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ASYMPTOTIC NORMALITY OF NONPARAMETRIC ESTIMATORS FOR DISCRETE-TIME SEMI-MARKOV CHAINS IN HIGHER DIMENSIONS*

BY

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Abstract. The asymptotic normality in higher dimensions of the nonparametric estimator of the transition probabilities of a Markov renewal chain is proved, and is applied to that of other nonparametric estimators for the associated semi-Markov chain.

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1. INTRODUCTION

In the case of continuous time, asymptotic normality of the nonparametric estimator for the transition probabilities of a continuous-time Markov renewal chain has been studied by Moore–Pyke [14] in one dimension, and was generalized by Gill [6] to higher dimensions, together with a characterization of the asymptotic correlation between the estimators considered. Ouhbi–Limnios [16, 17] studied the asymptotic normality of other nonparametric estimators related to the associated semi-Markov chain. Several statistical properties of those estimators were studied in [11, 1, 9, 7, 2, 13, 12, 3], and some applications were discussed in [8].

In the case of discrete time, the asymptotic normality in one dimension of various nonparametric estimators has been investigated by Barbu–Limnios [4]. Our aim is to establish asymptotic normality in higher dimensions of the nonparametric estimator of the transition probabilities of the underlying Markov renewal chain,

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and to provide a systematic and greatly simplified method of deriving the asymptotic normality results of Barbu–Limnios [4].

Let E be a finite state space and $\mathbb{N} = \{0, 1, 2, \ldots\}$. We write $\widetilde{E} := E \times E \times \mathbb{N}$. Let $\boldsymbol{q} = (\boldsymbol{q}(k))_{k\in\mathbb{N}} = (q_{ij}(k))_{(i,j,k)\in\widetilde{E}}$ be a matrix sequence with nonnegative entries such that $q_{ij}(0) = 0$ for $i, j \in E$ and

$$\sum_{j\in E}\sum_{k\in\mathbb{N}} q_{ij}(k) = 1 \quad \text{for all } i \in E.$$

A *Markov renewal chain* is a time-homogeneous Markov chain

$$(J, S) = (J_n, S_n)_{n=0}^{\infty}$$

with values in $E \times \mathbb{N}$ defined on probability spaces $(\Omega, \mathcal{F}, \mathbb{P}_{i_0})$ for $i_0 \in E$ such that the process $(J, X) = (J_n, X_n)_{n=0}^{\infty}$ defined by $J_0 = i_0$, $X_0 := S_0 := 0$ and $X_n := S_n - S_{n-1}$ for $n \geqslant 1$ is a Markov chain whose transition probabilities are given by

$$\mathbb{P}_{i_0}(J_{n+1} = j,\ X_{n+1} = k \mid J_n = i,\ X_n) = q_{ij}(k) \quad \text{for all } i, j \in E \text{ and } n, k \in \mathbb{N}.$$

(The quantity $q_{ij}(k)$ is often called the *semi-Markov kernel*, although it describes the transition of the Markov renewal chain, not of the associated semi-Markov chain.) Note that the *embedded Markov chain* $J = (J_n)_{n=0}^{\infty}$ is an E-valued Markov chain whose transition probabilities are

$$\mathbb{P}_{i_0}(J_{n+1} = j \mid J_n = i) = q_{ij}^J := \sum_{k\in\mathbb{N}} q_{ij}(k) \quad \text{for all } i, j \in E \text{ and } n \in \mathbb{N}.$$

For $j \in E$, we write $\tau_j = \inf\{n \geqslant 1 : J_n = j\}$. We assume the following conditions:

(A1) The embedded Markov chain J is *irreducible*, i.e., $\mathbb{P}_i(\tau_j < \infty) > 0$ for all $i, j \in E$.

(A2) The Markov renewal chain (J, S) is *aperiodic*, i.e., the greatest common divisor of $\{k \in \mathbb{N} : \mathbb{P}_i(S_{\tau_i} = k) > 0\}$ is 1 for all $i \in E$.

(A3) The Markov renewal chain (J, S) is *positive recurrent*, i.e., $\mu_{ii} := \mathbb{E}_i S_{\tau_i} < \infty$ for all $i \in E$.

We write **(A)** for all of the above assumptions being satisfied. In this case (see, e.g., [15, Theorem 1.7.7]) the embedded Markov chain J has a unique invariant probability $\boldsymbol{\nu} = (\nu_i)_{i\in E}$, and moreover (see [4, Proposition 3.6])

$$\overline{m} := \sum_{i\in E} \nu_i \mathbb{E}_i S_1 = \mu_{jj}\nu_j \quad \text{for all } j \in E. \tag{1.1}$$

For $M \in \mathbb{N}$ and $i \in E$, we set

$$N(M) := \max\{n \in \mathbb{N} : S_n \leqslant M\} \quad \text{and} \quad N_i(M) := \sum_{n=1}^{N(M)} 1_{\{J_{n-1}=i\}},$$

so that $N(M) = \sum_{i\in E} N_i(M)$. For the unknown quantity $\boldsymbol{q}$, we consider the nonparametric estimator

$$\widehat{\boldsymbol{q}}(M) = (\widehat{\boldsymbol{q}}(k,M))_{k\in\mathbb{N}} = (\widehat{q}_{ij}(k,M))_{(i,j,k)\in\widetilde{E}} \tag{1.2}$$

defined by

$$\widehat{q}_{ij}(k,M) := \frac{1}{N_i(M)} \sum_{n=1}^{N(M)} 1_{\{J_{n-1}=i,\ J_n=j,\ X_n=k\}}$$

so that $\sum_{j\in E}\sum_{k\in\mathbb{N}} \widehat{q}_{ij}(k,M) = 1$ for all $i \in E$. The strong consistency of $\widehat{\boldsymbol{q}}(M)$ is known:

THEOREM 1.1 (Barbu–Limnios [4, Theorem 4.1]). *Suppose* **(A)** *is satisfied. Then*

$$\widehat{\boldsymbol{q}}(M) \xrightarrow[M\to\infty]{\text{a.s.}} \boldsymbol{q},$$

or equivalently

$$\widehat{q}_{ij}(k,M) \xrightarrow[M\to\infty]{\text{a.s.}} q_{ij}(k) \quad \textit{for all } (i,j,k) \in \widetilde{E},$$

under $\mathbb{P}_{i_0}$ *for all initial values* $i_0 \in E$.

Let $\mathcal{N}(0,v)$ stand for the centered normal distribution whose variance is v. The asymptotic normality of $\widehat{\boldsymbol{q}}(M)$ in one dimension has been obtained as follows.

THEOREM 1.2 (Barbu–Limnios [4, Theorem 4.2]). *Suppose* **(A)** *is satisfied. Then, for any fixed* $(i,j,k) \in \widetilde{E}$, *we have*

$$\sqrt{M}(\widehat{q}_{ij}(k,M) - q_{ij}(k)) \xrightarrow[M\to\infty]{\text{d}} \mathcal{N}(0, v^{\boldsymbol{q}}(i,j,k))$$

under $\mathbb{P}_{i_0}$ *for all initial values* $i_0 \in E$, *where the asymptotic variance* $v^{\boldsymbol{q}}(i,j,k)$ *is given by*

$$v^{\boldsymbol{q}}(i,j,k) := \mu_{ii} q_{ij}(k)(1 - q_{ij}(k)).$$

Let us discuss convergence in the multi-dimensional case. For a nonnegative definite matrix $V = (V_{ee'})_{e,e'\in\widetilde{E}}$, we write $\mathcal{N}(\mathbf{0}, V)$ for the distribution of a centered Gaussian system $Y = (Y_e)_{e\in\widetilde{E}}$ whose covariances are given by $\mathbb{E}Y_eY_{e'} = V_{ee'}$, and we call V the *variance matrix* of $\mathcal{N}(\mathbf{0}, V)$. We now state our main theorem about the asymptotic normality of $\widehat{\boldsymbol{q}}(M)$ in the multi-dimensional case.

THEOREM 1.3. *Suppose* **(A)** *is satisfied. Then, on the product space*

$$\mathbb{R}^{\widetilde{E}} = \{\boldsymbol{x} = (x_e)_{e\in\widetilde{E}} = (x_{ij}(k))_{(i,j,k)\in\widetilde{E}} : x_e \in \mathbb{R} \text{ for } e \in \widetilde{E}\}$$

equipped with the product topology, we have

$$\sqrt{M}(\widehat{\boldsymbol{q}}(M) - \boldsymbol{q}) \xrightarrow[M\to\infty]{\mathrm{d}} \mathcal{N}(\boldsymbol{0}, V^{\boldsymbol{q}}) \tag{1.3}$$

under $\mathbb{P}_{i_0}$ for all initial values $i_0 \in E$, where the asymptotic variance matrix $V^{\boldsymbol{q}} = (V^{\boldsymbol{q}}_{ee'})_{e,e'\in\widetilde{E}}$ is given by

$$V^{\boldsymbol{q}}_{(i,j,k)(i',j',k')} = \mu_{ii} 1_{\{i=i'\}} q_{ij}(k)\big(1_{\{j=j',\ k=k'\}} - q_{i'j'}(k')\big).$$

Noting that $V^{\boldsymbol{q}}_{(i,j,k)(i,j,k)} = v^{\boldsymbol{q}}(i,j,k)$, we can derive the result of Theorem 1.2 directly from Theorem 1.3.

Let us consider the *semi-Markov chain* $Z = (Z_k)_{k\in\mathbb{N}}$ associated to (J,S), which is defined as the time-changed process $Z_k := J_{N(k)}$ for $k \in \mathbb{N}$ with $N(k) := \max\{n \in \mathbb{N} : S_n \leqslant k\}$. Note that $Z_0 = J_0$. We set

$$P_{ij}(k) := \mathbb{P}_i(Z_k = j),$$

and we call $\boldsymbol{P} = (\boldsymbol{P}(k))_{k\in\mathbb{N}} = (P_{ij}(k))_{(i,j,k)\in\widetilde{E}}$ the *distribution matrix sequence* of the semi-Markov chain Z.

Let $E = U \cup D$ be a nontrivial partition of the finite state space E and write

$$T_D = \inf\{k \in \mathbb{N} : Z_k \in D\}.$$

We define

$$R_i(k) = \mathbb{P}_i(T_D > k) \quad \text{for } i \in U,$$

and call $\boldsymbol{R} = (\boldsymbol{R}(k))_{k\in\mathbb{N}} = (R_i(k))_{(i,k)\in U\times\mathbb{N}}$ the *reliability vector sequence*.

As applications of our Theorem 1.3, we shall generalize the asymptotic normality results in one dimension to higher dimensions for certain estimators $\widehat{\boldsymbol{P}}(M)$ and $\widehat{\boldsymbol{R}}(M)$ of the distribution matrix sequence $\boldsymbol{P}$ and the reliability vector sequence $\boldsymbol{R}$, respectively.

This paper is organized as follows. In Section 2, we prove Theorem 1.3 about the asymptotic normality of $\widehat{\boldsymbol{q}}(M)$. In Section 3, we discuss matrix convolution inverse. Sections 4 and 5 are devoted to the asymptotic normality of the estimators $\widehat{\boldsymbol{P}}(M)$ and $\widehat{\boldsymbol{R}}(M)$, respectively. In Section 6, we discuss applications and give several examples. In Appendix, we deduce the asymptotic variances for the asymptotic normality in one dimension from our results in higher dimensions.

2. ESTIMATION OF THE TRANSITION PROBABILITIES

We write $\boldsymbol{a} \cdot \boldsymbol{b}$ for the inner product in $\mathbb{R}^{\widetilde{E}}$: for $\boldsymbol{a} = (a_e)_{e\in\widetilde{E}}$ and $\boldsymbol{b} = (b_e)_{e\in\widetilde{E}}$, set

$$\boldsymbol{a} \cdot \boldsymbol{b} := \sum_{e\in\widetilde{E}} a_e b_e.$$

We denote

$$\mathbb{R}_0^{\widetilde{E}} := \{\boldsymbol{x} = (x_e)_{e\in\widetilde{E}} \in \mathbb{R}^{\widetilde{E}} : x_e \neq 0 \text{ only for a finite number of } e \in \widetilde{E}\}.$$

It is well-known (see, e.g., [10, Theorem 4.29]) that, for $\mathbb{R}^{\widetilde{E}}$-valued processes, convergence in distribution on $\mathbb{R}^{\widetilde{E}}$,

$$\boldsymbol{X}(M) = (X_e(M))_{e\in\widetilde{E}} \xrightarrow[M\to\infty]{\mathrm{d}} \boldsymbol{X} = (X_e)_{e\in\widetilde{E}},$$

is equivalent to convergence of finite-dimensional distributions,

$$(X_e(M))_{e\in I} \xrightarrow[M\to\infty]{\mathrm{d}} (X_e)_{e\in I}$$

for all finite subsets I of $\widetilde{E}$. By the Cramér–Wold device, the latter convergence is reduced to the convergence

$$\boldsymbol{a} \cdot \boldsymbol{X}(M) \xrightarrow[M\to\infty]{\mathrm{d}} \boldsymbol{a} \cdot \boldsymbol{X}$$

for all constant $\boldsymbol{a} = (a_e)_{e\in\widetilde{E}} \in \mathbb{R}_0^{\widetilde{E}}$.

Let us proceed to the proof of our main theorem.

Proof of Theorem 1.3. In order to obtain the desired convergence

$$\widetilde{\boldsymbol{q}}(M) := \sqrt{M}(\widehat{\boldsymbol{q}}(M) - \boldsymbol{q}) \xrightarrow[M\to\infty]{\mathrm{d}} \mathcal{N}(\boldsymbol{0}, V^{\boldsymbol{q}}),$$

it suffices to show that the convergence

$$\boldsymbol{a} \cdot \widetilde{\boldsymbol{q}}(M) \xrightarrow[M\to\infty]{\mathrm{d}} \mathcal{N}(0, \boldsymbol{a} \cdot V^{\boldsymbol{q}}\boldsymbol{a}) \tag{2.1}$$

holds for all constant $\boldsymbol{a} = (a_e)_{e\in\widetilde{E}} \in \mathbb{R}_0^{\widetilde{E}}$, where we define

$$\boldsymbol{a} \cdot V^{\boldsymbol{q}}\boldsymbol{a} := \sum_{e,e'\in\widetilde{E}} a_e V^{\boldsymbol{q}}_{ee'} a_{e'}.$$

We may represent

$$\boldsymbol{a} \cdot \widetilde{\boldsymbol{q}}(M) = \boldsymbol{A}(M) \cdot \boldsymbol{Y}(M)$$

if we define $\boldsymbol{A}(M) = (A_e(M))_{e\in\widetilde{E}}$ and $\boldsymbol{Y}(M) = (Y_e(M))_{e\in\widetilde{E}}$ by setting

$$A_{(i,j,k)}(M) := a_{(i,j,k)}\frac{\sqrt{MN(M)}}{N_i(M)},$$
$$Y_{(i,j,k)}(M) := \frac{1}{\sqrt{N(M)}}\sum_{n=1}^{N(M)} 1_{\{J_{n-1}=i\}}(1_{\{J_n=j,\ X_n=k\}} - q_{ij}(k)).$$

By (1.1) and by [4, Propositions 3.7 and 3.8], we see that

$$\frac{N_i(M)}{M} \xrightarrow[M\to\infty]{\text{a.s.}} \frac{1}{\mu_{ii}}, \quad \frac{N(M)}{M} \xrightarrow[M\to\infty]{\text{a.s.}} \frac{1}{\overline{m}},$$

and hence

$$A_{(i,j,k)}(M) \xrightarrow[M\to\infty]{\text{a.s.}} a_{(i,j,k)}\frac{\mu_{ii}}{\sqrt{\overline{m}}}.$$

By Slutsky's theorem, the convergence (2.1) for all constant $\boldsymbol{a} = (a_e)_{e\in\widetilde{E}} \in \mathbb{R}_0^{\widetilde{E}}$ is reduced to the convergence

$$\boldsymbol{b}\cdot\boldsymbol{Y}(M) \xrightarrow[M\to\infty]{\text{d}} \mathcal{N}(0, \boldsymbol{b}\cdot V^{\boldsymbol{Y}}\boldsymbol{b}) \tag{2.2}$$

for all constant $\boldsymbol{b} = (b_e)_{e\in\widetilde{E}} \in \mathbb{R}_0^{\widetilde{E}}$, where, by (1.1),

$$V^{\boldsymbol{Y}}_{ee'} := \frac{\overline{m}}{\mu_{ii}\mu_{i'i'}}V^{\boldsymbol{q}}_{ee'} = \nu_i 1_{\{i=i'\}}q_{ij}(k)\big(1_{\{j=j',\ k=k'\}} - q_{i'j'}(k')\big)$$

for $e = (i,j,k)$ and $e' = (i',j',k')$. Since $N(M)/M \xrightarrow[M\to\infty]{\text{a.s.}} 1/\overline{m} > 0$, the convergence (2.2) is reduced to the convergence

$$Z(M) := \frac{1}{\sqrt{M}}\sum_{n=1}^{M}\xi_n \xrightarrow[M\to\infty]{\text{d}} \mathcal{N}(0, \boldsymbol{b}\cdot V^{\boldsymbol{Y}}\boldsymbol{b}), \tag{2.3}$$

where

$$\xi_n := \sum_{e=(i,j,k)\in\widetilde{E}} b_e 1_{\{J_{n-1}=i\}}\big(1_{\{J_n=j,\,X_n=k\}} - q_{ij}(k)\big).$$

Let us apply the martingale CLT (see, e.g., [5, Theorem 8.2.8]) to prove (2.3). Set $\mathcal{F}_n = \sigma(J_l, X_l; l \leqslant n)$ for $n \in \mathbb{N}$. Then we see that $\{\xi_n\}_{n=1}^{\infty}$ is a martingale difference with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$, since

$$\mathbb{E}_{i_0}[\xi_n \mid \mathcal{F}_{n-1}] = \sum_{e=(i,j,k)\in\widetilde{E}} b_e 1_{\{J_{n-1}=i\}}\mathbb{E}_{i_0}[1_{\{J_n=j,\,X_n=k\}} - q_{ij}(k) \mid \mathcal{F}_{n-1}] = 0.$$

Since $|\xi_n| \leqslant \sum_e |b_e| < \infty$, it is obvious that

$$\frac{1}{M}\sum_{n=1}^{M} \mathbb{E}_{i_0}[\xi_n^2\,;\ |\xi_n| > \varepsilon\sqrt{M}] = 0 \quad \text{for large enough } M \text{ for all } \varepsilon > 0.$$

We have now only to prove that

$$F(M) := \frac{1}{M}\sum_{n=1}^{M} \mathbb{E}_{i_0}[\xi_n^2 \mid \mathcal{F}_{n-1}] \xrightarrow[M\to\infty]{\text{a.s.}} \boldsymbol{b}\cdot V^{\boldsymbol{Y}}\boldsymbol{b}.$$

Expanding the square, we obtain

$$F(M) = \frac{1}{M}\sum_{n=1}^{M} \sum_{\substack{e=(i,j,k)\in\widetilde{E}\\ e'=(i',j',k')\in\widetilde{E}}} b_e b_{e'} 1_{\{J_{n-1}=i\}} 1_{\{i=i'\}} E_{ee'}(n),$$

where, for $i = i'$,

$$\begin{aligned} E_{ee'}(n) &:= \mathbb{E}_{i_0}\big[(1_{\{J_n=j,\,X_n=k\}} - q_{ij}(k))\big(1_{\{J_n=j',\,X_n=k'\}} - q_{i'j'}(k')\big) \,\big|\, \mathcal{F}_{n-1}\big] \\ &= \mathbb{E}_{i_0}[1_{\{J_n=j,\,X_n=k\}} 1_{\{J_n=j',\,X_n=k'\}} \mid \mathcal{F}_{n-1}] - q_{ij}(k)q_{i'j'}(k') \\ &= \mathbb{E}_{i_0}[1_{\{J_n=j,\,X_n=k\}} \mid \mathcal{F}_{n-1}] 1_{\{j=j',\,k=k'\}} - q_{ij}(k)q_{i'j'}(k') \\ &= q_{ij}(k)(1_{\{j=j',\,k=k'\}} - q_{i'j'}(k')). \end{aligned}$$

Since from the ergodic theorem (see, e.g., [15, Theorem 1.10.2]) we know that

$$\frac{1}{M}\sum_{n=1}^{M} 1_{\{J_{n-1}=i\}} \xrightarrow[M\to\infty]{\text{a.s.}} \nu_i,$$

we obtain

$$\begin{aligned} &F(M) \\ &= \sum_{\substack{ce=(i,j,k)\in\widetilde{E}\\ e'=(i',j',k')\in\widetilde{E}}} b_e b_{e'} \left(\frac{1}{M}\sum_{n=1}^{M} 1_{\{J_{n-1}=i\}}\right) 1_{\{i=i'\}} q_{ij}(k)(1_{\{j=j',\,k=k'\}} - q_{i'j'}(k')) \\ &\xrightarrow[M\to\infty]{\text{a.s.}} \sum_{\substack{e=(i,j,k)\in\widetilde{E}\\ e'=(i',j',k')\in\widetilde{E}}} b_e b_{e'} \nu_i q_{ij}(k)(1_{\{j=j',\,k=k'\}} - q_{i'j'}(k')) = \boldsymbol{b}\cdot V^{\boldsymbol{Y}}\boldsymbol{b}. \end{aligned}$$

The proof is now complete. ∎

3. ESTIMATION OF THE MATRIX CONVOLUTION INVERSE

For two matrix sequences

$$\boldsymbol{A} = (\boldsymbol{A}(k))_{k\in\mathbb{N}} = (A_{ij}(k))_{(i,j,k)\in\widetilde{E}} \quad \text{and} \quad \boldsymbol{B} = (\boldsymbol{B}(k))_{k\in\mathbb{N}} = (B_{ij}(k))_{(i,j,k)\in\widetilde{E}},$$

we define the *matrix convolution* of $\boldsymbol{A}$ and $\boldsymbol{B}$ by $\boldsymbol{A} * \boldsymbol{B} = ((\boldsymbol{A} * \boldsymbol{B})(k))_{k\in\mathbb{N}} = ((\boldsymbol{A} * \boldsymbol{B})_{ij}(k))_{(i,j,k)\in\widetilde{E}}$ with

$$(\boldsymbol{A} * \boldsymbol{B})(k) := \sum_{l=0}^{k} \boldsymbol{A}(l)\boldsymbol{B}(k-l),$$

or equivalently

$$(\boldsymbol{A} * \boldsymbol{B})_{ij}(k) := \sum_{l=0}^{k} \sum_{u\in E} A_{iu}(l)B_{uj}(k-l).$$

Define $\boldsymbol{\delta I} = (\boldsymbol{\delta I}(k))_{k\in\mathbb{N}}$ by

$$\boldsymbol{\delta I}(k) = \begin{cases} \boldsymbol{I} & \text{for } k = 0, \\ \boldsymbol{0} & \text{for } k \geqslant 1, \end{cases}$$

where $\boldsymbol{I}$ and $\boldsymbol{0}$ stand for the identity and zero matrices, respectively, so that $\boldsymbol{\delta I}$ is an identity of matrix convolution:

$$\boldsymbol{\delta I} * \boldsymbol{A} = \boldsymbol{A} * \boldsymbol{\delta I} = \boldsymbol{A}.$$

We denote the n-fold matrix convolution of $\boldsymbol{A}$ by $\boldsymbol{A}^{(n)}$, that is, $\boldsymbol{A}^{(0)} = \boldsymbol{\delta I}$ and $\boldsymbol{A}^{(n)} = \boldsymbol{A} * \boldsymbol{A}^{(n-1)}$ for $n \geqslant 1$. For any matrix-valued sequence $\boldsymbol{A}$ such that $\boldsymbol{A}^{(n)}(k) = \boldsymbol{0}$ whenever $n > k$, we may define

$$(\boldsymbol{\delta I} - \boldsymbol{A})^{(-1)} := \sum_{n=0}^{\infty} \boldsymbol{A}^{(n)},$$

which is the inverse of $\boldsymbol{\delta I} - \boldsymbol{A}$ with respect to matrix convolution:

$$(\boldsymbol{\delta I} - \boldsymbol{A})^{(-1)} * (\boldsymbol{\delta I} - \boldsymbol{A}) = (\boldsymbol{\delta I} - \boldsymbol{A}) * (\boldsymbol{\delta I} - \boldsymbol{A})^{(-1)} = \boldsymbol{\delta I}$$

(see [4, Proposition 3.4]).

Note that the n-fold matrix convolution of the transition probabilities, which we denote by $\boldsymbol{q}^{(n)} = (\boldsymbol{q}^{(n)}(k))_{k\in\mathbb{N}} = (q_{ij}^{(n)}(k))_{(i,j,k)\in\widetilde{E}}$, satisfies

$$\mathbb{P}_i(J_n = j,\, S_n = k) = q_{ij}^{(n)}(k).$$

By the assumption $\boldsymbol{q}(0) = \mathbf{0}$, it is obvious that $\boldsymbol{q}^{(n)}(k) = \mathbf{0}$ whenever $n > k$, and hence the matrix convolution inverse

$$\boldsymbol{\psi} := (\boldsymbol{\delta I} - \boldsymbol{q})^{(-1)} = \sum_{n=0}^{\infty} \boldsymbol{q}^{(n)}$$

is well-defined. Since $(\boldsymbol{\delta I} - \boldsymbol{q}) * \boldsymbol{\psi} = \boldsymbol{\psi} * (\boldsymbol{\delta I} - \boldsymbol{q}) = \boldsymbol{\delta I}$, we have the renewal equation

$$\boldsymbol{\psi} = \boldsymbol{\delta I} + \boldsymbol{q} * \boldsymbol{\psi} = \boldsymbol{\delta I} + \boldsymbol{\psi} * \boldsymbol{q}. \tag{3.1}$$

For the estimator $\widehat{\boldsymbol{q}}(M)$ introduced in (1.2) and its n-fold matrix convolution $\widehat{\boldsymbol{q}}^{(n)}(M)$, the inverse of the matrix convolution

$$\widehat{\boldsymbol{\psi}}(M) := (\boldsymbol{\delta I} - \widehat{\boldsymbol{q}}(M))^{(-1)} = \sum_{n=0}^{\infty} \widehat{\boldsymbol{q}}^{(n)}(M)$$

is well-defined and satisfies the renewal equation

$$\widehat{\boldsymbol{\psi}}(M) = \boldsymbol{\delta I} + \widehat{\boldsymbol{q}}(M) * \widehat{\boldsymbol{\psi}}(M) = \boldsymbol{\delta I} + \widehat{\boldsymbol{\psi}}(M) * \widehat{\boldsymbol{q}}(M). \tag{3.2}$$

The strong consistency of $\widehat{\boldsymbol{\psi}}(M)$ is known:

THEOREM 3.1 (Barbu–Limnios [4, Theorem 4.4]). *Suppose* **(A)** *is satisfied. Then*

$$\widehat{\boldsymbol{\psi}}(M) \xrightarrow[M\to\infty]{\text{a.s.}} \boldsymbol{\psi} \tag{3.3}$$

under $\mathbb{P}_{i_0}$ *for all initial values* $i_0 \in E$.

The asymptotic normality of $\widehat{\boldsymbol{\psi}}(M) = (\psi_{ij}(k, M))_{(i,j,k)\in\widetilde{E}}$ in one dimension has been obtained as follows.

THEOREM 3.2 (Barbu–Limnios [4, Theorem 4.5]). *Suppose* **(A)** *is satisfied. Then, for any fixed* $(i, j, k) \in \widetilde{E}$, *we have*

$$\sqrt{M}\big(\widehat{\psi}_{ij}(k, M) - \psi_{ij}(k)\big) \xrightarrow[M\to\infty]{\text{d}} \mathcal{N}(0, v^{\boldsymbol{\psi}}(i, j, k))$$

under $\mathbb{P}_{i_0}$ *for all initial values* $i_0 \in E$, *where the asymptotic variance* $v^{\boldsymbol{\psi}}(i, j, k)$ *is given by*

$$\begin{aligned} &v^{\boldsymbol{\psi}}(i, j, k) \\ &\quad := \sum_{\widetilde{i}\in E} \mu_{\widetilde{i}\,\widetilde{i}} \Big\{ \sum_{\widetilde{j}\in E} [(\psi_{i\widetilde{i}} * \psi_{\widetilde{j}j})^2 * q_{\widetilde{i}\,\widetilde{j}}](k) - \Big[\sum_{\widetilde{j}\in E} (\psi_{i\widetilde{i}} * q_{\widetilde{i}\,\widetilde{j}} * \psi_{\widetilde{j}j})(k) \Big]^2 \Big\}. \end{aligned} \tag{3.4}$$

Let us generalize Theorem 3.2 to higher dimensions, as a corollary of Theorem 1.3. Define the matrix sequences $\overline{\boldsymbol{q}} = (\overline{q}_{ij}(k))_{(i,j,k)\in\widetilde{E}}$ and $\boldsymbol{q}^{\widetilde{i}} =$

$(q_{ij}^{\widetilde{i}}(k))_{(i,j,k)\in\widetilde{E}}$ for $\widetilde{i}\in E$ by

$$\overline{q}_{ij}(k)=\mu_{ii}q_{ij}(k),\quad q_{ij}^{\widetilde{i}}(k)=1_{\{i=\widetilde{i}\}}q_{ij}(k),$$

and define the matrix sequence $\boldsymbol{\delta}^{\widetilde{e}}=(\delta_e^{\widetilde{e}})_{e\in\widetilde{E}}$ for $\widetilde{e}\in\widetilde{E}$ by

$$\delta_e^{\widetilde{e}}=1_{\{e=\widetilde{e}\}}$$

for $e=(i,j,k)\in\widetilde{E}$. For two matrix sequences $\boldsymbol{A}$ and $\boldsymbol{B}$, we define their tensor product $\boldsymbol{A}\otimes\boldsymbol{B}$ to be the $\widetilde{E}\times\widetilde{E}$-matrix $(A_eB_{e'})_{e,e'\in\widetilde{E}}$.

THEOREM 3.3. *Suppose* **(A)** *is satisfied. Then*

$$\sqrt{M}(\widehat{\boldsymbol{\psi}}(M)-\boldsymbol{\psi})\xrightarrow[M\to\infty]{\mathrm{d}}\mathcal{N}(\mathbf{0},V^{\boldsymbol{\psi}})$$

under $\mathbb{P}_{i_0}$ *for all initial values* $i_0\in E$*, where the asymptotic variance matrix* $V^{\boldsymbol{\psi}}=(V_{ee'}^{\boldsymbol{\psi}})_{e,e'\in\widetilde{E}}$ *is given by*

$$V^{\boldsymbol{\psi}}=\sum_{\widetilde{e}\in\widetilde{E}}\overline{q}_{\widetilde{e}}(\boldsymbol{\psi}*\boldsymbol{\delta}^{\widetilde{e}}*\boldsymbol{\psi})^{\otimes2}-\sum_{\widetilde{i}\in E}\mu_{\widetilde{i}\,\widetilde{i}}(\boldsymbol{\psi}*\boldsymbol{q}^{\widetilde{i}}*\boldsymbol{\psi})^{\otimes2}. \tag{3.5}$$

Proof. In what follows, we sometimes omit writing M in estimators, say $\widehat{\boldsymbol{q}}$ for $\widehat{\boldsymbol{q}}(M)$.

By the renewal equations (3.1) and (3.2), we have

$$\boldsymbol{\delta I}=(\boldsymbol{\delta I}-\boldsymbol{q})*\boldsymbol{\psi}=\widehat{\boldsymbol{\psi}}*(\boldsymbol{\delta I}-\widehat{\boldsymbol{q}}),$$

and hence

$$\begin{aligned}\widehat{\boldsymbol{\psi}}-\boldsymbol{\psi}&=\widehat{\boldsymbol{\psi}}*\boldsymbol{\delta I}-\boldsymbol{\delta I}*\boldsymbol{\psi}\\&=\widehat{\boldsymbol{\psi}}*(\boldsymbol{\delta I}-\boldsymbol{q})*\boldsymbol{\psi}-\widehat{\boldsymbol{\psi}}*(\boldsymbol{\delta I}-\widehat{\boldsymbol{q}})*\boldsymbol{\psi}\\&=\widehat{\boldsymbol{\psi}}*(\widehat{\boldsymbol{q}}-\boldsymbol{q})*\boldsymbol{\psi},\end{aligned}$$

which leads to the identity

$$\sqrt{M}(\widehat{\boldsymbol{\psi}}-\boldsymbol{\psi})=\boldsymbol{\psi}*\{\sqrt{M}(\widehat{\boldsymbol{q}}-\boldsymbol{q})\}*\boldsymbol{\psi}+(\widehat{\boldsymbol{\psi}}-\boldsymbol{\psi})*\{\sqrt{M}(\widehat{\boldsymbol{q}}-\boldsymbol{q})\}*\boldsymbol{\psi}. \tag{3.6}$$

By the strong consistency (3.3), the asymptotic normality (1.3) and Slutsky's theorem, we have

$$(\widehat{\boldsymbol{\psi}}-\boldsymbol{\psi})*\{\sqrt{M}(\widehat{\boldsymbol{q}}-\boldsymbol{q})\}*\boldsymbol{\psi}\xrightarrow[M\to\infty]{\mathrm{d}}\mathbf{0}. \tag{3.7}$$

Hence, by the asymptotic normality (1.3) and Slutsky's theorem, we obtain

$$\sqrt{M}(\widehat{\boldsymbol{\psi}}-\boldsymbol{\psi})\xrightarrow[M\to\infty]{\mathrm{d}}\boldsymbol{\psi}*\boldsymbol{Z}^{\boldsymbol{q}}*\boldsymbol{\psi},$$

where $\boldsymbol{Z}^{\boldsymbol{q}} \overset{\mathrm{d}}{=} \mathcal{N}(\boldsymbol{0}, V^{\boldsymbol{q}})$. Since $\boldsymbol{Z}^{\boldsymbol{q}}$ is a centered Gaussian system, we see that so is $\boldsymbol{Z}^{\boldsymbol{\psi}} := \boldsymbol{\psi} * \boldsymbol{Z}^{\boldsymbol{q}} * \boldsymbol{\psi}$. The asymptotic variance matrix $V^{\boldsymbol{\psi}}$ of $\boldsymbol{Z}^{\boldsymbol{\psi}}$ is

$$\begin{aligned} V^{\boldsymbol{\psi}} &= \mathbb{E}[(\boldsymbol{\psi} * \boldsymbol{Z}^{\boldsymbol{q}} * \boldsymbol{\psi}) \otimes (\boldsymbol{\psi} * \boldsymbol{Z}^{\boldsymbol{q}} * \boldsymbol{\psi})] \\ &= \sum_{\widetilde{e},\widetilde{e}' \in \widetilde{E}} \mathbb{E}[Z^{\boldsymbol{q}}_{\widetilde{e}} Z^{\boldsymbol{q}}_{\widetilde{e}'}](\boldsymbol{\psi} * \boldsymbol{\delta}^{\widetilde{e}} * \boldsymbol{\psi}) \otimes (\boldsymbol{\psi} * \boldsymbol{\delta}^{\widetilde{e}'} * \boldsymbol{\psi}) \\ &= \sum_{\widetilde{e},\widetilde{e}' \in \widetilde{E}} V^{\boldsymbol{q}}_{\widetilde{e}\widetilde{e}'}(\boldsymbol{\psi} * \boldsymbol{\delta}^{\widetilde{e}} * \boldsymbol{\psi}) \otimes (\boldsymbol{\psi} * \boldsymbol{\delta}^{\widetilde{e}'} * \boldsymbol{\psi}) \\ &= \sum_{\widetilde{e} \in \widetilde{E}} \overline{q}_{\widetilde{e}} (\boldsymbol{\psi} * \boldsymbol{\delta}^{\widetilde{e}} * \boldsymbol{\psi})^{\otimes 2} - \sum_{\widetilde{i} \in E} \mu_{\widetilde{i}\,\widetilde{i}} (\boldsymbol{\psi} * \boldsymbol{q}^{\widetilde{i}} * \boldsymbol{\psi})^{\otimes 2}. \end{aligned}$$

The proof is complete. ∎

4. ESTIMATION OF THE DISTRIBUTION MATRIX SEQUENCE

Let $\mathcal{M}$ denote the set of all random matrix sequences $\boldsymbol{A} = (A_{ij}(k))_{(i,j,k) \in \widetilde{E}}$ and set

$$\mathcal{D} = \Big\{ \boldsymbol{A} \in \mathcal{M} : \lim_{K \to \infty} \sum_{j \in E} \sum_{k \leqslant K} A_{ij}(k) \text{ exists in } L^2 \text{ for all } i \in E \Big\}.$$

We define the linear operator $\mathcal{S} : \mathcal{D} \to \mathcal{M}$ by

$$(\mathcal{S}\boldsymbol{A})_{ij}(k) := \lim_{K \to \infty} 1_{\{i=j\}} \sum_{j' \in E} \sum_{k < k' \leqslant K} A_{ij'}(k') \quad \text{in } L^2$$

with $\mathcal{S}\boldsymbol{A} = ((\mathcal{S}\boldsymbol{A})_{ij}(k))_{(i,j,k) \in \widetilde{E}}$.

For the semi-Markov chain $Z_k = J_{N(k)}$ with $N(k) = \max\{n \in \mathbb{N} : S_n \leqslant k\}$, its distribution matrix sequence $\boldsymbol{P} = (P_{ij}(k))_{(i,j,k) \in \widetilde{E}}$ is given by

$$P_{ij}(k) := \mathbb{P}_i(Z_k = j) = \mathbb{P}_i(Z_k = j,\, X_1 > k) + \mathbb{P}_i(Z_k = j,\, X_1 \leqslant k).$$

Note that

$$\begin{aligned} \mathbb{P}_i(Z_k = j,\, X_1 > k) &= 1_{\{i=j\}} \mathbb{P}_i(X_1 > k) = 1_{\{i=j\}} \sum_{j' \in E} \sum_{k' > k} q_{ij'}(k') \\ &= (\mathcal{S}\boldsymbol{q})_{ij}(k) \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}_i(Z_k = j,\, X_1 \leqslant k) &= \sum_{j' \in E} \sum_{k' \leqslant k} \mathbb{P}_i(Z_k = j,\, J_1 = j',\, X_1 = k') \\ &= \sum_{j' \in E} \sum_{k' \leqslant k} q_{ij'}(k') \mathbb{P}_{j'}(Z_{k-k'} = j) \\ &= \sum_{j' \in E} \sum_{k' \leqslant k} q_{ij'}(k') P_{j'j}(k - k') = (\boldsymbol{q} * \boldsymbol{P})_{ij}(k). \end{aligned}$$

We thus obtain the renewal equation

$$\boldsymbol{P} = \mathcal{S}\boldsymbol{q} + \boldsymbol{q} * \boldsymbol{P},$$

which can be solved as

$$\boldsymbol{P} = (\boldsymbol{\delta I} - \boldsymbol{q})^{(-1)} * \mathcal{S}\boldsymbol{q} = \boldsymbol{\psi} * \mathcal{S}\boldsymbol{q}.$$

Let us introduce the estimator of the distribution matrix sequence by

$$\widehat{\boldsymbol{P}}(M) := \widehat{\boldsymbol{\psi}}(M) * \mathcal{S}\widehat{\boldsymbol{q}}(M).$$

The strong consistency of $\widehat{\boldsymbol{P}}(M)$ is known:

THEOREM 4.1 (Barbu–Limnios [4, Theorem 4.6]). *Suppose* **(A)** *is satisfied. Then*

$$\widehat{\boldsymbol{P}}(M) \xrightarrow[M\to\infty]{\text{a.s.}} \boldsymbol{P}$$

under $\mathbb{P}_{i_0}$ *for all initial values* $i_0 \in E$.

The asymptotic normality of $\widehat{\boldsymbol{P}}(M) = (P_{ij}(k,M))_{(i,j,k)\in\widetilde{E}}$ in one dimension has also been obtained.

THEOREM 4.2 (Barbu–Limnios [4, Theorem 4.7]). *Suppose* **(A)** *is satisfied. Then, for any fixed* $(i,j,k) \in \widetilde{E}$, *we have*

$$\sqrt{M}\big(\widehat{P}_{ij}(k,M) - P_{ij}(k)\big) \xrightarrow[M\to\infty]{\text{d}} \mathcal{N}(0, v^{\boldsymbol{P}}(i,j,k))$$

under $\mathbb{P}_{i_0}$ *for all initial values* $i_0 \in E$, *where the asymptotic variance* $v^{\boldsymbol{P}}(i,j,k)$ *is given by*

$$v^{\boldsymbol{P}}(i,j,k) := \sum_{\widetilde{i}\in E} \mu_{\widetilde{i}\,\widetilde{i}} \Big\{ \sum_{\widetilde{j}\in E} [(C^{ij}_{\widetilde{i}\,\widetilde{j}} - 1_{\{\widetilde{i}=j\}}\Psi_{ij})^2 * q_{\widetilde{i}\,\widetilde{j}}](k) - \Big[\sum_{\widetilde{j}\in E} C^{ij}_{\widetilde{i}\,\widetilde{j}} * q_{\widetilde{i}\,\widetilde{j}} - 1_{\{\widetilde{i}=j\}}\psi_{ij} * H_j\Big]^2(k)\Big\}, \tag{4.1}$$

where

$$\Psi_{ij}(k) = \sum_{k'\leqslant k} \psi_{ij}(k'), \quad H_j(k) = \sum_{j'\in E}\sum_{k'\leqslant k} q_{jj'}(k'),$$
$$C^{ij}_{\widetilde{i}\,\widetilde{j}} = \psi_{i\widetilde{i}} * \psi_{\widetilde{j}j} * (1 - H_j).$$

Let us generalize Theorem 4.2 to higher dimensions, as a corollary of Theorem 1.3.

THEOREM 4.3. *Suppose* **(A)** *is satisfied. Then*

$$\sqrt{M}(\widehat{\boldsymbol{P}}(M)-\boldsymbol{P}) \xrightarrow[M\to\infty]{\mathrm{d}} \mathcal{N}(\mathbf{0}, V^{\boldsymbol{P}})$$

under $\mathbb{P}_{i_0}$ *for all initial values* $i_0 \in E$*, where the asymptotic variance matrix* $V^{\boldsymbol{P}} = (V^{\boldsymbol{P}}_{ee'})_{e,e'\in\widetilde{E}}$ *is given by*

$$\begin{aligned} V^{\boldsymbol{P}} = {} & \sum_{\widetilde{e}\in\widetilde{E}} \overline{q}_{\widetilde{e}}(\boldsymbol{\psi} * \boldsymbol{\delta}^{\widetilde{e}} * \boldsymbol{\psi} * \mathcal{S}\boldsymbol{q} + \boldsymbol{\psi} * \mathcal{S}\boldsymbol{\delta}^{\widetilde{e}})^{\otimes 2} \\ & - \sum_{\widetilde{i}\in E} \mu_{\widetilde{i}\,\widetilde{i}}(\boldsymbol{\psi} * \boldsymbol{q}^{\widetilde{i}} * \boldsymbol{\psi} * \mathcal{S}\boldsymbol{q} + \boldsymbol{\psi} * \mathcal{S}\boldsymbol{q}^{\widetilde{i}})^{\otimes 2}. \end{aligned} \tag{4.2}$$

We need the following lemma.

LEMMA 4.1. *Let* $\boldsymbol{Z}^{\boldsymbol{q}} = (Z^{\boldsymbol{q}}_e)_{e\in\widetilde{E}} = (Z^{\boldsymbol{q}}_{ij}(k))_{(i,j,k)\in\widetilde{E}} \overset{\mathrm{d}}{=} \mathcal{N}(\mathbf{0}, V^{\boldsymbol{q}})$*. Then*

$$\lim_{K\to\infty} \sum_{j\in E} \sum_{k\leqslant K} Z^{\boldsymbol{q}}_{ij}(k) = 0 \quad \text{in } L^2 \tag{4.3}$$

for all $i \in E$*. Consequently,* $\boldsymbol{Z}^{\boldsymbol{q}} \in \mathcal{D}$ *and*

$$(\mathcal{S}\boldsymbol{Z}^{\boldsymbol{q}})_{ij}(k) = -1_{\{i=j\}} \sum_{j'\in E} \sum_{k'\leqslant k} Z^{\boldsymbol{q}}_{ij'}(k') \tag{4.4}$$

for all $(i,j,k) \in \widetilde{E}$*.*

Proof. Expanding the square, we obtain

$$\begin{aligned} \mathbb{E}\Big(\sum_{j\in E}\sum_{k\leqslant K} Z^{\boldsymbol{q}}_{ij}(k)\Big)^2 &= \sum_{j\in E}\sum_{k\leqslant K}\sum_{j'\in E}\sum_{k'\leqslant K} V_{(i,j,k)(i,j',k')} \\ &= \mu_{ii}\sum_{j\in E}\sum_{k\leqslant K} q_{ij}(k)\sum_{j'\in E}\sum_{k'\leqslant K}(1_{\{j=j',\,k=k'\}} - q_{ij'}(k')) \\ &= \mu_{ii}\sum_{j\in E}\sum_{k\leqslant K} q_{ij}(k)\Big(1 - \sum_{j'\in E}\sum_{k'\leqslant K} q_{ij'}(k')\Big) \\ &\leqslant \mu_{ii}\sum_{j\in E}\sum_{k\in\mathbb{N}} q_{ij}(k)\sum_{j'\in E}\sum_{k'>K} q_{ij'}(k') \xrightarrow[K\to\infty]{} 0 \end{aligned}$$

by the dominated convergence theorem. ∎

We also need the following lemma.

LEMMA 4.2. *Suppose* **(A)** *is satisfied. Then*

$$\big(\sqrt{M}(\widehat{\boldsymbol{q}}-\boldsymbol{q}),\ \mathcal{S}\sqrt{M}(\widehat{\boldsymbol{q}}-\boldsymbol{q})\big) \xrightarrow[M\to\infty]{\mathrm{d}} (\boldsymbol{Z}^{\boldsymbol{q}},\ \mathcal{S}\boldsymbol{Z}^{\boldsymbol{q}})$$

in the product topology.

Proof. Set $\widetilde{\boldsymbol{q}} := \sqrt{M}(\widehat{\boldsymbol{q}}-\boldsymbol{q})$, so that $\widetilde{\boldsymbol{q}} \xrightarrow[M\to\infty]{\mathrm{d}} \boldsymbol{Z}^{\boldsymbol{q}}$ by the asymptotic normality (1.3). It suffices to show that

$$\boldsymbol{a}\cdot\widetilde{\boldsymbol{q}} + \boldsymbol{b}\cdot\mathcal{S}\widetilde{\boldsymbol{q}} \xrightarrow[M\to\infty]{\mathrm{d}} \boldsymbol{a}\cdot\boldsymbol{Z}^{\boldsymbol{q}} + \boldsymbol{b}\cdot\mathcal{S}\boldsymbol{Z}^{\boldsymbol{q}}$$

for all constants $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}_0^{\widetilde{E}}$. Since

$$\begin{aligned}\sum_{j'\in E}\sum_{k'\in\mathbb{N}} \widetilde{q}_{ij'}(k') &= \sqrt{M}\sum_{j'\in E}\sum_{k'\in\mathbb{N}} \widehat{q}_{ij'}(k') - \sqrt{M}\sum_{j'\in E}\sum_{k'\in\mathbb{N}} q_{ij'}(k') \\ &= \sqrt{M}\cdot 1 - \sqrt{M}\cdot 1 = 0,\end{aligned}$$

we obtain

$$\begin{aligned}\boldsymbol{a}\cdot\widetilde{\boldsymbol{q}} + \boldsymbol{b}\cdot\mathcal{S}\widetilde{\boldsymbol{q}} &= \sum_{e\in\widetilde{E}} a_e\widetilde{q}_e + \sum_{e=(i,j,k)\in\widetilde{E}} b_e 1_{\{i=j\}} \sum_{j'\in E}\sum_{k'>k} \widetilde{q}_{ij'}(k') \\ &= \sum_{e\in\widetilde{E}} a_e\widetilde{q}_e - \sum_{e=(i,j,k)\in\widetilde{E}} b_e 1_{\{i=j\}} \sum_{j'\in E}\sum_{k'\leqslant k} \widetilde{q}_{ij'}(k') \\ &\xrightarrow[M\to\infty]{\mathrm{d}} \sum_{e\in\widetilde{E}} a_e Z_e^{\boldsymbol{q}} - \sum_{e=(i,j,k)\in\widetilde{E}} b_e 1_{\{i=j\}} \sum_{j'\in E}\sum_{k'\leqslant k} Z_{ij'}^{\boldsymbol{q}}(k') \\ &= \boldsymbol{a}\cdot\boldsymbol{Z}^{\boldsymbol{q}} + \boldsymbol{b}\cdot\mathcal{S}\boldsymbol{Z}^{\boldsymbol{q}},\end{aligned} \tag{4.5}$$

where $(4.5)_3$ is obtained by applying (4.3) to the finite sum, and where $(4.5)_4$ is obtained from (4.4). The proof is complete. ∎

Proof of Theorem 4.3. By (3.6) and (3.7), we have

$$\sqrt{M}(\widehat{\boldsymbol{\psi}} - \boldsymbol{\psi}) = \boldsymbol{\psi} * \sqrt{M}(\widehat{\boldsymbol{q}} - \boldsymbol{q}) * \boldsymbol{\psi} + o(1);$$

here and in the sequel, $o(1)$ stands for an error term which vanishes in distribution as $M \to \infty$. Hence we have

$$\begin{aligned}\sqrt{M}(\widehat{\boldsymbol{P}} - \boldsymbol{P}) &= \sqrt{M}(\widehat{\boldsymbol{\psi}} * \mathcal{S}\widehat{\boldsymbol{q}} - \boldsymbol{\psi} * \mathcal{S}\boldsymbol{q}) \\ &= \sqrt{M}(\widehat{\boldsymbol{\psi}} - \boldsymbol{\psi}) * \mathcal{S}\widehat{\boldsymbol{q}} + \sqrt{M}\boldsymbol{\psi} * \mathcal{S}(\widehat{\boldsymbol{q}} - \boldsymbol{q}) \\ &= \sqrt{M}(\widehat{\boldsymbol{\psi}} - \boldsymbol{\psi}) * \mathcal{S}\boldsymbol{q} + o(1) + \boldsymbol{\psi} * \mathcal{S}\sqrt{M}(\widehat{\boldsymbol{q}} - \boldsymbol{q}) \\ &= \boldsymbol{\psi} * \sqrt{M}(\widehat{\boldsymbol{q}} - \boldsymbol{q}) * \boldsymbol{\psi} * \mathcal{S}\boldsymbol{q} + \boldsymbol{\psi} * \mathcal{S}\sqrt{M}(\widehat{\boldsymbol{q}} - \boldsymbol{q}) + o(1).\end{aligned}$$

By the asymptotic normality (1.3) and by Slutsky's theorem, we obtain

$$\sqrt{M}(\widehat{\boldsymbol{P}} - \boldsymbol{P}) \xrightarrow[M\to\infty]{\mathrm{d}} \boldsymbol{\psi} * \boldsymbol{Z}^{\boldsymbol{q}} * \boldsymbol{\psi} * \mathcal{S}\boldsymbol{q} + \boldsymbol{\psi} * \mathcal{S}\boldsymbol{Z}^{\boldsymbol{q}}.$$

The remainder of the proof is the same as that of Theorem 3.3. ∎

5. ESTIMATION OF THE RELIABILITY VECTOR SEQUENCE

Recall that the partition $E = U \cup D$ and the reliability vector sequence $\boldsymbol{R}$ have been introduced at the end of Introduction.

Let $\widetilde{U} := U \times U \times \mathbb{N}$ and let $\boldsymbol{q}_{UU} = ((q_{UU})_{ij}(k))_{(i,j,k)\in\widetilde{U}}$ denote the restriction of $\boldsymbol{q}$ to $\widetilde{U}$, i.e.,

$$(q_{UU})_{ij}(k) := q_{ij}(k) \quad \text{for } (i,j,k) \in \widetilde{U}.$$

Let $\boldsymbol{q}_{UU}^{(n)} = ((q_{UU}^{(n)})_{ij}(k))_{(i,j,k)\in\widetilde{U}}$ denotes the n-fold matrix convolution of $\boldsymbol{q}_{UU}$, which satisfies

$$\mathbb{P}_i(J_n = j,\, S_n = k,\, T_D > k) = (q_{UU}^{(n)})_{ij}(k).$$

We denote the matrix convolution inverse of $\boldsymbol{\delta I} - \boldsymbol{q}_{UU}$ by

$$\boldsymbol{\psi}_{UU} := (\boldsymbol{\delta I} - \boldsymbol{q}_{UU})^{(-1)} = \sum_{n=0}^{\infty} \boldsymbol{q}_{UU}^{(n)}.$$

Note that $\boldsymbol{q}_{UU}^{(n)}$ for $n \geqslant 2$ and $\boldsymbol{\psi}_{UU}$ *differ* from the restriction of $\boldsymbol{q}^{(n)}$ and $\boldsymbol{\psi}$ to $\widetilde{U}$, respectively.

The reliability vector sequence $\boldsymbol{R} = (R_i(k))_{(i,k)\in U\times\mathbb{N}}$ is given by

$$R_i(k) := \mathbb{P}_i(T_D > k) = \mathbb{P}_i(T_D > k,\, X_1 > k) + \mathbb{P}_i(T_D > k,\, X_1 \leqslant k).$$

Note that

$$\mathbb{P}_i(T_D > k,\, X_1 > k) = \mathbb{P}_i(X_1 > k) = (\mathcal{S}\boldsymbol{q})_{ii}(k) = ((\mathcal{S}\boldsymbol{q})_U)_i(k),$$

where we write $\boldsymbol{A}_U = ((\boldsymbol{A}_U)_i(k))_{(i,k)\in U\times\mathbb{N}} := (A_{ii}(k))_{(i,k)\in U\times\mathbb{N}}$ for a matrix sequence $\boldsymbol{A} = (A_{ij}(k))_{(i,j,k)\in\widetilde{E}}$, and

$$\begin{aligned}
\mathbb{P}_i(T_D > k,\, X_1 \leqslant k) &= \sum_{j'\in U}\sum_{k'\leqslant k} \mathbb{P}_i(T_D > k,\, J_1 = j',\, X_1 = k') \\
&= \sum_{j'\in U}\sum_{k'\leqslant k} q_{ij'}(k')\mathbb{P}_{j'}(T_D > k - k') \\
&= \sum_{j'\in U}\sum_{k'\leqslant k} q_{ij'}(k')R_{j'}(k - k') = (\boldsymbol{q}_{UU} * \boldsymbol{R})_i(k).
\end{aligned}$$

We thus obtain the renewal equation

$$\boldsymbol{R} = (\mathcal{S}\boldsymbol{q})_U + \boldsymbol{q}_{UU} * \boldsymbol{R},$$

which can be solved as

$$\boldsymbol{R} = (\boldsymbol{\delta I} - \boldsymbol{q}_{UU})^{(-1)} * (\mathcal{S}\boldsymbol{q})_U = \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q})_U.$$

Define $\widehat{\boldsymbol{q}}_{UU} = ((\widehat{q}_{UU})_{ij}(k))_{(i,j,k)\in\widetilde{U}}$ by $(\widehat{q}_{UU})_{ij}(k) := \widehat{q}_{ij}(k)$ for $(i,j,k) \in \widetilde{U}$ and

$$\widehat{\boldsymbol{\psi}}_{UU} := (\boldsymbol{\delta}\boldsymbol{I} - \widehat{\boldsymbol{q}}_{UU})^{(-1)} = \sum_{n=0}^{\infty} \widehat{\boldsymbol{q}}_{UU}^{(n)},$$

where $\widehat{\boldsymbol{q}}_{UU}^{(n)} = ((\widehat{q}_{UU}^{(n)})_{ij}(k))_{(i,j,k)\in\widetilde{U}}$ denotes the n-fold matrix convolution of $\widehat{\boldsymbol{q}}_{UU}$. Let us introduce an estimator of the reliability vector sequence by

$$\widehat{\boldsymbol{R}} := \widehat{\boldsymbol{\psi}}_{UU} * (\mathcal{S}\widehat{\boldsymbol{q}})_U.$$

The strong consistency of $\widehat{\boldsymbol{R}}$ is known:

THEOREM 5.1 (Barbu–Limnios [4, Theorem 5.1]). *Suppose* **(A)** *is satisfied. Then*

$$\widehat{\boldsymbol{R}} \xrightarrow[M\to\infty]{\text{a.s.}} \boldsymbol{R}$$

under $\mathbb{P}_{i_0}$ *for all initial values* $i_0 \in U$.

The asymptotic normality of $\widehat{\boldsymbol{R}} = (\widehat{R}_i(k,M))_{(i,k)\in U\times\mathbb{N}}$ in one dimension is as follows, which corrects the statement of Barbu–Limnios [4, Theorem 5.1].

THEOREM 5.2. *Suppose* **(A)** *is satisfied. Then, for any fixed* $(i,k) \in U \times \mathbb{N}$, *we have*

$$\sqrt{M}\big(\widehat{R}_i(k,M) - R_i(k)\big) \xrightarrow[M\to\infty]{\text{d}} \mathcal{N}(0, v^{\boldsymbol{R}}(i,k))$$

under $\mathbb{P}_{i_0}$ *for all initial values* $i_0 \in U$, *where the asymptotic variance* $v^{\boldsymbol{R}}(i,k)$ *is given by*

$$\begin{aligned} v^{\boldsymbol{R}}(i,k) := \sum_{\widetilde{i}\in U} \mu_{\widetilde{i}\,\widetilde{i}} \Big\{ & \sum_{\widetilde{j}\in U} [(D^{i}_{\widetilde{i}\,\widetilde{j}} - (\Psi_{UU})_{i\widetilde{i}})^2 * q_{\widetilde{i}\,\widetilde{j}}](k) \\ & - \sum_{\widetilde{j}\in U} [D^{i}_{\widetilde{i}\,\widetilde{j}} * q_{\widetilde{i}\,\widetilde{j}} - (\psi_{UU})_{i\widetilde{i}} * H_{\widetilde{i}}]^2(k) \Big\} \\ & + \sum_{\widetilde{i}\in U} \mu_{\widetilde{i}\,\widetilde{i}} \sum_{\widetilde{j}\in D} \big[((\Psi_{UU})_{i\widetilde{i}}(k) - (\Psi_{UU})_{i\widetilde{i}})^2 * q_{\widetilde{i}\,\widetilde{j}}\big](k), \end{aligned} \tag{5.1}$$

where

$$(\Psi_{UU})_{ij}(k) = \sum_{k'\leqslant k} (\psi_{UU})_{ij}(k')$$

and

$$D^{i}_{\widetilde{i}\,\widetilde{j}} = \sum_{j\in U} (\psi_{UU})_{i\widetilde{i}} * (\psi_{UU})_{\widetilde{j}j} * (1 - H_j).$$

Let us generalize Theorem 5.2 to higher dimensions, as a corollary of Theorem 1.3.

THEOREM 5.3. *Suppose* **(A)** *is satisfied. Then*

$$\sqrt{M}(\widehat{\boldsymbol{R}}-\boldsymbol{R}) \xrightarrow[M\to\infty]{\mathrm{d}} \mathcal{N}(\boldsymbol{0}, V^{\boldsymbol{R}}) \tag{5.2}$$

under $\mathbb{P}_{i_0}$ *for all initial values* $i_0 \in U$*, where the asymptotic variance matrix* $V^{\boldsymbol{R}} = (V^{\boldsymbol{R}}_{ee'})_{e,e'\in U\times\mathbb{N}}$ *is given by*

$$\begin{aligned} V^{\boldsymbol{R}} = {} & \sum_{\widetilde{e}\in\widetilde{U}} \overline{q}_{\widetilde{e}}\big(\boldsymbol{\psi}_{UU} * \boldsymbol{\delta}^{\widetilde{e}} * \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q})_U + \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{\delta}^{\widetilde{e}})_U\big)^{\otimes 2} \\ & - \sum_{\widetilde{i}\in U} \mu_{\widetilde{i}\,\widetilde{i}}\big(\boldsymbol{\psi}_{UU} * \boldsymbol{q}^{\widetilde{i}}_{UU} * \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q})_U + \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q}^{\widetilde{i}})_U\big)^{\otimes 2} \\ & + \sum_{\widetilde{e}\in U\times D\times\mathbb{N}} \overline{q}_{\widetilde{e}}\big(\boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{\delta}^{\widetilde{e}})_U\big)^{\otimes 2}. \end{aligned} \tag{5.3}$$

Proof. By the same arguments as in (3.6) and (3.7), we have

$$\sqrt{M}(\widehat{\boldsymbol{\psi}}_{UU} - \boldsymbol{\psi}_{UU}) = \boldsymbol{\psi}_{UU} * \sqrt{M}(\widehat{\boldsymbol{q}}_{UU} - \boldsymbol{q}_{UU}) * \boldsymbol{\psi}_{UU} + o(1).$$

Hence we have

$$\begin{aligned} \sqrt{M}(\widehat{\boldsymbol{R}}-\boldsymbol{R}) &= \sqrt{M}(\widehat{\boldsymbol{\psi}}_{UU} * (\mathcal{S}\widehat{\boldsymbol{q}})_U - \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q})_U) \\ &= \sqrt{M}(\widehat{\boldsymbol{\psi}}_{UU} - \boldsymbol{\psi}_{UU}) * (\mathcal{S}\widehat{\boldsymbol{q}})_U + \sqrt{M}\boldsymbol{\psi}_U * (\mathcal{S}(\widehat{\boldsymbol{q}} - \boldsymbol{q}))_U \\ &= \sqrt{M}(\widehat{\boldsymbol{\psi}}_{UU} - \boldsymbol{\psi}_{UU}) * (\mathcal{S}\boldsymbol{q})_U + o(1) + \boldsymbol{\psi}_{UU} * (\mathcal{S}\sqrt{M}(\widehat{\boldsymbol{q}} - \boldsymbol{q}))_U \\ &= \boldsymbol{\psi}_{UU} * \sqrt{M}(\widehat{\boldsymbol{q}}_{UU} - \boldsymbol{q}_{UU}) * \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q})_U \\ &\quad + \boldsymbol{\psi}_{UU} * (\mathcal{S}\sqrt{M}(\widehat{\boldsymbol{q}} - \boldsymbol{q}))_U + o(1). \end{aligned}$$

By the asymptotic normality (1.3) and Slutsky's theorem, we obtain

$$\sqrt{M}(\widehat{\boldsymbol{R}}-\boldsymbol{R}) \xrightarrow[M\to\infty]{\mathrm{d}} \boldsymbol{\psi}_{UU} * \boldsymbol{Z}^{\boldsymbol{q}}_{UU} * \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q})_U + \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{Z}^{\boldsymbol{q}})_U.$$

By the same arguments as in the proof of Theorem 3.3, we obtain (5.2) with the asymptotic covariance matrix given by

$$\begin{aligned} V^{\boldsymbol{R}} = {} & \sum_{\widetilde{e}\in\widetilde{E}} \overline{q}_{\widetilde{e}}\big(\boldsymbol{\psi}_{UU} * \boldsymbol{\delta}^{\widetilde{e}}_{UU} * \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q})_U + \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{\delta}^{\widetilde{e}})_U\big)^{\otimes 2} \\ & - \sum_{\widetilde{i}\in E} \mu_{\widetilde{i}\,\widetilde{i}}\big(\boldsymbol{\psi}_{UU} * \boldsymbol{q}^{\widetilde{i}}_{UU} * \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q})_U + \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q}^{\widetilde{i}})_U\big)^{\otimes 2}, \end{aligned}$$

where $\boldsymbol{\delta}^{\widetilde{e}}_{UU}$ stands for the restriction of $\boldsymbol{\delta}^{\widetilde{e}}$ on $\widetilde{U}$. Since we easily see that

$$\begin{aligned}
&\boldsymbol{\psi}_{UU} * \boldsymbol{\delta}^{\widetilde{e}}_{UU} * \boldsymbol{\psi}_{UU} = \mathbf{0} \quad \text{for } \widetilde{e} \notin \widetilde{U},\\
&\boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{\delta}^{\widetilde{e}})_U = \mathbf{0} \quad \text{for } \widetilde{e} \in D \times E \times \mathbb{N},\\
&\boldsymbol{\psi}_{UU} * \boldsymbol{q}^{\widetilde{i}}_{UU} * \boldsymbol{\psi}_{UU} = \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q}^{\widetilde{i}})_U = \mathbf{0} \quad \text{for } \widetilde{i} \in D,
\end{aligned}$$

we obtain (5.3). ∎

6. APPLICATION AND EXAMPLES

6.1. Asymptotic confidence regions. Let $\boldsymbol{X} = (X_e)_{e\in E}$ be a family of unknown parameters indexed by a finite set E. Suppose that we have estimators $\widehat{\boldsymbol{X}}(M) = (\widehat{X}_e(M))_{e\in E}$ of $\boldsymbol{X}$ and obtain asymptotic normality in higher dimensions:

$$\sqrt{M}(\widehat{\boldsymbol{X}}(M) - \boldsymbol{X}) \xrightarrow[M\to\infty]{\mathrm{d}} \mathcal{N}(\mathbf{0}, V) \tag{6.1}$$

with an asymptotic variance matrix $V = (V_{ee'})_{e,e'\in E}$. Let us discuss asymptotic confidence regions for the parameter $\boldsymbol{X}$.

Let $\boldsymbol{N} = (N_e)_{e\in E}$ denote the family of independent standard normal variables. For $u > 0$, set

$$\begin{aligned}
B(u) &= \{\boldsymbol{x} = (x_e)_{e\in E} : x_e \in [-u, u] \text{ for all } e \in E\},\\
\tau(u) &= \mathbb{P}(\boldsymbol{N} \notin B(u)) = 1 - \left(\int_{-u}^{u} \mathrm{e}^{-x^2/2}\,\frac{\mathrm{d}x}{\sqrt{2\pi}}\right)^{\#E},
\end{aligned}$$

where $\#E$ denotes the number of elements of E. Then the asymptotic normality formula (6.1) yields

$$\boldsymbol{P}\big(\sqrt{M}(\boldsymbol{X} - \widehat{\boldsymbol{X}}(M)) \notin \sqrt{V}B(u)\big) \xrightarrow[M\to\infty]{} \boldsymbol{P}(\boldsymbol{N} \notin B(u)) = \tau(u),$$

where $\sqrt{V}$ denotes the matrix square-root of the nonnegative-definite symmetric matrix V. We can now conclude that, for any $\alpha \in [0, 1]$, the region

$$\widehat{\boldsymbol{X}}(M) + \frac{1}{\sqrt{M}}\sqrt{V}B(u)$$

with $u = \tau^{-1}(\alpha)$ and M being large enough is a $100(1-\alpha)\%$ asymptotic confidence region of the parameter $\boldsymbol{X}$.

6.2. Several examples of our main theorems. Let $E = \{1, 2\}$ and suppose that $\boldsymbol{q}(k) = (q_{ij}(k))_{(i,j,k)\in\widetilde{E}}$ is of the form

$$\begin{pmatrix} q_{11}(k) & q_{12}(k) \\ q_{21}(k) & q_{22}(k) \end{pmatrix} = \begin{pmatrix} a_{11}x_k & a_{12}x_k \\ a_{21}x_k & a_{22}x_k \end{pmatrix} \quad \text{for all } i, j \in E \text{ and } k \in \mathbb{N},$$

with $x_0 = 0$, and that the parameters $a_{11}, a_{12}, a_{21}, a_{22}, x_1, x_2, \dots$ are positive and satisfy

$$a_{11} + a_{12} = a_{21} + a_{22} = 1, \quad \sum_{k\in\mathbb{N}} x_k = 1, \quad \overline{x} := \sum_{k\in\mathbb{N}} k x_k < \infty.$$

The transition probabilities of the embedded Markov chain J are given by

$$(q_{ij}^J)_{i,j\in E} = \boldsymbol{A} := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

so that assumption **(A1)** is satisfied. Since, for all $i \in E$ and $k \geqslant 1$, we have

$$\mathbb{P}_i(S_{\tau_i} = k) \geqslant \mathbb{P}_i(J_1 = i,\ X_1 = k) = q_{ii}(k) = a_{ii}x_k > 0,$$

we see that assumption **(A2)** is satisfied. For $i, j \in E$, we have

$$\begin{aligned} \mu_{ij} = \mathbb{E}_i S_{\tau_j} &= \sum_{k\in\mathbb{N}} k p_{ij}(k) + \sum_{r\neq j} \sum_{k\in\mathbb{N}} p_{ir}(k)\mathbb{E}_r[S_{\tau_j} + k] \\ &= a_{ij}\overline{x} + \sum_{r\neq j} a_{ir}(\mu_{rj} + \overline{x}) = \overline{x} + \sum_{r\neq j} a_{ir}\mu_{rj}. \end{aligned}$$

This shows

$$\mu_{12} = \frac{\overline{x}}{a_{12}}, \quad \mu_{21} = \frac{\overline{x}}{a_{21}}, \quad \mu_{11} = \frac{a_{12}+a_{21}}{a_{21}} \cdot \overline{x}, \quad \mu_{22} = \frac{a_{12}+a_{21}}{a_{12}} \cdot \overline{x},$$

and hence assumption **(A3)** is satisfied. Since $\overline{m} = \mu_{11}\nu_1 = \mu_{22}\nu_2$ and $\nu_1 + \nu_2 = 1$, we obtain

$$\overline{m} = \overline{x}, \quad \nu_1 = \frac{a_{21}}{a_{12}+a_{21}}, \quad \nu_2 = \frac{a_{12}}{a_{12}+a_{21}}.$$

If we write $(x_k^{(n)})_{k\in\mathbb{N}}$ for the n-fold convolution of $(x_k)_{k\in\mathbb{N}}$, i.e., $x_k^{(1)} = x_k$ and

$$x_k^{(n)} = \sum_{l=0}^{k} x_l^{(n-1)} x_{k-l} \quad \text{for } n \geqslant 2 \text{ and } k \in \mathbb{N},$$

we can see by induction that

$$\boldsymbol{q}^{(n)}(k) = x_k^{(n)} \boldsymbol{A}^n \quad \text{for } n \geqslant 1 \text{ and } k \in \mathbb{N}.$$

The matrix convolution inverse $\boldsymbol{\psi}$ is then given by

$$\boldsymbol{\psi}(0) = I, \quad \boldsymbol{\psi}(k) = \sum_{n=1}^{\infty} x_k^{(n)} \boldsymbol{A}^n = \sum_{n=1}^{k} x_k^{(n)} \boldsymbol{A}^n \quad \text{for } k \geqslant 1.$$

Since

$$(\mathcal{S}\boldsymbol{q})_{ij}(k) = 1_{\{i=j\}} \sum_{k'>k} x_{k'} \quad \text{for } i,j \in E,\ k \in \mathbb{N},$$

the distribution matrix sequence $\boldsymbol{P}$ is given by $\boldsymbol{P}(0) = \boldsymbol{I}$ and

$$P_{ij}(k) = 1_{\{i=j\}} \sum_{k'>k} x_{k'} + \sum_{l=1}^{k} \Big(\sum_{n=1}^{l} A_{ij}^{n} x_l^{(n)} \Big) \sum_{k'>k-l} x_{k'} \quad \text{for } i,j \in E,\ k \geqslant 1,$$

where $\boldsymbol{A}^n = (A_{ij}^n)_{i,j\in E}$ (so that $A_{ij}^1 = a_{ij}$). In particular, we obtain

$$\begin{aligned} &\psi_{11}(1) = a_{11}x_1, && \psi_{22}(1) = a_{22}x_1, \\ &P_{11}(1) = 1 - x_1 + a_{11}x_1, && P_{22}(1) = 1 - x_1 + a_{22}x_1. \end{aligned}$$

If we apply Theorem 1.3 to the two-dimensional estimator $(\widehat{q}_{11}(1), \widehat{q}_{22}(1))^\top$, we obtain

$$\sqrt{M} \begin{pmatrix} \widehat{q}_{11}(1,M) - q_{11}(1) \\ \widehat{q}_{22}(1,M) - q_{22}(1) \end{pmatrix} \xrightarrow[M\to\infty]{\mathrm{d}} \mathcal{N}(\mathbf{0}, W^1),$$

where the asymptotic variance matrix W^1 is given by

$$W^1 = \begin{pmatrix} V^{\boldsymbol{q}}_{(1,1,1),(1,1,1)} & V^{\boldsymbol{q}}_{(1,1,1),(2,2,1)} \\ V^{\boldsymbol{q}}_{(2,2,1),(1,1,1)} & V^{\boldsymbol{q}}_{(2,2,1),(2,2,1)} \end{pmatrix} = \begin{pmatrix} v^{\boldsymbol{q}}(1,1,1) & 0 \\ 0 & v^{\boldsymbol{q}}(2,2,1) \end{pmatrix} \tag{6.2}$$

with

$$v^{\boldsymbol{q}}(1,1,1) = \mu_{11}a_{11}x_1(1 - a_{11}x_1), \quad v^{\boldsymbol{q}}(2,2,1) = \mu_{22}a_{22}x_1(1 - a_{22}x_1).$$

Formula (6.2) shows that $\widehat{q}_{11}(1)$ and $\widehat{q}_{22}(1)$ are asymptotically independent. If we apply Theorem 3.3 to $(\widehat{\psi}_{11}(1), \widehat{\psi}_{22}(1))^\top$, we obtain

$$\sqrt{M} \begin{pmatrix} \widehat{\psi}_{11}(1,M) - \psi_{11}(1) \\ \widehat{\psi}_{22}(1,M) - \psi_{22}(1) \end{pmatrix} \xrightarrow[M\to\infty]{\mathrm{d}} \mathcal{N}(\mathbf{0}, W^2),$$

where the asymptotic variance matrix W^2 is given by

$$W^2 = \begin{pmatrix} V^{\boldsymbol{\psi}}_{(1,1,1),(1,1,1)} & V^{\boldsymbol{\psi}}_{(1,1,1),(2,2,1)} \\ V^{\boldsymbol{\psi}}_{(2,2,1),(1,1,1)} & V^{\boldsymbol{\psi}}_{(2,2,1),(2,2,1)} \end{pmatrix} = \begin{pmatrix} v^{\boldsymbol{\psi}}(1,1,1) & 0 \\ 0 & v^{\boldsymbol{\psi}}(2,2,1) \end{pmatrix}$$

with

$$v^{\boldsymbol{\psi}}(1,1,1) = \mu_{11}a_{11}x_1(1 - a_{11}x_1), \quad v^{\boldsymbol{\psi}}(2,2,1) = \mu_{22}a_{22}x_1(1 - a_{22}x_1).$$

If we apply Theorem 4.3 to $(\widehat{P}_{11}(1), \widehat{P}_{22}(1))^\top$, we obtain

$$\sqrt{M}\begin{pmatrix}\widehat{P}_{11}(1,M) - P_{11}(1)\\ \widehat{P}_{22}(1,M) - P_{22}(1)\end{pmatrix} \xrightarrow[M\to\infty]{\mathrm{d}} \mathcal{N}(\mathbf{0}, W^3),$$

where the asymptotic variance matrix W^3 is given by

$$W^3 = \begin{pmatrix} V^{\boldsymbol{P}}_{(1,1,1),(1,1,1)} & V^{\boldsymbol{P}}_{(1,1,1),(2,2,1)} \\ V^{\boldsymbol{P}}_{(2,2,1),(1,1,1)} & V^{\boldsymbol{P}}_{(2,2,1),(2,2,1)}\end{pmatrix} = \begin{pmatrix} v^{\boldsymbol{P}}(1,1,1) & 0 \\ 0 & v^{\boldsymbol{P}}(2,2,1)\end{pmatrix}$$

with

$$v^{\boldsymbol{P}}(1,1,1) = \mu_{11}\{(1-a_{11})x_1\}^2, \quad v^{\boldsymbol{P}}(2,2,1) = \mu_{22}\{(1-a_{22})x_1\}^2.$$

If we apply Theorem 1.3 to $(\widehat{q}_{11}(1), \widehat{q}_{11}(2))^\top$, we obtain

$$\sqrt{M}\begin{pmatrix}\widehat{q}_{11}(1,M) - q_{11}(1)\\ \widehat{q}_{11}(2,M) - q_{11}(2)\end{pmatrix} \xrightarrow[M\to\infty]{\mathrm{d}} \mathcal{N}(\mathbf{0}, W^4),$$

where the asymptotic variance matrix W^4 is given by

$$\begin{aligned} W^4 &= \begin{pmatrix} V^{\boldsymbol{q}}_{(1,1,1),(1,1,1)} & V^{\boldsymbol{q}}_{(1,1,1),(1,1,2)} \\ V^{\boldsymbol{q}}_{(1,1,2),(1,1,1)} & V^{\boldsymbol{q}}_{(1,1,2),(1,1,2)}\end{pmatrix} \\ &= \begin{pmatrix} v^{\boldsymbol{q}}(1,1,1) & -\mu_{11}a_{11}^2 x_1 x_2 \\ -\mu_{11}a_{11}^2 x_1 x_2 & v^{\boldsymbol{q}}(1,1,2)\end{pmatrix} \end{aligned} \tag{6.3}$$

with

$$v^{\boldsymbol{q}}(1,1,2) = \mu_{11}a_{11}x_2(1-a_{11}x_2).$$

Formula (6.3) shows that $\widehat{q}_{11}(1)$ and $\widehat{q}_{22}(1)$ are asymptotically in negative correlation. If we apply Theorem 3.3 to $(\widehat{\psi}_{11}(1), \widehat{\psi}_{11}(2))^\top$, we obtain

$$\sqrt{M}\begin{pmatrix}\widehat{\psi}_{11}(1,M) - \psi_{11}(1)\\ \widehat{\psi}_{11}(2,M) - \psi_{11}(2)\end{pmatrix} \xrightarrow[M\to\infty]{\mathrm{d}} \mathcal{N}(\mathbf{0}, W^5),$$

where the asymptotic variance matrix W^5 is given by

$$\begin{aligned} W^5 &= \begin{pmatrix} V^{\boldsymbol{\psi}}_{(1,1,1),(1,1,1)} & V^{\boldsymbol{\psi}}_{(1,1,1),(1,1,2)} \\ V^{\boldsymbol{\psi}}_{(1,1,2),(1,1,1)} & V^{\boldsymbol{\psi}}_{(1,1,2),(1,1,2)}\end{pmatrix} \\ &= \begin{pmatrix} v^{\boldsymbol{\psi}}(1,1,1) & 2\psi_{11}(1)v^{\boldsymbol{\psi}}(1,1,1) \\ 2\psi_{11}(1)v^{\boldsymbol{\psi}}(1,1,1) & v^{\boldsymbol{\psi}}(1,1,2)\end{pmatrix} \end{aligned} \tag{6.4}$$

with

$$\begin{aligned} v^{\boldsymbol{\psi}}(1,1,2) = {} & \mu_{11}a_{11}x_1 \cdot (2\psi_{11}(1))^2 + \mu_{11}a_{11}x_2 \\ & - \mu_{11} \cdot (2\psi_{11}(1) \cdot a_{11}x_1 + a_{11}x_2)^2. \end{aligned}$$

Formula (6.4) shows that $\widehat{\psi}_{11}(1)$ and $\widehat{\psi}_{22}(1)$ are asymptotically in positive correlation.

7. APPENDIX: COMPUTATION OF ASYMPTOTIC VARIANCES

7.1. The asymptotic variance for the matrix convolution inverse. Let us check that $V^{\boldsymbol{\psi}}_{ee'}$ in (3.5) for $e = e' = (i, j, k) \in \widetilde{E}$ coincides with $v^{\boldsymbol{\psi}}(i, j, k)$ in (3.4). Set

$$V^1 := \sum_{\widetilde{e}\in\widetilde{E}} \overline{q}_{\widetilde{e}}(\boldsymbol{\psi} * \boldsymbol{\delta}^{\widetilde{e}} * \boldsymbol{\psi})^{\otimes 2}, \quad V^2 := \sum_{\widetilde{i}\in E} \mu_{\widetilde{i}\,\widetilde{i}}(\boldsymbol{\psi} * \boldsymbol{q}^{\widetilde{i}} * \boldsymbol{\psi})^{\otimes 2},$$

so that $V^{\boldsymbol{\psi}} = V^1 - V^2$. For V^1_{ee}, we have

$$\begin{aligned} V^1_{ee} &= \sum_{\widetilde{e}\in\widetilde{E}} \overline{q}_{\widetilde{e}}[(\boldsymbol{\psi} * \boldsymbol{\delta}^{\widetilde{e}} * \boldsymbol{\psi})_e]^2 = \sum_{\widetilde{e}=(\widetilde{i},\widetilde{j},\widetilde{k})\in\widetilde{E}} q_{\widetilde{e}}[(\psi_{i\widetilde{i}} * \psi_{\widetilde{j}j})(k - \widetilde{k})]^2 \\ &= \sum_{\widetilde{i}\in E} \mu_{\widetilde{i}\,\widetilde{i}} \sum_{\widetilde{j}\in E} \sum_{\widetilde{k}\in\mathbb{N}} q_{\widetilde{i}\,\widetilde{j}}(\widetilde{k})[(\psi_{i\widetilde{i}} * \psi_{\widetilde{j}j})(k - \widetilde{k})]^2 \\ &= \sum_{\widetilde{i}\in E} \mu_{\widetilde{i}\,\widetilde{i}} \sum_{\widetilde{j}\in E} [(\psi_{i\widetilde{i}} * \psi_{\widetilde{j}j})^2 * q_{\widetilde{i}\,\widetilde{j}}](k). \end{aligned}$$

For V^2_{ee}, we have

$$V^2_{ee} = \sum_{\widetilde{i}\in E} \mu_{\widetilde{i}\,\widetilde{i}}[(\boldsymbol{\psi} * \boldsymbol{q}^{\widetilde{i}} * \boldsymbol{\psi})_e]^2 = \sum_{\widetilde{i}\in E} \mu_{\widetilde{i}\,\widetilde{i}}\Big[\sum_{\widetilde{j}\in E} (\psi_{i\widetilde{i}} * q_{\widetilde{i}\,\widetilde{j}} * \psi_{\widetilde{j}j})(k)\Big]^2.$$

Hence $V^{\boldsymbol{\psi}}_{ee} = v^{\boldsymbol{\psi}}(i, j, k)$.

7.2. The asymptotic variance for the distribution matrix sequence. Let us check that $V^{\boldsymbol{P}}_{ee'}$ in (4.2) for $e = e' = (i, j, k) \in \widetilde{E}$ coincides with $v^{\boldsymbol{P}}(i, j, k)$ in (4.1). Set

$$\begin{aligned} V^1 &:= \sum_{\widetilde{e}\in\widetilde{E}} \overline{q}_{\widetilde{e}}(\boldsymbol{\psi} * \boldsymbol{\delta}^{\widetilde{e}} * \boldsymbol{\psi} * \mathcal{S}\boldsymbol{q} + \boldsymbol{\psi} * \mathcal{S}\boldsymbol{\delta}^{\widetilde{e}})^{\otimes 2}, \\ V^2 &:= \sum_{\widetilde{i}\in E} \mu_{\widetilde{i}\,\widetilde{i}}(\boldsymbol{\psi} * \boldsymbol{q}^{\widetilde{i}} * \boldsymbol{\psi} * \mathcal{S}\boldsymbol{q} + \boldsymbol{\psi} * \mathcal{S}\boldsymbol{q}^{\widetilde{i}})^{\otimes 2}, \end{aligned}$$

so that $V^{\boldsymbol{P}} = V^1 - V^2$. Since

$$\begin{aligned} (\boldsymbol{\psi} * \boldsymbol{\delta}^{\widetilde{e}} * \boldsymbol{\psi} * \mathcal{S}\boldsymbol{q})_e &= \sum_{k''\in\mathbb{N}:\ \widetilde{k}\leqslant k''\leqslant k} (\psi_{i\widetilde{i}} * \psi_{\widetilde{j}j})(k'' - \widetilde{k}) \sum_{j'\in E} \sum_{k'>k-k''} q_{jj'}(k') \\ &= \sum_{k''\in\mathbb{N}:\ \widetilde{k}\leqslant k''\leqslant k} (\psi_{i\widetilde{i}} * \psi_{\widetilde{j}j})(k'' - \widetilde{k}) \cdot (1 - H_j(k - k'')) \\ &= \big(\psi_{i\widetilde{i}} * \psi_{\widetilde{j}j} * (1 - H_j)\big)(k - \widetilde{k}) = C^{ij}_{\widetilde{i}\,\widetilde{j}}(k - \widetilde{k}) \end{aligned}$$

and

$$\begin{aligned} (\boldsymbol{\psi} * \mathcal{S}\boldsymbol{\delta}^{\widetilde{e}})_e &= 1_{\{\widetilde{i}=j\}} \sum_{k'\leqslant k} \psi_{i\widetilde{i}}(k') 1_{\{k'>k-\widetilde{k}\geqslant 0\}} \\ &= 1_{\{\widetilde{i}=j\}}\big(\Psi_{ij}(k) - \Psi_{ij}(k - \widetilde{k})\big) 1_{\{\widetilde{k}\leqslant k\}}, \end{aligned}$$

we have

$$\begin{aligned} V_{ee}^1 &= \sum_{\widetilde{i}\in E} \mu_{\widetilde{i}\,\widetilde{i}} \sum_{\widetilde{j}\in E} \sum_{\widetilde{k}\in\mathbb{N}} q_{\widetilde{i}\,\widetilde{j}}(\widetilde{k})[(\boldsymbol{\psi} * \boldsymbol{\delta}^{\widetilde{e}} * \boldsymbol{\psi} * \mathcal{S}\boldsymbol{q})_e + (\boldsymbol{\psi} * \mathcal{S}\boldsymbol{\delta}^{\widetilde{e}})_e]^2 \\ &= \sum_{\widetilde{i}\in E} \mu_{\widetilde{i}\,\widetilde{i}} \sum_{\widetilde{j}\in E} [(C_{\widetilde{i}\,\widetilde{j}}^{ij} + 1_{\{\widetilde{i}=j\}}(\Psi_{ij}(k) - \Psi_{ij}))^2 * q_{\widetilde{i}\,\widetilde{j}}](k). \end{aligned}$$

Since

$$\begin{aligned} (\boldsymbol{\psi} * \boldsymbol{q}^{\widetilde{i}} * \boldsymbol{\psi} * \mathcal{S}\boldsymbol{q})_e &= \sum_{\widetilde{j}\in E} \sum_{k''\in\mathbb{N}} (\psi_{i\widetilde{i}} * q_{\widetilde{i}\,\widetilde{j}} * \psi_{\widetilde{j}j})(k'') \sum_{j'\in E} \sum_{k'>k-k''} q_{jj'}(k') \\ &= \sum_{\widetilde{j}\in E} [\psi_{i\widetilde{i}} * q_{\widetilde{i}\,\widetilde{j}} * \psi_{\widetilde{j}j} * (1 - H_j)](k) \\ &= \sum_{\widetilde{j}\in E} [C_{\widetilde{i}\,\widetilde{j}}^{ij} * q_{\widetilde{i}\,\widetilde{j}}](k) \end{aligned}$$

and

$$\begin{aligned} (\boldsymbol{\psi} * \mathcal{S}\boldsymbol{q}^{\widetilde{i}})_e &= 1_{\{\widetilde{i}=j\}} \sum_{k''\leqslant k} \psi_{i\widetilde{i}}(k'') \sum_{j'\in E} \sum_{k'>k-k''} q_{jj'}(k') \\ &= 1_{\{\widetilde{i}=j\}}[\psi_{ij} * (1 - H_j)](k), \end{aligned}$$

we have

$$\begin{aligned} V_{ee}^2 &= \sum_{\widetilde{i}\in E} \mu_{\widetilde{i}\,\widetilde{i}}[(\boldsymbol{\psi} * \boldsymbol{q}^{\widetilde{i}} * \boldsymbol{\psi} * \mathcal{S}\boldsymbol{q})_e + (\boldsymbol{\psi} * \mathcal{S}\boldsymbol{q}^{\widetilde{i}})_e]^2, \\ &= \sum_{\widetilde{i}\in E} \mu_{\widetilde{i}\,\widetilde{i}}\Big[\Big(\sum_{\widetilde{j}\in E} C_{\widetilde{i}\,\widetilde{j}}^{ij} * q_{\widetilde{i}\,\widetilde{j}} + 1_{\{\widetilde{i}=j\}}\psi_{ij} * (1 - H_j)\Big)^2\Big](k). \end{aligned}$$

Since $\psi_{ij} * 1 = \Psi_{ij}$, $\sum_{\widetilde{j}\in E} q_{i\widetilde{j}} * 1 = 1$ and $\sum_{\widetilde{j}\in E} \Psi_{ij} * q_{\widetilde{i}\,\widetilde{j}} = \psi_{ij} * H_{\widetilde{i}}$, we have

$$\begin{aligned} &\sum_{\widetilde{j}\in E} [(C_{\widetilde{i}\,\widetilde{j}}^{ij} + 1_{\{\widetilde{i}=j\}}(\Psi_{ij}(k) - \Psi_{ij}))^2 * q_{\widetilde{i}\,\widetilde{j}}](k) \\ &\qquad - \Big[\Big(\sum_{\widetilde{j}\in E} C_{\widetilde{i}\,\widetilde{j}}^{ij} * q_{\widetilde{i}\,\widetilde{j}} + 1_{\{\widetilde{i}=j\}}\psi_{ij} * (1 - H_j)\Big)^2\Big](k) \\ &\quad = \sum_{\widetilde{j}\in E} [(C_{\widetilde{i}\,\widetilde{j}}^{ij} - 1_{\{\widetilde{i}=j\}}\Psi_{ij})^2 * q_{\widetilde{i}\,\widetilde{j}}](k) \\ &\qquad - \Big[\Big(\sum_{\widetilde{j}\in E} C_{\widetilde{i}\,\widetilde{j}}^{ij} * q_{\widetilde{i}\,\widetilde{j}} - 1_{\{\widetilde{i}=j\}}\psi_{ij} * H_j\Big)^2\Big](k). \end{aligned}$$

Hence $V_{ee}^{\boldsymbol{P}} = v^{\boldsymbol{P}}(i, j, k)$.

7.3. The asymptotic variance for the reliability vector sequence. Let us check that $V^{\boldsymbol{R}}_{ee'}$ in (5.3) for $e = e' = (i,k) \in U \times \mathbb{N}$ coincides with $v^{\boldsymbol{R}}(i,k)$ in (5.1). Set

$$V^1 := \sum_{\widetilde{e}\in\widetilde{U}} \overline{q}_{\widetilde{e}} \big(\boldsymbol{\psi}_{UU} * \boldsymbol{\delta}^{\widetilde{e}} * \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q})_U + \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{\delta}^{\widetilde{e}})_U\big)^{\otimes 2},$$
$$V^2 := \sum_{\widetilde{i}\in U} \mu_{\widetilde{i}\,\widetilde{i}} \big(\boldsymbol{\psi}_{UU} * \boldsymbol{q}^{\widetilde{i}}_{UU} * \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q})_U + \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q}^{\widetilde{i}})_U\big)^{\otimes 2},$$
$$V^3 := \sum_{\widetilde{e}\in U\times D\times\mathbb{N}} \overline{q}_{\widetilde{e}} \big(\boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{\delta}^{\widetilde{e}})_U\big)^{\otimes 2},$$

so that $V^{\boldsymbol{R}} = V^1 - V^2 + V^3$. Since

$$\begin{aligned}
&\big(\boldsymbol{\psi}_{UU} * \boldsymbol{\delta}^{\widetilde{e}} * \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q})_U\big)_e \\
&\quad = \sum_{j\in U} \sum_{k''\in\mathbb{N}:\,\widetilde{k}\leqslant k''\leqslant k} \big((\boldsymbol{\psi}_{UU})_{i\widetilde{i}} * (\boldsymbol{\psi}_{UU})_{\widetilde{j}j}\big)(k''-\widetilde{k}) \sum_{j'\in E}\sum_{k'>k-k''} q_{jj'}(k') \\
&\quad = \sum_{j\in U} \big((\boldsymbol{\psi}_{UU})_{i\widetilde{i}} * (\boldsymbol{\psi}_{UU})_{\widetilde{j}j} * (1-H_j)\big)(k-\widetilde{k}) = D^i_{\widetilde{i}\,\widetilde{j}}(k-\widetilde{k})
\end{aligned}$$

and

$$\begin{aligned}
(\boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{\delta}^{\widetilde{e}})_U)_e &= \sum_{k'\leqslant k} (\psi_{UU})_{i\widetilde{i}}(k') 1_{\{k'>k-\widetilde{k}\geqslant 0\}} \\
&= \big((\Psi_{UU})_{i\widetilde{i}}(k) - (\Psi_{UU})_{i\widetilde{i}}(k-\widetilde{k})\big) 1_{\{\widetilde{k}\leqslant k\}},
\end{aligned}$$

we have

$$\begin{aligned}
V^1_{ee} &= \sum_{\widetilde{i}\in U} \mu_{\widetilde{i}\,\widetilde{i}} \sum_{\widetilde{j}\in U}\sum_{\widetilde{k}\in\mathbb{N}} q_{\widetilde{i}\,\widetilde{j}}(\widetilde{k}) \big[\big(\boldsymbol{\psi}_{UU} * \boldsymbol{\delta}^{\widetilde{e}} * \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q})_U\big)_e + (\boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{\delta}^{\widetilde{e}})_U)_e\big]^2 \\
&= \sum_{\widetilde{i}\in U} \mu_{\widetilde{i}\,\widetilde{i}} \sum_{\widetilde{j}\in U} \big[\big(D^i_{\widetilde{i}\,\widetilde{j}} + (\Psi_{UU})_{i\widetilde{i}}(k) - (\Psi_{UU})_{i\widetilde{i}}\big)^2 * q_{\widetilde{i}\,\widetilde{j}}\big](k)
\end{aligned}$$

and

$$\begin{aligned}
V^3_{ee} &= \sum_{\widetilde{i}\in U} \mu_{\widetilde{i}\,\widetilde{i}} \sum_{\widetilde{j}\in D}\sum_{\widetilde{k}\in\mathbb{N}} q_{\widetilde{i}\,\widetilde{j}}(\widetilde{k}) [(\boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{\delta}^{\widetilde{e}})_U)_e]^2, \\
&= \sum_{\widetilde{i}\in U} \mu_{\widetilde{i}\,\widetilde{i}} \sum_{\widetilde{j}\in D} \big[\big((\Psi_{UU})_{i\widetilde{i}}(k) - (\Psi_{UU})_{i\widetilde{i}}\big)^2 * q_{\widetilde{i}\,\widetilde{j}}\big](k).
\end{aligned}$$

Since

$$\begin{aligned}
&\big(\boldsymbol{\psi}_{UU} * \boldsymbol{q}^{\widetilde{i}}_{UU} * \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q})_U\big)_e \\
&\quad = \sum_{\widetilde{j}\in U} \sum_{k''\in\mathbb{N}} \sum_{j\in U} \big((\psi_{UU})_{i\widetilde{i}} * q_{\widetilde{i}\,\widetilde{j}} * (\psi_{UU})_{\widetilde{j}j}\big)(k'') \sum_{j'\in E}\sum_{k'>k-k''} q_{jj'}(k')
\end{aligned}$$

$$= \sum_{\widetilde{j}\in U} [(\psi_{UU})_{i\widetilde{i}} * q_{\widetilde{i}\,\widetilde{j}} * (\psi_{UU})_{\widetilde{j}j} * (1 - H_j)](k)$$
$$= \sum_{\widetilde{j}\in U} [D^i_{\widetilde{i}\,\widetilde{j}} * q_{\widetilde{i}\,\widetilde{j}}](k)$$

and

$$(\boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q}^{\widetilde{i}})_U)_e = \sum_{k''\leqslant k} (\psi_{UU})_{i\widetilde{i}}(k'') \sum_{j'\in E} \sum_{k'>k-k''} q_{\widetilde{i}j'}(k')$$
$$= [(\psi_{UU})_{i\widetilde{i}} * (1 - H_{\widetilde{i}})](k),$$

we have

$$V^2_{ee} = \sum_{\widetilde{i}\in U} \mu_{\widetilde{i}\,\widetilde{i}}\big[\big(\boldsymbol{\psi}_{UU} * \boldsymbol{q}^{\widetilde{i}}_{UU} * \boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q})_U\big)_e + \big(\boldsymbol{\psi}_{UU} * (\mathcal{S}\boldsymbol{q}^{\widetilde{i}})_U\big)_e\big]^2$$
$$= \sum_{\widetilde{i}\in U} \mu_{\widetilde{i}\,\widetilde{i}}\Big[\Big(\sum_{\widetilde{j}\in U} D^i_{\widetilde{i}\,\widetilde{j}} * q_{\widetilde{i}\,\widetilde{j}} + (\psi_{UU})_{i\widetilde{i}} * (1 - H_{\widetilde{i}})\Big)^2\Big](k).$$

Since $(\psi_{UU})_{ij} * 1 = (\Psi_{UU})_{ij}$, $\sum_{\widetilde{j}\in E} q_{i\widetilde{j}} * 1 = 1$ and $\sum_{\widetilde{j}\in E}(\Psi_{UU})_{ij} * q_{\widetilde{i}\,\widetilde{j}} = (\psi_{UU})_{ij} * H_{\widetilde{i}}$, we have

$$\sum_{\widetilde{j}\in U} \big[\big(D^i_{\widetilde{i}\,\widetilde{j}} + (\Psi_{UU})_{i\widetilde{i}}(k) - (\Psi_{UU})_{i\widetilde{i}}\big)^2 * q_{\widetilde{i}\,\widetilde{j}}\big](k)$$
$$- \Big[\Big(\sum_{\widetilde{j}\in U} D^i_{\widetilde{i}\,\widetilde{j}} * q_{\widetilde{i}\,\widetilde{j}} + (\psi_{UU})_{i\widetilde{i}} * (1 - H_{\widetilde{i}})\Big)^2\Big](k)$$
$$= \sum_{\widetilde{j}\in U} [(D^i_{\widetilde{i}\,\widetilde{j}} - (\Psi_{UU})_{i\widetilde{i}})^2 * q_{\widetilde{i}\,\widetilde{j}}](k)$$
$$- \Big[\Big(\sum_{\widetilde{j}\in U} D^i_{\widetilde{i}\,\widetilde{j}} * q_{\widetilde{i}\,\widetilde{j}} - (\psi_{UU})_{i\widetilde{i}} * H_{\widetilde{i}}\Big)^2\Big](k).$$

Hence $V^{\boldsymbol{R}}_{ee} = v^{\boldsymbol{R}}(i,k)$.

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