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SEMILINEAR FRACTIONAL ELLIPTIC PDES WITH GRADIENT NONLINEARITIES ON OPEN BALLS: EXISTENCE OF SOLUTIONS AND PROBABILISTIC REPRESENTATION

BY

GUILLAUME PENENT (SINGAPORE) AND NICOLAS PRIVAULT (SINGAPORE)

Abstract. We provide sufficient conditions for the existence of classical solutions of fractional semilinear elliptic PDEs of index $\alpha \in (1,2)$ with polynomial gradient nonlinearities on d-dimensional balls, $d \geqslant 2$. Our approach uses a tree-based probabilistic representation of solutions and their partial derivatives using α-stable branching processes, and allows us to take into account gradient nonlinearities not covered by deterministic finite difference methods so far. In comparison with the existing literature on the regularity of solutions, no polynomial order condition is imposed on gradient nonlinearities. Numerical illustrations demonstrate the accuracy of the method in dimension $d = 10$, solving a challenge encountered with the use of deterministic finite difference methods in high-dimensional settings.

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1. INTRODUCTION

The study of solutions of nonlocal and fractional elliptic partial differential equations (PDEs) is an active research topic which has attracted significant attention over the past decades. In the case of the classical (local) Laplacian, viscosity solutions of fully nonlinear second-order elliptic PDEs have been constructed in [19] by the Perron method.

On the other hand, nonlocal elliptic PDEs can be solved using weak solutions (see [28, Definition 2.1]) or viscosity solutions (see [31] and [28, Remark 2.11]). Weak solutions can be obtained from the Riesz representation or Lax–Milgram theorems as in [13, 27]. See also [3] for the use of the Perron method, and [11] for semigroup methods applied to second-order elliptic integro-differential PDEs.

Given $d \geqslant 1$, let

$$\Delta_\alpha u = -(-\Delta)^{\alpha/2} u = \frac{4^{\alpha/2}\Gamma(d/2+\alpha/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|} \lim_{r\to 0^+} \int_{\mathbb{R}^d\setminus B(x,r)} \frac{u(\cdot+z)-u(z)}{|z|^{d+\alpha}}\,dz$$

denote the fractional Laplacian on $\mathbb{R}^d$ with parameter $\alpha \in (0,2)$ (see, e.g., [21]), where $\Gamma(p) := \int_0^\infty e^{-\lambda x}\lambda^{p-1}\,d\lambda$ is the gamma function and $|z|$ is the Euclidean norm of $z \in \mathbb{R}^d$.

For problems of the form

$$\Delta_\alpha u(x) + f(x) = 0$$

with $u = \phi$ on $\mathbb{R}^d \setminus D$, where D is an open bounded domain in $\mathbb{R}^d$, the Hölder regularity of viscosity solutions has been proved in [20] when D is a ball and f, ϕ are bounded functions. Existence of viscosity solutions has been derived in [31] under smoothness assumptions on f, ϕ, and the existence of classical Hölder regular solutions has been proved in [29] when ϕ is bounded continuous and f is Hölder continuous. See also [13], resp. [23], for the existence of weak solutions, resp. viscosity solutions, with nonlocal operators. Regarding problems of the form

$$\Delta_\alpha u(x) + f(x, u(x)) = 0,$$

existence of nontrivial solutions with $u = 0$ outside an open bounded domain D with Lipschitz boundary in $\mathbb{R}^d$ has been considered in [30] using the mountain pass theorem when f is a Carathéodory function on $D\times\mathbb{R}^d$ satisfying a polynomial growth condition of order $m \in (1, (d+\alpha)/(d-\alpha))$.

The regularity of viscosity solutions of semilinear elliptic PDEs of the form

$$\Delta_\alpha u(x) - b(x)\|\nabla u(x)\|_{\mathbb{R}^d}^{\kappa+\tau} - \|\nabla u(x)\|_{\mathbb{R}^d}^r = 0, \qquad x \in D, \tag{1.1}$$

where D is an open domain in $\mathbb{R}^d$ and b is in the space $C^\tau(\mathbb{R}^d)$ of τ-Hölder continuous functions on $\mathbb{R}^d$ for some $\tau \in (0,1)$, has been considered in [4, §4.3]. Namely, from Theorem 3.1 therein, if b is in $C^\tau(\mathbb{R}^d)$ and $\kappa, r \in (0,2)$, then any bounded viscosity solution u of (1.1) is β-Hölder continuous for small enough β; see also [5, §4.1.2] for Lipschitz regularity in the case of mixed local and fractional Laplacians.

More recently, the Lipschitz regularity of viscosity solutions of

$$\Delta_\alpha u(x) + f(x, \nabla u(x)) = 0$$

on $D = B(0,R)$, the open ball of radius $R > 0$ in $\mathbb{R}^d$, has been obtained in [7, Theorem 2.1], provided that $f \in \mathcal{C}(\mathbb{R}^d \times \mathbb{R}^d)$ satisfies a power-type growth condition of order $m \in (0, \alpha+1)$ in $\nabla u(x)$, while this bound can be lifted under an extra coercivity condition on H.

In this paper, we consider the class of semilinear elliptic problems on $B(0,R)$ of the form

$$\begin{cases} \Delta_\alpha u(x) + f(x,u(x),\nabla u(x)) = 0, & x \in B(0,R), \\ u(x) = \phi(x), & x \in \mathbb{R}^d \setminus B(0,R), \end{cases} \tag{1.2}$$

where

- $f(x,y,z)$ is a polynomial nonlinearity on $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$, of the form

$$f(x,y,z) = \sum_{l=(l_0,\ldots,l_m)\in\mathcal{L}_m} c_l(x) y^{l_0} \prod_{i=1}^{m} (b_i(x)\cdot z)^{l_i}, \tag{1.3}$$

  where $\mathcal{L}_m$ is a finite subset of $\mathbb{N}^{m+1}$ for some $m \geqslant 0$, and $(c_l(x))_{l=(l_0,\ldots,l_m)\in\mathcal{L}_m}$, $(b_i(x))_{i=1,\ldots,m}$ are bounded continuous functions of $x \in \mathbb{R}^d$, with $x\cdot z := x_1 z_1 + \cdots + x_d z_d$,

- $\phi : \mathbb{R}^d \to \mathbb{R}$ is a bounded Lipschitz function on $\mathbb{R}^d \setminus B(0,R)$.

Using a probabilistic approach, we prove the existence of regular viscosity solutions to (1.2) under the following conditions. We note that, in comparison to the literature quoted above on the regularity of solutions, no coercivity or maximum growth order condition in z is imposed on $f(x,y,z)$.

ASSUMPTION (A)

(1) The boundary condition ϕ belongs to the fractional Sobolev space

$$H^\alpha(\mathbb{R}^d) := \left\{ u \in L^2(\mathbb{R}^d) : \frac{|u(x)-u(y)|}{|x-y|^{d/2+\alpha/2}} \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \right\}$$

and is bounded on $\mathbb{R}^d \setminus B(0,R)$.

(2) The coefficients $c_l(x)$, $l \in \mathcal{L}_m$, are uniformly bounded functions, i.e.,

$$\|c_l\|_\infty := \sup_{x\in B(0,R)} |c_l(x)| < \infty, \qquad l = (l_0,\ldots,l_m) \in \mathcal{L}_m.$$

(3) The coefficients $b_i(x)$, $i = 0,\ldots,m$, are such that

$$\sup_{x\in B(0,R)} \frac{|b_i(x)|}{R-|x|} < \infty, \qquad i = 1,\ldots,m.$$

Theorem 1.1 is the main result of this paper. It is implied by Theorem 4.1, in which we prove the existence of a classical solution for fractional elliptic problems of the form (1.2).

THEOREM 1.1. *Let $\alpha \in (1,2)$ and $d \geqslant 2$. Under Assumption (A), the semilinear elliptic PDE* (1.2) *admits a classical solution in $C^{\alpha+\epsilon}(B(0,R)) \cap \mathcal{C}^0(\overline{B}(0,R))$ for some $\epsilon > 0$, provided that R and $\max_{l\in\mathcal{L}_m} \|c_l\|_\infty$ are sufficiently small.*

Our method of proof relies on the probabilistic representation of PDE solutions using stochastic branching processes, as introduced in [32, 18]. Probabilistic representations have been applied to the blow-up and existence of solutions for parabolic PDEs in [24, 22]. They have also been recently extended in [1] to treat polynomial nonlinearities in gradient terms in elliptic PDEs with (local) diffusion generators, following the approach of [15] in the parabolic case. In this construction, gradient terms are associated to tree branches to which a Malliavin integration by parts is applied. In [25], this approach has been extended to the treatment of nonlocal pseudo-differential operators of the form $-\eta(-\Delta/2)$ using random branching trees constructed from a Lévy subordinator, with application to parabolic PDEs with fractional Laplacians.

The existence of viscosity solutions in Theorem 1.1 is obtained through a probabilistic representation of the form

$$u(x) := \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,0})], \quad x \in B(0,R), \tag{1.4}$$

where $\mathcal{H}_\phi(\mathcal{T}_{x,0})$ (see (4.2)) is a functional of a random branching tree $\mathcal{T}_{x,0}$ started at $x \in \mathbb{R}^d$, and constructed in Section 3. The proof of Theorem 1.1 also makes use of existence results for nonlinear elliptic PDEs with fractional Laplacians derived in [26, Theorem 1.2, Proposition 3.5].

To prove Theorem 1.1, in Proposition 4.1 we construct for each $i = 0,\dots,m$ a sufficiently integrable functional $\mathcal{H}_\phi(\mathcal{T}_{x,i})$ of a random tree $\mathcal{T}_{x,i}$ such that the probabilistic representation

$$u(x) = \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,0})], \quad x \in \mathbb{R}^d,$$

yields a viscosity solution of (1.2) in $\mathcal{C}^1(B(0,R)) \cap \mathcal{C}^0(\overline{B}(0,R))$, where the gradients $b_i(x)\cdot\nabla u(x)$, $x \in B(0,R)$, $i = 0,\dots,m$, can be represented as

$$u(x) = \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,0})], \quad b_i(x)\cdot\nabla u(x) = \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,i})], \quad x \in \mathbb{R}^d,$$

under integrability assumptions on $(\mathcal{H}_\phi(\mathcal{T}_{x,i}))_{x\in B(0,R)}$.

Then, in Proposition 4.2 we show that for any $d \geqslant 2$ and $p \geqslant 1$, the collection $(\mathcal{H}_\phi(\mathcal{T}_{x,i}))_{x\in B(0,R)}$ is bounded in $L^p(\Omega)$ uniformly in $x \in B(0,R)$, and therefore uniformly integrable, $i = 0,\dots,m$. We conclude the proof of Theorem 1.1 by showing, using results of [20, 29], that the $\mathcal{C}^1$ viscosity solution of (1.2) is in $C^{\alpha+\epsilon}(B(0,R)) \cap \mathcal{C}^0(\overline{B}(0,R))$ for some $\epsilon > 0$.

For this, we extend the arguments of [1] from the standard Laplacian Δ and Brownian motion to the fractional Laplacian $\Delta_\alpha := -(-\Delta)^{\alpha/2}$ and its associated stable process. There are, however, significant differences from the Brownian case. In particular, in the stable setting we rely on sharp gradient estimates for fractional

Green and Poisson kernel proved in [10], and on integrability results for stable process hitting times (see [9]). The behavior of the negative moments of stable processes (see (2.5)) requires a more involved treatment of integrability in small time when showing the boundedness of $(\mathcal{H}_\phi(\mathcal{T}_{x,i}))_{x\in B(0,R)}$ in $L^p(\Omega)$, for $p \geqslant 1$.

In addition, we present a Monte Carlo numerical implementation of the probabilistic representation (1.4) on specific examples. In comparison with deterministic finite difference methods (see e.g. [16, §6.3] for the one-dimensional Dirichlet problem), our approach allows us to take into account gradient nonlinearities. We also note that our tree-based Monte Carlo implementation applies to high-dimensional problems (see Figures 4 and 6 in dimension $d = 10$), whereas the application of deterministic finite difference methods to the fractional Laplacian in higher dimensions is challenging (see e.g. [16, p. 3082]).

This paper is organized as follows. Section 3 presents the description of the branching mechanism, following the preliminaries on stable processes and kernel introduced in Section 2. In Section 4 we state and prove our main existence result, Theorem 4.1, for the probabilistic representation of the solution of (1.2). Section 5 presents a Monte Carlo numerical implementation of our method on specific examples.

2. PRELIMINARIES AND NOTATION

Before proceeding further, we recall some preliminary results on fractional Laplacians on the ball $B(0,R)$ in $\mathbb{R}^d$.

2.1. Poisson and Green kernels. Given an $\mathbb{R}^d$-valued α-stable process $(X_t)_{t\geqslant 0}$, $\alpha \in (0,2)$, we consider the process

$$X_{t,x} := x + X_t, \qquad t \in \mathbb{R}_+,$$

started at $x \in \mathbb{R}^d$ (see e.g. [2, §1.3.1]), and the first hitting time

$$\tau_R(x) := \inf\{t \geqslant 0 : X_{t,x} \notin B(0,R)\}$$

of $\mathbb{R}^d \setminus B(0,R)$ by $(X_{t,x})_{t\geqslant 0}$. Note that by the bound [9, (1.4)] we have $\mathbb{E}[\tau_R(x)] < \infty$, and therefore $\tau_R(x)$ is almost surely finite for all $x \in B(0,R)$. The Green kernel $G_R(x,y)$ satisfies

$$\mathbb{E}\Big[\int_0^{\tau_R(x)} f(X_{t,x})\,dt\Big] = \int_{B(0,R)} G_R(x,y) f(y)\,dy, \qquad x \in B(0,R), \tag{2.1}$$

for f a nonnegative measurable function on $\mathbb{R}^d$. If $\alpha \in (0,2)\setminus\{d\}$, we also have

$$G_R(x,y) = \frac{\kappa_\alpha^d}{|x-y|^{d-\alpha}} \int_0^{r_0(x,y)} \frac{t^{\alpha/2-1}}{(1+t)^{d/2}}\,dt, \qquad x,y \in B(0,R),$$

(see [12, Theorem 3.1]), where

$$r_0(x,y) := \frac{(R^2-|x|^2)(R^2-|y|^2)}{R^2|x-y|^2} \quad \text{and} \quad \kappa_\alpha^d := \frac{2^{-\alpha}\Gamma(d/2)}{\pi^{d/2}(\Gamma(\alpha/2))^2}.$$

The Poisson kernel $P_R(x,y)$ of the harmonic measure $\mathbb{P}^x(X_{\tau_R(x),x} \in dy)$ satisfies

$$\mathbb{E}[f(X^x_{\tau_R(x)})] = \int_{\mathbb{R}^d \setminus B(0,R)} P_R(x,y) f(y)\, dy, \qquad x \in B(0,R), \tag{2.2}$$

for f a nonnegative measurable function on $\mathbb{R}^d$, and is given by

$$P_R(x,y) = \mathcal{A}(d,-\alpha) \int_{B(0,R)} \frac{G_R(x,z)}{|y-z|^{d+\alpha}}\, dz,$$

where

$$\mathcal{A}(d,-\alpha) := \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|}.$$

In particular, when $|x| < R$ and $|y| > R$ we have

$$P_R(x,y) = \frac{\mathcal{C}(\alpha,d)}{|x-y|^d}\left(\frac{R^2-|x|^2}{|y|^2-R^2}\right)^{\alpha/2},$$

with $\mathcal{C}(\alpha,d) := \Gamma(d/2)\pi^{-d/2-1}\sin(\pi\alpha/2)$. In addition, we have the bounds

$$|\nabla_x P_R(x,y)| \leqslant (d+\alpha)\frac{P_R(x,y)}{R-|x|}, \qquad x \in B(0,R),\ y \in \mathbb{R}^d \setminus \overline{B}(0,R), \tag{2.3}$$

where $\overline{B}(0,R)$ denotes the closed ball of radius $R > 0$ in $\mathbb{R}^d$ (see [10, Lemma 3.1]), and

$$|\nabla_x G_R(x,y)| \leqslant d\frac{G_R(x,y)}{\min(|x-y|, R-|x|)}, \qquad x,y \in B(0,R),\ x \neq y, \tag{2.4}$$

(see [10, Corollary 3.3]).

2.2. Moments of stable processes. We will need to estimate the negative moments $\mathbb{E}[|X_t|^{-p}]$ of an α-stable process $(X_t)_{t\geqslant 0}$ represented as the subordinated Brownian motion $(X_t)_{t\geqslant 0} = (B_{S_t})_{t\geqslant 0}$, where the subordinator $(S_t)_{t\geqslant 0}$ is an $\alpha/2$-stable process with Laplace exponent $\eta(\lambda) = (2\lambda)^{\alpha/2}$, i.e.

$$\mathbb{E}[e^{-\lambda S_t}] = e^{-t(2\lambda)^{\alpha/2}}, \qquad \lambda, t \geqslant 0,$$

(see, e.g., [2, Theorem 1.3.23 and pp. 55–56]). Using the fact that $B_{S_t}/\sqrt{S_t}$ follows the normal distribution $\mathcal{N}(0,1)$ given S_t, for $d \geqslant 1$ and $p \in (0,d)$ we have

$$\begin{aligned}
(2.5)\qquad \mathbb{E}[|X_t|^{-p}] &= \mathbb{E}[|B_{S_t}|^{-p}] = \mathbb{E}\left[S_t^{-p/2}\mathbb{E}\left[\frac{S_t^{p/2}}{|B_{S_t}|^p} \;\middle|\; S_t\right]\right] \\
&= \mathbb{E}\left[S_t^{-p/2}\int_{\mathbb{S}^{d-1}} \mu_d(d\sigma)\int_0^\infty r^{d-1-p}\frac{e^{-r^2/2}}{(2\pi)^{d/2}}\,dr\right] \\
&= 2\frac{2^{(d-p-2)/2}}{2^{d/2}\Gamma(d/2)}\Gamma((d-p)/2)\mathbb{E}[S_t^{-p/2}] \\
&= \frac{C_{\alpha,d,p}}{t^{p/\alpha}}, \qquad t>0,\ \alpha\in(1,2),
\end{aligned}$$

where μ_d denotes the surface measure on the d-dimensional sphere $\mathbb{S}^{d-1}$,

$$C_{\alpha,d,p} := 2^{1-p}\frac{\Gamma(p/\alpha)\Gamma((d-p)/2)}{\alpha\Gamma(p/2)\Gamma(d/2)},$$

and we have used the relation $\mathbb{E}[S_t^{-p}] = \alpha^{-1}2^{1-p}t^{-2p/\alpha}\Gamma(2p/\alpha)/\Gamma(p)$, $p,t>0$ (see, e.g., [25, (1.10)]).

2.3. Integration by parts. The stochastic representation of the gradient $\nabla u(x)$ will rely on an integration by parts argument. For this, we will use the weight functions $\mathcal{W}_{B(0,R)}(x,y)$ and $\mathcal{W}_{\partial B(0,R)}(x,y)$ defined as

$$(2.6)\quad \mathcal{W}_{B(0,R)}(x,y) := \frac{\nabla_x G_R(x,y)}{G_R(x,y)} \quad\text{and}\quad \mathcal{W}_{\partial B(0,R)}(x,y) := \frac{\nabla_x P_R(x,y)}{P_R(x,y)},$$

for $x,y\in B(0,R)$.

LEMMA 2.1. *Let $\alpha\in(1,2)$ and $d\geqslant 2$.*

(a) *Given a bounded measurable function ϕ on $\mathbb{R}^d\setminus B(0,R)$, the function*

$$(2.7)\quad \chi_1^\phi(x) := \mathbb{E}[\phi(X_{\tau_R(x),x})] = \int_{\mathbb{R}^d\setminus B(0,R)} P_R(x,y)\phi(y)\,dy, \quad x\in\overline{B}(0,R),$$

belongs to $\mathcal{C}^1(B(0,R))\cap\mathcal{C}^0(\overline{B}(0,R))$, with

$$\nabla\chi_1^\phi(x) = \mathbb{E}[\mathcal{W}_{\partial B(0,R)}(x,X_{\tau_R(x),x})\phi(X_{\tau_R(x),x})], \quad x\in B(0,R).$$

(b) *Given a bounded continuous function h on $\overline{B}(0,R)$, the function*

$$\chi_2^h(x) := \mathbb{E}\left[\int_0^{\tau_R(x)} h(X_{t,x})\,dt\right] = \int_{B(0,R)} G_R(x,y)h(y)\,dy, \quad x\in\overline{B}(0,R),$$

belongs to $\mathcal{C}^1(B(0,R))\cap\mathcal{C}^0(\overline{B}(0,R))$, with

$$\nabla\chi_2^h(x) = \mathbb{E}\left[\int_0^{\tau_R(x)} \mathcal{W}_{B(0,R)}(x,X_{t,x})h(X_{t,x})\,dt\right], \quad x\in B(0,R).$$

Proof. (a) Using (2.2) and the boundedness of ϕ on $\mathbb{R}^d \setminus B(0,R)$, we differentiate (2.7) under the integral sign to find that χ_1^ϕ is in $\mathcal{C}^1(B(0,R)) \cap \mathcal{C}^0(\overline{B}(0,R))$ with

$$\nabla \chi_1^\phi(x) = \int_{\mathbb{R}^d \setminus B(0,R)} \nabla_x P_R(x,y)\phi(y)dy = \mathbb{E}\left[\frac{\nabla_x P_R(x, X_{\tau_R(x),x})}{P_R(x, X_{\tau_R(x),x})}\phi(X_{\tau_R(x),x})\right]$$

for $x \in B(0,R)$.

(b) Using (2.1), the condition $d \geqslant 2$ and the relation

$$\begin{aligned}\chi_2^h(x) &= \int_{B(0,R)} G_R(x,y)h(y)\,dy \\ &= \int_{B(x,R)} \frac{\kappa_\alpha^d}{|z|^{d-\alpha}} \int_0^{r_0(x,z-x)} \frac{t^{\alpha/2-1}}{(1+t)^{d/2}}\,dt\,h(z-x)\,dz, \qquad x \in \overline{B}(0,R),\end{aligned}$$

we differentiate (2.7) under the integral sign and integrate by parts to obtain

$$\nabla \chi_2^h(x) = \int_{B(0,R)} \nabla_x G_R(x,y)h(y)\,dy = \mathbb{E}\left[\int_0^{\tau_R(x)} \frac{\nabla_x G_R(x, X_{t,x})}{G_R(x, X_{t,x})} h(X_{t,x})\,dt\right],$$

first for h a $\mathcal{C}^1$ function with compact support in $B(0,R)$, then by uniform approximation for h continuous with compact support in $\overline{B}(0,R)$, and finally by pointwise approximation for h bounded continuous on $\overline{B}(0,R)$, using the bound (2.4). ∎

3. MARKED BRANCHING PROCESS

Let $(q_{l_0,\ldots,l_m})_{(l_0,\ldots,l_m)\in\mathcal{L}_m}$ be a strictly positive probability mass function on $\mathcal{L}_m$, and let $\rho : \mathbb{R}^+ \to (0,\infty)$ be a probability density function on $\mathbb{R}_+$. We consider

- an i.i.d. family $(\tau^{i,j})_{i,j\geqslant 1}$ of random variables with distribution $\rho(t)dt$ on $\mathbb{R}_+$ and tail distribution function $\overline{F}(t) = \int_t^\infty \rho(ds)\,ds$, $t \geqslant 0$,
- an i.i.d. family $(I^{i,j})_{i,j\geqslant 1}$ of discrete random variables with distribution

$$\mathbb{P}(I^{i,j} = (l_0,\ldots,l_m)) = q_{l_0,\ldots,l_m} > 0, \qquad (l_0,\ldots,l_m) \in \mathcal{L}_m,$$

- an independent family $(X^{(i,j)})_{i,j\geqslant 1}$ of symmetric α-stable processes.

In addition, the families of random variables $(\tau^{i,j})_{i,j\geqslant 1}$, $(I^{i,j})_{i,j\geqslant 1}$ and $(X^{(i,j)})_{i,j\geqslant 1}$ are assumed to be mutually independent.

The probabilistic representation for the solution of (1.2) uses a branching process started from a particle $x \in B(0,R)$ with label $\bar{1} = (1)$ and mark $i \in \{0,\ldots,m\}$, which evolves according to the process $X_{s,x}^{\bar{1}} = x + X_s^{(1,1)}$, $s \in [0, T_{\bar{1}}]$, with $T_{\bar{1}} = \tau^{1,1} \wedge \tau_R(x) = \min(\tau^{1,1}, \tau_R(x))$, where in the notation

$$\tau_R(x) := \inf\{t \geqslant 0 : x + X_t^{(1,1)} \notin B(0,R)\},$$

we omit the reference to the label $(1,1)$.

If $\tau^{1,1} < \tau_R(x)$, then the process branches at time $\tau^{1,1}$ into new independent copies of $(X_t)_{t\geqslant 0}$, each of them started at $X^{\bar{1}}_{\tau^{1,1},x}$, and determined by a random sample $(l_0,\dots,l_m) \in \mathcal{L}_m$ of $I^{1,1}$. Namely, $|l| := l_0 + \cdots + l_m$ new branches carrying respectively the marks $i = 0,\dots,m$ are created with probability $q_{l_0,\dots,l_m}$, where

(1) the first l_0 branches carry the mark 0 and are indexed by $(1,1),(1,2),\dots,(1,l_0)$,

(2) for $i = 1,\dots,m$, the next l_i branches carry the mark i and are indexed by $(1, l_0 + \cdots + l_{i-1} + 1),\dots,(1, l_0 + \cdots + l_i)$.

Each new particle then follows independently the above mechanism in such a way that particles at generation $n \geqslant 1$ are assigned a label of the form $\bar{k} = (1,k_2,\dots,k_n) \in \mathbb{N}^n$, and every branch stops when it leaves the domain $B(0,R)$.

More precisely, the particle with label $\bar{k} = (1,k_2,\dots,k_n) \in \mathbb{N}^n$ is born at time $T_{\bar{k}-}$, where $\bar{k}- := (1,k_2,\dots,k_{n-1})$ represents the label of its parent, and its lifetime $\tau^{n,\pi_n(\bar{k})}$ is the element of index $\pi_n(\bar{k})$ in the i.i.d. sequence $(\tau^{n,j})_{j\geqslant 1}$, which defines an injection

$$\pi_n : \mathbb{N}^n \to \mathbb{N}, \qquad n \geqslant 1.$$

The random evolution of the particle of label $\bar{k}$ is given by

$$X^{\bar{k}}_{t,x} := X^{\bar{k}-}_{T_{\bar{k}-},x} + X^{n,\pi_n(\bar{k})}_{t-T_{\bar{k}-}}, \qquad t \in [T_{\bar{k}-}, T_{\bar{k}}],$$

where $T_{\bar{k}} := T_{\bar{k}-} + \tau^{n,\pi_n(\bar{k})} \wedge \tau_R(X^{\bar{k}-}_{T_{\bar{k}-},x})$ and

$$\tau_R(X^{\bar{k}-}_{T_{\bar{k}-},x}) := \inf\{t \geqslant 0 : X^{\bar{k}-}_{T_{\bar{k}-},x} + X_t^{n,\pi_n(\bar{k})} \notin B(0,R)\}.$$

If $\tau^{n,\pi_n(\bar{k})} < \tau_R(X^{\bar{k}-}_{T_{\bar{k}-},x})$, we draw a random sample $(l_0,\dots,l_m)$ of $I_{\bar{k}} := I^{n,\pi_n(\bar{k})}$ with probability $q_{l_0,\dots,l_m}$, and the particle $\bar{k}$ branches into $|I^{n,\pi_n(\bar{k})}| = l_0 + \cdots + l_m$ offsprings, indexed by $(1,\dots,k_n,j)$, $j = 1,\dots,|I^{n,\pi_n(\bar{k})}|$, and respectively carrying the marks $i = 0,\dots,m$, as in point (b) above. Namely, the particles whose index ends with an integer between 1 and l_0 will carry the mark 0, and those with index ending with an integer between $l_0 + \cdots + l_{i-1} + 1$ and $l_0 + \cdots + l_i$ will carry a mark $i \in \{1,\dots,m\}$. Finally, the mark of the particle $\bar{k}$ will be denoted by $\theta_{\bar{k}} \in \{0,\dots,m\}$.

The set of particles dying inside the ball $B(0,R)$ is denoted by $\mathcal{K}^\circ$, whereas those dying outside of $B(0,R)$ form a set denoted by $\mathcal{K}^\partial$. For $n \geqslant 1$, the set of nth generation particles that die inside $B(0,R)$ is denoted by $\mathcal{K}^\circ_n$, the set of nth generation particles which die outside $B(0,R)$ is denoted by $\mathcal{K}^\partial_n$, and we let $\mathcal{K}_n = \mathcal{K}^\circ_n \cup \mathcal{K}^\partial_n$.

DEFINITION 3.1. We denote by $\mathcal{T}_{x,i}$ the marked branching process, or random marked tree constructed above after starting from position $x \in \mathbb{R}^d$ and mark $i \in \{0,\ldots,m\}$ on its first branch.

The tree $\mathcal{T}_{x,0}$ will be used for the stochastic representation of the solution $u(x)$ of the PDE (1.2), while the trees $\mathcal{T}_{x,i}$ will be used for the stochastic representation of $b_i(x) \cdot \nabla u(x)$, $i = 1,\ldots,m$. Table 1 summarizes the notation introduced so far.

TABLE 1

Object	Notation
Initial position	x
Tree rooted at x with initial mark $\theta_{\bar{1}} = i$	$\mathcal{T}_{x,i}$
Particle (or label) of generation $n \geqslant 1$	$\bar{k} = (1, k_2, \ldots, k_n)$
First branching time	$T_{\bar{1}}$
Lifespan of a particle	$T_{\bar{k}} - T_{\bar{k}-}$
Birth time of the particle $\bar{k}$	$T_{\bar{k}-}$
Death time of the particle $\bar{k} \in \mathcal{K}^\circ$	$T_{\bar{k}} = \mathrm{T}_{\bar{k}-} + \tau^{n,\pi_n(\bar{k})}$
Death time of the particle $\bar{k} \in \mathcal{K}^\partial$	$T_{\bar{k}} = T_{\bar{k}-} + \tau_R(X^{\bar{k}-}_{T_{\bar{k}-},x})$
Position at birth of the particle $\bar{k}$	$X^{\bar{k}}_{T_{\bar{k}-},x}$
Position at death of the particle $\bar{k}$	$X^{\bar{k}}_{T_{\bar{k}},x}$
Mark of the particle $\bar{k}$	$\theta_{\bar{k}}$
Exit time starting from $x \in B(0,R)$	$\tau_R(x) := \inf\{t \geqslant 0 : x + X_t \notin B(0,R)\}$

Figure 1 represents the marking and labeling conventions used for the graphical representation of random marked trees, in which different colors represent different ways of branching.

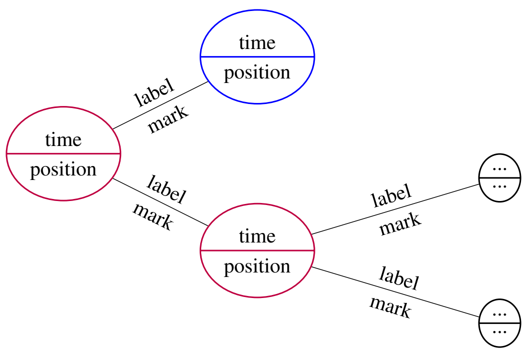


FIGURE 1. Tree labeling and marking conventions.

A sample tree for the PDE

$$\Delta_\alpha u(t,x) + c_{(0,0)}(x) + c_{(0,1)}(x)u(t,x)\frac{\partial u}{\partial x}(t,x) = 0$$

in dimension $d = 1$ is presented in Figure 2. Absence of branching is represented in blue, branching into two branches, one bearing the mark 0 and the other bearing the mark 1, is represented in purple, and the black color is used for leaves, i.e. for particles that die outside of the domain $B(0,R)$.

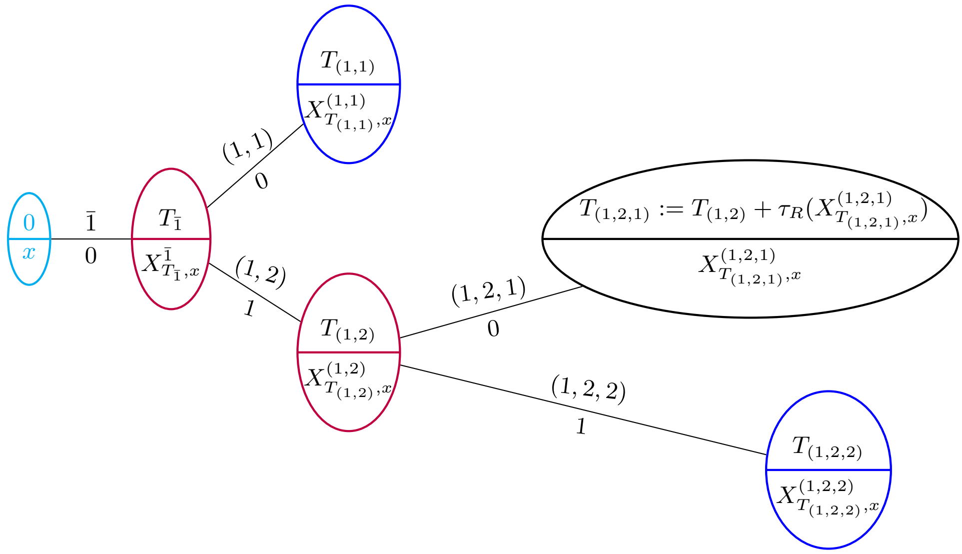


FIGURE 2. Tree labeling and marking conventions.

In Figure 2 we have $\mathcal{K}^\circ = \{\bar{1}, (1,1), (1,2), (1,2,2)\}$ and $\mathcal{K}^\partial = \{(1,2,1)\}$.

4. PROBABILISTIC REPRESENTATION OF PDE SOLUTIONS

We consider the weight function $\mathcal{W}(t,x,X)$ defined as

$$\mathcal{W}(t,x,X) := \mathcal{W}_{B(0,R)}(x, X_{t,x})\mathbf{1}_{\{X_{t,x}\in B(0,R)\}} + \mathcal{W}_{\partial B(0,R)}(x, X_{\tau_R(x),x})\mathbf{1}_{\{X_{t,x}\notin B(0,R)\}} \tag{4.1}$$

for $x \in B(0,R)$. We note that the products involved in the definition (4.2) of $\mathcal{H}_\phi(\mathcal{T}_{x,i})$ below are almost surely finite since the interbranching times $T_{\bar{k}} - T_{\bar{k}-}$ are identically distributed and the number of offsprings at any branching time is bounded by a constant depending only on the finite set $\mathcal{L}_m$.

DEFINITION 4.1. We define the functional $\mathcal{H}_\phi$ of the random tree $\mathcal{T}_{x,i}$ with initial mark $\theta_{\bar{1}} = i \in \{0,\dots,m\}$ as

(4.2)

$$\mathcal{H}_\phi(\mathcal{T}_{x,i}) := \prod_{\bar{k}\in\mathcal{K}^\circ} \frac{c_{I_{\bar{k}}}(X^{\bar{k}}_{T_{\bar{k}},x})\mathcal{W}^{\bar{k}}_{T_{\bar{k}-},x}}{q_{I_{\bar{k}}}\rho(T_{\bar{k}}-T_{\bar{k}-})} \prod_{\bar{k}\in\mathcal{K}^\partial} \frac{\phi(X^{\bar{k}}_{T_{\bar{k}},x})\mathcal{W}^{\bar{k}}_{T_{\bar{k}-},x}}{\overline{F}(T_{\bar{k}}-T_{\bar{k}-})}, \qquad x \in B(0,R),$$

where for $\bar{k} \in \mathcal{K}^\circ \cup \mathcal{K}^\partial$ we let

(4.3)

$$\mathcal{W}^{\bar{k}}_{T_{\bar{k}-},x} := \begin{cases} 1 & \text{if } \theta_{\bar{k}} = 0, \\ b_{\theta_{\bar{k}}}(X^{\bar{k}}_{T_{\bar{k}-},x}) \cdot \mathcal{W}(T_{\bar{k}}-T_{\bar{k}-}, X^{\bar{k}}_{T_{\bar{k}-},x}, X^{\bar{k}}) & \text{if } \theta_{\bar{k}} = 1,\ldots,m, \end{cases}$$

where $\theta_{\bar{k}} \in \{0,\ldots,m\}$ denotes the mark of the particle $\bar{k}$.

ASSUMPTION (B). Let $\alpha \in (1,2)$ and $d \geqslant 2$. The common probability density function ρ and the tail distribution function $\overline{F}$ of the random times $\tau^{i,j}$ satisfy the conditions

$$\sup_{t\in(0,1]} \frac{1}{\rho(t)t^{p/\alpha}} < \infty \quad \text{and} \quad \mathbb{E}[(\overline{F}(\tau_R(0)))^{1-p}] < \infty$$

for some $p \in (1,d)$.

When $\alpha \in (1,2)$ and R is sufficiently small, Assumption (B) is satisfied by any continuous probability density function $\rho(t)$ such that

$$\rho(t) \underset{t\to 0}{\sim} \kappa t^{\delta-1}$$

for some $\delta \in (0, 1-p/\alpha]$ and $\kappa > 0$, and $1/\overline{F}(x) \leqslant e^{\kappa x}$, $x \geqslant 0$, for some $\kappa > 0$ (see, e.g., [8, Lemma 6]). This includes for example a gamma distribution with shape parameter $\delta \in (0, 1-p/\alpha]$. The goal of this section is to prove the following result, which implies Theorem 1.1.

THEOREM 4.1. *Let* $\alpha \in (1,2)$ *and* $d \geqslant 2$. *Under Assumptions* (A)–(B), *if* $R > 0$ *and* $\max_{l\in\mathcal{L}_m} \|c_l\|_\infty$ *are sufficiently small, then the semilinear elliptic PDE* (1.2) *admits a classical solution in* $C^{\alpha+\epsilon}(B(0,R)) \cap \mathcal{C}^0(\overline{B}(0,R))$ *for some* $\epsilon > 0$, *which is the unique viscosity solution of* (1.2) *and can be represented as*

$$u(x) := \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,0})], \qquad x \in B(0,R). \tag{4.4}$$

Before giving the proof of Theorem 4.1 at the end of this section, we need to state and prove Propositions 4.1 and 4.2 below. First, in Proposition 4.1 we obtain a probabilistic representation for the solutions of semilinear elliptic PDEs of the form (1.2) under uniform integrability conditions on $(\mathcal{H}_\phi(\mathcal{T}_{x,i}))_{x\in B(0,R)}$, $i = 0,\ldots,m$. Then, in Proposition 4.2 we show that such conditions are satisfied under Assumptions (A)–(B).

PROPOSITION 4.1. *Let $\alpha \in (1,2)$ and $d \geqslant 2$, and assume that the family $(\mathcal{H}(\mathcal{T}_{x,i}))_{x \in B(0,R)}$ is uniformly integrable for $i = 0, \ldots, m$. Then the function $u(x)$ defined as*

$$u(x) := \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,0})], \quad x \in \overline{B}(0,R),$$

is a viscosity solution in $\mathcal{C}^1(B(0,R)) \cap \mathcal{C}^0(\overline{B}(0,R))$ of (1.2). In addition, the gradient $b_i(x) \cdot \nabla u(x)$ can be represented as the expected value

$$b_i(x) \cdot \nabla u(x) = \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,i})], \quad x \in B(0,R),\ i = 1, \ldots, m.$$

Proof. Let

$$v_i(x) := \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,i})], \quad x \in B(0,R),\ i = 1, \ldots, m.$$

By considering the first branching at time $T_{\bar{1}}$ and letting $\mathcal{T}^{(j)}_{X^{\bar{1}}_{T_{\bar{1}},x},i}$, $j = 1 + l_0 + \cdots + l_{i-1}, \ldots, l_0 + \cdots + l_i$, denote independent tree copies started at $X^{\bar{1}}_{T_{\bar{1}},x}$ with the mark $i \in \{0, \ldots, m\}$, we have

$$\begin{aligned}
&u(x) = \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,0})] \qquad (4.5)\\
&= \mathbb{E}\Bigg[\mathbf{1}_{\{T_{\bar{1}} = \tau_R(x)\}} \frac{\phi(X^{\bar{1}}_{\tau_R(x),x})}{\overline{F}(T_{\bar{1}})} \\
&\quad + \mathbf{1}_{\{T_{\bar{1}} < \tau_R(x)\}} \sum_{l \in \mathcal{L}_m} \mathbf{1}_{\{I_{\bar{1}} = (l_0, \ldots, l_m)\}} \frac{c_{I_{\bar{1}}}(X^{\bar{1}}_{T_{\bar{1}},x})}{q_{I_{\bar{1}}}\rho(T_{\bar{1}})} \prod_{i=0}^m \prod_{j=1+l_0+\cdots+l_{i-1}}^{l_0+\cdots+l_i} \mathcal{H}_\phi(\mathcal{T}^{(j)}_{X^{\bar{1}}_{T_{\bar{1}},x},i})\Bigg] \\
&= \mathbb{E}\Bigg[\phi(X^{\bar{1}}_{\tau_R(x),x}) + \int_0^{\tau_R(x)} \sum_{l \in \mathcal{L}_m} c_l(X^{\bar{1}}_{t,x}) u^{l_0}(X^{\bar{1}}_{t,x}) \prod_{i=1}^m v_i^{l_i}(X^{\bar{1}}_{t,x})\, dt\Bigg] \\
&= \mathbb{E}[\phi(X^{\bar{1}}_{\tau_R(x),x})] + \mathbb{E}\Bigg[\int_0^{\tau_R(x)} h(X^{\bar{1}}_{t,x})\, dt\Bigg],
\end{aligned}$$

where $u(x)$ and the function

$$h(x) := \sum_{l \in \mathcal{L}_m} c_l(x) u^{l_0}(x) \prod_{i=1}^m v_i^{l_i}(x), \quad x \in B(0,R),$$

are bounded continuous on $\overline{B}(0,R)$ by Lemma 4.3. Hence by Lemmas 2.1 and 4.3 the function u is differentiable in $B(0,R)$, with

$$\begin{aligned}
&\nabla u(x) = \nabla \mathbb{E}[\phi(X^{\bar{1}}_{\tau_R(x),x})] + \nabla \mathbb{E}\Bigg[\int_0^{\tau_R(x)} h(X^{\bar{1}}_{t,x})\, dt\Bigg] \\
&= \mathbb{E}[\mathcal{W}_{\partial B(0,R)}(x, X^{\bar{1}}_{\tau_R(x),x}) \phi(X^{\bar{1}}_{\tau_R(x),x})] + \mathbb{E}\Bigg[\int_0^{\tau_R(x)} \mathcal{W}_{B(0,R)}(x, X^{\bar{1}}_{t,x}) h(X^{\bar{1}}_{t,x})\, dt\Bigg] \\
&= \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,0}) \mathcal{W}(T_{\bar{1}}, x, X)],
\end{aligned}$$

and by (4.3)–(4.2) we have

$$\begin{aligned} b_i(x)\cdot\nabla u(x) &= \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,0})\, b_i(x)\cdot\mathcal{W}(T_{\bar{1}},x,X)] \\ &= \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,i})] \\ &= v_i(x), \quad x\in B(0,R),\ i=1,\ldots,m. \end{aligned}$$

Therefore, using (1.3), we can rewrite (4.5) as

$$u(x)=\mathbb{E}\Big[\phi(X^{\bar{1}}_{\tau_R(x),x})+\int_0^{\tau_R(x)} f\big(X^{\bar{1}}_{t,x},u(X^{\bar{1}}_{t,x}),\nabla u(X^{\bar{1}}_{t,x})\big)\,dt\Big],\quad x\in B(0,R).$$

It then follows from a classical argument that u is a viscosity solution of (1.2). Indeed, for any $\delta>0$, by the Markov property we also have

$$u(x)=\mathbb{E}\Big[u(X^{\bar{1}}_{\delta\wedge\tau_R(x),x})+\int_0^{\delta\wedge\tau_R(x)} f\big(X^{\bar{1}}_{t,x},u(X^{\bar{1}}_{t,x}),\nabla u(X^{\bar{1}}_{t,x})\big)\,dt\Big],\quad x\in B(0,R).$$

Next, let $\xi\in\mathcal{C}^2(B(0,R))$ be such that x is a maximum point of $u-\xi$ and $u(x)=\xi(x)$. By the Itô–Dynkin formula, we get

$$\mathbb{E}[\xi(X^{\bar{1}}_{\delta\wedge\tau_R(x),x})]=\xi(x)+\mathbb{E}\Big[\int_0^{\delta\wedge\tau_R(x)}\Delta_\alpha\xi(X^{\bar{1}}_{t,x})\,dt\Big].$$

Thus, since $u(x)=\xi(x)$ and $u\leqslant\xi$, we find

$$\mathbb{E}\Big[\int_0^{\delta\wedge\tau_R(x)}\big(\Delta_\alpha\xi(X^{\bar{1}}_{t,x})+f\big(X^{\bar{1}}_{t,x},u\big(X^{\bar{1}}_{t,x}\big),\nabla u(X^{\bar{1}}_{t,x})\big)\big)\,dt\Big]\geqslant 0.$$

Since $X_{t,x}$ converges in distribution to the constant $x\in\mathbb{R}^d$ as $t\to 0$, it admits an almost surely convergent subsequence, hence by continuity and boundedness of $f(\cdot,u(\cdot))$ together with the mean-value and dominated convergence theorems, we have

$$\Delta_\alpha\xi(x)+f(x,\xi(x),\nabla\xi(x))\geqslant 0,$$

hence u is a viscosity subsolution (and similarly a viscosity supersolution) of (1.2). ■

The proof of the next lemma uses the filtration $(\mathcal{F}_n)_{n\geqslant 1}$ defined by

$$\mathcal{F}_n:=\sigma\Big(T_{\bar{k}},I_{\bar{k}},X^{\bar{k}},\ \bar{k}\in\bigcup_{i=1}^n\mathbb{N}^i\Big),\quad n\geqslant 1.$$

Recall that $\mathcal{K}^\circ_i$ (resp. $\mathcal{K}^\partial_i$), $i=1,\ldots,n+1$, denotes the set of ith generation particles which die inside (resp. outside) the domain $B(0,R)$, and $\mathcal{K}_n=\mathcal{K}^\circ_n\cup\mathcal{K}^\partial_n$.

LEMMA 4.1. *Given $p \geqslant 1$, let $v : B(0,R) \to \mathbb{R}_+$ be a bounded measurable function satisfying*

$$v(x) \geqslant K_1 \mathbb{E}[(\overline{F}(\tau_R(x)))^{1-p}] \\ + \mathbb{E}\Bigg[\int_0^{\tau_R(x)} \big(K_2^p \mathbf{1}_{[0,1]}(t)\rho(t) + \widetilde{K}_2^p \mathbf{1}_{(1,\infty)}(t)\big) \sum_{l=(l_0,\dots,l_m)\in\mathcal{L}_m} \frac{v^{|l|}(X_{t,x}^{\bar{1}})}{q_l^{p-1}}\, dt\Bigg]$$

for all $x \in B(0,R)$ and some $K_1, K_2, \widetilde{K}_2 > 0$, where $|l| = l_0 + \cdots + l_m$. Then

(4.6)
$$v(x) \geqslant \mathbb{E}\Bigg[\prod_{\bar{k}\in\mathcal{K}^\partial} \frac{K_1}{\overline{F}^p(T_{\bar{k}} - T_{\bar{k}-})} \prod_{\substack{\bar{k}\in\mathcal{K}^\circ \\ T_{\bar{k}}-T_{\bar{k}-}\leqslant 1}} \frac{K_2^p}{q_{I_{\bar{k}}}^p} \prod_{\substack{\bar{k}\in\mathcal{K}^\circ \\ T_{\bar{k}}-T_{\bar{k}-}>1}} \frac{\widetilde{K}_2^p}{q_{I_{\bar{k}}}^p \rho^p(T_{\bar{k}} - T_{\bar{k}-})}\Bigg]$$

for all $x \in B(0,R)$.

Proof. Since $T_{\bar{1}}$ is independent of $(X_{s,x}^{\bar{1}})_{s\geqslant 0}$ and has probability density ρ, letting

$$g(y) := \sum_{l=(l_0,\dots,l_m)\in\mathcal{L}_m} \frac{y^{|l|}}{q_l^{p-1}}$$

we have

$$\begin{aligned}
&v(x) \\
&\geqslant \mathbb{E}\Bigg[K_1(\overline{F}(\tau_R(x)))^{1-p} + \int_0^{\tau_R(x)} \big(K_2^p \mathbf{1}_{[0,1]}(t)\rho(t) + \widetilde{K}_2^p \mathbf{1}_{(1,\infty)}(t)\big) g(v(X_{t,x}^{\bar{1}}))\, dt\Bigg] \\
&= \mathbb{E}\Bigg[\mathbb{E}\Bigg[K_1(\overline{F}(\tau_R(x)))^{1-p} \\
&\qquad + \int_0^{\tau_R(x)} \big(K_2^p \mathbf{1}_{[0,1]}(t)\rho(t) + \widetilde{K}_2^p \mathbf{1}_{(1,\infty)}(t)\big) g(v(X_{t,x}^{\bar{1}}))\, dt \,\Big|\, (X_{s,x}^{\bar{1}})_{s\geqslant 0}\Bigg]\Bigg] \\
&= \mathbb{E}\Bigg[\mathbb{E}\Bigg[\frac{K_1}{\overline{F}^p(\tau_R(x))} \mathbf{1}_{\{T_{\bar{1}}=\tau_R(x)\}} \\
&\qquad + \int_0^{\tau_R(x)} \Bigg(K_2^p \mathbf{1}_{[0,1]}(t) + \frac{\widetilde{K}_2^p}{\rho(t)} \mathbf{1}_{(1,\infty)}(t)\Bigg) g(v(X_{t,x}^{\bar{1}}))\rho(t)\, dt \,\Big|\, (X_{s,x}^{\bar{1}})_{s\geqslant 0}\Bigg]\Bigg] \\
&= \mathbb{E}\Bigg[\frac{K_1}{\overline{F}^p(\tau_R(x))} \mathbf{1}_{\{T_{\bar{1}}=\tau_R(x)\}} \\
&\qquad + K_2^p g(v(X_{T_{\bar{1}},x}^{\bar{1}})) \mathbf{1}_{\{T_{\bar{1}}\leqslant \min(1,\tau_R(x))\}} + \frac{\widetilde{K}_2^p}{\rho(T_{\bar{1}})} g(v(X_{T_{\bar{1}},x}^{\bar{1}})) \mathbf{1}_{\{1<T_{\bar{1}}<\tau_R(x)\}}\Bigg] \\
&= \mathbb{E}\Bigg[\frac{K_1}{\overline{F}^p(T_{\bar{1}})} \mathbf{1}_{\{T_{\bar{1}}=\tau_R(x)\}} \\
&\qquad + \frac{1}{q_{I_{\bar{1}}}^p} \Bigg(K_2^p \mathbf{1}_{\{T_{\bar{1}}\leqslant \min(1,\tau_R(x))\}} + \frac{\widetilde{K}_2^p}{\rho(T_{\bar{1}})} \mathbf{1}_{\{1<T_{\bar{1}}<\tau_R(x)\}}\Bigg) v^{|I_{\bar{1}}|}(X_{T_{\bar{1}},x}^{\bar{1}})\Bigg],
\end{aligned}$$

showing that

$$v(x) \geqslant \mathbb{E}\Bigg[\prod_{\bar{k}\in\mathcal{K}_1^\partial}\frac{K_1}{\overline{F}^p(T_{\bar{k}}-T_{\bar{k}-})}\prod_{\substack{\bar{k}\in\mathcal{K}_1^\circ\\ T_{\bar{k}}-T_{\bar{k}-}\leqslant 1}}\frac{K_2^p}{q_{I_{\bar{k}}}^p} \times \prod_{\substack{\bar{k}\in\mathcal{K}_1^\circ\\ T_{\bar{k}}-T_{\bar{k}-}>1}}\frac{\widetilde{K}_2^p}{q_{I_{\bar{k}}}^p\rho^p(T_{\bar{k}}-T_{\bar{k}-})}\prod_{\bar{k}\in\mathcal{K}_2}v(X^{\bar{k}}_{T_{\bar{k}-},x})\Bigg] \tag{4.7}$$

for $x \in B(0,R)$. By repeating this argument for the particles in $\bar{k} \in \mathcal{K}_2$, we find

$$v(X^{\bar{k}}_{T_{\bar{k}-},x}) \geqslant \mathbb{E}\Bigg[\frac{K_1}{\overline{F}^p(T_{\bar{k}}-T_{\bar{k}-})}\mathbf{1}_{\{X^{\bar{k}}_{T_{\bar{k}},x}\notin B(0,R)\}} + \frac{1}{q_{I_{\bar{k}}}^p}\left(K_2^p\mathbf{1}_{\{T_{\bar{k}}-T_{\bar{k}-}\leqslant\min(1,\tau_R(x))\}} + \frac{\widetilde{K}_2^p}{\rho^p(T_{\bar{k}}-T_{\bar{k}-})}\mathbf{1}_{\{1<T_{\bar{k}}-T_{\bar{k}-}<\tau_R(x)\}}\right) \times v^{|I_{\bar{k}}|}(X^{\bar{k}}_{T_{\bar{k}},x})\ \Bigg|\ \mathcal{F}_1\Bigg].$$

Plugging this expression in (4.7) above and using the tower property of the conditional expectation, we obtain

$$v(x) \geqslant \mathbb{E}\Bigg[\prod_{\bar{k}\in\bigcup_{i=1}^2\mathcal{K}_i^\partial}\frac{K_1}{\overline{F}^p(T_{\bar{k}}-T_{\bar{k}-})}\prod_{\substack{\bar{k}\in\bigcup_{i=1}^2\mathcal{K}_i^\circ\\ T_{\bar{k}}-T_{\bar{k}-}\leqslant 1}}\frac{K_2^p}{q_{I_{\bar{k}}}^p} \times \prod_{\substack{\bar{k}\in\bigcup_{i=1}^2\mathcal{K}_i^\circ\\ T_{\bar{k}}-T_{\bar{k}-}>1}}\frac{\widetilde{K}_2^p}{q_{I_{\bar{k}}}^p\rho^p(T_{\bar{k}}-T_{\bar{k}-})}\prod_{\bar{k}\in\mathcal{K}_4}v(X^{\bar{k}}_{T_{\bar{k}-},x})\Bigg],$$

and repeating this process inductively leads to

$$v(x) \geqslant \mathbb{E}\Bigg[\prod_{\bar{k}\in\bigcup_{i=1}^n\mathcal{K}_i^\partial}\frac{K_1}{\overline{F}^p(T_{\bar{k}}-T_{\bar{k}-})}\prod_{\substack{\bar{k}\in\bigcup_{i=1}^n\mathcal{K}_i^\circ\\ T_{\bar{k}}-T_{\bar{k}-}\leqslant 1}}\frac{K_2^p}{q_{I_{\bar{k}}}^p} \times \prod_{\substack{\bar{k}\in\bigcup_{i=1}^n\mathcal{K}_i^\circ\\ T_{\bar{k}}-T_{\bar{k}-}>1}}\frac{\widetilde{K}_2^p}{q_{I_{\bar{k}}}^p\rho^p(T_{\bar{k}}-T_{\bar{k}-})}\prod_{\bar{k}\in\mathcal{K}_{n+1}}v(X^{\bar{k}}_{T_{\bar{k}-},x})\Bigg]$$

for $n \geqslant 1$. Using Fatou's lemma as $n \to \infty$, since all particles become eventually extinct with probability 1, we obtain (4.6). ■

LEMMA 4.2. *Let $\alpha \in (1,2)$, $p \in [1,d)$, $d \geqslant 2$, and set*

$$b_{0,\infty} := \max_{1\leqslant i\leqslant m} \sup_{x\in B(0,R)} |b_i(x)|, \quad b_{1,\infty} := \max_{1\leqslant i\leqslant m} \sup_{x\in B(0,R)} \frac{|b_i(x)|}{R-|x|}.$$

Under Assumptions (A)–(B), we have

$$\mathbb{E}[|\mathcal{H}_\phi(\mathcal{T}_{x,i})|^p] \leqslant \mathbb{E}\Bigg[\prod_{\bar{k}\in\mathcal{K}^\partial} \frac{K_1}{\overline{F}^p(T_{\bar{k}}-T_{\bar{k}-})} \prod_{\substack{\bar{k}\in\mathcal{K}^\circ \\ T_{\bar{k}}-T_{\bar{k}-}\leqslant 1}} \frac{K_4 \max_{l\in\mathcal{L}_m} \|c_l\|_\infty^p}{q_{I_{\bar{k}}}^p} \times \prod_{\substack{\bar{k}\in\mathcal{K}^\circ \\ T_{\bar{k}}-T_{\bar{k}-}> 1}} \frac{K_3 \max_{l\in\mathcal{L}_m} \|c_l\|_\infty^p}{q_{I_{\bar{k}}}^p \rho^p(T_{\bar{k}}-T_{\bar{k}-})}\Bigg] \tag{4.8}$$

for $x \in B(0,R)$, $i = 0,\ldots,m$, where

$$K_1 := \|\phi\|_\infty^p (1+(d+\alpha)^p b_{1,\infty}^p), \qquad K_3 := 1 + d^p b_{1,\infty}^p + d^p b_{0,\infty}^p C_{\alpha,d,p}, \tag{4.9}$$

$$K_4 := \sup_{t\in[0,1]} \frac{1+d^p b_{1,\infty}^p}{\rho^p(t)} + d^p b_{0,\infty}^p \sup_{t\in[0,1]} \frac{C_{\alpha,d,p}}{\rho^p(t) t^{p/\alpha}}. \tag{4.10}$$

Proof. For $x \in B(0,R)$, let

$$\begin{aligned} w_i(x) &:= \mathbb{E}[|\mathcal{H}_\phi(\mathcal{T}_{x,i})|^p] \\ &= \mathbb{E}_i\Bigg[\prod_{\bar{k}\in\mathcal{K}^\circ} \frac{|c_{I_{\bar{k}}}(X^{\bar{k}}_{T_{\bar{k}},x})|^p |\mathcal{W}^{\bar{k}}_{T_{\bar{k}-},x}|^p}{q_{I_{\bar{k}}}^p \rho^p(T_{\bar{k}}-T_{\bar{k}-})} \prod_{\bar{k}\in\mathcal{K}^\partial} \frac{|\phi(X^{\bar{k}}_{T_{\bar{k}},x})|^p |\mathcal{W}^{\bar{k}}_{T_{\bar{k}-},x}|^p}{\overline{F}^p(T_{\bar{k}}-T_{\bar{k}-})}\Bigg], \end{aligned} \tag{4.11}$$

where $\mathbb{E}_i$ denotes the conditional expectation given that the tree $\mathcal{T}_{x,i}$ is started with the mark $i \in \{0,\ldots,m\}$. When $\bar{k} \in \mathcal{K}^\circ$ has mark $\theta_{\bar{k}} = 0$ we have $\mathcal{W}^{\bar{k}}_{T_{\bar{k}-},x} = 1$, whereas when $\theta_{\bar{k}} \neq 0$, using (2.4), (4.1)–(4.3) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |\mathcal{W}^{\bar{k}}_{T_{\bar{k}-},x}| &\leqslant \frac{d|b_{\theta_k}(X^{\bar{k}}_{T_{\bar{k}-},x})|}{\min(R-|X^{\bar{k}}_{T_{\bar{k}-},x}|, |X^{\bar{k}}_{T_{\bar{k}},x} - X^{\bar{k}}_{T_{\bar{k}-},x}|)} \\ &\leqslant d \max\left(\frac{|b_{\theta_k}(X^{\bar{k}}_{T_{\bar{k}-},x})|}{R-|X^{\bar{k}}_{T_{\bar{k}-},x}|}, \frac{|b_{\theta_k}(X^{\bar{k}}_{T_{\bar{k}-},x})|}{|X^{\bar{k}}_{T_{\bar{k}},x} - X^{\bar{k}}_{T_{\bar{k}-},x}|}\right) \\ &\leqslant d b_{1,\infty} + \frac{d b_{0,\infty}}{|X^{\bar{k}}_{T_{\bar{k}},x} - X^{\bar{k}}_{T_{\bar{k}-},x}|}. \end{aligned}$$

Similarly, when $\bar{k} \in \mathcal{K}^\partial$, the definition of $\mathcal{W}_{\partial B(0,R)}(x,y)$ in (2.6), together with the bound (2.3) and the Cauchy–Schwarz inequality, implies

$$|\mathcal{W}^{\bar{k}}_{T_{\bar{k}-},x}| \leqslant (d+\alpha) b_{1,\infty}. \tag{4.12}$$

Next, by conditional independence given $\mathcal{G} := \sigma(\tau^{i,j}, I^{i,j} : i,j \geqslant 1)$ of the terms in the product over $\bar{k} \in \mathcal{K}^\circ$ and $\bar{k} \in \mathcal{K}^\partial$, which involve random terms of the form $X^{\bar{k}}_{T_{\bar{k}},x} - X^{\bar{k}}_{T_{\bar{k}-},x}$ given $T_{\bar{k}} - T_{\bar{k}-}$, by (2.5) and (4.11)–(4.12) we have

$$
\begin{aligned}
&w_i(x) \\
&\leqslant \mathbb{E}\Bigg[\prod_{\bar{k}\in\mathcal{K}^\circ} \frac{\|c_{I_{\bar{k}}}\|_\infty^p}{q^p_{I_{\bar{k}}}} \mathbb{E}\Bigg[\frac{2^p}{\rho^p(T_{\bar{k}} - T_{\bar{k}-})}\left(1 + d^p b^p_{1,\infty} + \frac{d^p b^p_{0,\infty}}{|X^{\bar{k}}_{T_{\bar{k}},x} - X^{\bar{k}}_{T_{\bar{k}-},x}|^p}\right) \Bigg| \mathcal{G}\Bigg] \\
&\qquad\qquad \times \prod_{\bar{k}\in\mathcal{K}^\partial} \mathbb{E}\Bigg[\frac{K_1}{\overline{F}^p(T_{\bar{k}} - T_{\bar{k}-})} \Bigg| \mathcal{G}\Bigg]\Bigg] \\
&= \mathbb{E}\Bigg[\prod_{\bar{k}\in\mathcal{K}^\circ}\left(\frac{\|c_{I_{\bar{k}}}\|_\infty^p}{q^p_{I_{\bar{k}}}\rho^p(T_{\bar{k}} - T_{\bar{k}-})}\left(1 + d^p b^p_{1,\infty} + \frac{d^p b^p_{0,\infty} C_{\alpha,d,p}}{(T_{\bar{k}} - T_{\bar{k}-})^{p/\alpha}}\right)\right) \\
&\qquad\qquad \times \prod_{\bar{k}\in\mathcal{K}^\partial} \frac{K_1}{\overline{F}^p(T_{\bar{k}} - T_{\bar{k}-})}\Bigg].
\end{aligned}
$$

Splitting the terms in the product over $\bar{k} \in \mathcal{K}^\circ$ between small and large values of $T_{\bar{k}} - T_{\bar{k}-}$, we get

$$
\begin{aligned}
w_i(x) \leqslant \mathbb{E}\Bigg[&\prod_{\bar{k}\in\mathcal{K}^\circ} \frac{\|c_{I_{\bar{k}}}\|_\infty^p}{q^p_{I_{\bar{k}}}}\left(\frac{K_3}{\rho^p(T_{\bar{k}} - T_{\bar{k}-})}\mathbf{1}_{\{T_{\bar{k}} - T_{\bar{k}-} > 1\}} + K_4 \mathbf{1}_{\{T_{\bar{k}} - T_{\bar{k}-} \leqslant 1\}}\right) \\
&\times \prod_{\bar{k}\in\mathcal{K}^\partial} \frac{K_1}{\overline{F}^p(T_{\bar{k}} - T_{\bar{k}-})}\Bigg]
\end{aligned}
$$

for $x \in B(0,R)$, which yields (4.8), for $i = 0, \ldots, m$. ■

Proposition 4.2 provides sufficient conditions for the finiteness of the upper bound (4.8), and for $(\mathcal{H}_\phi(\mathcal{T}_{x,i}))_{x\in B(0,R)}$ to be bounded in $L^1(\Omega)$, uniformly in $x \in B(0,R)$, $i = 0, \ldots, m$, as required in Proposition 4.1.

PROPOSITION 4.2. *Let $\alpha \in (1,2)$, $p \in [1,d)$, and $d \geqslant 2$. Under Assumptions (A)–(B), suppose that ϕ is bounded on $\mathbb{R}^d$ and there exists a bounded strictly positive weak solution $v \in H^{\alpha/2}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ to the partial differential inequality*

$$
\begin{cases}
\Delta_\alpha v(x) + \widetilde{K}_2 \displaystyle\sum_{l\in\mathcal{L}_m} \frac{v^{|l|}(x)}{q_l^{p-1}} \leqslant 0, & x \in B(0,R), \\
v(x) \geqslant \widetilde{K}_1 > 0, & x \in \mathbb{R}^d \setminus B(0,R),
\end{cases}
\tag{4.13}
$$

where $\widetilde{K}_1 \geqslant K_1 \mathbb{E}[\overline{F}^{1-p}(\tau_R(0))]$, $K_1 > 0$ is given by (4.9), and $\widetilde{K}_2 > 0$. Then for sufficiently small $\max_{l\in\mathcal{L}_m}\|c_l\|_\infty$ we have the bound

$$
\mathbb{E}[|\mathcal{H}_\phi(\mathcal{T}_{x,i})|^p] \leqslant v(x) \leqslant \|v\|_\infty < \infty, \quad x \in B(0,R),\ i = 0, \ldots, m. \tag{4.14}
$$

Proof. We smooth out $v \in H^{\alpha/2}(\mathbb{R}^d)$ as

$$v_\varepsilon(x) := \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \psi\left(\frac{x-y}{\varepsilon}\right) v(y)\, dy, \qquad x \in \mathbb{R},\ \varepsilon > 0,$$

where $\psi : \mathbb{R} \to \mathbb{R}_+$ is a mollifier such that $\int_{-\infty}^{\infty} \psi(y)\, dy = 1$. By (4.13) and Jensen's inequality, we have

$$\begin{aligned}
&\Delta_\alpha v_\varepsilon(x) + \widetilde{K}_2 \sum_{l\in\mathcal{L}_m} \frac{v_\varepsilon^{|l|}(x)}{q_l^{p-1}} \\
&= \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \Delta_\alpha \psi\left(\frac{x-y}{\varepsilon}\right) v(y)\, dy + \widetilde{K}_2 \sum_{l\in\mathcal{L}_m} \frac{1}{q_l^{p-1}} \left(\frac{1}{\varepsilon}\int_{-\infty}^{\infty} \psi\left(\frac{x-y}{\varepsilon}\right) v(y)\, dy\right)^{|l|} \\
&\leqslant \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \psi\left(\frac{x-y}{\varepsilon}\right) \Delta_\alpha v(y)\, dy + \widetilde{K}_2 \sum_{l\in\mathcal{L}_m} \frac{1}{\varepsilon q_l^{p-1}} \int_{-\infty}^{\infty} \psi\left(\frac{x-y}{\varepsilon}\right) v^{|l|}(y)\, dy \\
&= \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \psi\left(\frac{x-y}{\varepsilon}\right) \left(\Delta_\alpha v(y) + \widetilde{K}_2 \sum_{l\in\mathcal{L}_m} \frac{v^{|l|}(y)}{q_l^{p-1}}\right) dy \\
&\leqslant 0, \qquad x \in B(0,R).
\end{aligned}$$

Applying the Itô–Dynkin formula to $v_\varepsilon(X_{s,x})$ with $v_\varepsilon \in H^\alpha(\mathbb{R}^d)$, by (4.13) we have

$$\begin{aligned}
v_\varepsilon(x) &= \mathbb{E}\left[v_\varepsilon(X^x_{\tau_R(x)}) - \int_0^{\tau_R(x)} \Delta_\alpha v_\varepsilon(X_{t,x})\, dt\right] \\
&\geqslant \mathbb{E}\left[\widetilde{K}_1 + \int_0^{\tau_R(x)} \widetilde{K}_2 \sum_{l\in\mathcal{L}_m} \frac{v_\varepsilon^{|l|}(X_{t,x})}{q_l^{p-1}}\, dt\right], \qquad x \in B(0,R).
\end{aligned}$$

Thus, passing to the limit as ε tends to zero, by dominated convergence and the facts that $\mathbb{E}[\tau_R(x)] < \infty$ and $v(x)$ is upper and lower bounded in $(0,\infty)$, for some sufficiently small $K_2 > 0$ we have

$$\begin{aligned}
v(x) &\geqslant \widetilde{K}_1 + \mathbb{E}\left[\int_0^{\tau_R(x)} \widetilde{K}_2 \sum_{l\in\mathcal{L}_m} \frac{v^{|l|}(X_{t,x})}{q_l^{p-1}}\, dt\right] \\
&\geqslant K_1 \mathbb{E}[\overline{F}^{1-p}(\tau_R(0))] \\
&\quad + \mathbb{E}\left[\int_0^{\tau_R(x)} (K_2^p \mathbf{1}_{[0,1]}(t)\rho(t) + \widetilde{K}_2^p \mathbf{1}_{(1,\infty)}(t)) \sum_{l\in\mathcal{L}_m} \frac{v^{|l|}(X_{t,x})}{q_l^{p-1}}\, dt\right], \\
&\geqslant K_1 \mathbb{E}[\overline{F}^{1-p}(\tau_R(x))] \\
&\quad + \mathbb{E}\left[\int_0^{\tau_R(x)} (K_2^p \mathbf{1}_{[0,1]}(t)\rho(t) + \widetilde{K}_2^p \mathbf{1}_{(1,\infty)}(t)) \sum_{l\in\mathcal{L}_m} \frac{v^{|l|}(X_{t,x})}{q_l^{p-1}}\, dt\right]
\end{aligned}$$

for $x \in B(0,R)$, as the function $\overline{F}^{1-p}$ is non-decreasing. Hence by Lemmas 4.1 and 4.2, for K_3, K_4 given in (4.9)–(4.10) we have, provided that $\max_{l\in\mathcal{L}_m}\|c_l\|_\infty$ is sufficiently small,

$$\begin{aligned} v(x) &\geqslant \mathbb{E}\Bigg[\prod_{\bar{k}\in\mathcal{K}^\partial}\frac{K_1}{\overline{F}^p(T_{\bar{k}}-T_{\bar{k}-})}\prod_{\substack{\bar{k}\in\mathcal{K}^\circ\\ T_{\bar{k}}-T_{\bar{k}-}\leqslant 1}}\frac{K_2^p}{q_{I_{\bar{k}}}^p}\prod_{\substack{\bar{k}\in\mathcal{K}^\circ\\ T_{\bar{k}}-T_{\bar{k}-}>1}}\frac{\widetilde{K}_2^p}{q_{I_{\bar{k}}}^p\rho^p(T_{\bar{k}}-T_{\bar{k}-})}\Bigg] \\ &\geqslant \mathbb{E}\Bigg[\prod_{\bar{k}\in\mathcal{K}^\partial}\frac{K_1}{\overline{F}^p(T_{\bar{k}}-T_{\bar{k}-})}\prod_{\substack{\bar{k}\in\mathcal{K}^\circ\\ T_{\bar{k}}-T_{\bar{k}-}\leqslant 1}}\frac{K_4\max_{l\in\mathcal{L}_m}\|c_l\|_\infty^p}{q_{I_{\bar{k}}}^p} \\ &\qquad\qquad \times\prod_{\substack{\bar{k}\in\mathcal{K}^\circ\\ T_{\bar{k}}-T_{\bar{k}-}>1}}\frac{K_3\max_{l\in\mathcal{L}_m}\|c_l\|_\infty^p}{q_{I_{\bar{k}}}^p\rho^p(T_{\bar{k}}-T_{\bar{k}-})}\Bigg] \\ &\geqslant \mathbb{E}[|\mathcal{H}_\phi(\mathcal{T}_{x,i})|^p] \end{aligned}$$

for $x \in B(0,R)$, $i = 0,\ldots,m$, which yields (4.14). ∎

Proof of Theorem 4.1. By [26, Theorem 1.2], the partial differential inequality (4.13) admits a positive (continuous) viscosity solution $v(x)$ on $\mathbb{R}^d$ when $R>0$ is sufficiently small. In addition, by [26, Proposition 3.5], $v \in H^{\alpha/2}(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ and v is a weak solution of (4.13). By Propositions 4.1 and 4.2, the PDE (1.2) admits a viscosity solution in $\mathcal{C}^1(B(0,R))\cap\mathcal{C}^0(\overline{B}(0,R))$, which can be represented as (4.4). Hence by [20, Theorem 1.2], ∇u and $f(u,\nabla u)$ are in $C^\epsilon(B(0,R))$ for some $\epsilon>0$, as the kernel of Δ_α satisfies (1.11) therein. Therefore, by [29, Theorem 1.3], the (unique) viscosity solution u is in $C^{\alpha+\epsilon}(B(0,R))\cap\mathcal{C}^0(\overline{B}(0,R))$. ∎

Lemma 4.3 extends [26, Lemma 3.3] from $i=0$ to $i\in\{1,\ldots,m\}$.

LEMMA 4.3. *Let $i \in \{0,\ldots,m\}$, and assume that $(\mathcal{H}(\mathcal{T}_{x,i}))_{x\in B(0,R)}$ is uniformly integrable. Then the function $v_i(x) := \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,i})]$ is continuous in $\overline{B}(0,R)$.*

Proof (given for completeness). Let $x\in\overline{B}(0,R)$. By [26, Lemma 3.2], for any sequence $(x_n)_{n\in\mathbb{N}}$ in $B(0,R)$ converging fast enough to $x\in\overline{B}(0,R)$ we have

$$\mathbb{P}\Big(\lim_{n\to\infty}\tau_R(x_n)=\tau_R(x)\Big)=1,$$

and letting $\tau_{\bar{k},x} := \tau_R(X^{\bar{k}-}_{T_{\bar{k}-},x})$ for $\bar{k}\in\mathcal{K}$, the event

$$A_{\bar{k}} := \Big\{\lim_{n\to\infty}\tau_{\bar{k},x_n}=\tau_{\bar{k},x}\Big\}\cap\Big\{\lim_{n\to\infty}X^{\bar{k}}_{\cdot,x_n}=X^{\bar{k}}_{\cdot,x}\Big\}$$

has probability 1. Again, by [26, Lemma 3.2(a)], for some $n_0(\omega)$ large enough we have

$$X^{\bar{k}}_{\tau_{\bar{k},x_n}} = X^{\bar{k}}_{\tau_{\bar{k},x}} + x_n - x,$$

and $\tau_{\bar{k},x_n} = \tau_{\bar{k},x}$ for $n \geqslant n_0(\omega)$. Therefore, using the continuity of the functions ϕ and $c_l, l \in \mathcal{L}$, we have

$$\lim_{n\to\infty} \phi(X^{\bar{k}}_{\tau_{\bar{k},x_n}})\mathcal{W}^{\bar{k}}_{T_{\bar{k}-},x_n}\mathbb{1}_{\{T_{\bar{k}}=\tau_{\bar{k},x_n}\}} = \phi(X^{\bar{k}}_{\tau_{\bar{k},x}})\mathcal{W}^{\bar{k}}_{T_{\bar{k}-},x}\mathbb{1}_{\{T_{\bar{k}}=\tau_{\bar{k},x}\}}, \qquad \mathbb{P}\text{-a.s.}$$

and

$$\lim_{n\to\infty} c_{I_{\bar{k}}}(X^{\bar{k}}_{T_{\bar{k}},x_n})\mathcal{W}^{\bar{k}}_{T_{\bar{k}-},x_n}\mathbb{1}_{\{T_{\bar{k}}<\tau_{\bar{k},x_n}\}} = \frac{c_{I_{\bar{k}}}(X^{\bar{k}}_{T_{\bar{k}},x})}{q_{I_{\bar{k}}}}\mathcal{W}^{\bar{k}}_{T_{\bar{k}-},x}\mathbb{1}_{\{T_{\bar{k}}<\tau_{\bar{k},x}\}}, \qquad \mathbb{P}\text{-a.s.}$$

Hence by (4.2), on the event $A := \bigcap_{\bar{k}\in\mathcal{K}} A_{\bar{k}}$ of probability 1, we have

$$\lim_{n\to\infty} \mathcal{H}_\phi(\mathcal{T}_{x_n,i}(\omega)) = \mathcal{H}_\phi(\mathcal{T}_{x,i}(\omega)).$$

Therefore, for any sequence $(x_n)_{n\geqslant 1}$ converging to $x \in \overline{B}(0,R)$ fast enough, we have

$$\mathbb{P}\Big(\lim_{n\to\infty} \mathcal{H}_\phi(\mathcal{T}_{x_n,i}) = \mathcal{H}_\phi(\mathcal{T}_{x,i}(\omega))\Big) = 1,$$

which implies that $\lim_{n\to\infty} v_i(x_n) = v_i(x)$ by uniform integrability of $(\mathcal{H}_\phi(\mathcal{T}_{x,i}(\omega)))_{x\in B(0,R)}$. ■

5. NUMERICAL EXAMPLES

In this section, we consider numerical applications of the probabilistic representation (4.4). The paths of the α-stable process $X_t = B_{S_t}$ are simulated by time discretization, by generating independent random samples of Brownian motion and of the $\alpha/2$-stable process $(S_t)_{t\in\mathbb{R}_+}$ using the identity in distribution

$$S_t \simeq 2t^{2/\alpha}\frac{\sin(\alpha(U+\pi/2)/2)}{\cos^{2/\alpha}(U)}\left(\frac{\cos(U-\alpha(U+\pi/2)/2)}{E}\right)^{-1+2/\alpha}$$

based on the Chambers–Mallows–Stuck (CMS) method, where U is uniform on $(-\pi/2,\pi/2)$, and E is exponential with unit parameter (see [33, (3.2)]). In order to keep computation times to a reasonable level, the probability density $\rho(t)$ of $\tau^{i,j}$, $i,j \geqslant 1$, is taken to be gamma with shape parameters ranging from 1.5 to 1.7. The C codes used to plot Figures 4 and 6 are available at `https://github.com/nprivaul/fractional_elliptic`.

Given $k \geqslant 0$, we consider the function

$$\Phi_{k,\alpha}(x) := (1-|x|^2)_+^{k+\alpha/2}, \qquad x \in \mathbb{R}^d,$$

which is Lipschitz if $k > 1 - \alpha/2$, and solves the Poisson problem $\Delta_\alpha \Phi_{k,\alpha} = -\Psi_{k,\alpha}$ on $\mathbb{R}^d$ with

$$\Psi_{k,\alpha}(x) := \begin{cases} \dfrac{\Gamma((d+\alpha)/2)\Gamma(k+1+\alpha/2)}{2^{-\alpha}\Gamma(k+1)\Gamma(d/2)} \, {}_2F_1\left(\dfrac{d+\alpha}{2}, -k; \dfrac{d}{2}; |x|^2\right), & |x| \leqslant 1, \\ \dfrac{2^\alpha \Gamma((d+\alpha)/2)\Gamma(k+1+\alpha/2)}{\Gamma(k+1+(d+\alpha)/2)\Gamma(-\alpha/2)|x|^{d+\alpha}} & \\ \quad \times {}_2F_1\left(\dfrac{d+\alpha}{2}, \dfrac{2+\alpha}{2}; k+1+\dfrac{d+\alpha}{2}; \dfrac{1}{|x|^2}\right), & |x| > 1, \end{cases}$$

for $x \in \mathbb{R}^d$, where ${}_2F_1(a, b; c; y)$ is Gauss's hypergeometric function (see [14, (5.2)], [6, Lemma 4.1], and [17, (36)].

5.1. Linear gradient term. We take $R = 1$, $m = 1$, $\mathcal{L}_1 = \{(0,0), (0,1)\}$, and

$$c_{(0,0)}(x) := \Psi_{k,\alpha}(x) + (2k+\alpha)|x|^2(1-|x|^2)^{k+\alpha/2},$$
$$c_{(0,1)}(x) := 1, \quad b_1(x) := (1-|x|^2)x,$$

and consider the PDE

$$\Delta_\alpha u(x) + \Psi_{k,\alpha}(x) + (2k+\alpha)|x|^2(1-|x|^2)^{k+\alpha/2} + (1-|x|^2)x \cdot \nabla u(x) = 0 \tag{5.1}$$

for $x \in B(0,1)$, with $u(x) = 0$ for $x \in \mathbb{R}^d \setminus B(0,1)$, and explicit solution

$$u(x) = \Phi_{k,\alpha}(x) = (1-|x|^2)_+^{k+\alpha/2}, \quad x \in \mathbb{R}^d.$$

The random tree associated to (5.1) starts at the point $x \in B(0,1)$, and branches into 0 or 1 branch as in the random tree samples of Figure 3.

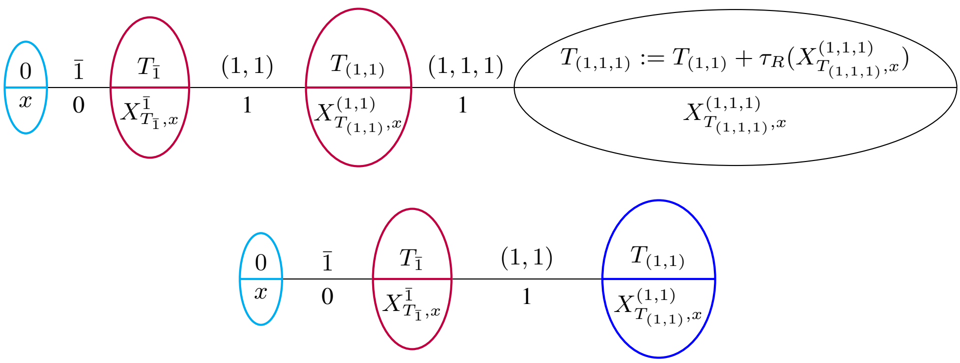
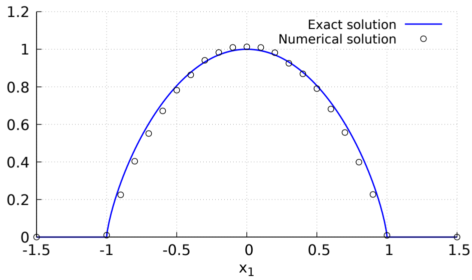
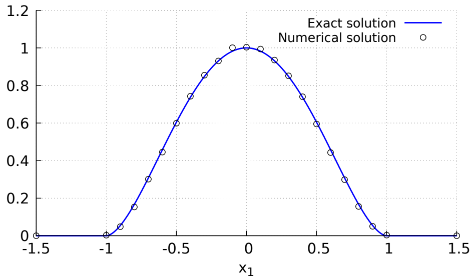


FIGURE 3. Random tree samples for the PDE (5.1).

The simulations of Figures 4(a) and 4(b) use respectively 10^7 and 2×10^7 Monte Carlo samples.



(a) Numerical solution of (5.1) with $k = 0$.



(b) Numerical solution of (5.1) with $k = 1$.

FIGURE 4. Numerical solution of (5.1) in dimension $d = 10$ with $\alpha = 1.75$.

5.2. Nonlinear gradient term. In this example we take $\mathcal{L}_1 = \{(0,0),(0,2)\}$,

$$c_{(0,0)}(x) := \Psi_{k,\alpha}(x) + (2k+\alpha)^2|x|^4(1-|x|^2)^{2k+\alpha},$$
$$c_{(0,2)}(x) := -1, \qquad b_1(x) := (1-|x|^2)x,$$

and consider the PDE with nonlinear gradient term

(5.2)
$$\Delta_\alpha u(x) + \Psi_{k,\alpha}(x) + (2k+\alpha)^2|x|^4(1-|x|^2)^{2k+\alpha} - ((1-|x|^2)x \cdot \nabla u(x))^2 = 0$$

for $x \in B(0,1)$, with $u(x) = 0$ for $x \in \mathbb{R}^d \setminus B(0,R)$, and explicit solution

$$u(x) = \Phi_{k,\alpha}(x) = (1-|x|^2)_+^{k+\alpha/2}, \qquad x \in \mathbb{R}^d.$$

The random tree associated to (5.2) starts at a point $x \in B(0,1)$ and branches into 0 or 2 branches as in the random tree sample of Figure 5.

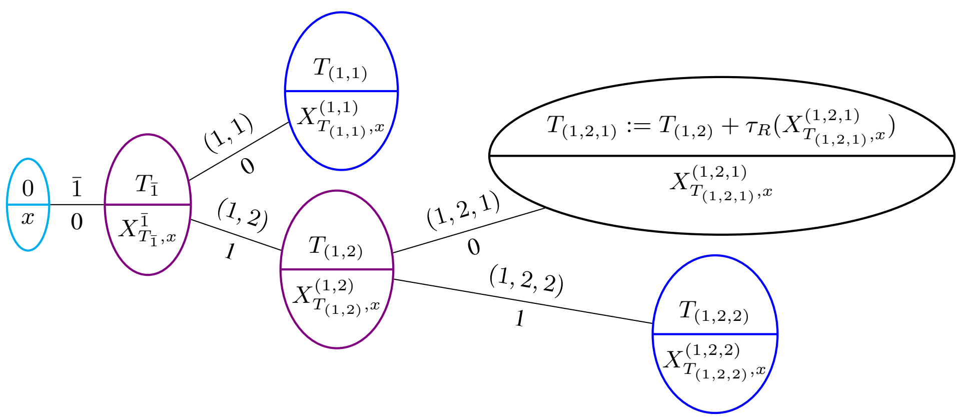
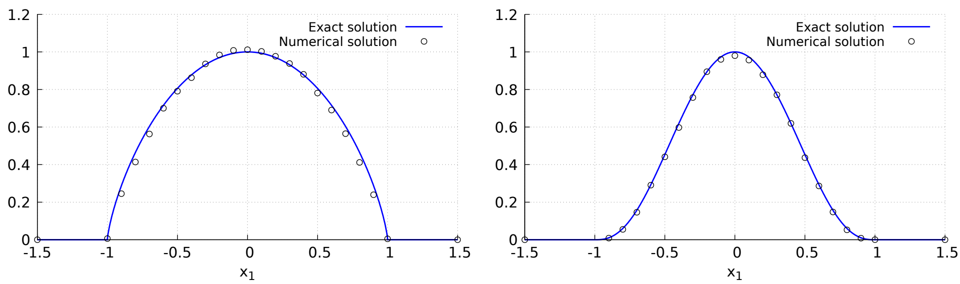


FIGURE 5. Random tree sample for the PDE (5.2).

The simulations of Figure 6 use five million Monte Carlo samples.



(a) Numerical solution of (5.2) with $k = 0$. (b) Numerical solution of (5.2) with $k = 2$.

FIGURE 6. Numerical solution of (5.2) in dimension $d = 10$ with $\alpha = 1.75$.

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Guillaume Penent, Nicolas Privault
Division of Mathematical Sciences
School of Physical and Mathematical Sciences
Nanyang Technological University
Singapore 637371
E-mail: pene0001@e.ntu.edu.sg
nprivault@ntu.edu.sg

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