of the semigroup defined by the Markov process which is the solution to (1.1).

Next, for a given vector $v \neq 0$ in $\mathbb{R}^d$ let us introduce

$$\mathcal{K}_v f(x) = \int_{\mathbb{R}} [f(x + vw) - f(x)] \frac{dw}{|w|^{1+\alpha}}, \quad x \in \mathbb{R}^d,$$

again as a principal value integral. Let $v^* = v/|v|$. By a simple change of variables we have

$$\mathcal{K}_v f(x) = |v|^\alpha \int_{\mathbb{R}} [f(x + v^* w) - f(x)] \frac{dw}{|w|^{1+\alpha}} = |v|^\alpha \mathcal{K}_{v^*} f(x). \tag{2.1}$$

For $r > 0$ we define

$$S_r(x) = c_\alpha (r^2 - |x|^2)_+^{\alpha/2}, \quad x \in \mathbb{R}^d,$$

where

$$c_\alpha = \frac{\alpha}{2\Gamma(1-\alpha/2)\Gamma(1+\alpha/2)}.$$

The following simple lemma is a key observation for the rest of this note.

LEMMA 2.1. *If $v \in \mathbb{R}^d$, then*

$$\mathcal{K}_v S_r(x) = -|v|^\alpha, \quad |x| < r.$$

Proof. By (2.1), it is enough to prove the lemma for $|v| = 1$. Next, we claim that we can assume that v can be taken to be $e_1 = (1, 0, \ldots, 0)$. Indeed, if $\mathcal{R}$ is a rotation such that $\mathcal{R}v = e_1$, then we have

$$\mathcal{K}_v S_r(x) = \mathcal{K}_v (S_r \circ \mathcal{R})(x) = \mathcal{K}_{e_1} S_r(\mathcal{R}x), \quad |x| < r.$$

Now, we recall the one-dimensional version of the result obtained by Getoor [1, Theorem 5.2.]. For $r > 0$ let $s_r(u) = c_\alpha (r^2 - |u|^2)_+^{\alpha/2}, u \in \mathbb{R}$. Then for $-r < u < r$, we have

$$\int_{\mathbb{R}} [s_r(u+w) - s_r(u)] \frac{dw}{|w|^{1+\alpha}} = -1. \tag{2.2}$$

Actually in [1] the above equality was proved almost surely, but it is not difficult to show that the left-hand side is a continuous function of $u \in (-r, r)$, hence it must hold pointwise on $(-r, r)$. Next, for $x = (x_1, \dots, x_d)$ with $|x| < r$ we have $|x_1| < \sqrt{r^2 - |\tilde{x}|^2}$, where $\tilde{x} = x - x_1 e_1$, and by (2.2) we obtain

$$\begin{aligned}
\mathcal{K}_{e_1} S_r(x) &= c_\alpha \int_{\mathbb{R}} [(r^2 - |x + we_1|^2)_+^{\alpha/2} - (r^2 - |x|^2)_+^{\alpha/2}] \frac{dw}{|w|^{1+\alpha}} \\
&= c_\alpha \int_{\mathbb{R}} [(r^2 - |\tilde{x}|^2 - |x_1 + w|^2)_+^{\alpha/2} - (r^2 - |\tilde{x}|^2 - |x_1|^2)_+^{\alpha/2}] \frac{dw}{|w|^{1+\alpha}} \\
&= \int_{\mathbb{R}} [s_{\sqrt{r^2 - |\tilde{x}|^2}}(x_1 + w) - s_{\sqrt{r^2 - |\tilde{x}|^2}}(x_1)] \frac{dw}{|w|^{1+\alpha}} = -1.
\end{aligned}$$

This implies that

$$\mathcal{K}_v S_r(x) = \mathcal{K}_{e_1} S_r(\mathcal{R}x) = -1,$$

since $|\mathcal{R}(x)| = |x| < r$. ∎

COROLLARY 2.1. *For any $|x| < r$ we have*

$$\mathcal{K}_\nu^{\mathcal{A}} S_r(x) = -\int_{\mathcal{S}} |A(x)z|^\alpha \, \mu(dz).$$

If $A(x)$ is an isometry for all $|x| < r$, then

$$\mathcal{K}_\nu^{\mathcal{A}} S_r(x) = -\mu(\mathcal{S}), \qquad |x| < r.$$

Proof. Since the function S_r is locally C^2, for any x with $|x| < r$ we can write

$$\mathcal{K}_\nu^{\mathcal{A}} S_r(x) = \int_{\mathcal{S}} \mathcal{K}_{A(x)z} S_r(x) \, \mu(dz) = -\int_{\mathcal{S}} |A(x)z|^\alpha \, \mu(dz),$$

where the last step follows from Lemma 2.1. ∎

The next two lemmas provide us with some smooth approximation of S_r.

LEMMA 2.2. *For $\beta > 0$ let $h_\beta(z) = z^\beta$, $z \geqslant 1$. There is an C^2 extension $\bar{h}_\beta$ of h_β to $\mathbb{R}$ such that $\bar{h}_\beta(z) = \bar{h}'_\beta(z) = \bar{h}''_\beta(z) = 0$, $z \leqslant 0$.*

Proof. Let $w(z) = z^3(a + b(z-1) + c(z-1)^2)$, $z \in \mathbb{R}$. We define

$$\bar{h}_\beta(z) = \begin{cases} 0 & \text{for } z < 0, \\ w(z) & \text{for } 0 \leqslant z < 1, \\ z^\beta & \text{for } z \geqslant 1, \end{cases}$$

where a, b, c are chosen in such a way that $w(1) = 1, w'(1) = \beta$ and $w''(1) = (\beta - 1)\beta$. It is clear that $\bar{h}_\beta$ is the desired extension. ∎

LEMMA 2.3. *Let $0 < R < R_0$. There exists a function $g_R : \mathbb{R}^d \to \mathbb{R}$ such that such that*

(a) *g_R is of class $C^2_\infty(\mathbb{R}^d)$,*

(b) *$g_R(x) = S_{R_0}(x)$ for $|x| \leqslant R$,*

(c) *there is a constant $C(\alpha)$ such that*

$$|g_R(x)| \leqslant C(\alpha)(R_0^2 - (R \wedge |x|)^2)_+^{\alpha/2}, \quad x \in \mathbb{R}^d.$$

Proof. Let

$$g_R(x) = c_\alpha (R_0^2 - R^2)^{\alpha/2} \bar{h}_{\alpha/2}\left(\frac{R_0^2 - |x|^2}{R_0^2 - R^2}\right), \quad x \in \mathbb{R}^d.$$

Then all the above properties of g_R follow from Lemma 2.2. ∎

LEMMA 2.4. *For any $r > 0$ the mean exit time τ_r^Y satisfies*

$$\mathbb{E}^x \tau_r^Y < \infty, \quad x \in \mathbb{R}^d.$$

Proof. For $t, v > 0$, let

$$N_X(t, v) = \sum_{s \leqslant t} \mathbf{1}_{B_v^c}(\Delta X_s)$$

be the number of jumps in the interval $[0, t]$ of the process X of length at least v. It is enough to show that

$$\sup_x \mathbb{P}^x(\tau_r^Y > t_0) < 1$$

for some $t_0 > 0$. Let $|x| < r$. It is clear that if $\{\tau_r^Y > t_0\}$ then the process Y does not make jumps of length at least $2r$ in $[0, t_0]$. Since the jumps of Y are of the form

$$\Delta Y_t = A(X_{t-})\Delta X_t,$$

for $t \leqslant t_0$ we have

$$|\Delta X_t| \leqslant \sup_{|x| \leqslant r} |A^{-1}(x)|\, |\Delta Y_t| \leqslant \sup_{|x| \leqslant r} |A^{-1}(x)| 2r$$

on the event $\{\tau_r^Y > t_0\}$. Let $a^* = \sup_{|x|\leqslant r} |A^{-1}(x)|$. This implies that

$$\begin{aligned}\mathbb{P}^x(\tau_r^Y > t_0) &\leqslant \mathbb{P}(N_X(t_0, 2ra^*) = 0)\\ &= e^{-t_0\nu(y:|y|\geqslant 2ra^*)} < 1,\end{aligned}$$

since $N_X(t_0, 2ra^*)$ has the Poisson distribution with intensity parameter equal to $t_0\nu(y : |y| \geqslant 2ra^*)$. The proof is complete. ∎

Now, we are in a position to prove the main result.

Proof of Theorems 1.1 and 1.2 . Recall that the generator of the process Y_t which is the solution to (1.1) is given by

$$\mathcal{K}_\nu^{\mathcal{A}} f(x) = \text{p.v.} \int_{\mathbb{R}^d} [f(x + A(x)y) - f(x)]\, \nu(dy).$$

Let $0 < R < R_0$ be such that $R - (R_0 - R)^{1/3} > 0$. Pick $r = (R - (R_0 - R)^{1/3})$. Now, let $|x| \leqslant r$. Then from Lemma 2.3(b)–(c) we infer that

$$\begin{aligned}(2.3)\quad &|\mathcal{K}_\nu^{\mathcal{A}} S_{R_0}(x) - \mathcal{K}_\nu^{\mathcal{A}} g_R(x)|\\ &= \Big| \int_{|x+A(x)y|\geqslant R} [S_{R_0}(x + A(x)y) - g_R(x + A(x)y)]\, \nu(dy) \Big|\\ &\leqslant \int_{|x+A(x)y|\geqslant R} [S_{R_0}(x + A(x)y) + |g_R(x + A(x)y)|]\, \nu(dy)\\ &\leqslant C(\alpha)(R_0^2 - R^2)^{\alpha/2} \nu(y : |A(x)y| \geqslant R - r)\\ &\leqslant C(\alpha)(R_0^2 - R^2)^{\alpha/2} \nu(y : |y| \geqslant 1)|A(x)|^\alpha (R - r)^{-\alpha}\\ &\leqslant C(\alpha)(R_0 - R)^{\alpha/6} \nu(y : |y| \geqslant 1)|A(x)|^\alpha R_0^{\alpha/2}.\end{aligned}$$

Note that $\mathbb{E}^x \tau_r^Y < \infty$ by Lemma 2.4, so by Dynkin's lemma applied to g_R which is in the domain of the generator $\mathcal{K}_\nu^{\mathcal{A}}$, and by Lemma 2.3(b), we have

$$\begin{aligned}\mathbb{E}^x(g_R(Y_{\tau_r})) &= g_R(x) + \mathbb{E}^x \int_0^{\tau_r^Y} \mathcal{K}_\nu^{\mathcal{A}} g_R(Y_t)\, dt\\ &= S_{R_0}(x) + \mathbb{E}^x \int_0^{\tau_r^Y} \mathcal{K}_\nu^{\mathcal{A}} S_{R_0}(Y_t)\, dt\\ &\quad + \mathbb{E}^x \int_0^{\tau_r^Y} (\mathcal{K}_\nu^{\mathcal{A}} g_R(Y_t) - \mathcal{K}_\nu^{\mathcal{A}} S_{R_0}(Y_t))\, dt.\end{aligned}$$

Since, by Corollary 2.1,

$$\mathbb{E}^x \int_0^{\tau_r^Y} \mathcal{K}_\nu^{\mathcal{A}} S_{R_0}(Y_t)\, dt = -\mathbb{E}^x \int_0^{\tau_r^Y} \int_{\mathcal{S}} |A(Y_t)z|^\alpha\, \mu(dz)\, dt,$$

we obtain

$$S_{R_0}(x) - \mathbb{E}^x \int_0^{\tau_r^Y} \int_{\mathcal{S}} |A(Y_t)z|^\alpha \, \mu(dz)\, dt = \mathbb{E}^x(g_R(X_{\tau_r^Y})) + \mathbb{E}^x \int_0^{\tau_r^Y} (\mathcal{K}_\nu^{\mathcal{A}} S_{R_0}(Y_t) - \mathcal{K}_\nu^{\mathcal{A}} g_R(Y_t))\, dt.$$

We will show that the right-hand side of the above equality vanishes when $R \uparrow R_0$. By Lemma 2.3(b)–(c),

$$|\mathbb{E}^x(g_R(Y_{\tau_r^Y}))| \leqslant C(\alpha)(R_0^2 - r^2)^{\alpha/2} \leqslant C(\alpha) R_0^{\alpha/2} (R_0 - R)^{\alpha/6},$$

and by (2.3),

$$\mathbb{E}^x \int_0^{\tau_r^Y} |(\mathcal{K}_\nu^{\mathcal{A}} S(Y_t) - \mathcal{K}_\nu^{\mathcal{A}} g_R(Y_t))|\, dt \leqslant C(\alpha)(R_0 - R)^{\alpha/6} \nu(B_1^c) |A(x)|^\alpha R_0^{\alpha/2} \mathbb{E}^x \tau_r^Y.$$

Letting $R \uparrow R_0$ we obtain

$$S_{R_0}(x) = \lim_{R \uparrow R_0} \mathbb{E}^x \int_0^{\tau_r^Y} \int_{\mathcal{S}} |A(Y_t)z|^\alpha \, \mu(dz)\, dt = \mathbb{E}^x \int_0^{\tau_{R_0}^Y} \int_{\mathcal{S}} |A(Y_t)z|^\alpha \, \mu(dz)\, dt.$$

Let $A_{\max} = \max_{\{|x| \leqslant R_0,\, |z|=1\}} |A(x)z|^\alpha$ and $A_{\min} = \min_{\{|x| \leqslant R_0,\, |z|=1\}} |A(x)z|^\alpha$. Then from the above inequality we obtain

$$c \frac{S_{R_0}(x)}{A_{\max}} \leqslant \mathbb{E}^x \tau_{R_0}^Y \leqslant c \frac{S_{R_0}(x)}{A_{\min}},$$

where $c = 1/\mu(\mathcal{S}) = 2\alpha/\nu(B_1^c)$.

If we assume that $A(x)$ is an isometry for all $x \in B_{R_0}$ then $A_{\max} = A_{\min} = 1$ and we have equality in the conclusion of Theorem 1.2. In particular, if we consider $A(x) = I$, then we obtain the formula for the mean exit time of any symmetric stable process. ∎

REMARK 2.1. If a ball is replaced by an ellipsoid $E \subset \mathbb{R}^d$ with center at the origin, then we can easily extend the conclusions of the above theorems. For example, let L be the linear operator such that $L(E) = B_1$. Then

$$\mathbb{E}^x \tau_E^X = \mathbb{E}^{Lx} \tau_{B_1}^{LX} = \frac{\kappa_\alpha}{\nu(E^c)} (1 - |Lx|^2)_+^{\alpha/2}, \quad x \in \mathbb{R}^d,$$

where τ_E^X is the first exit time of the process X from E, and $\tau_{B_1}^{LX}$ is the first exit time of the process LX from the ball B_1. Note that LX is a symmetric α-stable process.

Acknowledgments. The author is grateful to K. Bogdan and T. Kulczycki for discussions on the problem treated in the paper.

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Received 5.9.2025;
accepted 5.11.2025