PROBABILITY
AND
MATHEMATICAL STATISTICS

Published online 10.1.2025
doi:10.37190/0208-4147.00206
Online First version

GROWING ODD GRAPHS AND ASYMPTOTIC DISTRIBUTION IN A DEFORMED VACUUM STATE

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Abstract. We investigate the asymptotic distribution of odd graphs in a deformed vacuum state, focusing on the spectral analysis of these graphs. We explore the adjacency matrices of odd graphs and derive explicit expressions for their mean and variance in the deformed vacuum state. Our main results provide the probability measures and the corresponding coherent states for the distribution of these graphs. We calculate the Jacobi coefficients and Cauchy transforms related to these distributions, which have not been addressed explicitly in the existing literature. Our findings contribute to a deeper understanding of the probabilistic and spectral properties of odd graphs in quantum state frameworks.

2020 Mathematics Subject Classification: Primary 46L53; Secondary: 05C50, 60F05.

Key words and phrases: adjacency matrix, spectral distribution, quantum decomposition, q-deformed state, two-sided Rayleigh distribution, odd graph.

1. INTRODUCTION

Quantum probability theory, first introduced by von Neumann [12], revolutionized our understanding of quantum statistics by employing self-adjoint operators and trace functions as analogs for random variables and probability measures, respectively. This foundational work paved the way for subsequent developments in the field, including the concept of *quantum decomposition*, which was introduced by Hashimoto [4] in the context of adjacency matrices of large Cayley graphs. Since then, quantum decomposition has been applied to a variety of graph structures, including Hamming graphs [6, 8], Johnson graphs [5, 8, 9], odd graphs [11], and homogeneous trees [3].

Despite its broad applications, most studies have focused on distributions with respect to vacuum states and deformed vacuum states, with the notable exception of odd graphs, which have primarily been studied in the vacuum state context. The comprehensive summary of these developments can be found in Hora & Obata [10].

The *quantum decomposition method* provides a powerful framework for analyzing the distribution of adjacency matrices through a three-term recurrence relation, establishing a crucial connection with interacting Fock probability spaces. This approach proves especially effective for asymptotic spectral analysis of growing graphs.

Consider a growing family of graphs $\{G^{(k)} = (V^{(k)}, E^{(k)})\}_{k\geqslant 1}$ and the limit

$$\lim_{k\to\infty} \frac{A_k}{Z_k},$$

where A_k is the adjacency matrix of $G^{(k)}$ and Z_k a normalizing constant. We define a stratification $V^{(k)} = \bigcup_{n=0}^{\infty} V_n^{(k)}$ based on the natural distance function of $G^{(k)}$ and decompose the adjacency matrix A_k into a sum of quantum components:

$$A_k = A_k^+ + A_k^-.$$

These operators asymptotically act in the Hilbert space $\Gamma(G^{(k)})$ associated with the stratification of $V^{(k)}$. Consequently, there exists an interacting Fock space $(\Gamma, \{\omega_n\}, B^+, B^-)$ where the limit

$$\tilde{B}^\pm = \lim_{k\to\infty} \frac{A_k^\pm}{Z_k}$$

is described, with $\tilde{B}^\pm$ being a linear combination of $B^\pm$ and a function of the number operator N.

In this paper, we extend the work of Igarashi and Obata [11], who studied odd graphs and found that the two-sided Rayleigh distribution emerged in the limit of the vacuum spectral distribution of the adjacency matrix. Our goal is to compute an explicit probability measure that describes the limit distribution of the normalized adjacency matrix for the same family of growing odd graphs, but now considering a deformed vacuum state within the framework of quantum probability theory.

The paper is structured as follows. Section 2 reviews the basics of deformed vacuum states and the tools needed to prove their positivity. Section 3 elaborates on quantum decomposition for distance-regular graphs. Section 4 presents the novel Quantum Central Limit Theorem (QCLT) for odd graphs in the context of deformed vacuum states, including its proof. Finally, Section 5 provides an explicit description of the probability measure obtained in the QCLT of Section 4.

2. DEFORMED VACUUM STATE

In this section, we will define the deformed vacuum state and discuss its properties.

Let $G = (V, E)$ be a graph and $\mathcal{A}(G)$ its adjacency algebra. The *vacuum state* at a fixed origin $o \in V$ is defined by

$$\langle a \rangle_o = \langle \delta_o, a\delta_o \rangle, \qquad a \in \mathcal{A}(G).$$

It is well known that $\langle A^m \rangle_o$ represents the number of m-step walks from $o \in V$ back to itself. More generally, we have

$$(A^m)_{xy} = \langle \delta_x, A^m \delta_y \rangle,$$

which corresponds to the number of m-step walks connecting y and x.

We are interested in a particular one-parameter deformation of the vacuum state. For $q \in \mathbb{R}$ (though $q \in \mathbb{C}$ can be considered, our focus is on the range $-1 \leqslant q \leqslant 1$, see [2]), we define a matrix $Q = Q_q$, called the *Q-matrix* of a graph $G = (V, E)$, by

$$Q = Q_q = (q^{\delta(x,y)})_{x,y \in V}.$$

For $q = 0$ we interpret $0^0 = 1$ and $Q = 1$ (the identity matrix). Thus we have

$$Q\delta_o = \sum_{x \in V} q^{\delta(x,o)} \delta_x.$$

We can define

$$\langle a \rangle_q = \sum_{x \in V} q^{\delta(x,o)} \langle \delta_x, a\delta_o \rangle = \langle Q\delta_o, a\delta_o \rangle, \qquad a \in \mathcal{A}(G). \tag{2.1}$$

A normalized linear function defined as above is called a *deformed vacuum state* on $\mathcal{A}(G)$.

3. QUANTUM DECOMPOSITION FOR DISTANCE-REGULAR GRAPHS

Let $G = (V, E)$ be a graph with a fixed origin $o \in V$. The graph is stratified into a disjoint union of strata:

$$V = \bigcup_{n=0}^{\infty} V_n, \qquad V_n = \{x \in V : \delta(o, x) = n\}.$$

This partition is known as the *stratification* (distance partition). For $\epsilon \in \{+, -, \circ\}$ we define A^ϵ as follows:

$$(A^\epsilon)_{xy} = \begin{cases} 1 & \text{if } x \sim y \text{ and } \delta(o,x) - \delta(o,y) = \epsilon, \\ 0 & \text{otherwise,} \end{cases}$$

where ϵ is assigned the values $+1, -1, 0$ according as $\epsilon = +, -, \circ$. The adjacency matrix A is decomposed into three parts:

$$A = A^+ + A^- + A^\circ. \tag{3.1}$$

We refer to (3.1) as the *quantum decomposition* of A with A^ϵ representing the *quantum components*.

For each $n = 0, 1, 2, \ldots$, we define a unit vector in $l^2(V)$ as

$$\Phi_n = |V_n|^{-1/2} \sum_{x \in V_n} \delta_x,$$

which is called the *nth number vector*. In particular, $\Phi_0 = \delta_0$ is known as the *vacuum vector*. Let $\Gamma(G)$ denote the closed subspace spanned by $\{\Phi_0, \Phi_1, \ldots\}$. Although $\Gamma(G)$ is not always invariant under the quantum components A^ϵ, the method of quantum decomposition is most effective when either $\Gamma(G)$ is invariant or $\Gamma(G)$ is "asymptotically" invariant under the quantum components.

Let $G = (V, E)$ be a graph. For non-negative integers i, j, k, the graph $G = (V, E)$ is called *distance-regular* if, for any choice of $x, y \in V$ the number

$$p_{ij}^k = |\{z \in V : \delta(x, z) = i,\ \delta(y, z) = j\}|,$$

where $\delta(x, y) = k$, depends only on i, j, k. We consider the deformed vacuum state defined by

$$\langle a \rangle_q = \langle Q\delta_0, a\delta_0 \rangle = \sum_{n=0}^{\infty} q^n |V_n|^{1/2} \langle \Phi_n, a\Phi_0 \rangle, \qquad a \in \mathcal{A}(G),$$

where $Q = (q^{\delta(x,y)})$ with $q \in \mathbb{R}$. Let $G = (V, E)$ be a distance-regular graph with intersection numbers $\{p_{ij}^k\}$, and let the degree be given by $\kappa = p_{11}^0$. The mean and the variance of the adjacency matrix A in the deformed vacuum state are given by $\langle A \rangle_q = q\kappa$ and $\Sigma_q^2(A) = \langle (A - \langle A \rangle_q)^2 \rangle_q = \kappa(1-q)(1+q+qp_{11}^1)$, respectively.

Now, consider a growing distance-regular graph $G^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$. Suppose each $G^{(\nu)}$ is endowed with a deformed vacuum state $\langle \cdot \rangle_q$, where q may depend on ν. The normalized adjacency matrix of interest is given by

$$\frac{A_\nu - \langle A_\nu \rangle_q}{\Sigma_q(A_\nu)}.$$

Taking the quantum decomposition $A_\nu = A_\nu^+ + A_\nu^- + A_\nu^\circ$ into account, we obtain

$$\frac{A_\nu - \langle A_\nu \rangle_q}{\Sigma_q(A_\nu)} = \frac{A_\nu^+}{\Sigma_q(A_\nu)} + \frac{A_\nu^-}{\Sigma_q(A_\nu)} + \frac{A_\nu^\circ - q\kappa(\nu)}{\Sigma_q(A_\nu)}.$$

For $n = 1, 2, \ldots$ we define

$$\bar{\omega}_n(\nu, q) = \frac{p_{1,n-1}^n(\nu) p_{1,n}^{n-1}(\nu)}{\Sigma_q^2(A_\nu)} \quad \text{and} \quad \bar{\alpha}_n(\nu, q) = \frac{p_{1,n-1}^{n-1}(\nu) - q\kappa(\nu)}{\Sigma_q(A_\nu)}.$$

It is well known that

$$\frac{A_\nu^+}{\Sigma_q(A_\nu)}\Phi_n = \sqrt{\bar{\omega}_{n+1}(\nu,q)}\Phi_{n+1}, \qquad n=0,1,2,\dots,$$

$$\frac{A_\nu^-}{\Sigma_q(A_\nu)}\Phi_0 = 0, \qquad \frac{A_\nu^-}{\Sigma_q(A_\nu)}\Phi_n = \sqrt{\bar{\omega}_n(\nu,q)}\Phi_{n-1}, \qquad n=1,2,\dots,$$

$$\frac{A_\nu^\circ - q\kappa(\nu)}{\Sigma_q(A_\nu)}\Phi_n = \bar{\alpha}_{n+1}(\nu,q)\Phi_n, \qquad n=0,1,2,\dots.$$

We consider the following limits:

$$\omega_n = \lim_{\nu,q}\bar{\omega}_n(\nu,q) = \lim_{\nu,q}\frac{p^n_{1,n-1}(\nu)p^{n-1}_{1,n}(\nu)}{\Sigma_q^2(A_\nu)}, \tag{3.2}$$

$$\alpha_n = \lim_{\nu,q}\bar{\alpha}_n(\nu,q) = \lim_{\nu,q}\frac{p^{n-1}_{1,n-1}(\nu) - q\kappa(\nu)}{\Sigma_q(A_\nu)}, \tag{3.3}$$

assuming they exist under appropriate scaling balance of ν and q. If the limits (3.2) and (3.3) exist, we obtain a Jacobi coefficient $(\{\omega_n\},\{\alpha_n\})$. Let $\Gamma_{\{\omega_n\}} = (\Gamma,\{\Psi_n\},B^+,B^-)$ be an interacting Fock space associated with $\{\omega_n\}$ and B° be the diagonal operator defined by $\{\alpha_n\}$. We set

$$\tilde{A}_\nu^\pm = A_\nu^\pm, \qquad \tilde{A}_\nu^\circ = A_\nu^\circ - q\kappa(\nu),$$

and we can now establish the Quantum Central Limit Theorem for a growing distance-regular graph in the deformed vacuum state. This theorem appears in the book [10] by Hora and Obata as Theorem 3.29.

THEOREM 3.1. *Let $G^{(\nu)} = (V^{(\nu)},E^{(\nu)})$ be a growing distance-regular graph with A_ν being the adjacency matrix, and each $\mathcal{A}(G^{(\nu)})$ be a given a deformed vacuum state $\langle\cdot\rangle_q$. Assume that the limits in* (3.2) *and* (3.3) *exist and they become a Jacobi coefficient, and that the limit*

$$c_n = \lim_{\nu,q} q^n|V_n^{(\nu)}|^{1/2} = \lim_{\nu,q} q^n\sqrt{p^0_{nn}(\nu)} \tag{3.4}$$

exists for all n for which $\{\alpha_n\}$ is defined. Let $\Gamma_{\{\omega_n\}} = (\Gamma,\{\Psi_n\},B^+,B^-)$ be an interacting Fock space associated with $\{\omega_n\}$, B° the diagonal operator defined by $\{\alpha_n\}$, and Υ the formal sum of vectors defined by

$$\Upsilon = \sum_{n=0}^\infty c_n\Psi_n.$$

Then

$$\lim_{\nu,q}\left\langle \frac{\tilde{A}_\nu^{\epsilon_m}}{\Sigma_q(A_\nu)}\cdots\frac{\tilde{A}_\nu^{\epsilon_1}}{\Sigma_q(A_\nu)}\right\rangle_q = \langle\Upsilon, B^{\epsilon_m}\cdots B^{\epsilon_1}\Psi_0\rangle$$

for any $\epsilon_1,\dots,\epsilon_m \in \{+,-,\circ\}$ and $m=1,2,\dots$.

4. QCLT FOR ODD GRAPHS IN A DEFORMED VACUUM STATE

In this section, we delve into the Quantum Central Limit Theorem (QCLT) for odd graphs in the context of a deformed vacuum state. Odd graphs, characterized by their unique regularity and distance-transitivity properties, present a fascinating case study for spectral analysis. Building on foundational results by Igarashi and Obata, this work investigates the asymptotic behavior of the adjacency matrix for odd graphs as the graph size grows.

Let $k \geqslant 2$ be an integer and set $S = \{1, \ldots, 2k-1\}$. The pair

$$V = \{x \subset S : |x| = k-1\}, \qquad E = \{(x, y) : x, y \in V,\, x \cap y = \emptyset\}$$

is called the *odd graph* and is denoted by O_k. Obviously, O_k is a regular graph of degree k.

Igarashi and Obata [11] proved that the distance between two vertices of an odd graph is characterized by the cardinality of their intersection. Set

$$I_n = \begin{cases} k-1-n/2 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n \text{ is odd,} \end{cases}$$

where $n = 0, 1, \ldots, k-1$. Then, for a pair of vertices x, y of the odd graph O_k, we have

$$|x \cap y| = I_n \iff \delta(x, y) = n.$$

As a direct consequence of this fact, odd graphs are *distance-transitive*, therefore *distance-regular*. In order to apply quantum probabilistic techniques to obtain the asymptotic spectral distribution of the adjacency matrix A_k as $k \to \infty$, in [11] the intersection numbers of O_k was computed.

PROPOSITION 4.1. *Let $\{p^h_{ij}\}$ be the intersection numbers of the odd graph O_k, $k \geqslant 2$. For $1 \leqslant n \leqslant k-1$,*

$$p^n_{1,n-1} = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

For $0 \leqslant n \leqslant k-2$,

$$p^n_{1,n+1} = \begin{cases} k-n/2 & \text{if } n \text{ is even,} \\ k-(n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

For $0 \leqslant n \leqslant k-1$,

$$p^n_{1,n} = \begin{cases} 0 & \text{if } 1 \leqslant n \leqslant k-2, \\ (k+1)/2 & \text{if } n = k-1 \text{ and } k \text{ is odd,} \\ k/2 & \text{if } n = k-1 \text{ and } k \text{ is even.} \end{cases}$$

From the last proposition the following central limit theorem for odd graphs (with respect to the vacuum state) was deduced.

THEOREM 4.2 (Igarashi & Obata [11]). *For the adjacency matrix A_k of the odd graph O_k we have*

$$\lim_{k\to\infty}\left\langle\left(\frac{A_k}{\sqrt{k}}\right)^m\right\rangle_o = \int_{-\infty}^{\infty} x^m|x|\exp(-x^2)\,dx, \qquad m=1,2,\dots.$$

In this paper, we focus on the asymptotic properties of the adjacency matrix A_k of odd graphs within a deformed vacuum state framework. The mean and the variance of A_k in this state are given by

$$\langle O_k\rangle_q = qk, \qquad \Sigma_q^2(O_k) = k(1-q^2),$$

respectively. This leads to the normalized adjacency matrix

$$\frac{O_k - \langle O_k\rangle_q}{\Sigma_q(O_k)} = \frac{O_k - qk}{\sqrt{k(1-q^2)}}.$$

To explore the asymptotic distribution in the deformed vacuum state, we analyze the sequences $\{\omega_n\}, \{\alpha_n\}, \{c_n\}$ defined in (3.2)–(3.4). This enables us to derive new results about the spectral behavior of A_k.

To determine ω_n, consider the following cases:

- For even n,

$$\omega_n = \lim_{k,q} \frac{\frac{n}{2}\left(k-\frac{n}{2}\right)}{k(1-q^2)}.$$

- For odd n,

$$\omega_n = \lim_{k,q} \frac{\frac{n+1}{2}\left(k-\frac{n-1}{2}\right)}{k(1-q^2)}.$$

Next, for α_n we have

$$\alpha_n = \lim_{k,q} \frac{p_{1,n-1}^{n-1}(k) - qk}{\Sigma_q(O_k)} = \lim_{k,q} \frac{-qk}{\sqrt{k(1-q^2)}} = \lim_{k,q} \frac{-q\sqrt{k}}{\sqrt{1-q^2}}.$$

Note that q may depend on k, so we need to balance q and k. An appropriate scenario is

$$\lim_{k\to\infty} q = 0, \qquad \lim_{k\to\infty} q\sqrt{k} = \gamma, \tag{4.1}$$

where $\gamma \in \mathbb{R}$ can be arbitrarily chosen. Under these conditions we get

$$\{\omega_n\}_{n\geqslant 1} = \{1,1,2,2,3,3,4,\dots\}, \qquad \{\alpha_n\}_{n\geqslant 1} = \{-\gamma,-\gamma,-\gamma,\dots\}.$$

To compute c_n, we evaluate $p^0_{nn}(k)$. For n even, we have

$$p^0_{nn} = \binom{k-1}{k-1-n/2}\binom{k}{n/2} = \binom{k-1}{n/2}\binom{k}{n/2},$$

thus

$$\begin{aligned} c_n &= \lim_{k,q} q^n \sqrt{\binom{k-1}{n/2}\binom{k}{n/2}} \\ &= \lim_{k,q} \frac{q^n}{(n/2)!} \sqrt{\left(1-\frac{1}{k}\right)\cdots\left(1-\frac{n/2}{k}\right)\left(1-\frac{n/2+1}{k}\right) k^{n/2+n/2}} \\ &= \frac{\gamma^n}{(n/2)!}. \end{aligned}$$

For odd n, we have

$$\begin{aligned} p^0_{nn}(k) &= \binom{k-1}{(n-1)/2}\binom{k}{k-1-(n-1)/2} \\ &= \binom{k-1}{k-1-(n-1)/2}\binom{k}{k-1-(n-1)/2}, \end{aligned}$$

thus

$$\begin{aligned} c_n &= \lim_{k,q} q^n \sqrt{\binom{k-1}{k-1-(n-1)/2}\binom{k}{k-1-(n-1)/2}} \\ &= \lim_{k,q} q^n \sqrt{\left(1-\frac{1}{k}\right)\cdots\left(1-\frac{n/2}{k}\right)\left(1-\frac{n/2+1}{k}\right) \frac{k^{(n-1)/2+(n-1)/2+1}}{\left(\frac{n-1}{2}+1\right)\left(\frac{n-1}{2}\right)!\left(\frac{n-1}{2}\right)!}} \\ &= \frac{\gamma^n}{\sqrt{\frac{n-1}{2}+1}\left(\frac{n-1}{2}\right)!}. \end{aligned}$$

The formal sum of vectors is given by

$$\Omega_\gamma = \sum_{n=0}^{\infty} c_n \Psi_n, \quad \text{where} \quad c_n = \begin{cases} \dfrac{\gamma^n}{(n/2)!} & \text{if } n \text{ is even,} \\ \dfrac{\gamma^n}{\sqrt{\frac{n-1}{2}+1}\left(\frac{n-1}{2}\right)!} & \text{if } n \text{ is odd.} \end{cases}$$

This corresponds to the coherent state of the Fock space $\Gamma_{\{\omega_n\}}$. We are now prepared to describe the asymptotic distribution of odd graphs in the deformed vacuum state.

THEOREM 4.3. *Let A_k be the adjacency matrix of the odd graph O_k, $k \geqslant 2$. Let $\Gamma_{\{\omega_n\}} = (\Gamma, \{\Psi_n\}, B^+, B^-)$ be the interacting Fock space associated with $\{\omega_n\} = \{1, 1, 2, 2, 3, 3, 4, \dots\}$. Then taking the limits as in (4.1) we have*

$$\lim_{k,q} \frac{A_k^{\pm}}{\Sigma_q(A_k)} = B^{\pm}, \qquad \lim_{k,q} \frac{A_k^{\circ} - \langle A_k \rangle_q}{\Sigma_q(A_k)} = -\gamma,$$

in the sense of stochastic convergence, where the left-hand sides are in the deformed vacuum state $\langle \cdot \rangle_q$ and the right-hand sides in the coherent state $\langle \cdot \rangle_{\Omega_\gamma}$. In particular, for $m = 1, 2, \dots,$

$$\lim_{k,q} \left\langle \left(\frac{A_k - \langle A_k \rangle_q}{\Sigma_q(A_k)} \right)^m \right\rangle_q = \langle (B^+ + B^- - \gamma)^m \rangle_{\Omega_\gamma}. \tag{4.2}$$

5. CALCULATING THE LIMIT MEASURE

In this section our objective is to determine a probability measure μ such that

$$\langle (B^+ + B^- - \gamma)^m \rangle_{\Omega_\gamma} = \int_{-\infty}^{\infty} t^m \, \mu(dt), \qquad m = 1, 2, \dots.$$

Throughout the remainder of this section, let $\Gamma_{\{\omega_n\}} = (\Gamma, \{\Psi_n\}, B^+, B^-)$ represent the interacting Fock space associated with $\{\omega_n\} = \{1, 1, 2, 2, 3, 3, 4, \dots\}$. It is worth noting that no existing literature provides an explicit calculation of this distribution.

We recall that Ω_γ is a coherent state with parameter $\gamma \in \mathbb{R}$. Therefore by combining [10, Proposition 4.17] with (4.2) we obtain

$$\langle \Omega_\gamma, (B^+ + B^- - \gamma)^m \Psi_0 \rangle = \langle \Psi_0, (B^+ + B^- - \gamma B^+ B^-)^m \Psi_0 \rangle.$$

Next, we need to find the measure μ such that

$$\langle \Psi_0, (B^+ + B^- - \gamma B^+ B^-)^m \Psi_0 \rangle = \int_{-\infty}^{\infty} t^m \, \mu(dt), \qquad m = 1, 2, \dots. \tag{5.1}$$

As $-\gamma B^+ B^-$ is a diagonal operator defined by the sequence $\{0, -\gamma, -\gamma, -\gamma, \dots\}$, the Jacobi coefficient of μ in (5.1) is $(\{\omega_n\}, \{0, -\gamma, -\gamma, -\gamma, \dots\})$. The Cauchy transform of μ is given by

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{\mu(dt)}{z - t} = \cfrac{1}{z - \cfrac{1}{z + \gamma - \cfrac{1}{z + \gamma - \cfrac{2}{z + \gamma - \cdots}}}},$$

where $\operatorname{Im} z \neq 0$.

Let ν be a measure such that $\nu(dt) = |t|e^{-t^2}dt$, which has the Jacobi coefficient $(\{\omega_n\}, \{\alpha_n \equiv 0\})$. Then we have

$$G_\mu(z-\gamma) = \cfrac{1}{z-\gamma-\cfrac{1}{z-\cfrac{1}{z-\cfrac{2}{z-\cdots}}}} = \frac{1}{z-\gamma-K_\nu(z)},$$

where $K_\nu(z) = z - 1/G_\nu(z)$. We shall apply Stieltjes' inversion formula to the right-hand side of the last equation, i.e. we need to compute

$$\begin{aligned}(5.2)\qquad -\frac{1}{\pi}\lim_{y\to+0}\operatorname{Im} G_\mu(x+iy-\gamma) &= -\frac{1}{\pi}\lim_{y\to+0}\frac{\operatorname{Im} G_\nu(x+iy)}{\|-\gamma G_\nu(x+iy)+1\|^2}\\ &= \frac{f_\nu(x)}{\|-\gamma\lim_{y\to+0} G_\nu(x+iy)+1\|^2},\end{aligned}$$

where $f_\nu = d\nu/dx$. Since ν is symmetric, we have $G_\nu(z) = zG_\rho(z^2)$ (see [1]), where $\rho(dx) = e^{-x}dx$. Thus,

$$\begin{aligned}(5.3)\qquad \lim_{y\to+0} G_\nu(z) &= \lim_{y\to+0} zG_\rho(z^2)\\ &= \lim_{y\to+0}\operatorname{Re} zG_\rho(z^2) + i\lim_{y\to+0}\operatorname{Im} zG_\rho(z^2)\\ &= \pi x Hf_\rho(x^2) + i\pi|x|f_\rho(x^2),\end{aligned}$$

where $Hf_\rho(x) = e^{-x}\operatorname{Ei}(x)$ is the Hilbert transform (see [7, Chapter 3]) of $f_\rho(x)$ $= e^{-x}$ and $\operatorname{Ei}(x)$ is the special function on the complex plane called the *exponential integral*, defined by

$$\operatorname{Ei}(x) = -\int_{-x}^{\infty}\frac{e^{-t}}{t}dt.$$

Combining (5.2) and (5.3) we obtain

$$-\frac{1}{\pi}\lim_{y\to+0}\operatorname{Im} G_\mu(x+iy-\gamma) = \frac{|x|e^{-x^2}}{(-\gamma\pi x e^{-x^2}\operatorname{Ei}(x^2)+1)^2 + (\gamma\pi|x|e^{-x^2})^2}.$$

Therefore the explicit form for μ is a translation of the above expression. Then we may restate Theorem 4.3.

THEOREM 5.1. *For the adjacency matrix A_k of the odd graph O_k we have*

$$\lim_{k,q}\left\langle\left(\frac{A_k - \langle A_k\rangle_q}{\Sigma_q(A_k)}\right)^m\right\rangle_q = \int_{-\infty}^{\infty} x^m\,\mu(dx), \qquad m = 1, 2, \dots,$$

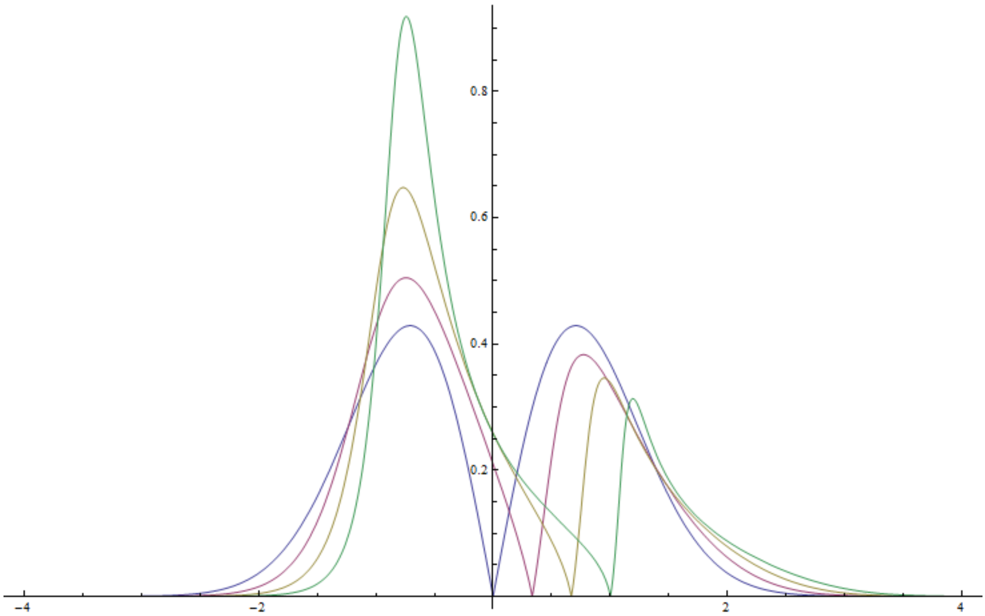

FIGURE 1. $\mu(dx)$ with $\gamma = 0, 1/3, 1/6, 1$ (Theorem 5.1)

where the explicit form of μ is

$$\mu(dx) = \frac{|x-\gamma|e^{-(x-\gamma)^2}}{(-\gamma\pi(x-\gamma)e^{-(x-\gamma)^2}\mathrm{Ei}((x-\gamma)^2)+1)^2 + (\gamma\pi|x-\gamma|e^{-(x-\gamma)^2})^2} dx.$$

(*See Fig.* 1.)

REMARK 5.2. The case $\gamma = 0$ in Theorem 5.1 is [11, Theorem 6.1].

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Received 30.8.2024;
accepted 14.12.2024