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ON CONSISTENCY OF WAVELET ESTIMATORS IN NONPARAMETRIC REGRESSION MODELS WITH WIDELY ORTHANT DEPENDENT SAMPLES*

BY

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Abstract. In this paper, we mainly investigate the nonparametric regression model based on widely orthant dependent samples. The weak consistency, strong consistency, complete consistency, and the rates of strong consistency and complete consistency for the wavelet estimator are established under some mild conditions, which generalize and improve the corresponding ones for other dependent cases. Some numerical simulations are presented to verify the validity of the theoretical results based on finite samples.

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Key words and phrases: nonparametric regression model, widely orthant dependent random variables, wavelet estimator, consistency, the rate of consistency.

1. INTRODUCTION

Consider the following nonparametric regression model:

$$Y_{ni} = f(t_{ni}) + \varepsilon_i, \quad 1 \leqslant i \leqslant n,\ n \geqslant 1, \tag{1.1}$$

where the regression function $f(\cdot)$ is an unknown Borel measurable function defined on [0,1], $\{t_{ni}\}$ are nonrandom design points with $0 \leqslant t_{n1} \leqslant \cdots \leqslant t_{nn} \leqslant 1$, and $\{\varepsilon_i\}$ are random errors.

It is known that the model (1.1) has many applications in filtering and prediction in communications and control systems, pattern recognition, classification and econometrics, and is an important tool of data analysis. In the past decades, many scholars studied weighted estimators of $f(\cdot)$. For more details, one can refer e.g. to Priestley and Chao [17], Georgiev and Greblicki [8], Müller [16], Georgiev [7],

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Fan [6], Roussas et al. [18], Tran et al. [22], Liang and Jing [14], Wang et al. [26], Thanh and Yin [21], Chen et al. [4], Yang et al. [32] and Tang et al. [20].

One of the most important weight function estimations is wavelet estimation. A wavelet estimator of $f(\cdot)$ was proposed by Antoniadis et al. [1] as follows:

$$f_n(t) = \sum_{i=1}^{n} Y_{ni} \int_{A_{ni}} E_m(t,s)\,ds, \qquad A_{ni} = [s_{n(i-1)}, s_{ni}), \tag{1.2}$$

where $A_{n1}, A_{n2}, \ldots, A_{nn}$ is a partition of the interval $[0,1]$ with $t_{ni} \in A_{ni}$. The wavelet kernel $E_m(t,s)$ is defined as follows:

$$E_0(t,s) = \sum_{j\in\mathbb{Z}} \varphi(t-j)\varphi(s-j), \qquad E_m(t,s) = 2^m E_0(2^m t, 2^m s),$$

where $m = m(n) > 0$ is an integer depending only on n, and φ is the scaling function in the Schwartz space. The wavelet method has the advantage of adaptability to the degree of smoothness of the underlying function and the ability to adapt to local features of unknown curves. Hence, many authors have applied wavelet procedures to estimate the unknown regression function $f(\cdot)$ in the nonparametric model (1.1). In the independent case, the consistency and asymptotic normality of estimator (1.2) have been established by Antoniadis et al. [1]. In various dependent cases, Chai and Liu [2] discussed weak convergence, strong convergence and the rate of convergence based on ρ-mixing random errors; Xue [30] obtained the uniform convergence rate and uniform mean convergence rate of order r of wavelet estimators for some mixing errors; Li et al. [13] obtained the weak consistency and uniformly asymptotic normality of wavelet estimators in regression models with associated samples; Li and Guo [12] established the asymptotic normality of a wavelet estimator for strong mixing errors; Zhou et al. [34] studied the asymptotic properties of wavelet-based estimators in nonparametric regression models for some weakly dependent processes; Ding et al. [5] investigated the mean consistency and complete consistency of a wavelet estimator for extended negatively dependent errors; Wang et al. [25] investigated the mean consistency, weak consistency, strong consistency, complete consistency and strong convergence rate of wavelet estimators in the nonparametric regression model with repeated measurements based on extended negatively dependent errors; and Tang et al. [20] considered the Berry–Esseen bounds of a wavelet estimator in the nonparametric regression model under linear process errors based on asymptotically negatively associated sequences.

In this paper, we will further study the wavelet estimator of $f(\cdot)$ in the nonparametric model (1.1) based on widely orthant dependent (WOD for short) random variables, the concept of which was introduced by Wang et al. [26] as follows.

DEFINITION 1.1. A finite collection of random variables $X_1, \ldots, X_n$ is said to be *widely upper orthant dependent* (WUOD for short) if there exists a finite real number $g_U(n)$ such that for all finite real numbers $x_i, 1 \leqslant i \leqslant n$,

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leqslant g_U(n) \prod_{i=1}^{n} P(X_i > x_i).$$

A finite collection of random variables $X_1, \ldots, X_n$ is said to be *widely lower orthant dependent* (WLOD for short) if there exists a finite real number $g_L(n)$ such that for all finite real numbers $x_i, 1 \leqslant i \leqslant n$,

$$P(X_1 \leqslant x_1, X_2 \leqslant x_2, \ldots, X_n \leqslant x_n) \leqslant g_L(n) \prod_{i=1}^{n} P(X_i \leqslant x_i).$$

If $X_1 \ldots, X_n$ are both WUOD and WLOD, we say that they are WOD, and $g_U(n), g_L(n)$ are called *dominating coefficients*. A sequence $\{X_n,\ n \geqslant 1\}$ of random variables is said to be WOD if every finite subcollection is WOD.

It can be easily checked that $g_U(n) \geqslant 1$ and $g_L(n) \geqslant 1$. If $g_U(n) = g_L(n) = M \geqslant 1$ for all $n \geqslant 1$, then the sequence $\{X_n,\ n \geqslant 1\}$ of random variables is said to be *extended negatively dependent* (END), a notation proposed by Liu [15]. If $g_U(n) = g_L(n) = 1$ for all $n \geqslant 1$, then $\{X_n,\ n \geqslant 1\}$ is *be negatively orthant dependent* (NOD). This notion was introduced by Lehmann [11], and carefully studied by Joag-Dev and Proschan [10] among others. Joag-Dev and Proschan [10] pointed out that negatively associated (NA) random variables are NOD. Furthermore, Hu [9] pointed out that negatively supper-additive dependent (NSD) random variables are NOD. Hence, the class of WOD random variables includes independent sequences, NA sequences, NSD sequences, NOD sequences, and END sequences as special cases. Wang et al. [23] presented some examples to show that WOD random variables contain some negatively dependent random variables, positively dependent random variables, and some other classes of dependent random variables. Thus, the study of WOD random variables is of general interest. There have been a number of works on WOD random variables. For more details, we refer the readers e.g. to Wang and Cheng [27], Chen et al. [3], Shen [19], Wang et al. [26], Yang et al. [31], Wang and Hu [24], Xia et al. [29] and Zhang et al. [33].

While abundant asymptotic results exist for wavelet estimators, a systematic investigation under the general WOD framework is still worth in-depth research. In this work, some consistency results for the wavelet estimator (1.2) based on WOD errors are established. Weak consistency, strong consistency, complete consistency and the convergence rates of strong and complete consistencies are obtained under some mild conditions. Moreover, the restriction on the dominating coefficients is rather weak. Unlike most results in the cited references, the dominating coefficient $g(n)$ in our work can increase polynomially or even geometrically. These results extend and improve some corresponding ones based on independent or dependent errors. Some simulation studies are also provided to confirm the theoretical results.

The outline of this paper is as follows. The main results and simulation studies are presented in Section 2. Some lemmas are provided in Section 3. The proofs of the results are given in Section 4. Throughout this paper, C, C_i, $i \geqslant 1$, represent some positive constants, which may vary in different places. Denote $a^+ = aI(a \geqslant 0)$, $a^- = -aI(a < 0)$ and $g(n) = \max\{g_U(n), g_L(n)\}$. Let $\log x = \ln \max(x, e)$, and $I(A)$ be the indicator function of the set A. $\lfloor x \rfloor$ stands for the in-

teger part of x; $a_n = o(b_n)$ means $a_n/b_n \to 0$; and $a_n = O(b_n)$ means $a_n \leqslant Cb_n$ for all n large enough.

2. MAIN RESULTS AND SIMULATIONS

2.1. Main results. Based on the definitions above, the assumptions for the main results are listed as follows.

$(\boldsymbol{A_1})$ $f(\cdot)$ satisfies the Lipschitz condition of order $\gamma > 0$, and $f(\cdot) \in H^v$ for some $v > 1/2$, where H^v is the Sobolev space of order v, i.e., $\int |\hat{f}(\omega)|^2(1+\omega^2)^v d\omega < \infty$, where $\hat{f}$ is the Fourier transform of f.

$(\boldsymbol{A_2})$ $\varphi(\cdot)$ satisfies the Lipschitz condition of order 1 on a compact support, and is τ-regular with $\tau \geqslant v$, i.e., for any $l \leqslant \tau$, and for any integer k, one has $\left|\frac{d^l\varphi}{dx^l}\right| \leqslant C_k(1+|x|)^{-k}$, where C_k is a generic constant depending only on k. Furthermore, $|\hat{\varphi}(\xi) - 1| = O(\xi)$ as $\xi \to 0$.

$(\boldsymbol{A_3})$ $\max_{1\leqslant i\leqslant n} |s_{ni} - s_{n(i-1)}| = O(1/n)$, and $2^m = O(n^{1-\mu})$ for some $0 < \mu < 1$.

$(\boldsymbol{A_4})$ $f(\cdot) \in H^v$ for some $v > 3/2$, and $f(\cdot)$ satisfies the Lipschitz condition of order $\gamma \geqslant \mu/2$.

REMARK 2.1. $(\boldsymbol{A_1})$ ensures that the function $f(\cdot)$ possesses smoothness and controllability within the Sobolev space H^v, enabling the wavelet transform to effectively capture local features of the function. $(\boldsymbol{A_2})$ provides control over the function $\varphi(\cdot)$, ensuring that its variation on the compact support is manageable and that it exhibits a certain degree of regularity. $(\boldsymbol{A_3})$ addresses the convergence of sequences for proving asymptotic properties in wavelet estimation. $(\boldsymbol{A_4})$ emphasizes the requirement of higher smoothness of $f(\cdot)$ within the Sobolev space, enhancing the effectiveness of the wavelet basis. The assumptions above are standard for wavelet estimation and have been widely used in many papers. For more details, we refer the readers e.g. to Antoniadis et al. [1], Xue [30], Li et al. [13], Li and Guo [12], Zhou et al. [34] and Ding et al. [5].

Now we present our main results. The first one is the weak consistency of the wavelet estimator (1.2) based on WOD errors.

THEOREM 2.1. *Suppose that* $(\boldsymbol{A_1})$*–*$(\boldsymbol{A_3})$ *hold. Assume that* $\{\varepsilon_n, n \geqslant 1\}$ *is a sequence of zero mean WOD random errors with* $\sup_{i\geqslant 1} E|\varepsilon_i|^{\frac{\mu p+1}{2\mu}} < \infty$ *for some* $p > 1/\mu$ *and the dominating coefficients* $g(n) = O(e^{n^\varrho})$ *for some* $0 \leqslant \varrho < (\mu p - 1)/(p-1)$. *Then for any* $t \in [0,1]$,

$$f_n(t) \to f(t) \quad \textit{in probability as } n \to \infty.$$

The second one is the strong consistency of the wavelet estimator (1.2) based on WOD errors.

THEOREM 2.2. *Suppose that* $(\boldsymbol{A_1})$–$(\boldsymbol{A_3})$ *hold. Assume that* $\{\varepsilon_n,\ n \geqslant 1\}$ *is a sequence of zero mean WOD random errors with* $\sup_{i\geqslant 1} E|\varepsilon_i|^p < \infty$ *for some* $p > 1/\mu$ *and the dominating coefficients* $g(n) = O(e^{n^{\varrho}})$ *for some* $0 \leqslant \varrho < (\mu p - 1)/(p - 1)$. *Then for any* $t \in [0, 1]$,

$$f_n(t) \to f(t) \quad \textit{a.s. as } n \to \infty.$$

REMARK 2.2. It is obvious that $\frac{\mu p+1}{2\mu} < p$ since $p > 1/\mu$ and $0 < \mu < 1$ in Theorem 2.1. Thus we get weak consistency under weaker assumptions than those in Theorem 2.2.

REMARK 2.3. Li et al. [13] obtained the weak consistency result for the estimator (1.2) based on NA errors with $\mu = 2/3$. Note that strong consistency is much stronger than weak consistency, μ is released to $0 < \mu < 1$ in Theorem 2.2, and NA implies WOD. Therefore, our result markedly improves and extends the corresponding one of Li et al. [13] from the NA setting to the WOD setting. Moreover, it is worth mentioning that the dominating coefficients are allowed to increase geometrically and the moment conditions are unrelated to the dominating coefficients.

REMARK 2.4. In sequential analysis, researchers need to update estimates in real-time based on continuously accumulating observed data. Weak consistency only guarantees that under a fixed sample size, as the number of repeated trials increases, the probability of small estimation errors increases. In contrast, strong consistency ensures that along a single sample path, the estimator will eventually stabilize near the true parameter, which is crucial for sequential decision-making. In addition, the importance of complete convergence cannot be ignored. For example, in the law of iterated logarithm, complete convergence is the basis for establishing accurate asymptotic results.

By imposing slightly stronger condition on the moment in Theorem 2.2, we can obtain the following result on complete consistency of the wavelet estimator.

THEOREM 2.3. *Suppose that* $(\boldsymbol{A_1})$–$(\boldsymbol{A_3})$ *hold. Assume that* $\{\varepsilon_n,\ n \geqslant 1\}$ *is a sequence of zero mean WOD random errors with* $\sup_{i\geqslant 1} E|\varepsilon_i|^{2p} < \infty$ *for some* $p > 1/\mu$ *and the dominating coefficients* $g(n) = O(e^{n^{\varrho}})$ *for some* $0 \leqslant \varrho < (\mu p - 1)/(p - 1/2)$. *Then for any* $t \in [0, 1]$,

$$f_n(t) \to f(t) \quad \textit{completely as } n \to \infty,$$

i.e., for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P(|f_n(t) - f(t)| > \epsilon) < \infty.$$

REMARK 2.5. Ding et al. [5] established complete consistency for the estimator (1.2) based on END errors. Comparing Theorem 2.3 with that of Ding et al. [5], we have the following improvements or generalizations:

- The restrictions $v > 3/2$ and $\gamma = 1$ in Ding et al. [5] are released to $v > 1/2$ and $\gamma > 0$.
- Ding et al. [5] assumed that $2^m = O(n^{1/3})$, i.e., $\mu = 2/3$, while here we release μ to $0 < \mu < 1$.
- They required that $\{\varepsilon_n,\ n \geqslant 1\}$ is stochastically dominated by a random variable X, while Theorem 2.3 does not need this assumption.
- The moment condition $EX^4 < \infty$ in Ding et al. [5] is relaxed here. For example, if we let $2^m = O(n^{1/3})$, then the moment condition in Theorem 2.3 is $\sup_{i\geqslant 1} E|\varepsilon_i|^\kappa < \infty$ for some $\kappa > 3$, which is weaker than that of Ding et al. [5].
- The END structure is generalized to the WOD structure, and the dominating coefficients are also allowed to increase geometrically.

The following two results concern the rates of strong consistency and the complete consistency for the estimator (1.2) based on WOD errors.

THEOREM 2.4. *Suppose that* $(\boldsymbol{A_2})$–$(\boldsymbol{A_4})$ *hold. Assume that* $\{\varepsilon_n,\ n \geqslant 1\}$ *is a sequence of zero mean WOD random errors with* $\sup_{i\geqslant 1} E|\varepsilon_i|^{2p} < \infty$ *for some* $p > 1/\mu$ *and the dominating coefficients* $g(n) = O(n^\tau)$ *for some* $\tau \geqslant 0$. *If* $n^{1-\mu} = O(2^m)$, *then for any* $t \in [0, 1]$,

$$f_n(t) - f(t) = O(n^{\mu-1}) + o(n^{-\mu/2}\log n) \quad \textit{a.s.}$$

THEOREM 2.5. *Suppose that* $(\boldsymbol{A_2})$–$(\boldsymbol{A_4})$ *hold. Assume that* $\{\varepsilon_n,\ n \geqslant 1\}$ *is a sequence of zero mean WOD random errors with* $\sup_{i\geqslant 1} E|\varepsilon_i|^{4p} < \infty$ *for some* $p > 1/\mu$ *and the dominating coefficients* $g(n) = O(n^\tau)$ *for some* $\tau \geqslant 0$. *If* $n^{1-\mu} = O(2^m)$, *then for any* $t \in [0, 1]$,

$$f_n(t) - f(t) = O(n^{\mu-1}) + o(n^{-\mu/2}\log n) \quad \textit{completely}.$$

REMARK 2.6. It is easy to check that if we choose $\mu = 2/3$, then the optimal rates of strong consistency and complete consistency are $o(n^{-1/3}\log n)$. Moreover, the dominating coefficients can increase polynomially no matter how large τ is. Ding et al. [5] obtained the rate $O(n^{-1/4})$ of complete consistency under the moment condition $EX^6 < \infty$. Obviously, the rate obtained in Theorem 2.5 is sharper. Therefore, our result extends and improves the corresponding one of Ding et al. [5] from the END setting to the WOD setting.

2.2. Numerical simulation. In this section, we will carry out some numerical simulations to study the numerical performance for the estimator of $f(\cdot)$. The data are generated from (1.1). According to different types of random errors, we mainly consider the following two cases.

CASE 1. For fixed $n \geqslant 3$ and $1 \leqslant i \leqslant n$, $(\varepsilon_1, \ldots, \varepsilon_n) \sim N(\mathbf{0}, \mathbf{\Sigma})$, where $\mathbf{0}$ represents the zero vector and

$$\mathbf{\Sigma} = \begin{pmatrix} 1+\theta^2 & -\theta & 0 & \cdots & 0 & 0 & 0 \\ -\theta & 1+\theta^2 & -\theta & \cdots & 0 & 0 & 0 \\ 0 & -\theta & 1+\theta^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\theta^2 & -\theta & 0 \\ 0 & 0 & 0 & \cdots & -\theta & 1+\theta^2 & -\theta \\ 0 & 0 & 0 & \cdots & 0 & -\theta & 1+\theta^2 \end{pmatrix}_{n\times n},$$

where $0 < \theta < 1$. It follows from Joag-Dev and Proschan [10] that $(\varepsilon_1, \ldots, \varepsilon_n)$ is a NA vector and thus a WOD vector with finite moment of any order and $g(n) = 1$. For simplicity, choose the scale function $\varphi(x) = I(0 \leqslant x \leqslant 1)$, $\theta = 0.3$, $2^m = \lfloor n^{1/3} \rfloor$, $t_{ni} = (i - 0.5)/n$, and $s_{ni} = i/n$ for $i = 1, \ldots, n$. Taking 200 points t at equal intervals in the interval $(0, 1)$, and the sample sizes $n = 100, 200, 400, 600$, we use R software to compute $f_n(t) - f_i(t)$, $i = 1, 2, 3$, with $f_1(t) = t^2 - 2t$, $f_2(t) = \sin 2t$ and

$$f_3(t) = \begin{cases} -t^3/3 & \text{if } 0 \leqslant t < 0.4, \\ t & \text{if } 0.4 \leqslant t \leqslant 0.6, \\ t^2/2 & \text{if } 0.6 < t \leqslant 1, \end{cases}$$

respectively, and obtain the fine performance of the estimator $f_n(t)$ for 1000 replications.

CASE 2. The marginal distributions of $\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_n$ are the uniform distribution on $[0, 1]$, while $(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2), (\tilde{\varepsilon}_3, \tilde{\varepsilon}_4), \ldots, (\tilde{\varepsilon}_{2m-1}, \tilde{\varepsilon}_{2m}), \ldots$ each follow a joint distribution of the FGM copula

$$C_{\theta_n}(u, v) = uv + \theta_n uv(1-u)(1-v), \quad (u, v) \in [0, 1] \times [0, 1]. \tag{2.1}$$

Let $\varepsilon_n = \tilde{\varepsilon}_n - E\tilde{\varepsilon}_n$. It follows from Wu et al. [28] that $\{\varepsilon_n,\ n \geqslant 1\}$ is a sequence of zero mean WOD random variables. Choose $\theta_n = n^{-1/2}$. It is easy to check from Wu et al. [28] that $g(n) = O(\exp\{cn^{1/2}\})$ for some $c > 0$. Other parameters are the same as those in Case 1.

The global mean square errors (GMSE) of $f_n(t)$ are calculated based on the settings in Case 1 and Case 2 and shown in Figures 2.1–2.3, which vividly illustrate a striking trend: whether $f(\cdot)$ is continuous or not, the GMSEs exhibit a pronounced decline toward zero as the sample size n increases. This observation holds true across both Case 1 and Case 2. Such findings resonate harmoniously with the theoretical outcomes delineated in Section 2.1, providing validation to the foundational theory.

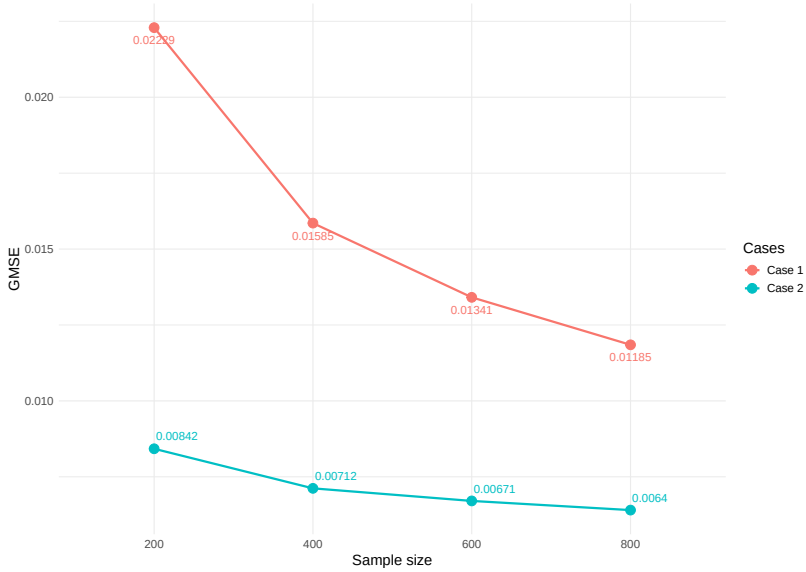


FIGURE 2.1. The GMSEs of $f_n(t)$ as $f(t) = f_1(t)$ in Case 1 and Case 2

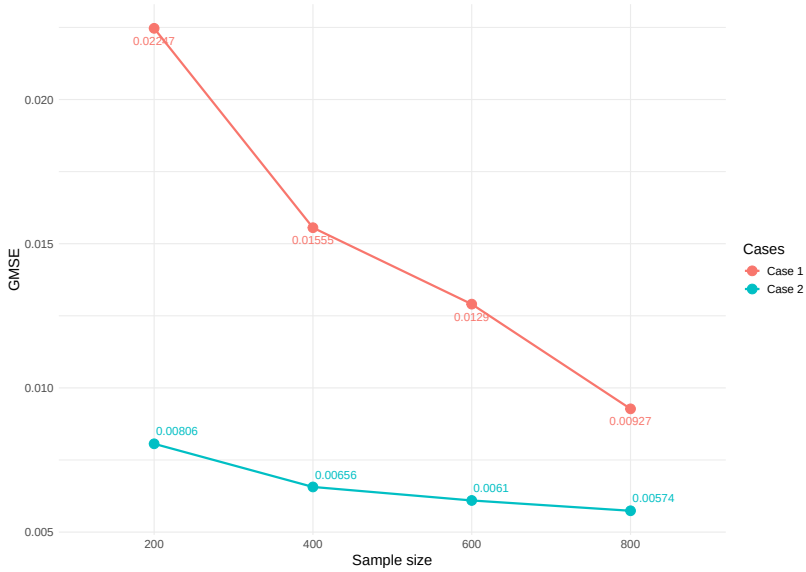


FIGURE 2.2. The GMSEs of $f_n(t)$ as $f(t) = f_2(t)$ in Case 1 and Case 2

Figures 2.1–2.3 suggest that in the setting of continuous functions, the fitting in Case 2 may be better than in Case 1, while in the setting of discontinuous functions, the opposite may be true. In addition, it must be pointed out that the fitting of the estimator in the discontinuous case is not as good as in the continuous case, which

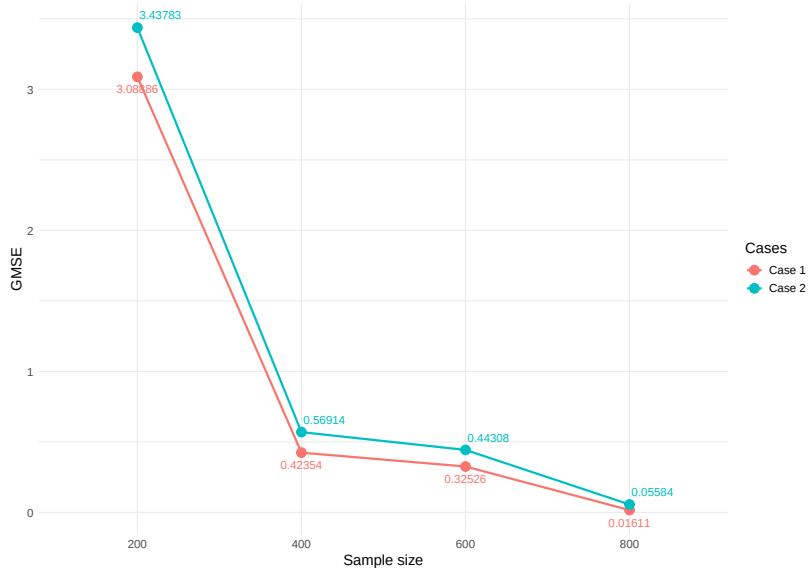


FIGURE 2.3. The GMSEs of $f_n(t)$ as $f(t) = f_3(t)$ in Case 1 and Case 2

may be due to the lack of gradual transition between data points like continuous data, and the information is relatively sparse.

In particular, this experiment also attempts to discuss the asymptotic normality of the estimator. The unbiased estimates of variance and mean are used to participate in the calculation, and display the performance at the single point ($t = 0.5$) and different sample sizes ($n = 400$ and $n = 800$) in Case 2, as shown in Figures 2.4–2.5. The results show that the data may be asymptotically normal, which is of great significance in statistical analysis as it enables the use of well-established normal-based methods for large sample inferences.

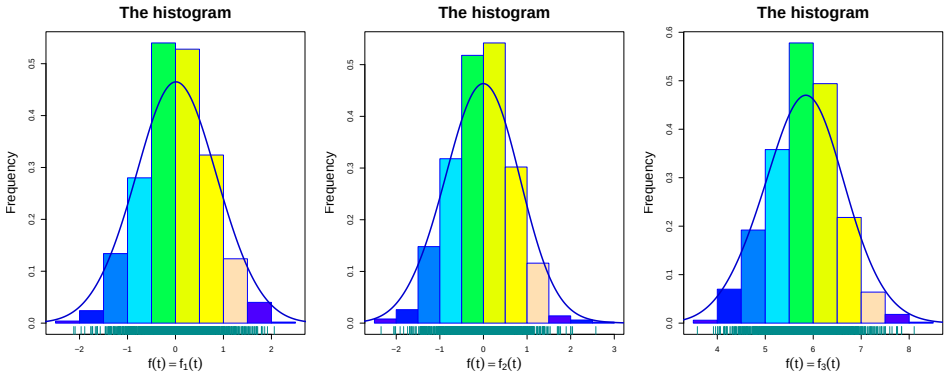


FIGURE 2.4. The histograms of $f_n(t)$ as $f(t) = f_k(t)$ $(k = 1, 2, 3)$ and $n = 400$ in Case 2

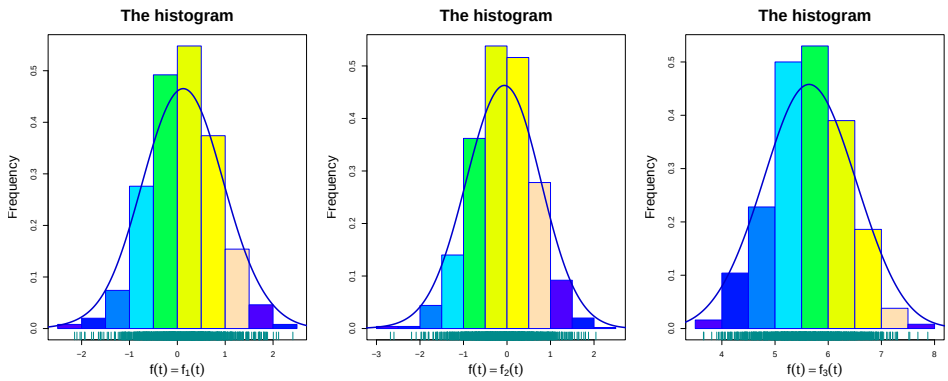


FIGURE 2.5. The histograms of $f_n(t)$ as $f(t) = f_k(t)$ $(k = 1, 2, 3)$ and $n = 800$ in Case 2

3. SOME LEMMAS

To prove the main results of the paper, we need the following important lemmas. The first one is a basic property of WOD random variables, presented by Wang et al. [26].

LEMMA 3.1. *Let* $\{X_n,\ n \geqslant 1\}$ *be a sequence of WOD random variables. If* $\{h_n(\cdot),\ n \geqslant 1\}$ *are all nondecreasing* (*or nonincreasing*), *then* $\{h_n(X_n),\ n \geqslant 1\}$ *are still WOD.*

The next lemma can be found in Li and Guo [12].

LEMMA 3.2. *Under Assumptions* $(\boldsymbol{A_1})$–$(\boldsymbol{A_3})$, *we have*

$$\text{(i)}\ \Big|\int\limits_{A_{ni}} E_m(t,s)\,ds\Big| = O\Big(\frac{2^m}{n}\Big), \qquad i = 1,\dots,n;$$

$$\text{(ii)}\ \sum_{i=1}^{n}\Big(\int\limits_{A_{ni}} E_m(t,s)\,ds\Big)^2 = O\Big(\frac{2^m}{n}\Big);$$

$$\text{(iii)}\ \sup_{0\leqslant t\leqslant 1}\int\limits_0^1 |E_m(t,s)|\,ds \leqslant C;$$

$$\text{(iv)}\ \sum_{i=1}^{n}\Big|\int\limits_{A_{ni}} E_m(t,s)\,ds\Big| \leqslant C.$$

The following was proved by Antoniadis et al. [1].

LEMMA 3.3. *Suppose that* $(\boldsymbol{A_1})$–$(\boldsymbol{A_3})$ *hold. Then*

$$Ef_n(t) - f(t) = O(n^{-\gamma}) + O(\tau_m),$$

where

$$\tau_m = \begin{cases} (1/2^m)^{v-1/2} & \text{if } 1/2 < v < 3/2, \\ \sqrt{m}/2^m & \text{if } v = 3/2, \\ 1/2^m & \text{if } v > 3/2. \end{cases}$$

The last lemma is a Bernstein type inequality for WOD random variables, established by Xia et al. [29].

LEMMA 3.4. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of WOD random variables with $EX_n = 0$ and $|X_n| \leqslant d_n$ a.s. for each $n \geqslant 1$, where $\{d_n,\ n \geqslant 1\}$ is a sequence of positive constants. Denote $b_n = \max_{1\leqslant i\leqslant n} d_i$ and $\Delta_n^2 = \sum_{i=1}^n EX_i^2$ for each $n \geqslant 1$. Then for every $\varepsilon > 0$,*

$$P\Big(\Big|\sum_{i=1}^n X_i\Big| \geqslant \varepsilon\Big) \leqslant 2g(n)\exp\Big\{-\frac{\varepsilon^2}{2(2\Delta_n^2 + b_n\varepsilon)}\Big\}.$$

4. PROOFS OF THE RESULTS

With the lemmas above at hand, we now present the proofs of the main results.

4.1. Proof of Theorem 2.1. It follows from (1.1) and (1.2) that

$$\begin{aligned} |f_n(t) - f(t)| &\leqslant |f_n(t) - Ef_n(t)| + |Ef_n(t) - f(t)| \\ &= \Big|\sum_{i=1}^n \int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i\Big| + |Ef_n(t) - f(t)|. \end{aligned} \tag{4.1}$$

By Lemma 3.3 we know that $|Ef_n(t) - f(t)| \to 0$ as $n \to \infty$. Therefore, it suffices to prove

$$\sum_{i=1}^n \int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i \to 0 \quad \text{in probability as } n \to \infty. \tag{4.2}$$

Since $\int_{A_{ni}} E_m(t,s)\,ds = (\int_{A_{ni}} E_m(t,s)\,ds)^+ - (\int_{A_{ni}} E_m(t,s)\,ds)^-$, without loss of generality we may assume that $\int_{A_{ni}} E_m(t,s)\,ds > 0$. Choose appropriate q such that $0 \leqslant \varrho < q < [(\mu p + 1)\mu - 2\mu]/(\mu p + 1 - 2\mu) (< (\mu p - 1)/(p-1) < \mu)$ and denote, for fixed $n \geqslant 1$ and $1 \leqslant i \leqslant n$,

$$\begin{aligned} X_{n,i}(1) = {} & -n^{-q} I\Big(\int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i < -n^{-q}\Big) \\ & + \int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i I\Big(\Big|\int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i\Big| \leqslant n^{-q}\Big) \\ & + n^{-q} I\Big(\int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i > n^{-q}\Big), \end{aligned}$$

$$X_{n,i}(2) = \Big(\int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i - n^{-q} \Big) I\Big(\int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i > n^{-q} \Big) + \Big(\int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i + n^{-q} \Big) I\Big(\int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i < -n^{-q} \Big).$$

Obviously,

$$\int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i = X_{n,i}(1) + X_{n,i}(2). \tag{4.3}$$

By Lemma 3.1 we can see that $\{X_{n,i}(1), 1 \leqslant i \leqslant n, n \geqslant 1\}$ is an array of rowwise WOD random variables. We will deduce from (4.3) that

$$P\Big(\Big| \sum_{i=1}^{n} X_{n,i}(1) \Big| > \epsilon \Big) \to 0 \quad \text{as } n \to \infty,$$
$$P\Big(\Big| \sum_{i=1}^{n} X_{n,i}(2) \Big| > \epsilon \Big) \to 0 \quad \text{as } n \to \infty.$$

It follows from $E\varepsilon_i = 0$, Lemma 3.2 and $q < [(\mu p+1)\mu - 2\mu]/(\mu p+1-2\mu) < (\mu p-1)/(p-1) < \mu$ that

$$\begin{aligned} &\Big| \sum_{i=1}^{n} EX_{n,i}(1) \Big| \\ &\leqslant \sum_{i=1}^{n} E \Big| \int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i \Big| I\Big(\Big| \int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i \Big| > n^{-q} \Big) \\ &\leqslant n^{(\frac{\mu p+1}{2\mu}-1)q} \sum_{i=1}^{n} \Big| \int\limits_{A_{ni}} E_m(t,s)\,ds \Big| \cdot \Big| \int\limits_{A_{ni}} E_m(t,s)\,ds \Big|^{\frac{\mu p+1}{2\mu}-1} \sup_{i\geqslant 1} E|\varepsilon_i|^{\frac{\mu p+1}{2\mu}} \\ &\leqslant C n^{(\frac{\mu p+1}{2\mu}-1)(q-\mu)} \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{4.4}$$

Note that $|X_{n,i}(1) - EX_{n,i}(1)| \leqslant 2n^{-q}$ for each $1 \leqslant i \leqslant n$. On the other hand, if $(\mu p+1)/(2\mu) \geqslant 2$ then

$$\begin{aligned} &\sum_{i=1}^{n} E|X_{n,i}(1) - EX_{n,i}(1)|^2 \\ &\qquad = \sum_{i=1}^{n} \Big(\int\limits_{A_{ni}} E_m(t,s)\,ds \Big)^2 \sup_{i\geqslant 1} E|\varepsilon_i|^2 = o(n^{-q}), \end{aligned} \tag{4.5}$$

and if $1/\mu < (\mu p+1)/(2\mu) < 2$ then

$$\begin{aligned} &\sum_{i=1}^{n} E|X_{n,i}(1) - EX_{n,i}(1)|^2 \\ &\leqslant 2n^{-q(2-\frac{\mu p+1}{2\mu})} \sum_{i=1}^{n} \Big| \int\limits_{A_{ni}} E_m(t,s)\,ds \Big| \cdot \Big| \int\limits_{A_{ni}} E_m(t,s)\,ds \Big|^{\frac{\mu p+1}{2\mu}-1} \sup_{i\geqslant 1} E|\varepsilon_i|^{\frac{\mu p+1}{2\mu}} \\ &= o(n^{-q}). \end{aligned} \tag{4.6}$$

Hence, from Lemma 3.4 and (4.4)–(4.6) we deduce that for any $\epsilon > 0$,

$$\begin{aligned} P\Big(\Big|\sum_{i=1}^{n} X_{n,i}(1)\Big| > \epsilon\Big) &\leqslant CP\Big(\Big|\sum_{i=1}^{n}(X_{n,i}(1) - EX_{n,i}(1))\Big| > \epsilon/2\Big) \\ &\leqslant Cg(n)\exp\Big\{-\frac{\epsilon^2/4}{2\,[o(n^{-q}) + \epsilon n^{-q}]}\Big\} \\ &\leqslant C\exp\{n^{\varrho} - \epsilon n^{q}/16\} \leqslant C\exp\{-\epsilon n^{q}/32\} \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

It is easy to check that $q < ([(\mu p+1)\mu - 2\mu]/(\mu p+1-2\mu) < (\mu p-1)/(p-1)$ is equivalent to $-\mu+1+\big(1-\frac{\mu p+1}{2\mu}\big)(\mu-q) < 0$. Hence, it follows from Markov's inequality that

$$\begin{aligned} (4.7)\quad P\Big(\Big|\sum_{i=1}^{n} X_{n,i}(2)\Big| > \epsilon\Big) &\leqslant CE\Big|\sum_{i=1}^{n} X_{n,i}(2)\Big| \\ &\leqslant C\sum_{i=1}^{n} E\Big|\int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i\Big| I\Big(\Big|\int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i\Big| > n^{-q}\Big) \\ &\leqslant Cn^{-\mu}\sum_{i=1}^{n} E|\varepsilon_i| I(|\varepsilon_i| > Cn^{\mu-q}) \\ &\leqslant Cn^{-\mu+1+(1-\frac{\mu p+1}{2\mu})(\mu-q)} \sup_{i\geqslant 1} E|\varepsilon_i|^{\frac{\mu p+1}{2\mu}} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

The proof is complete. ∎

4.2. Proof of Theorem 2.2. We use the same notations as in the proof of Theorem 2.1, with $q < (\mu p - 1)/(p-1) < \mu$. In view of the proof of Theorem 2.1, it suffices to prove

$$(4.8)\qquad \sum_{i=1}^{n} \int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i \to 0 \quad \text{a.s. as } n \to \infty.$$

It follows from $E\varepsilon_i = 0$, Lemma 3.2 and $q < (\mu p - 1)/(p-1) < \mu$ that

$$\begin{aligned} (4.9)\quad &\Big|\sum_{i=1}^{n} EX_{n,i}(1)\Big| \\ &\leqslant \sum_{i=1}^{n} E\Big|\int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i\Big| I\Big(\Big|\int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i\Big| > n^{-q}\Big) \\ &\leqslant n^{(p-1)q}\sum_{i=1}^{n}\Big|\int\limits_{A_{ni}} E_m(t,s)\,ds\Big| \cdot \Big|\int\limits_{A_{ni}} E_m(t,s)\,ds\Big|^{p-1} \sup_{i\geqslant 1} E|\varepsilon_i|^p \\ &\leqslant Cn^{(p-1)(q-\mu)} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Note that $|X_{n,i}(1) - EX_{n,i}(1)| \leqslant 2n^{-q}$ for each $1 \leqslant i \leqslant n$. On the other hand, if $p \geqslant 2$ then

$$\sum_{i=1}^{n} E|X_{n,i}(1) - EX_{n,i}(1)|^2 = o(n^{-q}),$$

and if $1/\mu < p < 2$ then

$$\begin{aligned}
&\sum_{i=1}^{n} E|X_{n,i}(1) - EX_{n,i}(1)|^2 \\
&\qquad \leqslant 2n^{-q(2-p)} \sum_{i=1}^{n} \Big| \int\limits_{A_{ni}} E_m(t,s)\,ds \Big| \cdot \Big| \int\limits_{A_{ni}} E_m(t,s)\,ds \Big|^{p-1} \sup_{i \geqslant 1} E|\varepsilon_i|^p \\
&\qquad = o(n^{-q}).
\end{aligned}$$

Hence, by Lemma 3.4 we find that for any $\epsilon > 0$,

$$\begin{aligned}
(4.10) \qquad &\sum_{n=1}^{\infty} P\Big(\Big| \sum_{i=1}^{n} X_{n,i}(1) \Big| > \epsilon \Big) \\
&\qquad \leqslant C \sum_{n=1}^{\infty} P\Big(\Big| \sum_{i=1}^{n} (X_{n,i}(1) - EX_{n,i}(1)) \Big| > \epsilon/2 \Big) \\
&\qquad \leqslant C \sum_{n=1}^{\infty} g(n) \exp\left\{ -\frac{\epsilon^2/4}{2\,[o(n^{-q}) + \epsilon n^{-q}]} \right\} \\
&\qquad \leqslant C \sum_{n=1}^{\infty} \exp\{n^{\varrho} - \epsilon n^q/16\} \leqslant C \sum_{n=1}^{\infty} \exp\{-\epsilon n^q/32\} < \infty,
\end{aligned}$$

which together with the Borel–Cantelli lemma yields $\sum_{i=1}^{n} X_{n,i}(1) \to 0$ a.s.

Now we turn to proving $\sum_{i=1}^{n} X_{n,i}(2) \to 0$ a.s. Note by $q < (\mu p - 1)/(p-1)$ that

$$\sum_{i=1}^{\infty} i^{-\mu} E|\varepsilon_i| I(|\varepsilon_i| > Ci^{\mu-q}) \leqslant C \sum_{i=1}^{\infty} i^{-\mu} \cdot i^{(\mu-q)(1-p)} \sup_{i \geqslant 1} E|\varepsilon_i|^p < \infty,$$

which implies

$$\sum_{i=1}^{\infty} i^{-\mu} |\varepsilon_i| I(|\varepsilon_i| > Ci^{\mu-q}) < \infty \quad \text{a.s.}$$

Therefore, by the Kronecker lemma we derive

$$\begin{aligned}
\Big| \sum_{i=1}^{n} X_{n,i}(2) \Big| &\leqslant \sum_{i=1}^{n} \Big| \int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i \Big| I\Big(\Big| \int\limits_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i \Big| > n^{-q} \Big) \\
&\leqslant Cn^{-\mu} \sum_{i=1}^{n} |\varepsilon_i| I(|\varepsilon_i| > Cn^{\mu-q}) \\
&\leqslant Cn^{-\mu} \sum_{i=1}^{n} |\varepsilon_i| I(|\varepsilon_i| > Ci^{\mu-q}) \to 0 \quad \text{a.s.}
\end{aligned}$$

The proof is complete. ∎

4.3. Proof of Theorem 2.3. We use the same notations as in the proof of Theorem 2.1, with $0 \leqslant \varrho < q < (\mu p - 1)/(p - 1/2) < (\mu p - 1)/(p - 1)$. In view of the proof of Theorem 2.1, we only need to show

$$\sum_{i=1}^{n} \int\limits_{A_{ni}} E_m(t,s)\, ds\, \varepsilon_i \to 0 \quad \text{completely as } n \to \infty,$$

or equivalently, for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P\Big(\Big|\sum_{i=1}^{n} \int\limits_{A_{ni}} E_m(t,s)\, ds\, \varepsilon_i\Big| > 2\epsilon\Big) < \infty. \tag{4.11}$$

Note that in (4.10) it has been proved that $\sum_{n=1}^{\infty} P(|\sum_{i=1}^{n} X_{n,i}(1)| > \epsilon) < \infty$. We will prove $\sum_{n=1}^{\infty} P(|\sum_{i=1}^{n} X_{n,i}(2)| > \epsilon) < \infty$. It is easy to check that $q < (\mu p - 1)/(p - 1/2)$ is equivalent to $-\mu + 1 + (1 - 2p)(\mu - q) < -1$. Hence, it follows from (4.7) that

$$\sum_{n=1}^{\infty} P\Big(\Big|\sum_{i=1}^{n} X_{n,i}(2)\Big| > \epsilon\Big) \leqslant C \sum_{n=1}^{\infty} n^{-\mu+1+(1-2p)(\mu-q)} \sup_{i \geqslant 1} E|\varepsilon_i|^{2p} < \infty,$$

completing the proof. ∎

4.4. Proof of Theorem 2.4. It follows from Lemma 3.3 and $n^{1-\mu} = O(2^m)$ that

$$|Ef_n(t) - f(t)| = O(n^{-\gamma}) + O(n^{\mu-1}). \tag{4.12}$$

Hence, from (4.1), it suffices to prove

$$\Big|\sum_{i=1}^{n} \int\limits_{A_{ni}} E_m(t,s)\, ds\, \varepsilon_i\Big| = o(n^{-\mu/2} \log n) \quad \text{a.s.} \tag{4.13}$$

We use the same notations as in the proof of Theorem 2.1, with $\mu/2 < q < \mu - (1 - \mu/2)/(2p - 1)$. It follows from $E\varepsilon_i = 0$, Lemma 3.2 and $q < \mu - (1 - \mu/2)/(2p - 1)$ that

$$\begin{aligned}
&n^{\mu/2} \log^{-1} n \Big|\sum_{i=1}^{n} EX_{n,i}(1)\Big| \\
&\quad \leqslant n^{\mu/2} \log^{-1} n \sum_{i=1}^{n} E\Big| \int\limits_{A_{ni}} E_m(t,s)\, ds\, \varepsilon_i\Big| I\Big(\Big| \int\limits_{A_{ni}} E_m(t,s)\, ds\, \varepsilon_i\Big| > n^{-q}\Big) \\
&\quad \leqslant n^{\mu/2+(2p-1)q} \log^{-1} n \sum_{i=1}^{n} \Big| \int\limits_{A_{ni}} E_m(t,s)\, ds\Big| \cdot \Big| \int\limits_{A_{ni}} E_m(t,s)\, ds\Big|^{2p-1} \sup_{i \geqslant 1} E|\varepsilon_i|^{2p} \\
&\quad \leqslant C n^{\mu/2+(2p-1)(q-\mu)} \log^{-1} n \leqslant C n^{\mu-1} \log^{-1} n \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Note that $|X_{n,i}(1) - EX_{n,i}(1)| \leqslant 2n^{-q}$ for each $1 \leqslant i \leqslant n$. On the other hand,

from $\sup_{i\geqslant 1} E|\varepsilon_i|^2 < \infty$ and Lemma 3.2 we have

$$\sum_{i=1}^{n} E|X_{n,i}(1) - EX_{n,i}(1)|^2 \leqslant \sum_{i=1}^{n} \Big(\int_{A_{ni}} E_m(t,s)\,ds \Big)^2 \sup_{i\geqslant 1} E|\varepsilon_i|^2 = O(n^{-\mu}).$$

Hence, by Lemma 3.4 we find that for any $\epsilon > 0$,

$$\begin{aligned}(4.14)\qquad &\sum_{n=1}^{\infty} P\Big(\Big| \sum_{i=1}^{n} X_{n,i}(1) \Big| > \epsilon n^{-\mu/2} \log n \Big) \\ &\qquad \leqslant C \sum_{n=1}^{\infty} P\Big(\Big| \sum_{i=1}^{n} (X_{n,i}(1) - EX_{n,i}(1)) \Big| > \epsilon n^{-\mu/2} \log n/2 \Big) \\ &\qquad \leqslant C \sum_{n=1}^{\infty} g(n) \exp\Big\{ -\frac{\epsilon^2 n^{-\mu} \log^2 n/4}{2\left[O(n^{-\mu}) + \epsilon n^{-\mu/2-q} \log n \right]} \Big\} \\ &\qquad \leqslant C \sum_{n=1}^{\infty} n^{\tau} \exp\{-(\tau+2)\log n\} \leqslant C \sum_{n=1}^{\infty} n^{-2} < \infty,\end{aligned}$$

which together with the Borel–Cantelli lemma yields

$$\sum_{i=1}^{n} X_{n,i}(1) = o(n^{-\mu/2} \log n) \quad \text{a.s.}$$

Now we turn to proving

$$\sum_{i=1}^{n} X_{n,i}(2) = o(n^{-\mu/2} \log n) \quad \text{a.s.}$$

Note by $q < \mu - (1-\mu/2)/(2p-1)$ that

$$\begin{aligned}&\sum_{i=1}^{\infty} i^{-\mu/2} \log^{-1} i E|\varepsilon_i| I(|\varepsilon_i| > C i^{\mu-q}) \\ &\qquad \leqslant C \sum_{i=1}^{\infty} i^{-\mu/2} \log^{-1} i \cdot i^{(\mu-q)(1-2p)} \sup_{i\geqslant 1} E|\varepsilon_i|^{2p} < \infty,\end{aligned}$$

which implies

$$\sum_{i=1}^{\infty} i^{-\mu/2} \log^{-1} i |\varepsilon_i| I(|\varepsilon_i| > C i^{\mu-q}) < \infty \quad \text{a.s.}$$

Therefore, by Kronecker's lemma we obtain

$$\begin{aligned}&n^{\mu/2} \log^{-1} n \Big| \sum_{i=1}^{n} X_{n,i}(2) \Big| \\ &\quad \leqslant n^{\mu/2} \log^{-1} n \sum_{i=1}^{n} \Big| \int_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i \Big| I\Big(\Big| \int_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i \Big| > n^{-q} \Big) \\ &\quad \leqslant C n^{-\mu/2} \log^{-1} n \sum_{i=1}^{n} |\varepsilon_i| I(|\varepsilon_i| > C n^{\mu-q}) \\ &\quad \leqslant C n^{-\mu/2} \log^{-1} n \sum_{i=1}^{n} |\varepsilon_i| I(|\varepsilon_i| > C i^{\mu-q}) \to 0 \quad \text{a.s.},\end{aligned}$$

which is equivalent to $\sum_{i=1}^{n} X_{n,i}(2) = o(n^{-\mu/2}\log n)$ a.s. Hence, (4.13) is proved, which together with (4.12) and the assumption $\gamma \geqslant \mu/2$ yields the desired result. ∎

4.5. Proof of Theorem 2.5. We use the same notations as in the proof of Theorem 2.4, with $\mu/2 < q < \mu - (2-\mu/2)/(4p-1) < \mu - (1-\mu/2)/(2p-1)$. According to (4.12), we only need to prove that for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P\Big(\Big|\sum_{i=1}^{n} \int_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i\Big| > 2\epsilon n^{-\mu/2}\log n\Big) < \infty. \tag{4.15}$$

Note that in (4.14) it has been proved that

$$\sum_{n=1}^{\infty} P\Big(\Big|\sum_{i=1}^{n} X_{n,i}(1)\Big| > \epsilon n^{-\mu/2}\log n\Big) < \infty.$$

We will prove

$$\sum_{n=1}^{\infty} P\Big(\Big|\sum_{i=1}^{n} X_{n,i}(2)\Big| > \epsilon n^{-\mu/2}\log n\Big) < \infty.$$

It is easy to check that $q < \mu - (2-\mu/2)/(4p-1)$ is equivalent to $-\mu/2 + 1 + (1-4p)(\mu - q) < -1$. Hence, it follows from Markov's inequality that

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\Big(\Big|\sum_{i=1}^{n} X_{n,i}(2)\Big| > \epsilon n^{-\mu/2}\log n\Big) \leqslant C\sum_{n=1}^{\infty} n^{\mu/2}\log^{-1} n E\Big|\sum_{i=1}^{n} X_{n,i}(2)\Big| \\
&\quad\leqslant C\sum_{n=1}^{\infty} n^{\mu/2}\log^{-1} n \sum_{i=1}^{n} E\Big|\int_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i\Big| I\Big(\Big|\int_{A_{ni}} E_m(t,s)\,ds\,\varepsilon_i\Big| > n^{-q}\Big) \\
&\quad\leqslant C\sum_{n=1}^{\infty} n^{-\mu/2}\log^{-1} n \sum_{i=1}^{n} E|\varepsilon_i| I(|\varepsilon_i| > Cn^{\mu-q}) \\
&\quad\leqslant C\sum_{n=1}^{\infty} n^{-\mu/2+1+(1-4p)(\mu-q)}\log^{-1} n \sup_{i\geqslant 1} E|\varepsilon_i|^{4p} < \infty.
\end{aligned}$$

Therefore, (4.15) has been proved and this completes the proof of the theorem. ∎

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