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COMPLETE f-MOMENT CONVERGENCE FOR m-WOD LINEAR PROCESSES WITH RANDOM COEFFICIENTS AND ITS APPLICATION*

BY

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Abstract. In this paper, we study complete f-moment convergence, complete sth moment convergence and complete convergence for m-widely orthant dependent linear processes with random coefficients under some general conditions. The results presented extend and improve some known results. As an application, we present conclusions on complete f-moment consistency and complete consistency for a weighted estimator in a nonparametric regression model based on widely orthant dependent errors. Moreover, we perform a numerical simulation to verify the validity of the theoretical results.

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1. INTRODUCTION

The assumption that random variables are independent is commonly made in many statistical applications. In practical applications, however, the assumption of independence is unreasonable. Therefore, in recent decades, many scholars have replaced this assumption by various dependence structures. One of the most important dependence structures is widely orthant dependence (WOD), which was first introduced by Wang et al. [29] in the following way.

DEFINITION 1.1. Random variables $\{X_n, n \geqslant 1\}$ are called *widely upper orthant dependent* (WUOD) if there exists a sequence $g_U(n)$ such that for every

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$n \geqslant 1$ and all $x_i \in (-\infty, \infty)$,

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leqslant g_U(n) \prod_{i=1}^{n} P(X_i > x_i);$$

they are called *widely lower orthant dependent* (WLOD) if there exists a sequence $g_L(n)$ such that for every $n \geqslant 1$ and all $x_i \in (-\infty, \infty)$,

$$P(X_1 \leqslant x_1, X_2 \leqslant x_2, \ldots, X_n \leqslant x_n) \leqslant g_L(n) \prod_{i=1}^{n} P(X_i \leqslant x_i);$$

and they are WOD if they are both WUOD and WLOD.

In Definition 1.1, $g_U(n)$ and $g_L(n)$, $n \geqslant 1$, are called the *dominating coefficients*. It is easily seen that if $g_U(n) = g_L(n) = M > 0$, then the definition of WOD random variables is equivalent to the definition of extended negatively dependent (END) random variables introduced by Liu [17]. Furthermore, if $M = 1$, then the concept of WOD random variables reduces to negatively orthant dependent (NOD) random variables proposed by Joag-Dev and Proschan [12]. Therefore, the class of WOD random variables includes END random variables, NOD random variables, negatively associated (NA) random variables, and independent random variables.

Based on the above definition of WOD random variables, Fang et al. [7] introduced the following definition of m-WOD random variables.

DEFINITION 1.2. Let $m \geqslant 1$ be a fixed integer. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be *m-WOD* if for any $n \geqslant 2$ and any $i_1, \ldots, i_n$ such that $|i_k - i_j| \geqslant m$ for all $1 \leqslant k \neq j \leqslant n$, the random variables $X_{i_1}, \ldots, X_{i_n}$ are WOD.

It is obvious that if $m = 1$, then the concept of m-WOD random variables is equivalent to that of WOD random variables. Therefore, the class of m-WOD random variables includes m-END random variables, m-NOD random variables, m-NA random variables, and m-dependent random variables. However, there are relatively few studies of m-WOD random variables; the interested readers can refer e.g. to Fang et al. [7], Guan et al. [9], and Wang and Shen [33].

The concept of complete convergence was introduced by Hsu and Robbins [11] as follows: A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to *converge completely* to a constant θ if

$$\sum_{n=1}^{\infty} P(|X_n - \theta| > \epsilon) < \infty \quad \text{for all } \epsilon > 0.$$

By the Borel–Cantelli lemma, this implies that $X_n \to \theta$ almost surely. Hence complete convergence implies a.s. convergence. The concept of complete moment convergence was introduced by Chow [4], and this convergence is stronger than

complete convergence: Let $\{X_n,\, n \geqslant 1\}$ be a sequence of random variables; if there are $a_n, b_n, q > 0$ such that

$$\sum_{n=1}^{\infty} a_n E\left\{b_n^{-1}|X_n| - \epsilon\right\}_+^q < \infty \quad \text{for all } \epsilon > 0,$$

where $a_+ = \max\{0, a\}$, then we say that $\{X_n,\, n \geqslant 1\}$ *converges moment completely*.

Recently, complete moment convergence has been extended to complete f-moment convergence introduced by Wu et al. [38]. This concept is defined as follows.

DEFINITION 1.3. Let $\{X_n,\, n \geqslant 1\}$ be a sequence of random variables, and $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing continuous function with $f(0) = 0$. Then we say that $\{X_n,\, n \geqslant 1\}$ *converges f-moment completely* if there exists a sequence $\{c_n,\, n \geqslant 1\}$ of positive constants such that

$$\sum_{n=1}^{\infty} c_n E f(\{|X_n| - \epsilon\}_+) < \infty \quad \text{for all } \epsilon > 0.$$

Obviously, complete f-moment convergence implies complete moment convergence if $f(t) = t^q$, and complete convergence if $c_n = 1, n \geqslant 1$ and $f(t) = t$. At present, research on complete f-moment convergence is still relatively scarce. For more details about complete f-moment convergence, one can refer to Wu et al. [38], Lu et al. [18], Wang and Wang [36], Yao et al. [42], Wang et al. [30], and Wang et al. [32].

Next, we recall the definition of moving average processes.

Let $\{Y_j,\, -\infty < j < \infty\}$ be a sequence of random variables and $\{a_j,\, -\infty < j < \infty\}$ be an absolutely summable sequence of real numbers, that is,

$$\sum_{j=-\infty}^{\infty} |a_j| < \infty.$$

A *linear process* or an *infinite moving average process* generated by $\{Y_j,\, -\infty < j < \infty\}$ is defined as

$$X_n = \sum_{j=-\infty}^{\infty} a_j Y_{n-j}.$$

The properties of linear processes have been widely studied by many researchers. For example, McLeish [20] proved the asymptotic convergence for time series involving linear processes and he checked limit theorems for dependent random variables in various cases; Ko [13] provided the asymptotic properties of linear processes generated by associated random variables in Hilbert space; Tao et al. [25] obtained complete convergence of moving average processes for WOD sequences; Li and Zhang [16], Zhou and Lin [43], Ko [14], Guan et al. [9] and

Wang and Shen [33] studied complete moment convergence of moving average processes based on some dependent random variables; Kulik [15] derived limit theorems for moving averages with random coefficients and heavy-tailed noise; Thanh et al. [26] demonstrated the state observers with random sampling times and convergence analysis of double-indexed and randomly weighted sums of mixing processes; Saavedra et al. [22] gave an estimation of population spectrum for linear processes with random coefficients; Hosseini and Nezakati [10], Tang and Zhang [24], Lu and Wang [19] and Meng et al. [21] established complete moment convergence of linear processes for some dependent random variables with random coefficients.

Now, let us recall the notion of linear processes with random coefficients introduced by Hosseini and Nezakati [10].

DEFINITION 1.4. Let $\{Y_j,\ -\infty < j < \infty\}$ and $\{A_j,\ -\infty < j < \infty\}$ be two sequences of random variables, and

$$X_n = \sum_{j=-\infty}^{\infty} A_j Y_{n-j}.$$

Then $\{X_n,\ n \geqslant 1\}$ is a *linear process* or an *infinite moving average process with random coefficients* $\{A_j,\ -\infty < j < \infty\}$.

Inspired by the above literature, we further investigate complete f-moment convergence, complete sth moment convergence and complete convergence for m-WOD linear processes with random coefficients. In contrast to the conclusions of Lu and Wang [19], we generalize WOD random variables to m-WOD random variables, and extend complete moment convergence to complete f-moment convergence and complete sth moment convergence. In this paper, we further consider the convergence properties of m-WOD random variables in the case $q > 2$. Finally, we derive complete f-moment consistency and complete consistency for a weighted estimator in nonparametric models based on WOD linear processes with random coefficients, and carry out a numerical simulation to verify the validity of our theoretical results.

The following definition of stochastic domination will play an important role throughout the paper.

DEFINITION 1.5. A sequence $\{Y_n,\ n \geqslant 1\}$ of random variables is said to be *stochastically dominated* by a random variable Y if there exists a positive constant C such that

$$P(|Y_n| > x) \leqslant CP(|Y| > x)$$

for all $x \geqslant 0$ and $n \geqslant 1$.

The structure of this article is as follows. Some preliminary lemmas are stated in Section 2. The main results and their proofs are provided in Section 3. An application is given in Section 4.

Throughout this paper, the symbol C represents positive constants which may be different in various places and do not depend on n. Let $I(A)$ be the indicator function of the set A, $\lfloor x \rfloor$ be the integer part of x and $\log x = \ln \max\{x, e\}$.

2. SOME LEMMAS

In this section, we present some basic lemmas needed to prove the main results. The first one is a fundamental property of m-WOD random variables, which can be found in Fang et al. [7].

LEMMA 2.1. *Let $\{Y_n,\ n \geqslant 1\}$ be a sequence of m-WOD random variables. If $\{f_n(\cdot),\ n \geqslant 1\}$ are all nondecreasing (or nonincreasing) functions, then the random variables $\{f_n(Y_n), n \geqslant 1\}$ are still m-WOD with the same dominating coefficients.*

By Definition 1.5, we obtain the following lemma, which can be found in Adler and Rosalsky [1] and Adler et al. [2] for instance.

LEMMA 2.2. *Let $\{Y_n,\ n \geqslant 1\}$ be a sequence of random variables which is stochastically dominated by a random variable Y. Then for any $\alpha, b > 0$,*

$$E|Y_n|^\alpha I(|Y_n| \leqslant b) \leqslant C_1[E|Y|^\alpha I(|Y| \leqslant b) + b^\alpha P(|Y| > b)],$$
$$E|Y_n|^\alpha I(|Y_n| > b) \leqslant C_2 E|Y|^\alpha I(|Y| > b),$$

where C_1 and C_2 are positive constants. Thus, $E|Y_n|^\alpha \leqslant C_3 E|Y|^\alpha$, where C_3 is a positive constant.

The next lemma comes from Wu et al. [37] and is useful for our later proof.

LEMMA 2.3. *Let $\{Y_i,\ 1 \leqslant i \leqslant n\}$ and $\{Z_i,\ 1 \leqslant i \leqslant n\}$ be two sequences of random variables. Then for any $q > r > 0$ and $\epsilon, a > 0$,*

$$E\Big(\Big|\sum_{i=1}^n (Y_i + Z_i)\Big| - \epsilon a\Big)_+^r \leqslant C_r\left(\epsilon^{-q} + \frac{r}{q-r}\right) a^{r-q} E\Big|\sum_{i=1}^n Y_i\Big|^q + C_r E\Big|\sum_{i=1}^n Z_i\Big|^r$$

and

$$E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^k (Y_i + Z_i)\Big| - \epsilon a\Big)_+^r$$
$$\leqslant C_r\left(\epsilon^{-q} + \frac{r}{q-r}\right) a^{r-q} E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^k Y_i\Big|^q\Big) + C_r E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^k Z_i\Big|^r\Big),$$

where $C_r = 1$ if $0 < r \leqslant 1$, while $C_r = 2^{r-1}$ if $r > 1$.

The following lemma gives Marcinkiewicz–Zygmund-type inequalities for WOD random variables, as presented by Wang et al. [34] and Ding et al. [6].

LEMMA 2.4. *Let $1 \leqslant p \leqslant 2$ and $\{Y_n, n \geqslant 1\}$ be a sequence of WOD random variables with dominating coefficients $g(n)$. Assume that $EY_n = 0$ and $E|Y_n|^p < \infty$ for each $n \geqslant 1$. Then there exist positive constants $C_1(p)$, $C_2(p)$, $C_3(p)$ and $C_4(p)$ depending only on p such that for each $n \geqslant 1$,*

$$E\Big|\sum_{i=1}^{n} Y_i\Big|^p \leqslant [C_1(p) + C_2(p)g(n)] \sum_{i=1}^{n} E|Y_i|^p$$

and

$$E\Big(\max_{1\leqslant k\leqslant n} \Big|\sum_{i=1}^{k} Y_i\Big|^p\Big) \leqslant [C_3(p) + C_4(p)g(n)] \log^p n \sum_{i=1}^{n} E|Y_i|^p.$$

Applying Lemma 2.4, we can get the following Marcinkiewicz–Zygmund-type inequalities for m-WOD random variables.

LEMMA 2.5. *Let $1 \leqslant p \leqslant 2$ and $\{Y_n, n \geqslant 1\}$ be a sequence of m-WOD random variables with dominating coefficients $g(n)$. Assume that $EY_n = 0$ and $E|Y_n|^p < \infty$ for each $n \geqslant 1$. Then there exist positive constants $C_1(m,p)$, $C_2(m,p)$, $C_3(m,p)$ and $C_4(m,p)$ depending only on m and p such that for each $n \geqslant 1$,*

$$E\Big|\sum_{i=1}^{n} Y_i\Big|^p \leqslant [C_1(m,p) + C_2(m,p)g(n)] \sum_{i=1}^{n} E|Y_i|^p \tag{2.1}$$

and

$$E\Big(\max_{1\leqslant k\leqslant n} \Big|\sum_{i=1}^{k} Y_i\Big|^p\Big) \leqslant [C_3(m,p) + C_4(m,p)g(n)] \log^p n \sum_{i=1}^{n} E|Y_i|^p. \tag{2.2}$$

Proof. For fixed $n \geqslant m$, let $r = \lfloor n/m \rfloor$. Define

$$Z_i = \begin{cases} Y_i, & 1 \leqslant i \leqslant n, \\ 0, & i > n. \end{cases}$$

Noting that $\sum_{i=1}^{n} Y_i = \sum_{j=1}^{m} \sum_{i=0}^{r} Z_{mi+j}$, we have

$$\Big|\sum_{i=1}^{n} Y_i\Big| \leqslant \sum_{j=1}^{m} \Big|\sum_{i=0}^{r} Z_{mi+j}\Big|$$

and $\{Z_{mi+j},\ 0 \leqslant i \leqslant r\}$ are WOD random variables with dominating coefficients $g(n)$ for each $1 \leqslant j \leqslant m$ by the definition of m-WOD random variables. By the above inequality, the C_r-inequality and Lemma 2.4, we get

$$E\Big(\Big|\sum_{i=1}^{n} Y_i\Big|^p\Big) \leqslant E\Big(\sum_{j=1}^{m} \Big|\sum_{i=0}^{r} Z_{mi+j}\Big|\Big)^p \leqslant m^{p-1} \sum_{j=1}^{m} E\Big(\Big|\sum_{i=0}^{r} Z_{mi+j}\Big|^p\Big)$$

$$\leqslant m^{p-1}[C_1(p)+C_2(p)g(n)]\sum_{j=1}^{m}\sum_{i=0}^{r}E|Z_{mi+j}|^p$$
$$=[C_1(m,p)+C_2(m,p)g(n)]\sum_{i=1}^{n}E|Y_i|^p.$$

Thus, (2.1) has been proved.

Next, we will prove (2.2). For $1\leqslant k\leqslant n$, let $r'=\lfloor k/m\rfloor$. Define

$$Z_i'=\begin{cases}Y_i, & 1\leqslant i\leqslant k,\\ 0, & i>k.\end{cases}$$

Obviously, we have

$$\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k}Y_i\Big|\leqslant\sum_{j=1}^{m}\max_{0\leqslant r'\leqslant r}\Big|\sum_{i=0}^{r'}Z'_{mi+j}\Big|.$$

By the above inequality, the C_r-inequality and Lemma 2.4, we get

$$\begin{aligned}E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k}Y_i\Big|^p\Big)&\leqslant E\Big(\sum_{j=1}^{m}\max_{0\leqslant r'\leqslant r}\Big|\sum_{i=0}^{r'}Z'_{mi+j}\Big|\Big)^p\\&\leqslant m^{p-1}\sum_{j=1}^{m}E\Big(\max_{0\leqslant r'\leqslant r}\Big|\sum_{i=0}^{r'}Z'_{mi+j}\Big|^p\Big)\\&\leqslant m^{p-1}[C_3(p)+C_4(p)g(n)]\log^p n\sum_{j=1}^{m}\sum_{i=0}^{r}E|Z'_{mi+j}|^p\\&=[C_3(m,p)+C_4(m,p)g(n)]\log^p n\sum_{i=1}^{n}E|Y_i|^p.\ \blacksquare\end{aligned}$$

According to Fang et al. [7] and Wang and Shen [33], the following Rosenthal-type inequalities hold for m-WOD random variables.

LEMMA 2.6. *Let $p>2$ and $\{Y_n,\ n\geqslant 1\}$ be a sequence of m-WOD random variables with dominating coefficients $g(n)$. Assume that $EY_n=0$ and $E|Y_n|^p<\infty$ for each $n\geqslant 1$. Then there exist positive constants $C_5(m,p)$, $C_6(m,p)$, $C_7(m,p)$ and $C_8(m,p)$ depending only on m and p such that for each $n\geqslant 1$,*

$$E\Big(\Big|\sum_{i=1}^{n}Y_i\Big|^p\Big)\leqslant C_5(m,p)\sum_{i=1}^{n}E|Y_i|^p+C_6(m,p)g(n)\Big(\sum_{i=1}^{n}EY_i^2\Big)^{p/2}$$

and

$$\begin{aligned}E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k}Y_i\Big|^p\Big)&\leqslant C_7(m,p)\log^p n\sum_{i=1}^{n}E|Y_i|^p\\&\quad+C_8(m,p)g(n)\log^p n\Big(\sum_{i=1}^{n}EY_i^2\Big)^{p/2}.\end{aligned}$$

3. MAIN RESULTS AND THEIR PROOFS

First, we introduce a function

$$f : \mathbb{R}^+ \to \mathbb{R}^+,$$

which is strictly increasing and continuous with $f(0) = 0$.

Let

$$h : \mathbb{R}^+ \to \mathbb{R}^+$$

be the inverse function of $f(t)$, i.e., $h(f(t)) = t,\ t \geqslant 0$. Assume that for some positive constants s and δ, the function f satisfies the condition

$$\int_{f(\delta)}^{\infty} h^{-s}(t)\, dt < \infty. \tag{3.1}$$

The following theorem discusses complete f-moment convergence for m-WOD linear processes with random coefficients when $1 < p < 2$.

THEOREM 3.1. *Let* $\{X_n = \sum_{j=-\infty}^{\infty} A_j Y_{n-j},\ n \geqslant 1\}$ *be a linear process with random coefficients* $\{A_j,\ -\infty < j < \infty\}$*, and* $\{Y_j,\ -\infty < j < \infty\}$ *be a sequence of zero mean* m*-WOD random variables with dominating coefficients* $g(n) = \max\{g_U(n),\ g_L(n)\}$*, which is stochastically dominated by a random variable* Y*. Assume that* $\{Y_j,\ -\infty < j < \infty\}$ *is independent of* $\{A_j,\ -\infty < j < \infty\}$*. Suppose that* $\alpha > 0$*,* $1 < p < 2$ *and* $p < q \leqslant 2$ *are such that*

$$E\Big(\sum_{j=-\infty}^{\infty} |A_j|\Big)^q < \infty.$$

Assume further that

$$g(n) = O(n^{\alpha\lambda}) \quad \text{for some } 0 \leqslant \lambda < q - p$$

and there exists some $1 \leqslant s < p + \lambda$ *such that* (3.1) *holds. Then we have the following two results:*

(i) *If* $E|Y|^{p+\lambda} < \infty$*, then for all* $\epsilon > 0$*,*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} E f\Big(\Big\{n^{-\alpha}\Big|\sum_{i=1}^{n} X_i\Big| - \epsilon\Big\}_+\Big) < \infty. \tag{3.2}$$

(ii) *If* $E|Y|^{p+\lambda}\log^q(1 + |Y|) < \infty$*, then for all* $\epsilon > 0$*,*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} E f\Big(\Big\{n^{-\alpha} \cdot \max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k} X_i\Big| - \epsilon\Big\}_+\Big) < \infty. \tag{3.3}$$

Proof. For each $n \geqslant 1$ and $-\infty < i < \infty$ denote

$$Y'_{ni} = -n^\alpha I(Y_i < -n^\alpha) + Y_i I(|Y_i| \leqslant n^\alpha) + n^\alpha I(Y_i > n^\alpha),$$
$$Y''_{ni} = (Y_i + n^\alpha) I(Y_i < -n^\alpha) + (Y_i - n^\alpha) I(Y_i > n^\alpha).$$

Since $EY_i = 0$ for $-\infty < i < \infty$, for each $n \geqslant 1$ we have

$$\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \sum_{j=-\infty}^{\infty} A_j Y_{i-j} = \sum_{j=-\infty}^{\infty} \sum_{i=1-j}^{n-j} A_j Y_i$$
$$= \sum_{j=-\infty}^{\infty} \sum_{i=1-j}^{n-j} A_j (Y'_{ni} - EY'_{ni}) + \sum_{j=-\infty}^{\infty} \sum_{i=1-j}^{n-j} A_j (Y''_{ni} - EY''_{ni}).$$

It follows from Markov's inequality and (3.1) that

$$\sum_{n=1}^{\infty} n^{\alpha p-2} E f\Big(\Big\{n^{-\alpha}\Big|\sum_{i=1}^{n} X_i\Big| - \epsilon\Big\}_+\Big)$$
$$= \sum_{n=1}^{\infty} n^{\alpha p-2} \int_0^{f(\delta)} P\Big(\Big|\sum_{i=1}^{n} X_i\Big| - \epsilon n^\alpha > h(t) n^\alpha\Big)\, dt$$
$$+ \sum_{n=1}^{\infty} n^{\alpha p-2} \int_{f(\delta)}^{\infty} P\Big(\Big|\sum_{i=1}^{n} X_i\Big| - \epsilon n^\alpha > h(t) n^\alpha\Big)\, dt$$
$$\leqslant f(\delta) \sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\Big|\sum_{i=1}^{n} X_i\Big| > \epsilon n^\alpha\Big)$$
$$+ \sum_{n=1}^{\infty} n^{\alpha p-\alpha s-2} E\Big(\Big|\sum_{i=1}^{n} X_i\Big| - \epsilon n^\alpha\Big)_+^s \int_{f(\delta)}^{\infty} h^{-s}(t)\, dt$$
$$\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\Big|\sum_{i=1}^{n} X_i\Big| > \epsilon n^\alpha\Big) + C \sum_{n=1}^{\infty} n^{\alpha p-\alpha s-2} E\Big(\Big|\sum_{i=1}^{n} X_i\Big| - \epsilon n^\alpha\Big)_+^s$$
$$=: I_1 + I_2.$$

For I_1, by Markov's inequality we have

$$I_1 = C \sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\Big| \sum_{j=-\infty}^{\infty} \sum_{i=1-j}^{n-j} A_j (Y'_{ni} - EY'_{ni})$$
$$+ \sum_{j=-\infty}^{\infty} \sum_{i=1-j}^{n-j} A_j (Y''_{ni} - EY''_{ni})\Big| > \epsilon n^\alpha\Big)$$
$$\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\Big| \sum_{j=-\infty}^{\infty} \sum_{i=1-j}^{n-j} A_j (Y'_{ni} - EY'_{ni})\Big| > \frac{\epsilon n^\alpha}{2}\Big)$$
$$+ C \sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\Big| \sum_{j=-\infty}^{\infty} \sum_{i=1-j}^{n-j} A_j (Y''_{ni} - EY''_{ni})\Big| > \frac{\epsilon n^\alpha}{2}\Big)$$

$$\begin{aligned}
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}E\Big|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{n-j} A_j(Y'_{ni}-EY'_{ni})\Big|^q\\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}E\Big|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{n-j} A_j(Y''_{ni}-EY''_{ni})\Big|\\
&=: I_3+I_4,
\end{aligned}$$

while for I_2, it follows from Lemma 2.3 that

$$\begin{aligned}
I_2 &= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha s-2}E\Big(\Big|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{n-j} A_j(Y'_{ni}-EY'_{ni})\\
&\quad + \sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{n-j} A_j(Y''_{ni}-EY''_{ni})\Big| - \epsilon n^{\alpha}\Big)_+^s\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}E\Big|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{n-j} A_j(Y'_{ni}-EY'_{ni})\Big|^q\\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha s-2}E\Big|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{n-j} A_j(Y''_{ni}-EY''_{ni})\Big|^s\\
&=: I_3+I_5.
\end{aligned}$$

Hence, to prove (3.2), we only need to show that $I_3, I_4, I_5 < \infty$.

Obviously, we can easily see that $\{Y'_{ni}-EY'_{ni},\ -\infty<i<\infty\}$ and $\{Y''_{ni}-EY''_{ni},\ -\infty<i<\infty\}$ are both sequences of m-WOD random variables by Lemma 2.1. By Hölder's inequality, Lemma 2.5, the C_r-inequality and Lemma 2.2, we have

$$\begin{aligned}
I_3 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}E\Big(\sum_{j=-\infty}^{\infty}|A_j|\Big|\sum_{i=1-j}^{n-j}(Y'_{ni}-EY'_{ni})\Big|\Big)^q\\
&= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}E\Big(\sum_{j=-\infty}^{\infty}|A_j|^{1-1/q}|A_j|^{1/q}\Big|\sum_{i=1-j}^{n-j}(Y'_{ni}-EY'_{ni})\Big|\Big)^q\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}E\Big[\Big(\sum_{j=-\infty}^{\infty}|A_j|\Big)^{q-1}\Big(\sum_{j=-\infty}^{\infty}|A_j|\Big|\sum_{i=1-j}^{n-j}(Y'_{ni}-EY'_{ni})\Big|^q\Big)\Big]\\
&= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\sum_{l=1}^{\infty}E\Big[\Big(\sum_{j=-\infty}^{\infty}|A_j|\Big)^{q-1}\Big(\sum_{j=-\infty}^{\infty}|A_j|\Big|\sum_{i=1-j}^{n-j}(Y'_{ni}-EY'_{ni})\Big|^q\Big)\\
&\quad\times \underbrace{I\Big(l-1\leqslant\sum_{j=-\infty}^{\infty}|A_j|<l\Big)}_{\mathcal{A}_l}\Big]\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\sum_{l=1}^{\infty} l^{q-1}\sum_{j=-\infty}^{\infty}E|A_j|I(\mathcal{A}_l)E\Big|\sum_{i=1-j}^{n-j}(Y'_{ni}-EY'_{ni})\Big|^q
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} g(n) \sum_{l=1}^{\infty} l^{q-1} \sum_{j=-\infty}^{\infty} E|A_j| I(\mathcal{A}_l) \sum_{i=1-j}^{n-j} E|Y'_{ni}|^q \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\alpha\lambda-1} \sum_{l=1}^{\infty} l^{q-1} \sum_{j=-\infty}^{\infty} E|A_j| I(\mathcal{A}_l) \\
&\quad \times [E|Y|^q I(|Y| \leqslant n^\alpha) + n^{\alpha q} P(|Y| > n^\alpha)] \\
&= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\alpha\lambda-1} [E|Y|^q I(|Y| \leqslant n^\alpha) + n^{\alpha q} P(|Y| > n^\alpha)] \\
&\quad \times \Big[\sum_{j=-\infty}^{\infty} E|A_j| I\Big(\sum_{j=-\infty}^{\infty} |A_j| < 1 \Big) + \sum_{l=2}^{\infty} l^{q-1} \sum_{j=-\infty}^{\infty} E|A_j| I(\mathcal{A}_l) \Big] \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\alpha\lambda-1} [E|Y|^q I(|Y| \leqslant n^\alpha) + n^{\alpha q} P(|Y| > n^\alpha)] \\
&\quad \times \Big[1 + \sum_{l=2}^{\infty} \Big(\frac{l}{l-1} \Big)^{q-1} E\Big(\sum_{j=-\infty}^{\infty} |A_j| \Big)^q I(\mathcal{A}_l) \Big] \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\alpha\lambda-1} \Big(1 + 2^{q-1} E\Big(\sum_{j=-\infty}^{\infty} |A_j| \Big)^q \Big) \\
&\quad \times [E|Y|^q I(|Y| \leqslant n^\alpha) + n^{\alpha q} P(|Y| > n^\alpha)] \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\alpha\lambda-1} E|Y|^q I(|Y| \leqslant n^\alpha) + C\sum_{n=1}^{\infty} n^{\alpha p+\alpha\lambda-1} P(|Y| > n^\alpha) \\
&=: I_6 + I_7.
\end{aligned}
$$

By the slicing technique, we get

$$
\begin{aligned}
I_6 &= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\alpha\lambda-1} \sum_{i=1}^{n} E|Y|^q I((i-1)^\alpha < |Y| \leqslant i^\alpha) \\
&\leqslant C\sum_{i=1}^{\infty} i^{\alpha p-\alpha q+\alpha\lambda} E|Y|^q I((i-1)^\alpha < |Y| \leqslant i^\alpha) \\
&\leqslant CE|Y|^{p+\lambda} < \infty
\end{aligned}
$$

and

$$
\begin{aligned}
I_7 &= C\sum_{n=1}^{\infty} n^{\alpha p+\alpha\lambda-1} \sum_{i=n}^{\infty} P(i^\alpha < |Y| \leqslant (i+1)^\alpha) \\
&\leqslant C\sum_{i=1}^{\infty} i^{\alpha p+\alpha\lambda} P(i^\alpha < |Y| \leqslant (i+1)^\alpha) \\
&\leqslant CE|Y|^{p+\lambda} < \infty.
\end{aligned}
$$

Next, we will prove $I_5 < \infty$ in the following two cases.

CASE I: $1 < s < p + \lambda$. By Hölder's inequality and Lemma 2.5, we obtain

$$\begin{aligned}
I_5 &\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha s-2} E\Big(\sum_{j=-\infty}^{\infty} |A_j| \Big| \sum_{i=1-j}^{n-j} (Y''_{ni} - EY''_{ni}) \Big| \Big)^s \\
&= C \sum_{n=1}^{\infty} n^{\alpha p-\alpha s-2} E\Big(\sum_{j=-\infty}^{\infty} |A_j|^{1-1/s} |A_j|^{1/s} \Big| \sum_{i=1-j}^{n-j} (Y''_{ni} - EY''_{ni}) \Big| \Big)^s \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha s-2} \\
&\quad \times \sum_{l=1}^{\infty} E\Big[\Big(\sum_{j=-\infty}^{\infty} |A_j| \Big)^{s-1} \Big(\sum_{j=-\infty}^{\infty} |A_j| \Big| \sum_{i=1-j}^{n-j} (Y''_{ni} - EY''_{ni}) \Big|^s \Big) I(\mathcal{A}_l) \Big] \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha s-2} \sum_{l=1}^{\infty} l^{s-1} \sum_{j=-\infty}^{\infty} E|A_j| I(\mathcal{A}_l) E \Big| \sum_{i=1-j}^{n-j} (Y''_{ni} - EY''_{ni}) \Big|^s \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha s+\alpha\lambda-1} \sum_{l=1}^{\infty} l^{s-1} \sum_{j=-\infty}^{\infty} E|A_j| I(\mathcal{A}_l) E|Y|^s I(|Y| > n^{\alpha}) \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha s+\alpha\lambda-1} \Big(1 + 2^{s-1} E\Big(\sum_{j=-\infty}^{\infty} |A_j| \Big)^s \Big) E|Y|^s I(|Y| > n^{\alpha}) \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha s+\alpha\lambda-1} \Big(1 + 2^{s-1} \Big(E\Big(\sum_{j=-\infty}^{\infty} |A_j| \Big)^q \Big)^{s/q} \Big) E|Y|^s I(|Y| > n^{\alpha}) \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha s+\alpha\lambda-1} \sum_{i=n}^{\infty} E|Y|^s I(i^{\alpha} < |Y| \leqslant (i+1)^{\alpha}) \\
&\leqslant C \sum_{i=1}^{\infty} i^{\alpha p-\alpha s+\alpha\lambda} E|Y|^s I(i^{\alpha} < |Y| \leqslant (i+1)^{\alpha}) \\
&\leqslant C E|Y|^{p+\lambda} < \infty.
\end{aligned}$$

CASE II: $s = 1$. It is easy to see that

$$\begin{aligned}
I_5 &= C \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} E \Big| \sum_{j=-\infty}^{\infty} \sum_{i=1-j}^{n-j} A_j (Y''_{ni} - EY''_{ni}) \Big| \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} \sum_{j=-\infty}^{\infty} E|A_j| \sum_{i=1-j}^{n-j} E|Y''_{ni}| \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha-1} \sum_{j=-\infty}^{\infty} E|A_j| E|Y| I(|Y| > n^{\alpha}) \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha-1} \Big(E\Big(\sum_{j=-\infty}^{\infty} |A_j| \Big)^q \Big)^{1/q} \sum_{i=n}^{\infty} E|Y| I(i^{\alpha} < |Y| \leqslant (i+1)^{\alpha}) \\
&\leqslant C \sum_{i=1}^{\infty} i^{\alpha p-\alpha} E|Y| I(i^{\alpha} < |Y| \leqslant (i+1)^{\alpha}) \leqslant C E|Y|^p < \infty.
\end{aligned}$$

Thus we have verified that $I_4 < \infty$ and (3.2) holds.

In the following, we will prove that (3.3) holds. By Markov's inequality and (3.1), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} Ef\Big(\Big\{n^{-\alpha}\cdot \max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k} X_i\Big| - \epsilon\Big\}_+\Big) \\
&\qquad = \sum_{n=1}^{\infty} n^{\alpha p-2} \int_0^{f(\delta)} P\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k} X_i\Big| - \epsilon n^{\alpha} > h(t)n^{\alpha}\Big)\,dt \\
&\qquad\quad + \sum_{n=1}^{\infty} n^{\alpha p-2} \int_{f(\delta)}^{\infty} P\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k} X_i\Big| - \epsilon n^{\alpha} > h(t)n^{\alpha}\Big)\,dt \\
&\qquad \leqslant f(\delta)\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k} X_i\Big| > \epsilon n^{\alpha}\Big) \\
&\qquad\quad + \sum_{n=1}^{\infty} n^{\alpha p-\alpha s-2} E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k} X_i\Big| - \epsilon n^{\alpha}\Big)_+^{s} \int_{f(\delta)}^{\infty} h^{-s}(t)\,dt \\
&\qquad \leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k} X_i\Big| > \epsilon n^{\alpha}\Big) \\
&\qquad\quad + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha s-2} E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k} X_i\Big| - \epsilon n^{\alpha}\Big)_+^{s} \\
&\qquad =: J_1 + J_2.
\end{aligned}$$

For J_1, by Markov's inequality we have

$$\begin{aligned}
J_1 &= C\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{k-j} A_j(Y'_{ni} - EY'_{ni}) \\
&\quad + \sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{k-j} A_j(Y''_{ni} - EY''_{ni})\Big| > \epsilon n^{\alpha}\Big) \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{k-j} A_j(Y'_{ni} - EY'_{ni})\Big| > \frac{\epsilon n^{\alpha}}{2}\Big) \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{k-j} A_j(Y''_{ni} - EY''_{ni})\Big| > \frac{\epsilon n^{\alpha}}{2}\Big) \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{k-j} A_j(Y'_{ni} - EY'_{ni})\Big|^{q}\Big) \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{k-j} A_j(Y''_{ni} - EY''_{ni})\Big|\Big) \\
&=: J_3 + J_4,
\end{aligned}$$

and for J_2, it follows from Lemma 2.3 that

$$\begin{aligned}
J_2 &= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha s-2} E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{k-j} A_j(Y'_{ni}-EY'_{ni})\\
&\quad+\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{k-j} A_j(Y''_{ni}-EY''_{ni})\Big|-\epsilon n^{\alpha}\Big)_+^{s}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{k-j} A_j(Y'_{ni}-EY'_{ni})\Big|^{q}\Big)\\
&\quad+C\sum_{n=1}^{\infty} n^{\alpha p-\alpha s-2} E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{k-j} A_j(Y''_{ni}-EY''_{ni})\Big|^{s}\Big)\\
&=: J_3+J_5.
\end{aligned}$$

Hence, to prove (3.3), we only need to show that $J_3, J_4, J_5<\infty$.

Applying Hölder's inequality, Lemma 2.5, the C_r-inequality and Lemma 2.2, we derive

$$\begin{aligned}
J_3 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} E\Big(\sum_{j=-\infty}^{\infty}|A_j|\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1-j}^{k-j}(Y'_{ni}-EY'_{ni})\Big|\Big)^{q}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\\
&\quad\times E\Big[\Big(\sum_{j=-\infty}^{\infty}|A_j|\Big)^{q-1}\Big(\sum_{j=-\infty}^{\infty}|A_j|\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1-j}^{k-j}(Y'_{ni}-EY'_{ni})\Big|^{q}\Big)\Big]\\
&= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\\
&\quad\times\sum_{l=1}^{\infty} E\Big[\Big(\sum_{j=-\infty}^{\infty}|A_j|\Big)^{q-1}\Big(\sum_{j=-\infty}^{\infty}|A_j|\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1-j}^{k-j}(Y'_{ni}-EY'_{ni})\Big|^{q}\Big)I(\mathcal{A}_l)\Big]\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\sum_{l=1}^{\infty} l^{q-1}\sum_{j=-\infty}^{\infty} E|A_j|I(\mathcal{A}_l)\\
&\quad\times E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1-j}^{n-j}(Y'_{ni}-EY'_{ni})\Big|^{q}\Big)\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\alpha\lambda-1}\log^{q} n\sum_{l=1}^{\infty} l^{q-1}\sum_{j=-\infty}^{\infty} E|A_j|I(\mathcal{A}_l)\\
&\quad\times[E|Y|^{q}I(|Y|\leqslant n^{\alpha})+n^{\alpha q}P(|Y|>n^{\alpha})]\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\alpha\lambda-1}\log^{q} n\Big(1+2^{q-1}E\Big(\sum_{j=-\infty}^{\infty}|A_j|\Big)^{q}\Big)\\
&\quad\times[E|Y|^{q}I(|Y|\leqslant n^{\alpha})+n^{\alpha q}P(|Y|>n^{\alpha})]
\end{aligned}$$

$$\begin{aligned}
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\alpha\lambda-1}\log^q n\, E|Y|^q I(|Y|\leqslant n^\alpha)\\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p+\alpha\lambda-1}\log^q n\, P(|Y|>n^\alpha)\\
&=: J_6+J_7.
\end{aligned}$$

Using the slicing technique, we obtain

$$\begin{aligned}
J_6 &= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\alpha\lambda-1}\log^q n\sum_{i=1}^{n} E|Y|^q I((i-1)^\alpha<|Y|\leqslant i^\alpha)\\
&\leqslant C\sum_{i=1}^{\infty} i^{\alpha p-\alpha q+\alpha\lambda}\log^q i E|Y|^q I((i-1)^\alpha<|Y|\leqslant i^\alpha)\\
&\leqslant CE|Y|^{p+\lambda}\log^q(1+|Y|)<\infty
\end{aligned}$$

and

$$\begin{aligned}
J_7 &= C\sum_{n=1}^{\infty} n^{\alpha p+\alpha\lambda-1}\log^q n\sum_{i=n}^{\infty} P(i^\alpha<|Y|\leqslant (i+1)^\alpha)\\
&\leqslant C\sum_{i=1}^{\infty} i^{\alpha p+\alpha\lambda}\log^q i\, P(i^\alpha<|Y|\leqslant (i+1)^\alpha)\\
&\leqslant CE|Y|^{p+\lambda}\log^q(1+|Y|)<\infty.
\end{aligned}$$

Next, we will prove $J_5<\infty$ in the following two cases.

CASE I: $1<s<p+\lambda$. By Hölder's inequality and Lemma 2.5, we obtain

$$\begin{aligned}
J_5 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha s-2}E\Big(\sum_{j=-\infty}^{\infty}|A_j|\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1-j}^{k-j}(Y''_{ni}-EY''_{ni})\Big|\Big)^s\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha s-2}\sum_{l=1}^{\infty}E\Big[\Big(\sum_{j=-\infty}^{\infty}|A_j|\Big)^{s-1}\\
&\quad\times\Big(\sum_{j=-\infty}^{\infty}|A_j|\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1-j}^{k-j}(Y''_{ni}-EY''_{ni})\Big|^s\Big)I(\mathcal{A}_l)\Big]\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha s-2}\sum_{l=1}^{\infty} l^{s-1}\sum_{j=-\infty}^{\infty}E|A_j|I(\mathcal{A}_l)\\
&\quad\times E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1-j}^{k-j}(Y''_{ni}-EY''_{ni})\Big|^s\Big)\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha s+\alpha\lambda-1}\log^s n\sum_{l=1}^{\infty} l^{s-1}\sum_{j=-\infty}^{\infty}E|A_j|I(\mathcal{A}_l)E|Y|^s I(|Y|>n^\alpha)\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha s+\alpha\lambda-1}\log^s n\Big(1+2^{s-1}E\Big(\sum_{j=-\infty}^{\infty}|A_j|\Big)^s\Big)E|Y|^s I(|Y|>n^\alpha)
\end{aligned}$$

$$\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha s+\alpha\lambda-1}\log^s n\sum_{i=n}^{\infty} E|Y|^s I(i^\alpha<|Y|\leqslant(i+1)^\alpha)$$
$$\leqslant C\sum_{i=1}^{\infty} i^{\alpha p-\alpha s+\alpha\lambda}\log^s i\,E|Y|^s I(i^\alpha<|Y|\leqslant(i+1)^\alpha)$$
$$\leqslant CE|Y|^{p+\lambda}\log^s(1+|Y|)<\infty.$$

CASE II: $s=1$. Similarly to the proof of $I_5<\infty$ with $s=1$, we obtain

$$J_5 = C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{k-j} A_j(Y''_{ni}-EY''_{ni})\Big|\Big)$$
$$\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}\sum_{j=-\infty}^{\infty} E|A_j|\sum_{i=1-j}^{n-j} E|Y''_{ni}|$$
$$\leqslant CE|Y|^p<\infty.$$

Thus we have verified that $J_4<\infty$ and (3.3) holds. The proof is complete. ∎

Under the conditions of Theorem 3.1, the complete f-moment convergence for m-WOD linear processes has been studied in the case of $1<p<2$. The following theorem extends this analysis and gives the same results in the case of $p\geqslant 2$.

THEOREM 3.2. *Let* $\{X_n=\sum_{j=-\infty}^{\infty}A_jY_{n-j},\ n\geqslant 1\}$ *be a linear process with random coefficients* $\{A_j,\ -\infty<j<\infty\}$*, and* $\{Y_j,\ -\infty<j<\infty\}$ *be a sequence of zero mean* m*-WOD random variables with dominating coefficients* $g(n)=\max\{g_U(n),g_L(n)\}$*, which is stochastically dominated by a random variable* Y*. Assume that* $\{Y_j,\ -\infty<j<\infty\}$ *is independent of* $\{A_j,\ -\infty<j<\infty\}$*. Suppose that* $\alpha>1/2$*,* $p\geqslant 2$ *and* $q>2(\alpha p-1)/(2\alpha-1)$ *are such that*

$$E\Big(\sum_{j=-\infty}^{\infty}|A_j|\Big)^q<\infty.$$

Assume further that

$$g(n)=O(n^{\alpha\lambda}) \quad \textit{for some } 0\leqslant\lambda<q-p-q/(2\alpha)+1/\alpha$$

and there exists some $1\leqslant s<2$ *such that* (3.1) *holds. Then we have the following two results:*

(i) *If* $E|Y|^{p+\lambda}<\infty$*, then for all* $\epsilon>0$*,*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}Ef\Big(\Big\{n^{-\alpha}\Big|\sum_{i=1}^{n}X_i\Big|-\epsilon\Big\}_+\Big)<\infty. \tag{3.4}$$

(ii) *If* $E|Y|^{p+\lambda}\log^q(1+|Y|)<\infty$*, then for all* $\epsilon>0$*,*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}Ef\Big(\Big\{n^{-\alpha}\cdot\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k}X_i\Big|-\epsilon\Big\}_+\Big)<\infty. \tag{3.5}$$

Proof. In the following proof, we use the same symbols as in the proof of Theorem 3.1. According to the proof of Theorem 3.1, to prove (3.4), we only need to show that $I_3 < \infty$.

Note that

$$q > \frac{2(\alpha p - 1)}{2\alpha - 1} \geqslant p \geqslant 2.$$

By Hölder's inequality and Lemma 2.6, we have

$$\begin{aligned}
I_3 &\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} E\Big(\sum_{j=-\infty}^{\infty} |A_j| \Big| \sum_{i=1-j}^{n-j} (Y'_{ni} - EY'_{ni}) \Big| \Big)^q \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} \\
&\quad \times \sum_{l=1}^{\infty} E\Big[\Big(\sum_{j=-\infty}^{\infty} |A_j| \Big)^{q-1} \Big(\sum_{j=-\infty}^{\infty} |A_j| \Big| \sum_{i=1-j}^{n-j} (Y'_{ni} - EY'_{ni}) \Big|^q \Big) I(\mathcal{A}_l) \Big] \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} \sum_{l=1}^{\infty} l^{q-1} \sum_{j=-\infty}^{\infty} E|A_j| I(\mathcal{A}_l) E \Big| \sum_{i=1-j}^{n-j} (Y'_{ni} - EY'_{ni}) \Big|^q \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} \sum_{l=1}^{\infty} l^{q-1} \sum_{j=-\infty}^{\infty} E|A_j| I(\mathcal{A}_l) \sum_{i=1-j}^{n-j} E|Y'_{ni}|^q \\
&\quad + C \sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\alpha\lambda-2} \sum_{l=1}^{\infty} l^{q-1} \sum_{j=-\infty}^{\infty} E|A_j| I(\mathcal{A}_l) \Big(\sum_{i=1-j}^{n-j} E|Y'_{ni}|^2 \Big)^{q/2} \\
&=: I_{31} + I_{32}.
\end{aligned}$$

Similar to the proof of $I_3 < \infty$ in Theorem 3.1, we get $I_{31} < \infty$. Hence, we only need to prove $I_{32} < \infty$. We can easily derive

$$\begin{aligned}
I_{32} &\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\alpha\lambda-2} \sum_{l=1}^{\infty} l^{q-1} \sum_{j=-\infty}^{\infty} E|A_j| I(\mathcal{A}_l) (nE|Y|^2)^{q/2} \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\alpha\lambda+q/2-2} \sum_{l=1}^{\infty} l^{q-1} \sum_{j=-\infty}^{\infty} E|A_j| I(\mathcal{A}_l) \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\alpha\lambda+q/2-2} \Big(1 + 2^{q-1} E\Big(\sum_{j=-\infty}^{\infty} |A_j| \Big)^q \Big) \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\alpha\lambda+q/2-2} < \infty.
\end{aligned}$$

Thus, we have proved that (3.4) holds.

Finally, we will prove (3.5). From the proof of (3.3), it follows that to prove (3.5), we only need to show that $J_3 < \infty$. By Hölder's inequality and Lemma 2.6, we have

$$J_3 \leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} E\Big(\sum_{j=-\infty}^{\infty} |A_j| \max_{1\leqslant k\leqslant n} \Big| \sum_{i=1-j}^{k-j} (Y'_{ni} - EY'_{ni}) \Big| \Big)^q$$

$$
\begin{aligned}
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} \\
&\quad\times \sum_{l=1}^{\infty} E\Big[\Big(\sum_{j=-\infty}^{\infty}|A_j|\Big)^{q-1}\Big(\sum_{j=-\infty}^{\infty}|A_j|\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1-j}^{k-j}(Y'_{ni}-EY'_{ni})\Big|^q\Big)I(\mathcal{A}_l)\Big] \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} \\
&\quad\times \sum_{l=1}^{\infty} l^{q-1}\sum_{j=-\infty}^{\infty} E|A_j|I(\mathcal{A}_l)E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1-j}^{k-j}(Y'_{ni}-EY'_{ni})\Big|^q\Big) \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\log^q n\sum_{l=1}^{\infty} l^{q-1}\sum_{j=-\infty}^{\infty} E|A_j|I(\mathcal{A}_l)\sum_{i=1-j}^{n-j}E|Y'_{ni}|^q \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\alpha\lambda-2}\log^q n \\
&\quad\times \sum_{l=1}^{\infty} l^{q-1}\sum_{j=-\infty}^{\infty} E|A_j|I(\mathcal{A}_l)\Big(\sum_{i=1-j}^{n-j}E|Y'_{ni}|^2\Big)^{q/2} \\
&=: J_{31}+J_{32}.
\end{aligned}
$$

Similar to the proof of $J_3<\infty$ in Theorem 3.1, we get $J_{31}<\infty$. Hence, we only need to prove $J_{32}<\infty$. It is evident that

$$
\begin{aligned}
J_{32} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\alpha\lambda-2}\log^q n\sum_{l=1}^{\infty} l^{q-1}\sum_{j=-\infty}^{\infty} E|A_j|I(\mathcal{A}_l)(nEY^2)^{q/2} \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\alpha\lambda+q/2-2}\log^q n\Big(1+2^{q-1}E\Big(\sum_{j=-\infty}^{\infty}|A_j|\Big)^q\Big) \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\alpha\lambda+q/2-2}\log^q n<\infty. \quad\blacksquare
\end{aligned}
$$

Based on the proofs of Theorems 3.1 and 3.2, we record the following special cases.

COROLLARY 3.1. *Let* $\{X_n=\sum_{j=-\infty}^{\infty}A_jY_{n-j},\ n\geqslant 1\}$ *be a linear process with random coefficients* $\{A_j,\ -\infty<j<\infty\}$*, and* $\{Y_j,\ -\infty<j<\infty\}$ *be a sequence of zero mean* m*-WOD random variables with dominating coefficients* $g(n)=\max\{g_U(n),g_L(n)\}$*, which is stochastically dominated by a random variable* Y*. Assume that* $\{Y_j,\ -\infty<j<\infty\}$ *is independent of* $\{A_j,\ -\infty<j<\infty\}$*. Assume further that one of the following conditions hold:*

(a) *Suppose that* $\alpha>0$*,* $1<p<2$ *and* $p<q\leqslant 2$ *are such that*

$$E\Big(\sum_{j=-\infty}^{\infty}|A_j|\Big)^q<\infty$$

and

$$g(n) = O(n^{\alpha\lambda}) \quad \textit{for some } 0 \leqslant \lambda < q - p.$$

(b) *Suppose that $\alpha > 1/2$, $p \geqslant 2$ and $q > 2(\alpha p - 1)/(2\alpha - 1)$ are such that*

$$E\Big(\sum_{j=-\infty}^{\infty} |A_j|\Big)^q < \infty$$

and

$$g(n) = O(n^{\alpha\lambda}) \quad \textit{for some } 0 \leqslant \lambda < q - p - q/(2\alpha) + 1/\alpha.$$

Then we have the following two results:

(i) *If $E|Y|^{p+\lambda} < \infty$, then for all $\epsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p - \alpha s - 2} E\Big(\Big|\sum_{i=1}^{n} X_i\Big| - \epsilon n^{\alpha}\Big)_+^s < \infty$$

and thus

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\Big(\Big|\sum_{i=1}^{n} X_i\Big| > \epsilon n^{\alpha}\Big) < \infty.$$

(ii) *If $E|Y|^{p+\lambda} \log^q(1 + |Y|) < \infty$, then for all $\epsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p - \alpha s - 2} E\Big(\max_{1 \leqslant k \leqslant n} \Big|\sum_{i=1}^{k} X_i\Big| - \epsilon n^{\alpha}\Big)_+^s < \infty$$

and thus

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\Big(\max_{1 \leqslant k \leqslant n} \Big|\sum_{i=1}^{k} X_i\Big| > \epsilon n^{\alpha}\Big) < \infty. \tag{3.6}$$

REMARK 3.1. This work extends the theoretical results of Lu and Wang [19] in the following points:

(i) Complete moment convergence is generalized to complete f-moment convergence and complete sth moment convergence.
(ii) WOD random variables are extended to m-WOD random variables under the same conditions.
(iii) Complete f-moment convergence, complete sth moment convergence and complete convergence for m-WOD linear processes with random coefficients in the $p \geqslant 2$ case are obtained.

4. AN APPLICATION IN NONPARAMETRIC REGRESSION MODELS

In this section, we will apply the results obtained in (3.5) of Theorem 3.2 and (3.6) of Corollary 3.1 to a nonparametric regression model and give complete f-moment consistency and complete consistency for a weighted estimator based on a linear process with WOD errors. Finally, a numerical simulation is provided.

4.1. Theoretical results. In this subsection, we first introduce the following nonparametric regression model:

$$Y_{ni} = K(x_{ni}) + \varepsilon_{ni}, \quad i = 1, \dots, n,\ n \geqslant 1, \tag{4.1}$$

where x_{ni} are known fixed design points from A, $A \subset \mathbb{R}^d$ is a given compact set for some positive integer $d \geqslant 1$, $K(\cdot)$ is an unknown regression function defined on A and ε_{ni} are random errors such that $(\varepsilon_{n1}, \varepsilon_{n2}, \dots, \varepsilon_{nn})$ has the same distribution as $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ for each $n \geqslant 1$. Then, to estimate the unknown function K, we consider the following weighted regression estimator:

$$\hat{K}_n(x) = \sum_{i=1}^{n} W_{ni}(x) Y_{ni}, \quad x \in A \subset \mathbb{R}^d, \tag{4.2}$$

where $W_{ni}(x) = W_{ni}(x; x_{n1}, x_{n2}, \dots, x_{nn})$, $i = 1, \dots, n$, are weight functions.

The above estimator (4.2) was introduced by Stone [23], and many authors have studied its asymptotic properties. For more details about the above estimator with constant weight, one can refer to Georgiev and Greblicki [8], Tran et al. [28], Thanh and Yin [27], Chen et al. [3], Yan [41], Wu et al. [37], Lu et al. [18], Wu et al. [38], Xi et al. [40], Deng et al. [5], and Wang et al. [31].

The symbol $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$. Let $c(K)$ denote the set of continuity points of the function K on A. For any point $x \in A$, the weight function $W_{ni}(x)$ satisfies the following assumptions:

(A_1) $\sum_{i=1}^{n} W_{ni}(x) \to 1$ as $n \to \infty$;

(A_2) $\sum_{i=1}^{n} |W_{ni}(x)| \leqslant C < \infty$ for all n;

(A_3) $\sum_{i=1}^{n} |W_{ni}(x)| \cdot |K(x_{ni}) - K(x)| I(\|x_{ni} - x\| > a) \to 0$ as $n \to \infty$ for all $a > 0$.

REMARK 4.1. Assumptions (A_1)–(A_3) about the weight function are very general, and Wang et al. [35] pointed out that the nearest neighbor weights satisfy these conditions. Many scholars have used the above assumptions to prove the convergence of the estimator $\hat{K}_n(x)$, including Xi et al. [40], Deng et al. [5], Wu et al. [39], and Wang et al. [31].

Based on the above assumptions, we can establish the following result about complete consistency for the nonparametric regression estimator $\hat{K}_n(x)$.

THEOREM 4.1. *Assume that conditions (A_1)–(A_3) are satisfied. Let $\{\varepsilon_n = \sum_{j=-\infty}^{\infty} A_j\delta_{n-j},\ n \geqslant 1\}$ be a linear process with random coefficients $\{A_j,\ -\infty < j < \infty\}$, and $\{\delta_j,\ -\infty < j < \infty\}$ be a sequence of zero mean WOD random variables with dominating coefficients $g(n) = \max\{g_U(n), g_L(n)\}$, which is stochastically dominated by a random variable δ. Assume that $\{\delta_j,\ -\infty < j < \infty\}$ is independent of $\{A_j,\ -\infty < j < \infty\}$. Suppose that $2 < p < 4$ and $q > 2p/(4-p)$ are such that*

$$E\Big(\sum_{j=-\infty}^{\infty} |A_j|\Big)^q < \infty.$$

Assume further that

$$g(n) = O(n^{2\lambda/p}) \quad \text{for some } 0 \leqslant \lambda < q - p/2 - pq/4.$$

If $E|\delta|^{p+\lambda}\log^q(1+|\delta|) < \infty$ and

$$\max_{1\leqslant i\leqslant n} |W_{ni}(x)| = O(n^{-2/p}), \tag{4.3}$$

then for any $x \in c(K)$,

$$\hat{K}_n(x) \to K(x) \quad \text{completely}.$$

Proof. Note that

$$|\hat{K}_n(x) - K(x)| \leqslant |\hat{K}_n(x) - E\hat{K}_n(x)| + |E\hat{K}_n(x) - K(x)|.$$

Hence, for all $a \geqslant 0$ and for any $x \in c(K)$, we have

$$\begin{aligned} |E\hat{K}_n(x) - K(x)| &\leqslant \sum_{i=1}^{n} |W_{ni}(x)| \cdot |K(x_{ni}) - K(x)| I(\|x_{ni} - x\| \leqslant a) \\ &\quad + \sum_{i=1}^{n} |W_{ni}(x)| \cdot |K(x_{ni}) - K(x)| I(\|x_{ni} - x\| > a) \\ &\quad + |K(x)|\Big|\sum_{i=1}^{n} W_{ni}(x) - 1\Big|. \end{aligned} \tag{4.4}$$

Because $x \in c(K)$, we know that for all $\epsilon > 0$, there exists a constant $\varsigma > 0$ such that $|K(x_0) - K(x)| < \epsilon$ for all $x_0 \in A$ as long as $\|x_0 - x\| < \varsigma$. Take $0 < a < \varsigma$ in (4.4). Due to arbitrariness of $\epsilon > 0$, we get

$$
\begin{aligned}
&|E\hat{K}_n(x) - K(x)| \\
&\quad \leqslant \epsilon \sum_{i=1}^{n} |W_{ni}(x)| + \sum_{i=1}^{n} |W_{ni}(x)| \cdot |K(x_{ni}) - K(x)| I(\|x_{ni} - x\| > a) \\
&\qquad + |K(x)| \Big| \sum_{i=1}^{n} W_{ni}(x) - 1 \Big| \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

So we have $|E\hat{K}_n(x) - K(x)| \to 0$ as $n \to \infty$.

In the following, we only need to prove

$$
|\hat{K}_n(x) - E\hat{K}_n(x)| = \Big| \sum_{i=1}^{n} W_{ni}(x)\varepsilon_{ni} \Big| \to 0 \quad \text{completely.}
$$

Since $(\varepsilon_{n1}, \varepsilon_{n2}, \ldots, \varepsilon_{nn})$ has the same distribution as $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ for each $n \geqslant 1$, we only need to verify that for all $\epsilon > 0$,

$$
\sum_{n=1}^{\infty} P\Big(\Big| \sum_{i=1}^{n} W_{ni}(x)\varepsilon_{ni} \Big| > \epsilon \Big) = \sum_{n=1}^{\infty} P\Big(\Big| \sum_{i=1}^{n} W_{ni}(x)\varepsilon_i \Big| > \epsilon \Big) < \infty.
$$

Let $\{j_1, \ldots, j_n\}$ be a rearrangement of $\{1, \ldots, n\}$ such that $W_{nj_1}(x) \geqslant \cdots \geqslant W_{nj_n}(x)$. By (4.3), (A_1), (A_2) and Abel's inequality, we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} P\Big(\Big| \sum_{i=1}^{n} W_{ni}(x)\varepsilon_i \Big| > \epsilon \Big) &= \sum_{n=1}^{\infty} P\Big(\Big| \sum_{i=1}^{n} W_{nj_i}(x)\varepsilon_{j_i} \Big| > \epsilon \Big) \\
&\leqslant \sum_{n=1}^{\infty} P\Big(C \max_{1 \leqslant i \leqslant n} |W_{nj_i}(x)| \cdot \max_{1 \leqslant k \leqslant n} \Big| \sum_{i=1}^{k} \varepsilon_{j_i} \Big| > \epsilon \Big) \\
&\leqslant \sum_{n=1}^{\infty} P\Big(C n^{-2/p} \cdot \max_{1 \leqslant k \leqslant n} \Big| \sum_{i=1}^{k} \varepsilon_{j_i} \Big| > \epsilon \Big) \\
&= \sum_{n=1}^{\infty} P\Big(\max_{1 \leqslant k \leqslant n} \Big| \sum_{i=1}^{k} \varepsilon_{j_i} \Big| > \frac{\epsilon n^{2/p}}{C} \Big).
\end{aligned}
$$

We apply Corollary 3.1 with $m = 1$, $X_i = \varepsilon_{j_i}$, $Y_i = \delta_i$ and $\alpha = 2/p$, and $E|\delta|^{p+\lambda} \log^q(1 + |\delta|) < \infty$, to get

$$
\sum_{n=1}^{\infty} P\Big(\Big| \sum_{i=1}^{n} W_{ni}(x)\varepsilon_{ni} \Big| > \epsilon \Big) < \infty. \quad \blacksquare
$$

Next, we establish the following result about complete f-moment consistency for the nonparametric regression estimator $\hat{K}_n(x)$ based on WOD errors.

THEOREM 4.2. *Under the conditions of Theorem* 4.1*, if further there exists some* $1 \leqslant s < 2$ *such that* (3.1) *holds, then for any* $x \in c(K)$*,*

$$
\sum_{n=1}^{\infty} Ef(\{|\hat{K}_n(x) - K(x)| - \epsilon\}_+) < \infty.
$$

Proof. Similar to the proof of Theorem 4.1, we only need to prove

$$\sum_{n=1}^{\infty} Ef(\{|\hat{K}_n(x) - E\hat{K}_n(x)| - \epsilon\}_+) = \sum_{n=1}^{\infty} Ef\Big(\Big\{\Big|\sum_{i=1}^{n} W_{ni}(x)\varepsilon_{ni}\Big| - \epsilon\Big\}_+\Big) < \infty.$$

Since $(\varepsilon_{n1}, \dots, \varepsilon_{nn})$ has the same distribution as $(\varepsilon_1, \dots, \varepsilon_n)$ for each $n \geqslant 1$, we only need to verify that for all $\epsilon > 0$,

$$\sum_{n=1}^{\infty} Ef\Big(\Big\{\Big|\sum_{i=1}^{n} W_{ni}(x)\varepsilon_{ni}\Big| - \epsilon\Big\}_+\Big) = \sum_{n=1}^{\infty} Ef\Big(\Big\{\Big|\sum_{i=1}^{n} W_{ni}(x)\varepsilon_i\Big| - \epsilon\Big\}_+\Big) < \infty.$$

Let $\{j_1, \dots, j_n\}$ be a rearrangement of $\{1, \dots, n\}$ such that $W_{nj_1}(x) \geqslant \cdots \geqslant W_{nj_n}(x)$. By (4.3), (A_1), (A_2) and Abel's inequality, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} Ef\Big(\Big\{\Big|\sum_{i=1}^{n} W_{ni}(x)\varepsilon_i\Big| - \epsilon\Big\}_+\Big) &= \sum_{n=1}^{\infty} Ef\Big(\Big\{\Big|\sum_{i=1}^{n} W_{nj_i}(x)\varepsilon_{j_i}\Big| - \epsilon\Big\}_+\Big) \\
&\leqslant \sum_{n=1}^{\infty} Ef\Big(\Big\{C \max_{1\leqslant i\leqslant n} |W_{nj_i}(x)| \cdot \max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k} \varepsilon_{j_i}\Big| - \epsilon\Big\}_+\Big) \\
&\leqslant \sum_{n=1}^{\infty} Ef\Big(\Big\{Cn^{-2/p} \cdot \max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k} \varepsilon_{j_i}\Big| - \epsilon\Big\}_+\Big).
\end{aligned}$$

We apply Theorem 3.2 with $m = 1$, $X_i = \varepsilon_{j_i}$, $Y_i = \delta_i$ and $\alpha = 2/p$, and $E|\delta|^{p+\lambda} \log^q(1 + |\delta|) < \infty$, to get

$$\sum_{n=1}^{\infty} Ef(\{|\hat{K}_n(x) - K(x)| - \epsilon\}_+) < \infty. \quad \blacksquare$$

REMARK 4.2. Recently, Yao et al. [42] investigated complete consistency for a nonparametric regression estimator based on moving average processes with WOD errors. By contrast, we generalize the real number coefficients in the moving average process to the case of random coefficients. Moreover, we further investigate complete f-moment consistency of the nonparametric estimator $\hat{K}_n(x)$ based on WOD errors.

Now, we will show that assumptions (A_1)–(A_3) and (4.3) are satisfied for the nearest neighbor weights. Put $A = [0, 1]$ and $x_{ni} = i/n$, $1 \leqslant i \leqslant n$. For any $x \in A$, we rewrite

$$|x_{n1} - x|,\ |x_{n2} - x|, \dots, |x_{nn} - x|$$

as follows:

$$|x_{nR_1(x)} - x| \leqslant |x_{nR_2(x)} - x| \leqslant \cdots \leqslant |x_{nR_n(x)} - x|;$$

if $|x_{ni} - x| = |x_{nj} - x|$, then $|x_{ni} - x|$ is placed before $|x_{nj} - x|$ when $i < j$.

Let $1 \leqslant k_n \leqslant n$. The nearest neighbor weight function is defined as follows:

$$\tilde{W}_{ni}(x) = \begin{cases} 1/k_n & \text{if } |x_{ni} - x| \leqslant |x_{nR_{k_n}(x)} - x|, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we get the nearest neighbor weight estimator of $K(x)$:

$$\tilde{K}_n(x) = \sum_{i=1}^{n} \tilde{W}_{ni}(x) Y_{ni}.$$

COROLLARY 4.1. *Let $\{\varepsilon_n = \sum_{j=-\infty}^{\infty} A_j \delta_{n-j}, n \geqslant 1\}$ be a linear process with random coefficients $\{A_j, -\infty < j < \infty\}$, and $\{\delta_j, -\infty < j < \infty\}$ be a sequence of zero mean WOD random variables with dominating coefficients $g(n) = \max\{g_U(n), g_L(n)\}$, which is stochastically dominated by a random variable δ. Assume that $\{\delta_j, -\infty < j < \infty\}$ is independent of $\{A_j, -\infty < j < \infty\}$. Suppose that $2 < p < 4$ and $q > 2p/(4-p)$ are such that $E(\sum_{j=-\infty}^{\infty} |A_j|)^q < \infty$, and*

$$g(n) = O(n^{2\lambda/p}) \quad \text{for some } 0 \leqslant \lambda < q - p/2 - pq/4.$$

Assume further that $k_n = \lfloor n^{\varphi} \rfloor$ for some $2/p \leqslant \varphi < 1$, $E|\delta|^{p+\lambda} \log^q(1+|\delta|) < \infty$ and $K(x)$ is continuous on the compact set A. Then for all $x \in A$, $\tilde{K}_n(x) \to K(x)$ completely.

Proof. Obviously, to prove the above result, we only need to show that the conditions of Theorem 4.1 are satisfied.

For (A_1), it is easily checked that for any $x \in A$,

$$\sum_{i=1}^{n} \tilde{W}_{ni}(x) = \sum_{i=1}^{k_n} \tilde{W}_{nR_i(x)}(x) = \sum_{i=1}^{k_n} \frac{1}{k_n} = 1.$$

We can easily get (A_2) as $\tilde{W}_{ni}(x) \geqslant 0$. For (A_3), by $a > 0$ and $\varphi < 1$ we have

$$\begin{aligned} \sum_{i=1}^{n} |\tilde{W}_{ni}(x)| \cdot |K(x_{ni}) - K(x)| I(\|x_{ni} - x\| > a) & \\ \leqslant C \sum_{i=1}^{n} |\tilde{W}_{ni}(x)| \frac{(x_{ni} - x)^2}{a^2} & \\ \leqslant C \sum_{i=1}^{k_n} \frac{(x_{nR_i(x)} - x)^2}{k_n} & \\ \leqslant C \sum_{i=1}^{k_n} \frac{(\frac{i}{n})^2}{k_n} \leqslant C n^{2\varphi - 2} \to 0 \quad & \text{as } n \to \infty. \end{aligned}$$

Finally, we show (4.3): it follows from the definition of $\tilde{W}_{ni}(x)$ that

$$\max_{1 \leqslant i \leqslant n} |\tilde{W}_{ni}| = \frac{1}{k_n} \leqslant C n^{-\varphi} \leqslant C n^{-2/p}.$$

The proof is complete. ∎

COROLLARY 4.2. *Under the conditions of Corollary* 4.1*, if further there exists some* $1 \leqslant s < 2$ *such that* (3.1) *holds, then for all* $x \in A$*,*

$$\sum_{n=1}^{\infty} Ef(\{|\tilde{K}_n(x) - K(x)| - \epsilon\}_+) < \infty.$$

Proof. The proof is analogous to that for Corollary 4.1. ∎

4.2. Numerical simulations. In this subsection, we present a simulation study based on the model (4.1) and verify the consistency of $\tilde{K}_n(x)$. Let $A_0 = 1$ and $A_j \equiv 0$ for $-\infty < j < \infty$ and $j \neq 0$. Then for each $i \geqslant 1$, we have $\varepsilon_i = \sum_{j=-\infty}^{\infty} A_j \delta_{i-j} = \delta_i$. The marginal distributions of $\varepsilon_1, \ldots, \varepsilon_n$ are the uniform distribution on $[0, 1]$, while $(\varepsilon_1, \varepsilon_2), (\varepsilon_3, \varepsilon_4), \ldots, (\varepsilon_{2m-1}, \varepsilon_{2m}), \ldots$ respectively follow a joint distribution of the FGM copula

$$C_{\theta_n}(u, v) = uv + \theta_n uv(1 - u)(1 - v), \quad (u, v) \sim [0, 1] \times [0, 1],$$

with the parameter $\theta_n = n^{-1}$. We find that $\{\varepsilon_n,\ n \geqslant 1\}$ is a sequence of 1-WOD random variables with dominating coefficient $g(n) = O(n)$ (see Wang et al. [29]). Now we will centralize $\{\varepsilon_n,\ n \geqslant 1\}$ so that they have zero means. Let $k_n = \lfloor n^{0.7} \rfloor$. It is easy to verify that the conditions required in Corollary 4.1 are all satisfied. Taking the points $x = 0.25, 0.5, 0.75$ and the sample sizes $n = 100, 200, 400, 600$, respectively, we use R software to compute $\tilde{K}_n(x) - K(x)$ with $K(x) = 2x^2 - x$ and $K(x) = \sin x - \sqrt{x}$ for 1000 times and obtain the Mean Squared Errors (MSE) of $\tilde{K}_n(x)$ in Table 1 and the boxplots of $\tilde{K}_n(x) - K(x)$ in Figures 1–6.

TABLE 1. MSE of the estimator $\tilde{K}_n(x)$

$K(x)$	x	$n = 100$	$n = 200$	$n = 400$	$n = 600$
$2x^2 - x$	0.25	2.92735×10^{-3}	1.857004×10^{-3}	1.093461×10^{-3}	8.982386×10^{-4}
	0.5	2.685551×10^{-3}	1.865083×10^{-3}	1.182049×10^{-3}	7.995173×10^{-4}
	0.75	2.768019×10^{-3}	1.918095×10^{-3}	1.196158×10^{-3}	9.499686×10^{-4}
$\sin x - \sqrt{x}$	0.25	3.06745×10^{-3}	2.086873×10^{-3}	1.162709×10^{-3}	8.81945×10^{-4}
	0.5	2.66011×10^{-3}	1.865797×10^{-3}	1.17836×10^{-3}	8.940392×10^{-4}
	0.75	2.792211×10^{-3}	1.862983×10^{-3}	1.184583×10^{-3}	9.27663×10^{-4}

From Figures 1–6, we can find that no matter $K(x) = 2x^2 - x$ or $K(x) = \sin x - \sqrt{x}$ with $x = 0.25, 0.5, 0.75$, the differences $\tilde{K}_n(x) - K(x)$ fluctuate around zero and the variation range decreases markedly as n increases. Table 1 accurately reflects that the MSE of $\tilde{K}_n(x)$ decreases significantly as increases. These simulation results are consistent with theoretical results.

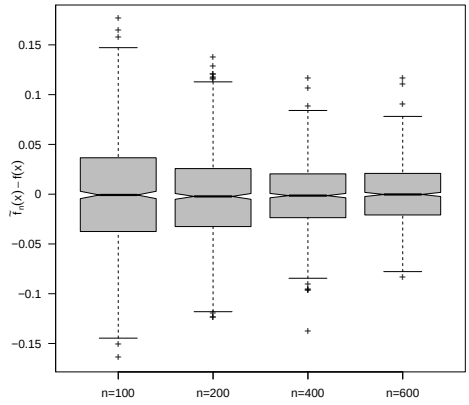


Fig 1: Boxplots of $\tilde{f}_n(x) - f(x)$ with x=0.25 and f(x)=2x^2−x

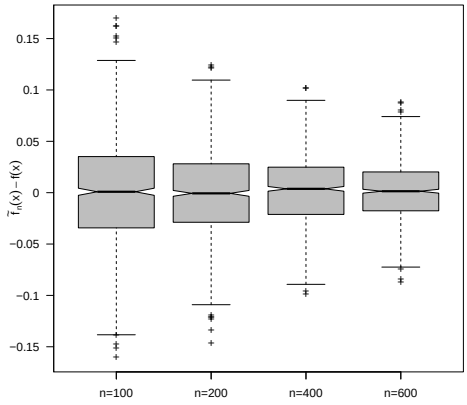


Fig 2: Boxplots of $\tilde{f}_n(x) - f(x)$ with x=0.5 and f(x)=2x^2−x

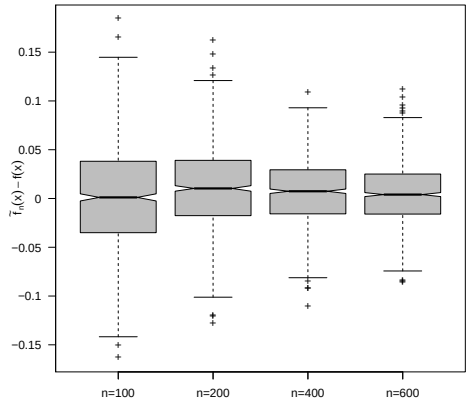


Fig 3: Boxplots of $\tilde{f}_n(x) - f(x)$ with x=0.75 and f(x)=2x^2−x

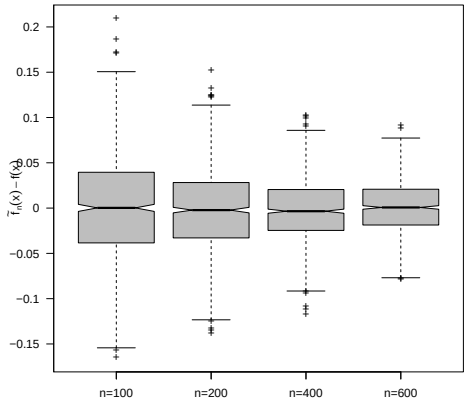


Fig 4: Boxplots of $\tilde{f}_n(x) - f(x)$ with x=0.25 and f(x)=sin x−x$^{1/2}$

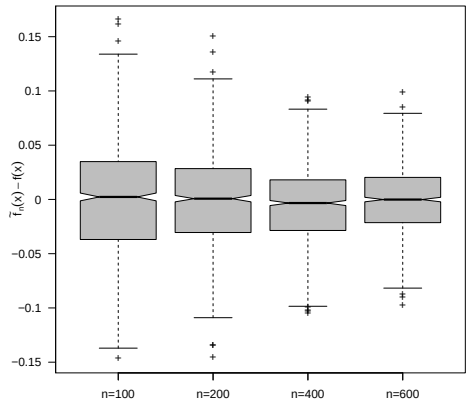


Fig 5: Boxplots of $\tilde{f}_n(x) - f(x)$ with x=0.5 and f(x)=sin x−x$^{1/2}$

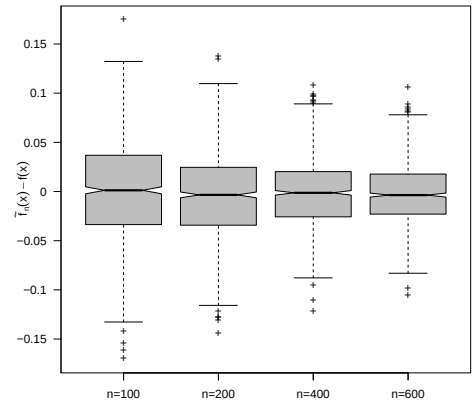


Fig 6: Boxplots of $\tilde{f}_n(x) - f(x)$ with x=0.75 and f(x)=sin x−x$^{1/2}$

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