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ASYMPTOTIC BEHAVIOR OF INTERMEDIATE ORDER STATISTICS AND RECORD VALUES FOR FOLDED DISTRIBUTIONS

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Abstract. The asymptotic behavior of intermediate order statistics and record values for folded distributions is studied. The relation between the weak convergence of a distribution function F_X and of its folded distribution function, $F_{|X|}$, is revealed in the cases of intermediate order statistics and record values. Additionally, a few illustrative examples are provided.

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1. INTRODUCTION

Measurements are frequently recorded without their algebraic sign. As a consequence, the underlying distribution of measurements is replaced by a distribution of absolute measurements, which results in what is known as the folded distribution. Hence a folded distribution arises when deviations are measured and only the magnitude is recorded. The directions (signs) of the deviations are considered unimportant and ignored in this case (see, [16, 18, 22]). For instance, in studying material strength, the force required to fracture a material may be directional (positive or negative), but only its absolute value is relevant in actually causing a fracture. In X-ray crystallography, only the magnitude of radiation penetration can be measured, even though the underlying physical process of diffraction involves phase (signed) information. Without exaggeration, we can say that a wide range of disciplines, including finance, economics, risk management, and tolerance design heavily utilize folded distributions (see [21]).

Folded distributions are commonly referred to as half-normal, half-Student, half-Cauchy distributions, etc. These distributions only retain (the positive) half of the distribution, as the name implies. The fact that these "half" distributions are centered at zero before folding is implied by their name. Naturally, though, a distribution that is not centered at zero can still be folded. This is the definition of a folded Student-t distribution, for example. A recent publication by Barakat et al. [11] studies other folded distributions.

Switching between signed distributions, including symmetric distributions, and their unsigned (folded) equivalents and seeing how the accompanying order statistics (OSs) and record values relate to one another is helpful in a variety of real-world scenarios. Mutangi and Matarise [20], for instance, studied the asymptotic maximum in the scenario of $Y = |X|$, when X has a normal distribution (e.g., a parent distribution function (DF)), based on the generated folded random variable (RV) Y. They provided a mechanism for figuring out norming constants or the folded normal distribution's maxima, and they showed that the Gumbel distribution is the limiting distribution of the linearly normalized maxima from the generated folded normal distribution. The relationship between the asymptotic theory of extremes, the folded normal distribution, and normal distribution has been utilized. This discovery is of significance for modeling extreme events, especially in time series where the identification of outliers is imperative. Recently, Barakat et al. [11] studied the asymptotic maxima for the folded RV $Y = |X|$, where X has an arbitrary continuous DF for which the asymptotic behavior of the extremes is known.

Apart from the X-folding technique as a potential transformation of X, a comparable problem was examined for other generated distributions, or generalized families of DFs (e.g., Y might be created by incorporating one or more shape parameters into the DF of X). For instance, the relationship between the weak convergence of the parent distribution of X and the generated family (the DF of Y) is revealed in the study conducted by Barakat et al. [9] on the asymptotic behavior of the OSs and record values based on the gamma and Kumaraswamy-generated distribution families. The class of beta-generated distributions (see [8]) and the Marshall–Olkin parameterization operation (see [6]) are two more distributions that have been studied.

The main object of this paper is to extend the results of [11] to the intermediate and record values models. There is a resemblance between these two models of ordered RVs because they have the same possible limit distributions. In the next section, we will shed some light on these two models. Throughout this paper, the symbols "$\underset{n}{\rightarrow}$" and "$\xrightarrow[n]{w}$" stand for convergence and weak convergence, as $n \to \infty$, respectively.

2. AUXILIARY RESULTS

In this section, we will discuss the intermediate OSs and record values models in more detail and highlight some of the requirements for this study.

2.1. Limit distributions of intermediate OSs. Let $X_1, \ldots, X_n$ be mutually independent RVs with a common continuous DF $F(x) = P(X \leqslant x)$. Let

$X_{k_n:n} := X_{k:n}$ represent the kth smallest OSs. A rank sequence $\{k\}$ is called an *intermediate rank* and the sequence $\{X_k\}$ is called an *intermediate OS* if $k/n \underset{n}{\to} 0$ or 1. Intermediate OSs have many applications. Value-at-risk, for instance, is a valuable metric in risk management. Intermediate quantiles are helpful when analyzing low-frequency, high-severity losses in insurance and finance. In this context, Csörgő and Steinebach [14] applied intermediate OSs to estimate the adjustment coefficient in risk theory. On the other hand, estimating tail values like the intermediate quantile and the tail index can be done using intermediate OSs. In statistical theory, they can be used to estimate probabilities of future extreme observations and to estimate tail quantiles of the underlying distribution that are extremes relative to the available sample size. Many authors (see, e.g., [19, 25]) have found estimators that are based, in part, on intermediate OSs. The theory of asymptotic intermediate OSs belongs to the theory of limit OSs with variable rank k, where $k/n \underset{n}{\to} \lambda,\ 0 \leqslant \lambda \leqslant 1$. The theory of limit OSs with variable rank k has been the subject of study by various researchers, including [3, 4, 5, 13, 24, 26]. Smirnov [24] demonstrated that, for any non-decreasing variable rank k, there are constants $a_n > 0$ and b_n such that

$$P(X_{k:n} \leqslant a_n x + b_n) = F_{k:n}(a_n x + b_n) = I_{F_X(a_n x+b_n)}(k, n-k+1) \underset{n}{\overset{w}{\to}} G(x) \tag{2.1}$$

for some DF $G(x)$, where $I_x(\cdot,\cdot)$ is the beta DF, if and only if

$$\frac{nF_X(a_n x + b_n) - k}{\sqrt{k(1-k/n)}} \underset{n}{\to} \Lambda(x). \tag{2.2}$$

The function $\Lambda(x)$ is a non-decreasing right continuous and extended-real function satisfying $\lim_{x\to-\infty} \Lambda(x) = -\infty$, $\lim_{x\to\infty} \Lambda(x) = \infty$, and the limit in (2.1) will be $G(x) = \Phi(\Lambda(x))$, where Φ is the standard normal DF. The variable ranks are classified into intermediate (lower or upper) and central ranks. Here, we will review some theorems for the case of intermediate OSs.

THEOREM 2.1 (Lower intermediate OSs value theorem, [13]). *Let $k^* := k/n \underset{n}{\to} 0$. Furthermore, let the intermediate rank k satisfy Chibisov's condition* (*see* [13]) *that for some $0 < \alpha < 1,\ \ell > 0$, and $\nu \in \mathbb{R}$ we have*

$$\lim_{n\to\infty}(\sqrt{k_{n+z_n(\nu)}} - \sqrt{k_n}) = \frac{\alpha\nu\ell}{2}$$

for any sequence of integer numbers z_n for which $\frac{z_n(\nu)}{n^{1-\alpha/2}} \underset{n}{\to} \nu$. Then (2.2) *is reduced to*

$$V_n(a_n x + b_n) = \frac{nF_X(a_n x + b_n) - k}{\sqrt{k}} = \frac{F_X(a_n x + b_n) - k^*}{\sqrt{k^*/n}} \underset{n}{\to} V(x). \tag{2.3}$$

According to [13], *the only possibilities for* $V(x)$ *in* (2.3) *are* $V_{i,\beta}(x),\ i = 1, 2, 3,$ *such that*

(2.4)
$$\begin{aligned} &\textbf{\textit{Type I:}} \quad V_{1,\beta}(x) = \begin{cases} -\beta \log |x|, & x < 0, \\ \infty, & x \geqslant 0, \end{cases} \\ &\textbf{\textit{Type II:}} \quad V_{2,\beta}(x) = \begin{cases} \beta \log x, & x > 0, \\ -\infty, & x \leqslant 0, \end{cases} \\ &\textbf{\textit{Type III:}} \quad V_{3,\beta}(x) = V_3(x) = x, \quad \forall x, \end{aligned}$$

where $\beta > 0$ *is some positive constant depending only on the type of* $F_X(x)$ *and the values of* α *and* ℓ.

REMARK 2.1. Chibisov [13] noted that the Chibisov condition implies $k \sim \ell^2 n^\alpha$ as $n \to \infty$. Barakat and Omar [10] showed that the latter condition implies Chibisov's condition. The result of [10] reveals that the class of intermediate rank sequences that satisfy Chibisov's condition is very wide, and consequently the Chibisov limit types are widely applicable.

In what follows, we write $(a_{i,n}, b_{i,n}),\ a_{i,n} > 0$, for the normalizing constants that are utilized, where the subscript i equals 1, 2, or 3, depending on whether there is weak convergence to the type $G_{1,\beta}(x) = \Phi(V_{1,\beta}(x))$, $G_{2,\beta}(x) = \Phi(V_{2,\beta}(x))$, or $G_3(x) = \Phi(V_3(x))$, respectively. We will say that F is *attracted to the domain of attraction of* $G_{i,\beta}(x)$ and write $F_X \in \mathcal{D}_L(G_{i,\beta}(x))$, or $F_X(a_{i,n}x + b_{i,n}) \in \mathcal{D}_L(G_{i,\beta}(x))$.

LEMMA 2.1 (cf. [13]).

(1) *If* $F_X(a_{1,n}x + b_{1,n}) \in \mathcal{D}_L(G_{1,\beta}(x))$, *then* $a_{1,n} \underset{n}{\to} \infty$ *and* $b_{1,n} = 0$.

(2) *If* $F_X(a_{2,n}x + b_{2,n}) \in \mathcal{D}_L(G_{2,\beta}(x))$, *then* $a_{2,n} \underset{n}{\to} 0$ *and* $b_{2,n} = x_0$, $F_X(x_0) = 0$.

(3) $F_X(a_{3,n}x + b_{3,n}) \in \mathcal{D}_L(G_3(x))$ *if and only if* $b_{3,n}$ *is the smallest number for which* $F_X(b_{3,n} - 0) \leqslant k/n \leqslant F_X(b_{3,n})$, *and* $b_{3,n}$ *satisfies the condition*

$$\frac{b_{3,n+z_n} - b_{3,n}}{b_{3,n+z'_n} - b_{3,n}} \underset{n}{\to} \frac{\nu}{\mu}, \quad \textit{where} \quad \frac{z_n}{n^{1-\alpha/2}} \underset{n}{\to} \nu \textit{ and } \frac{z'_n}{n^{1-\alpha/2}} \underset{n}{\to} \mu.$$

In this case, $a_{3,n}$ *may be chosen as the smallest number satisfying*

$$F_X(a_{3,n} + b_{3,n} - 0) \leqslant \frac{k + \sqrt{k}}{n} \leqslant F_X(a_{3,n} + b_{3,n}).$$

The corresponding possible non-degenerate limiting distributions for the upper intermediate term $X_{k':n}$, where $k' = n - k + 1$, are $G'(x) = 1 - \Phi(V_i(-x;\ \beta))$, $i = 1, 2, 3$. Moreover, we get the limit $G'(x)$ if and only if

$$U_n(c_n x + d_n) = \frac{n\overline{F}_X(c_n x + d_n) - k}{\sqrt{k}} = \frac{\overline{F}_X(c_n x + d_n) - k^*}{\sqrt{k^*/n}} \underset{n}{\to} U(x),$$

where $k^* = k/n$, $\overline{F}_X(x) = 1 - F_X(x)$, and $U(x)$ is defined by the equation

$$G'(x) = \Phi(-U(x)) = 1 - \Phi(U(x)).$$

THEOREM 2.2. *The proper limit distributions for the variables $X_{k';n}$ when $n \to \infty$, $k' = n - k + 1$, and k satisfies the Chibisov condition can be only one of the three types:*

$$\begin{aligned} &\textbf{\textit{Type I:}} \quad U_{1,\beta'}(x) = \begin{cases} -\beta' \log x, & x > 0, \\ \infty, & x \leqslant 0, \end{cases} \\ &\textbf{\textit{Type II:}} \quad U_{2,\beta'}(x) = \begin{cases} \beta' \log |x|, & x < 0, \\ -\infty, & x \geqslant 0, \end{cases} \\ &\textbf{\textit{Type III:}} \quad U_{3,\beta'}(x) = U_3(x) = -x, \quad \forall x, \end{aligned} \tag{2.5}$$

where $\beta' > 0$ is some positive constant depending only on the type of $F_X(x)$ and the values of α and ℓ.

In the following, we write $(c_{i,n}, d_{i,n}),\ c_{i,n} > 0$, for the normalizing constants that are utilized, where the subscript i equals 1, 2, or 3, and it depends on whether there is weak convergence to the type $G'_{1,\beta'}(x)$, $G'_{2,\beta'}(x)$, or $G'_3(x)$, respectively. We will say that $F_X(x)$ is *attracted to the domain of* $G'_{i,\beta'}(x)$ and write $F_X(x) \in \mathcal{D}_U(G'_{i,\beta'}(x))$, or $F_X(c_{i,n}x + d_{i,n}) \in \mathcal{D}_U(G'_{i,\beta'}(x))$.

LEMMA 2.2.

(1) *If $F_X(c_{1,n}x + d_{1,n}) \in \mathcal{D}_U(G'_{1,\beta'}(x))$, then $c_{1,n} \underset{n}{\to} \infty$ and $d_{1,n} = 0$.*

(2) *If $F_X(c_{2,n}x + d_{2,n}) \in \mathcal{D}_U(G'_{2,\beta'}(x))$, then $c_{2,n} \underset{n}{\to} 0$ and $d_{2,n} = x^0$, $\overline{F}_X(x^0) = 0$.*

(3) *$F_X(c_{3,n}x + d_{3,n}) \in \mathcal{D}_U(G'_3(x))$ if and only if $d_{3,n}$ is the smallest number for which $\overline{F}_X(d_{3,n} + 0) \leqslant k/n \leqslant \overline{F}_X(d_{3,n})$, and $d_{3,n}$ satisfies*

$$\frac{d_{3,n+r_n(\nu)} - d_{3,n}}{d_{3,n+r_n(\mu)} - d_{3,n}} \underset{n}{\to} \frac{\nu}{\mu}, \quad \text{where} \quad \frac{r_n(\nu)}{n^{1-\alpha/2}} \underset{n}{\to} \nu \text{ and } \frac{r_n(\mu)}{n^{1-\alpha/2}} \underset{n}{\to} \mu.$$

2.2. Some required results on record values. Record values naturally occur in many real-world scenarios, and only those may be recorded in certain circumstances, such as those involving hydrology, meteorology, and sports and athletic events. Chandler [12] is credited with introducing the concept of record values to model data of extreme weather situations. The record value model can also be used in other contexts, such as reliability theory, the greatest water level or temperature, and progressively larger insurance claims in non-life insurance. For more details about this model, see [1, 2, 7].

Let $X_1, X_2, \ldots$ be an infinite sequence of independent identically distributed (i.i.d.) RVs. If $X_j > X_i$ for each $i < j$, then an observation X_j is considered an upper record value; a similar definition applies to lower record values. X_1 is typically an upper record value as well as a lower record value. The upper record value sequence R_n can be characterized by $R_n = X_{N_n}$, where $N_n = \min\{j : j > N_{n-1}, X_j > X_{N_{n-1}}\}$ (note that $N_1 = 1$) is the upper record time sequence. The DF of the upper record value can be expressed in terms of the function $U_F(x) = -\log \overline{F}_X(x) = -\log(1 - F_X(x))$. Thus, we obtain the following general results.

LEMMA 2.3 (cf. [15]). *A DF $F_X(x)$ is said to belong to the domain of upper record value attraction of a non-degenerate DF $\Psi(x)$, written $F_X(x) \in \mathcal{D}_{\text{urec}}(\Psi(x))$, if there exist normalizing constants $A_n > 0$ and B_n such that*

$$P(R_n \leqslant A_n x + B_n) = F_{R_n}(A_n x + B_n) \underset{n}{\overset{w}{\to}} \Psi(x). \tag{2.6}$$

Condition (2.6) *is satisfied if and only if*

$$\frac{U_F(A_n x + b_n) - n}{\sqrt{n}} \underset{n}{\to} \Phi^{-1}(\Psi(x)) = \mathcal{R}(x). \tag{2.7}$$

The following result, dating back to Resnick [23], establishes different forms of the limit $\Psi(x)$, as well as a condition under which (2.6) is obtained. Additionally, three possible distribution types may appear as limiting distributions of suitable normalized record value, which are $\Psi_{i,\beta}(x) = \Phi(\mathcal{R}_{i,\beta}(x))$, $i = 1, 2, 3$, where $\Psi_{3,\beta}(x) = \Psi_3(x)$ and

$$\begin{aligned} &\textbf{Type I:} && \mathcal{R}_1(x,\beta) = \begin{cases} -\infty, & x \leqslant 0, \\ \beta \log x, & x > 0, \beta > 0, \end{cases} \\ &\textbf{Type II:} && \mathcal{R}_2(x,\beta) = \begin{cases} -\beta \log(-x), & x \leqslant 0, \beta > 0, \\ \infty, & x > 0, \end{cases} \\ &\textbf{Type III:} && \mathcal{R}_3(x,\beta) = \mathcal{R}_3(x) = x, \quad -\infty < x < \infty. \end{aligned} \tag{2.8}$$

REMARK 2.2. Sometimes we write $(A_{i,n}, B_{i,n}), A_{i,n} > 0$, for the normalizing constants, where the subscript i equals 1, 2, or 3, according to the weak convergence to the type $\Psi_{1,\beta}(x)$, $\Psi_{2,\beta}(x)$, or $\Psi_3(x)$, respectively. Moreover, when focusing on some specific normalizing constants $A_{i,n} > 0$ and $B_{i,n}$, we use the notation $F_X(A_{i,n}x + B_{i,n}) \in \mathcal{D}_{\text{urec}}(\Psi_{i,\beta}(x))$ instead of $F_X \in \mathcal{D}_{\text{urec}}(\Psi_{i,\beta}(x))$.

LEMMA 2.4 (cf. [2]).

(1) *If $F_X(A_{1,n}x + B_{1,n}) \in \mathcal{D}_{\text{urec}}(\Psi_{1,\beta}(x))$, then $A_{1,n} = F_X^{-1}(1 - e^{-n})$ and $B_{1,n} = 0$.*

(2) *If $F_X(A_{2,n}x + B_{2,n}) \in \mathcal{D}_{\text{urec}}(\Psi_{2,\beta}(x))$, then $A_{2,n} = F_X^{-1}(1) - F_X^{-1}(1 - e^{-n})$ and $B_{2,n} = F_X^{-1}(1)$, where $F_X(x^0) = 1$.*

(3) *If $F_X(A_{3,n}x + B_{3,n}) \in \mathcal{D}_{\text{urec}}(\Psi_3(x))$, then $A_{3,n} = F_X^{-1}(1 - e^{-n-\sqrt{n}}) - F_X^{-1}(1 - e^{-n})$ and $B_{3,n} = F_X^{-1}(1 - e^{-n})$.*

3. ASYMPTOTIC BEHAVIOR OF THE INTERMEDIATE OSs OF A FOLDED DISTRIBUTION

In this section, we study the asymptotic behavior of the intermediate OSs of a folded distribution. Here, we begin with a well-known basic lemma.

LEMMA 3.1. *Let $F_X(x)$ be the DF of the RV X with support $\{x : -\infty \leqslant x_0 \leqslant x \leqslant x^0 \leqslant \infty\}$. Then*

$$F_{|X|}(x) = P(|X| \leqslant x) = F_X(x) - F_X(-x).$$

THEOREM 3.1. *Let $-\infty \leqslant x_0 < 0 \leqslant x^0 \leqslant \infty$. Then we have the following implications:*

(1) *Let $F_X(a_{1,n}x) \in \mathcal{D}_L(G_{1,\beta}(x))$ and $F_X(c_{1,n}x) \in \mathcal{D}_U(G'_{1,\beta'}(x))$. Then*

$$F_{|X|}(c_{1,n}x) \in \mathcal{D}_U(G'_{1,\beta'}(x)) \quad \text{if } 0 \leqslant \lim_{n\to\infty} \frac{F_X(-c_{1,n}x)}{\sqrt{k^*/n}} < \infty,$$

$$F_{|X|}(a_{1,n}x) \in \mathcal{D}_U(G'_{1,\beta}(x)) \quad \text{if } 0 \leqslant \lim_{n\to\infty} \frac{\overline{F}_X(a_{1,n}x)}{\sqrt{k^*/n}} < \infty.$$

(2) *Let $F_X(a_{2,n}x+b_{2,n}) \in \mathcal{D}_L(G_{2,\beta}(x))$ and $F_X(c_{2,n}x+d_{2,n}) \in \mathcal{D}_U(G'_{2,\beta'}(x))$. Then*

(i) $F_{|X|}(c_{2,n}x + d_{2,n}) \in \mathcal{D}_U(G'_{2,\beta'}(x))$

$$\text{if } |x^0| > |x_0| \text{ and } 0 \leqslant \lim_{n\to\infty} \frac{F_X(-c_{2,n}x - x^0)}{\sqrt{k^*/n}} < \infty;$$

(ii) $F_{|X|}(a_{2,n}x - b_{2,n}) \in \mathcal{D}_U(G'_{2,\beta}(x))$

$$\text{if } |x_0| > |x^0| \text{ and } 0 \leqslant \lim_{n\to\infty} \frac{\overline{F}_X(a_{2,n}x - x_0)}{\sqrt{k^*/n}} < \infty.$$

(3) *Let $F_X(a_{3,n}x + b_{3,n}) \in \mathcal{D}_L(G_3(x))$ and $F_X(c_{3,n}x + d_{3,n}) \in \mathcal{D}_U(G'_3(x))$. Then*

(i) $F_{|X|}(c_{3,n}x + d_{3,n}) \in \mathcal{D}_U(G'_3(x)) \quad$ *if* $0 \leqslant \lim_{n\to\infty} \frac{F_X(-(c_{3,n}x+d_{3,n}))}{\sqrt{k^*/n}} < \infty$;

(ii) $F_{|X|}(a_{3,n}x - b_{3,n}) \in \mathcal{D}_U(G'_3(x)) \quad$ *if* $0 \leqslant \lim_{n\to\infty} \frac{\overline{F}_X(a_{3,n}x - b_{3,n})}{\sqrt{k^*/n}} < \infty$.

(4) (i) *Let $F_X(a_{2,n}x + b_{2,n}) \in \mathcal{D}_L(G_{2,\beta}(x))$ and $F_X(c_{1,n}x) \in \mathcal{D}_U(G'_{1,\beta'}(x))$. Then*

$$F_{|X|}(c_{1,n}x) \in \mathcal{D}_U(G'_{1,\beta'}(x)) \qquad \text{if } 0 \leqslant \lim_{n\to\infty} \frac{F_X(-c_{1,n}x)}{\sqrt{k^*/n}} < \infty,$$

$$F_{|X|}(a_{2,n}x - b_{2,n}) \in \mathcal{D}_U(G'_{2,\beta}(x)) \quad \text{if } 0 \leqslant \lim_{n\to\infty} \frac{\overline{F}_X(a_{2,n}x - x_0)}{\sqrt{k^*/n}} < \infty.$$

(ii) *Let* $F_X(a_{1,n}x) \in \mathcal{D}_L(G_{1,\beta}(x))$ *and* $F_X(c_{2,n}x + d_{2,n}) \in \mathcal{D}_U(G'_{2,\beta'}(x))$. *Then*

$$F_{|X|}(c_{2,n}x + d_{2,n}) \in \mathcal{D}_U(G'_{2,\beta'}(x)) \quad \textit{if } 0 \leqslant \lim_{n\to\infty} \frac{F_X(-c_{2,n}x - x^0)}{\sqrt{k^*/n}} < \infty,$$

$$F_{|X|}(a_{1,n}x) \in \mathcal{D}_U(G'_{1,\beta}(x)) \quad \textit{if } 0 \leqslant \lim_{n\to\infty} \frac{\overline{F}_X(a_{1,n}x)}{\sqrt{k^*/n}} < \infty.$$

(5) (i) *Let* $F_X(a_{1,n}x) \in \mathcal{D}_L(G_{1,\beta}(x))$ *and* $F_X(c_{3,n}x + d_{3,n}) \in \mathcal{D}_U(G'_3(x))$. *Then*

$$F_{|X|}(c_{3,n}x + d_{3,n}) \in \mathcal{D}_U(G'_3(x)) \quad \textit{if } 0 \leqslant \lim_{n\to\infty} \frac{F_X(-(c_{3,n}x + d_{3,n}))}{\sqrt{k^*/n}} < \infty,$$

$$F_{|X|}(a_{1,n}x) \in \mathcal{D}_U(G'_{1,\beta}(x)) \quad \textit{if } 0 \leqslant \lim_{n\to\infty} \frac{\overline{F}_X(a_{1,n}x)}{\sqrt{k^*/n}} < \infty.$$

(ii) *Let* $F_X(a_{3,n}x + b_{3,n}) \in \mathcal{D}_L(G_3(x))$ *and* $F_X(c_{1,n}x) \in \mathcal{D}_U(G'_{1,\beta'}(x))$. *Then*

$$F_{|X|}(c_{1,n}x) \in \mathcal{D}_U(G'_{1,\beta'}(x)) \quad \textit{if } 0 \leqslant \lim_{n\to\infty} \frac{F_X(-c_{1,n}x)}{\sqrt{k^*/n}} < \infty,$$

$$F_{|X|}(a_{3,n}x - b_{3,n}) \in \mathcal{D}_U(G'_3(x)) \quad \textit{if } 0 \leqslant \lim_{n\to\infty} \frac{\overline{F}_X(a_{3,n}x - b_{3,n})}{\sqrt{k^*/n}} < \infty.$$

(6) (i) *Let* $F_X(a_{2,n}x + b_{2,n}) \in \mathcal{D}_L(G_{2,\beta}(x))$ *and* $F_X(c_{3,n}x + d_{3,n}) \in \mathcal{D}_U(G'_3(x))$. *Then*

$$F_{|X|}(c_{3,n}x + d_{3,n}) \in \mathcal{D}_U(G'_3(x)) \quad \textit{if } 0 \leqslant \lim_{n\to\infty} \frac{F_X(-(c_{3,n}x + d_{3,n}))}{\sqrt{k^*/n}} < \infty,$$

$$F_{|X|}(a_{2,n}x - b_{2,n}) \in \mathcal{D}_U(G'_{2,\beta}(x)) \quad \textit{if } |x_0| > |x^0| \textit{ and } 0 \leqslant \lim_{n\to\infty} \frac{\overline{F}_X(a_{2,n}x - x_0)}{\sqrt{k^*/n}} < \infty.$$

(ii) *Let* $F_X(a_{3,n}x + b_{3,n}) \in \mathcal{D}_L(G_3(x))$ *and* $F_X(c_{2,n}x + d_{2,n}) \in \mathcal{D}_U(G'_{2,\beta'}(x))$. *Then*

$$F_{|X|}(c_{2,n}x + d_{2,n}) \in \mathcal{D}_U(G'_{2,\beta'}(x)) \quad \textit{if } |x^0| > |x_0| \textit{ and } 0 \leqslant \lim_{n\to\infty} \frac{F_X(-c_{2,n}x - x^0)}{\sqrt{k^*/n}} < \infty,$$

$$F_{|X|}(a_{3,n}x-b_{3,n})\in\mathcal{D}_U(G'_3(x)) \qquad \textit{if } 0\leqslant \lim_{n\to\infty}\frac{\overline{F}_X(a_{3,n}x-b_{3,n})}{\sqrt{k^*/n}}<\infty.$$

Proof. We prove only the first three statements since the proofs of the other statements are similar.

In view of Lemma 3.1 we get $\overline{F}_{|X|}(x) = \overline{F}_X(x) + F_X(-x)$. Therefore, for $i = 1, 2, 3$,

$$\frac{\overline{F}_{|X|}(c_{i,n}x+d_{i,n})-k^*}{\sqrt{k^*/n}} = \frac{\overline{F}_X(c_{i,n}x+d_{i,n})+F_X(-c_{i,n}x-d_{i,n})-k^*}{\sqrt{k^*/n}}. \tag{3.1}$$

Put $i = 1$ in (3.1). In view of (2.5) and Lemma 2.2, we get

$$\begin{aligned}\frac{\overline{F}_{|X|}(c_{1,n}x)-k^*}{\sqrt{k^*/n}} &= \frac{\overline{F}_X(c_{1,n}x)-k^*}{\sqrt{k^*/n}} + \frac{F_X(-c_{1,n}x)}{\sqrt{k^*/n}} \\ &\underset{n}{\to} U_{1,\beta'}(x) + \begin{cases}\lim\limits_{n\to\infty}\dfrac{F_X(-c_{1,n}x)}{\sqrt{k^*/n}} & \text{if } x>0,\\ \infty & \text{if } x\leqslant 0,\end{cases}\end{aligned}$$

where the possible limit points of the sequence $\frac{F_X(-c_{1,n}x)}{\sqrt{k^*/n}}$ are in $\{0, A, \infty\}$, where A is a positive finite constant. However, the only limit points which give a non-degenerate limit for $\frac{\overline{F}_{|X|}(c_{1,n}x)-k^*}{\sqrt{k^*/n}}$ are $\{0, A\}$, while in the case of $\frac{F_X(-c_{1,n}x)}{\sqrt{k^*/n}} \underset{n}{\to} 0$, we get

$$\frac{\overline{F}_{|X|}(c_{1,n}x)-k^*}{\sqrt{k^*/n}} \underset{n}{\to} \begin{cases}-\beta'\log x, & x>0,\\ \infty, & x\leqslant 0.\end{cases}$$

On the other hand, if $\frac{F_X(-c_{1,n}x)}{\sqrt{k^*/n}} \underset{n}{\to} A$, we get

$$\frac{\overline{F}_{|X|}(c_{1,n}x)-k^*}{\sqrt{k^*/n}} \underset{n}{\to} \begin{cases}-\beta'\log x + A, & x>0,\\ \infty, & x\leqslant 0.\end{cases}$$

Hence,

$$F_{|X|}(c_{1,n}x) \in \begin{cases}\mathcal{D}_U(G'_{1,\beta'}(x)) & \text{if } \lim\limits_{n\to\infty}\dfrac{F_X(-c_{1,n}x)}{\sqrt{k^*/n}} = 0,\\[2ex] \mathcal{D}_U\big(G'_{1,\beta'}\big(e^{-\frac{1}{\beta'}A}x\big)\big) & \text{if } \lim\limits_{n\to\infty}\dfrac{F_X(-c_{1,n}x)}{\sqrt{k^*/n}} = A.\end{cases}$$

On the other hand, due to (3.1) and Lemma 2.1 with normalizing constants $a_{1,n}$ and $b_{1,n}$, we get

$$\frac{\overline{F}_{|X|}(a_{1,n}x)-k^*}{\sqrt{\frac{k^*}{n}}}=\frac{F_X(a_{1,n}(-x))-k^*}{\sqrt{k^*/n}}+\frac{\overline{F}_X(a_{1,n}x)}{\sqrt{k^*/n}}$$

$$\underset{n}{\rightarrow} V_{1,\beta}(-x)+\begin{cases}\lim\limits_{n\to\infty}\dfrac{\overline{F}_X(a_{1,n}x)}{\sqrt{k^*/n}}, & x>0,\\ \infty, & x<0,\end{cases}$$

where the possible limit points of $\frac{\overline{F}_X(a_{1,n}x)}{\sqrt{k^*/n}}$ are in $\{0,A',\infty\}$, where A' is a positive finite constant. However, the only limit points which give a non-degenerate limit for $\frac{\overline{F}_{|X|}(a_{1,n}x)-k^*}{\sqrt{k^*/n}}$ are $\{0,A'\}$. If $\frac{\overline{F}_X(a_{1,n}x)}{\sqrt{k^*/n}}\underset{n}{\rightarrow}0$, then due to (2.4) we get

$$\frac{\overline{F}_{|X|}(a_{1,n}x)-k^*}{\sqrt{k^*/n}}\underset{n}{\rightarrow}\begin{cases}-\beta\log x, & x>0,\\ \infty, & x\leqslant 0.\end{cases}$$

If $\frac{\overline{F}_X(a_{1,n}x)}{\sqrt{k^*/n}}\underset{n}{\rightarrow}A'$, then due to (2.4), we get

$$\frac{\overline{F}_{|X|}(a_{1,n}x)-k^*}{\sqrt{k^*/n}}\underset{n}{\rightarrow}\begin{cases}-\beta\log x+A', & x>0,\\ \infty, & x\leqslant 0.\end{cases}$$

Hence,

$$F_{|X|}(a_{1,n}x)\in\begin{cases}\mathcal{D}_U(G'_{1,\beta}(x)) & \text{if } \lim\limits_{n\to\infty}\dfrac{\overline{F}_X(a_{1,n}x)}{\sqrt{k^*/n}}=0,\\ \mathcal{D}_U\big(G'_{1,\beta}\big(e^{-\frac{1}{\beta}A'}\ x\big)\big) & \text{if } \lim\limits_{n\to\infty}\dfrac{\overline{F}_X(a_{1,n}x)}{\sqrt{k^*/n}}=A'.\end{cases}$$

This completes the proof of part (1) of Theorem 3.1.

Turning to part (2)(i), from Lemma 2.2 we have $d_{2,n}=x^0<\infty$. Let $|x^0|>|x_0|$ and put $i=2$ in (3.1). We get

$$\frac{\overline{F}_{|X}(c_{2,n}x+d_{2,n})-k^*}{\sqrt{k^*/n}}=\frac{\overline{F}_X(c_{2,n}x+d_{2,n})-k^*}{\sqrt{k^*/n}}+\frac{F_X(-c_{2,n}x-x^0)}{\sqrt{k^*/n}}.$$

In view of (2.5) and Lemma 2.2, we get

$$\frac{\overline{F}_{|X|}(c_{2,n}x+d_{2,n})-k^*}{\sqrt{k^*/n}}\underset{n}{\rightarrow}U_{2,\beta'}(x)+\lim_{n\to\infty}\frac{F_X(-c_{2,n}x-x^0)}{\sqrt{k^*/n}},$$

where the possible limit points of $\frac{F_X(-c_{2,n}x-x^0)}{\sqrt{k^*/n}}$ are in $\{0,B,\infty\}$, where B is a finite constant. However, the only limit points which give a non-degenerate limit for $\frac{\overline{F}_{|X|}(c_{2,n}x+d_{2,n})-k^*}{\sqrt{k^*/n}}$ are in $\{0, B\}$. If $\frac{F_X(-c_{2,n}x-x^0)}{\sqrt{k^*/n}} \underset{n}{\rightarrow} 0$, then in view of (2.5), we get

$$\frac{\overline{F}_{|X|}(c_{2,n}x+d_{2,n})-k^*}{\sqrt{k^*/n}} \underset{n}{\rightarrow} \begin{cases} \beta' \log|x|, & x<0, \\ -\infty, & x \geqslant 0. \end{cases}$$

If $\frac{F_X(-c_{2,n}x-x^0)}{\sqrt{k^*/n}} \underset{n}{\rightarrow} B$, then in view of (2.5), we get

$$\frac{\overline{F}_{|X|}(c_{2,n}x+d_{2,n})-k^*}{\sqrt{k^*/n}} \underset{n}{\rightarrow} \begin{cases} \beta' \log|x|+B, & x<0, \\ -\infty, & x \geqslant 0. \end{cases}$$

Hence,

$$F_{|X|}(c_{2,n}x+d_{2,n}) \in \begin{cases} \mathcal{D}_U(G'_{2,\beta'}(x)) & \text{if } \lim\limits_{n\to\infty} \frac{F_X(-c_{2,n}x-x^0)}{\sqrt{k^*/n}} = 0, \\ \mathcal{D}_U\big(G'_{2,\beta'}\big(e^{\frac{1}{\beta'}B}x\big)\big) & \text{if } \lim\limits_{n\to\infty} \frac{F_X(-c_{2,n}x-x^0)}{\sqrt{k^*/n}} = B. \end{cases}$$

Thus, part (2)(i) of Theorem 3.1 is proved.

Now we prove part (2)(ii). Let $|x_0| > |x^0|$. Consider (3.1) with normalizing constants $a_{2,n}$ and $-b_{2,n}$. Using Lemma 2.1, we get

$$\frac{\overline{F}_{|X|}(a_{2,n}x-b_{2,n})-k^*}{\sqrt{k^*/n}} = \frac{F_X(a_{2,n}(-x)+b_{2,n})-k^*}{\sqrt{k^*/n}} + \frac{\overline{F}_X(a_{2,n}x-x_0)}{\sqrt{k^*/n}}.$$

Due to (2.4) and Lemma 2.1, we have

$$\frac{\overline{F}_{|X|}(a_{2,n}x-b_{2,n})-k^*}{\sqrt{k^*/n}} \underset{n}{\rightarrow} V_{2,\beta}(-x) + \lim_{n\to\infty} \frac{\overline{F}_X(a_{2,n}x-x_0)}{\sqrt{k^*/n}},$$

where the possible limit points of $\frac{\overline{F}_X(a_{2,n}x-x_0)}{\sqrt{k^*/n}}$ are in $\{0, B', \infty\}$, where B' is a finite constant. However, the only limit points which give a non-degenerate limit for $\frac{\overline{F}_{|X|}(a_{2,n}x-b_{2,n})-k^*}{\sqrt{k^*/n}}$ are in $\{0, B'\}$. If $\frac{\overline{F}_X(a_{2,n}x-x_0)}{\sqrt{k^*/n}} \underset{n}{\rightarrow} 0$, then in view of (2.4), we have

$$\frac{\overline{F}_{|X|}(a_{2,n}x-b_{2,n})-k^*}{\sqrt{k^*/n}} \underset{n}{\rightarrow} \begin{cases} \beta \log|x|, & x<0, \\ -\infty, & x \geqslant 0. \end{cases}$$

If $\frac{\overline{F}_X(a_{2,n}x-x_0)}{\sqrt{k^*/n}} \underset{n}{\rightarrow} B'$, then in view of (2.4), we have

$$\frac{\overline{F}_{|X|}(a_{2,n}x-b_{2,n})-k^*}{\sqrt{k^*/n}} \underset{n}{\rightarrow} \begin{cases} \beta \log|x|+B', & x<0, \\ -\infty, & x \geqslant 0. \end{cases}$$

Hence,

$$F_{|X|}(a_{2,n}x - b_{2,n}) \in \begin{cases} \mathcal{D}_U(G'_{2,\beta}(x)) & \text{if } \lim\limits_{n\to\infty} \dfrac{\overline{F}_X(a_{2,n}x - x_0)}{\sqrt{k^*/n}} = 0, \\ \mathcal{D}_U\big(G'_{2,\beta}\big(e^{\frac{1}{\beta}B'}x\big)\big) & \text{if } \lim\limits_{n\to\infty} \dfrac{\overline{F}_X(a_{2,n}x - x_0)}{\sqrt{k^*/n}} = B'. \end{cases}$$

This completes the proof of part (2)(ii) of Theorem 3.1.

Now, we prove part (3)(i). Putting $i = 3$ in (3.1), we have

$$\frac{\overline{F}_{|X|}(c_{3,n}x + d_{3,n}) - k^*}{\sqrt{k^*/n}} = \frac{\overline{F}_X(c_{3,n}x + d_{3,n}) - k^*}{\sqrt{k^*/n}} + \frac{F_X(-c_{3,n}x - d_{3,n})}{\sqrt{k^*/n}}.$$

If $F_X(-c_{3,n}x - d_{3,n}) \underset{n}{\to} 0$ then due to (2.5), we get

$$\frac{\overline{F}_{|X|}(c_{3,n}x + d_{3,n}) - k^*}{\sqrt{k^*/n}} \underset{n}{\to} U_3(x) + \lim_{n\to\infty} \frac{F_X(-c_{3,n}x - d_{3,n})}{\sqrt{k^*/n}},$$

where the the possible limit points of $\frac{F_X(-c_{3,n}x - d_{3,n})}{\sqrt{k^*/n}}$ are in $\{C, \infty\}$, where $C \geqslant 0$ is a finite constant. The only limit point that gives a non-degenerate limit for $\frac{\overline{F}_{|X|}(c_{3,n}x + d_{3,n}) - k^*}{\sqrt{k^*/n}}$ is C. Then

$$\frac{\overline{F}_{|X|}(c_{3,n}x + d_{3,n}) - k^*}{\sqrt{k^*/n}} \underset{n}{\to} -(x - C).$$

Hence, $F_{|X|}(c_{3,n}x + d_{3,n}) \in \mathcal{D}_U(G'_3(x - C))$. This proves part (3)(i) of Theorem 3.1.

Now, we turn to part (3)(ii). Considering (3.1) with normalizing constants $a_{3,n}$ and $-b_{3,n}$, we have

$$\frac{\overline{F}_{|X|}(a_{3,n}x - b_{3,n}) - k^*}{\sqrt{k^*/n}} = \frac{F_X(a_{3,n}(-x) + b_{3,n}) - k^*}{\sqrt{k^*/n}} + \frac{\overline{F}_X(a_{3,n}x - b_{3,n})}{\sqrt{k^*/n}}.$$

Assuming that $\overline{F}_X(a_{3,n}x - b_{3,n}) \underset{n}{\to} 0$, due to (2.4) we have

$$\frac{\overline{F}_{|X|}(a_{3,n}x - b_{3,n}) - k^*}{\sqrt{k^*/n}} \underset{n}{\to} V_3(-x) + \lim_{n\to\infty} \frac{F_X(a_{3,n}x - b_{3,n})}{\sqrt{k^*/n}}.$$

The possible limit points of $\frac{F_X(a_{3,n}x - b_{3,n})}{\sqrt{k^*/n}}$ are in $\{C', \infty\}$, where $C' \geqslant 0$ is a finite constant. The only limit point that gives a non-degenerate limit for

$\frac{\overline{F}_{|X|}(a_{3,n}x-b_{3,n})-k^*}{\sqrt{k^*/n}}$ is C'. Then

$$\frac{\overline{F}_{|X|}(a_{3,n}x-b_{3,n})-k^*}{\sqrt{k^*/n}} \underset{n}{\rightarrow} -(x-C').$$

Hence, $F_{|X|}(a_{3,n}x-b_{3,n}) \in \mathcal{D}_U(G_3'(x-C'))$. This completes the proof of part (3)(ii) of Theorem 3.1. ∎

REMARK 3.1. It can be shown that the sufficient conditions in parts (1)–(3) are also necessary by noting that the proof steps in those parts can be reversed. Moreover, the conditions given in all the parts of the theorem guarantee that the folded distribution belongs to the upper intermediate domain of attraction of the same upper, or lower, intermediate limit type to which the distribution itself belongs with the same normalizing constants, $c_{i,n}$ and $d_{i,n}, i=1,2,3$, or $a_{i,n}$ and $b_{i,n},\ i=1,2,3$. However, if these conditions are not met, the folded distribution might be of the same upper limit type but with different normalizing constants $\alpha_{i,n}$ and $\beta_{i,n},\ i=1,2,3$. In this case, the new normalizing constants would not meet the Khinchin criteria for the first group of the normalizing constants, e.g., at least one of the conditions

$$0<\lim_{n\to\infty}\frac{\alpha_{i,n}}{c_{i,n}}<\infty \quad \text{and} \quad -\infty<\lim_{n\to\infty}\frac{\beta_{i,n}-d_{i,n}}{c_{i,n}}<\infty,$$

or $0<\lim\limits_{n\to\infty}\frac{\alpha_{i,n}}{a_{i,n}}<\infty$ and $-\infty<\lim\limits_{n\to\infty}\frac{\beta_{i,n}-b_{i,n}}{a_{i,n}}<\infty$, is not met. The following example endorses this fact.

EXAMPLE 3.1. Let X be the uniform DF $F_X(x)=\frac{x+1}{2}$, $-1\leqslant x\leqslant 1$. The corresponding folded uniform DF is $F_{|X|}(x)=x,\ 0\leqslant x\leqslant 1$. It is easy to show that $F_X(a_{3,n}x+b_{3,n})\in\mathcal{D}_L(G_3(x))$ and $F_X(c_{3,n}x+d_{3,n})\in\mathcal{D}_U(G_3'(x))$, where $a_{3,n}=c_{3,n}=\frac{2\sqrt{k}}{n}$ and $b_{3,n}=-d_{3,n}=\frac{2k}{n}-1$. Then it is easy to check that

$$\frac{\overline{F}_X(a_{3,n}x-b_{3,n})}{\sqrt{k}/n}=\frac{1-\frac{a_{3,n}x-b_{3,n}+1}{2}}{\sqrt{k}/n}=-x+\sqrt{k}\underset{n}{\rightarrow}\infty$$

and

$$\frac{F_X(-c_{3,n}x-d_{3,n})}{\sqrt{k}/n}=\frac{\frac{-c_{3,n}x-d_{3,n}+1}{2}}{\sqrt{k}/n}=-x+\sqrt{k}\underset{n}{\rightarrow}\infty.$$

Hence, in view of Theorem 3.1(3), $F_{|X|}(a_{3,n}x-b_{3,n})\notin\mathcal{D}_U(G_3'(x))$ and $F_{|X|}(c_{3,n}x+d_{3,n})\notin\mathcal{D}_U(G_3'(x))$. On the other hand, we can easily show that $F_{|X|}(\alpha_{3,n}x+\beta_{3,n})\in\mathcal{D}_U(G_3'(x))$, where $\alpha_{3,n}=\frac{\sqrt{k}}{n}$, $\beta_{3,n}=1-\frac{k}{n}$, $\frac{\alpha_{3,n}}{a_{3,n}}=\frac{\alpha_{3,n}}{c_{3,n}}=\frac{1}{2}$, $\lim\limits_{n\to\infty}\frac{\beta_{3,n}-b_{3,n}}{a_{3,n}}=-\infty$, and $\lim\limits_{n\to\infty}\frac{\beta_{3,n}-d_{3,n}}{c_{3,n}}=\infty$.

4. ASYMPTOTIC BEHAVIOR OF RECORD VALUES FROM FOLDED DISTRIBUTIONS

In this section, the asymptotic behavior of univariate record values of a folded distribution is studied. In addition, some illustrative examples are provided.

THEOREM 4.1. *Let* $-\infty \leqslant x_0 < 0 \leqslant x^0 \leqslant \infty$*, where* x_0 *and* x^0 *are the left and right endpoints, respectively. Then we have the following implications:*

(1) *Let* $F_X(A_{1,n}x + B_{1,n}) \in \mathcal{D}_{\text{urec}}(\Psi_{1,\beta}(x))$*. Then*

$$F_{|X|}(A_{1,n}x + B_{1,n}) \in \mathcal{D}_{\text{urec}}(\Psi_{1,\beta}(x)) \quad \textit{if } 0 \leqslant \lim_{n\to\infty} \frac{F_X(-A_{1,n}x)}{1 - F_X(A_{1,n}x)} < \infty.$$

(2) *Let* $F_X(A_{2,n}x + B_{2,n}) \in \mathcal{D}_{\text{urec}}(\Psi_{2,\beta}(x))$ *and suppose* $|x^0| > |x_0|$*. Then*

$$F_{|X|}(A_{2,n}x + B_{2,n}) \in \mathcal{D}_{\text{urec}}(\Psi_{2,\beta}(x)) \quad \textit{if } 0 \leqslant \lim_{n\to\infty} \frac{F_X(-A_{2,n}x - x^0)}{1 - F_X(A_{2,n}x + x^0)} < \infty.$$

(3) *Let* $F_X(A_{3,n}x + B_{3,n}) \in \mathcal{D}_{\text{urec}}(\Psi_3(x))$*. Then* $F_{|X|}(A_{3,n}x + B_{3,n}) \in \mathcal{D}_{\text{urec}}(\Psi_3(x))$ *if*

$$\lim_{n\to\infty} F_X(-A_{3,n}x - B_{3,n}) = 0 \textit{ and } 0 \leqslant \lim_{n\to\infty} \frac{F_X(-A_{3,n}x - B_{3,n})}{1 - F_X(A_{3,n}x + B_{3,n})} < \infty.$$

Proof. In view of Lemma 3.1, we get

$$\begin{aligned} \frac{U_{|X|}(A_{i,n}x + B_{i,n}) - n}{\sqrt{n}} &= \frac{-\log(1 - F_{|X|}(A_{i,n}x + B_{i,n})) - n}{\sqrt{n}} \\ &= \frac{-\log(1 - F_X(A_{i,n}x + B_{i,n}) + F_X(-A_{i,n}x - B_{i,n})) - n}{\sqrt{n}} \\ &= \frac{-\log(1 - F_X(A_{i,n}x + B_{i,n})) - n}{\sqrt{n}} \\ &\quad + \frac{-\log\left(1 + \frac{F_X(-A_{i,n}x - B_{i,n})}{1 - F_X(A_{i,n}x + B_{i,n})}\right)}{\sqrt{n}}, \quad i = 1, 2, 3. \end{aligned} \tag{4.1}$$

Put $i = 1$ in (4.1). In view of (2.8) and Lemma 2.4, we get

$$\begin{aligned} &\frac{U_{|X|}(A_{1,n}x + B_{1,n}) - n}{\sqrt{n}} \\ &\quad = \frac{-\log(1 - F_X(A_{1,n}x)) - n}{\sqrt{n}} + \frac{-\log\left(1 + \frac{F_X(-A_{1,n}x)}{1 - F_X(A_{1,n}x)}\right)}{\sqrt{n}} \\ &\quad \underset{n}{\to} \begin{cases} \beta \log x & \text{if } x > 0 \\ -\infty & \text{if } x \leqslant 0 \end{cases} + \begin{cases} \lim\limits_{n\to\infty} \dfrac{-\log\left(1 + \frac{F_X(-A_{1,n}x)}{1 - F_X(A_{1,n}x)}\right)}{\sqrt{n}} & \text{if } x > 0, \\ 0 & \text{if } x \leqslant \infty, \end{cases} \end{aligned}$$

where the possible limit points of $\frac{F_X(-A_{1,n}x)}{1-F_X(A_{1,n}x)}$ are in $\{M,\infty\}$, where $M \geqslant 0$ is a finite constant. The only limit point that gives a non-degenerate limit for $\frac{U_{|X|}(A_{1,n}x+B_{1,n})-n}{\sqrt{n}}$ is M. Then

$$\frac{U_{|X|}(A_{1,n}x+B_{1,n})-n}{\sqrt{n}} \underset{n}{\rightarrow} \begin{cases} \beta\log x & \text{if } x>0, \\ -\infty & \text{if } x\leqslant 0. \end{cases}$$

Hence, $F_{|X|}(A_{1,n}x+B_{1,n}) \in \mathcal{D}_{\text{urec}}(\Psi_{1,\beta}(x))$. This completes the proof of part (1).

Now we prove (2). From Lemma 2.4, we have $B_{2,n}=x^0$. Suppose $|x^0|>|x_0|$. On putting $i=2$ in (4.1), due to (2.8) we get

$$\begin{aligned} &\frac{U_{|X|}(A_{2,n}x+B_{2,n})-n}{\sqrt{n}} \\ &\qquad = \frac{-\log(1-F_X(A_{2,n}x+B_{2,n}))-n}{\sqrt{n}} + \frac{-\log\left(1+\frac{F_X(-A_{2,n}x-x^0)}{1-F_X(A_{2,n}x+x^0)}\right)}{\sqrt{n}} \\ &\qquad \underset{n}{\rightarrow} \begin{cases} -\beta\log|x| & \text{if } x<0 \\ \infty & \text{if } x\geqslant 0 \end{cases} + \lim_{n\to\infty} \frac{-\log\left(1+\frac{F_X(-A_{2,n}x-x^0)}{1-F_X(A_{2,n}x+x^0)}\right)}{\sqrt{n}}, \end{aligned}$$

where the sequence $\frac{F_X(-A_{2,n}x-x^0)}{1-F_X(A_{2,n}x+x^0)}$ has limit points in $\{N,\infty\}$, where $N \geqslant 0$ is a finite constant. The only limit point that provides a non-degenerate limit for $\frac{U_{|X|}(A_{2,n}x+B_{2,n})-n}{\sqrt{n}}$ is N. Then

$$\frac{U_{|X|}(A_{2,n}x+B_{2,n})-n}{\sqrt{n}} \underset{n}{\rightarrow} \begin{cases} -\beta\log|x| & \text{if } x<0, \\ \infty & \text{if } x\geqslant 0. \end{cases}$$

Hence, $F_{|X|}(A_{2,n}x+B_{2,n}) \in \mathcal{D}_{\text{urec}}(\Psi_{2,\beta}(x))$. This completes the proof of part (2).

Turning to (3), consider $i=3$ with (4.1) due to (2.8). Assuming that $F_X(-A_{3,n}x-B_{3,n}) \underset{n}{\rightarrow} 0$, we get

$$\begin{aligned} \frac{U_{|X|}(A_{3,n}x+B_{3,n})-n}{\sqrt{n}} &= \frac{-\log(1-F_X(A_{3,n}x+B_{3,n}))-n}{\sqrt{n}} \\ &\quad + \frac{-\log\left(1+\frac{F_X(-A_{3,n}x-B_{3,n})}{1-F_X(A_{3,n}x+B_{3,n})}\right)}{\sqrt{n}} \\ &\underset{n}{\rightarrow} x + \lim_{n\to\infty} \frac{-\log\left(1+\frac{F_X(-A_{3,n}x-B_{3,n})}{1-F_X(A_{3,n}x+B_{3,n})}\right)}{\sqrt{n}}, \end{aligned}$$

where the sequence $\frac{F_X(-A_{3,n}x-B_{3,n})}{1-F_X(A_{3,n}x+B_{3,n})}$ has limit points in $\{L,\infty\}$, where $L \geqslant 0$ is a finite constant. The only limit point that provides a non-degenerate limit for

$\frac{U_{|X|}(A_{3,n}x+B_{3,n})-n}{\sqrt{n}}$ is L. Then

$$\frac{U_{|X|}(A_{3,n}x+B_{3,n})-n}{\sqrt{n}} \underset{n}{\rightarrow} x.$$

Hence, $F_{|X|}(A_{3,n}x+B_{3,n}) \in \mathcal{D}_{\text{urec}}(\Psi_3(x))$. This completes the proof of Theorem 4.1. ∎

4.1. Illustrative examples

EXAMPLE 4.1 (Logistic distribution). The logistic DF is defined as $F_X(x) = \frac{1}{1+e^{-x}}$, $-\infty \leqslant x \leqslant \infty$. It is known (see [2]) that $F_X(A_{3,n}x+B_{3,n}) \in \mathcal{D}_{\text{urec}}(\Psi_3)$ with normalizing constants $A_{3,n} = \log\left(\frac{e^{n+\sqrt{n}}-1}{e^n-1}\right)$ and $B_{3,n} = \log(e^n-1)$. It is easy to check that $F_X(-A_{3,n}x-B_{3,n}) \underset{n}{\rightarrow} 0$ and

$$\frac{F_X(-A_{3,n}x-B_{3,n})}{1-F_X(A_{3,n}x+B_{3,n})} \underset{n}{\rightarrow} \begin{cases} 0, \text{ or} \\ L>0. \end{cases}$$

Therefore,

$$\begin{aligned} F_X(-A_{3,n}x-B_{3,n}) &= \frac{1}{1+e^{-(-\log(\frac{e^{n+\sqrt{n}}-1}{e^n-1})x-\log(e^n-1))}} \\ &= \frac{1}{1+e^{\log(\frac{e^{n+\sqrt{n}}-1}{e^n-1}x(e^n-1))}} = \frac{1}{1+(e^{n+\sqrt{n}}-1)x} \underset{n}{\rightarrow} 0. \end{aligned}$$

Also, $F_X(A_{3,n}x+B_{3,n}) = \frac{1}{1+\frac{1}{e^{n+\sqrt{n}}-1}}$. Hence

$$\frac{F_X(-A_{3,n}x-B_{3,n})}{1-F_X(A_{3,n}x+B_{3,n})} = \frac{\frac{1}{1+(e^{n+\sqrt{n}}-1)x}}{1-\frac{1}{1+\frac{1}{(e^{n+\sqrt{n}}-1)x}}} = \frac{\frac{1}{1+(e^{n+\sqrt{n}}-1)x}}{1-\frac{1}{\frac{(e^{n+\sqrt{n}}-1)x+1}{(e^{n+\sqrt{n}}-1)x}}} = 1.$$

In view of Theorem 4.1(3), we have $\frac{F_X(-A_{3,n}x-B_{3,n})}{1-F_X(A_{3,n}x+B_{3,n})} \underset{n}{\rightarrow} 1$. Hence $F_{|X|}(A_{3,n}x+B_{3,n}) \in \mathcal{D}_{\text{urec}}(\Psi_3(x))$.

EXAMPLE 4.2 (Standard normal distribution). The standard normal distribution is defined as $F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$, $-\infty < x < \infty$. It is known (see [2]) that $F_X(A_{3,n}x+B_{3,n}) \in \mathcal{D}_{\text{urec}}(\Psi_3)$ with normalizing constants $A_{3,n} \sim \sqrt{2n+2\sqrt{n}}-\sqrt{2n} = \sqrt{2n}(\sqrt{1+1/\sqrt{n}}-1)$ and $B_{3,n} \sim \sqrt{2n}$. Therefore, it is easy to verify that $F_X(-A_{3,n}x-B_{3,n}) \underset{n}{\rightarrow} 0$ and

$$\frac{F_X(-A_{3,n}x-B_{3,n})}{\overline{F}_X(A_{3,n}x+B_{3,n})} \underset{n}{\rightarrow} \begin{cases} 0, \text{ or} \\ L>0. \end{cases}$$

Thus,

$$F_X(-A_{3,n}x - B_{3,n}) \sim \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{-\sqrt{2n}(\sqrt{1+1/\sqrt{n}}-1)x-\sqrt{2n}} e^{-t^2/2}\, dt$$
$$\underset{n}{\rightarrow} \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{-\infty} e^{-t^2/2}\, dt = 0.$$

Also, $\overline{F}_X(A_{3,n}x + B_{3,n}) \sim \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2n}(\sqrt{1+1/\sqrt{n}}-1)x+\sqrt{2n}}^{\infty} e^{-t^2/2}\, dt$. Thus

$$\lim_{n\to\infty} \frac{F_X(-A_{3,n}x - B_{3,n})}{\overline{F}_X(A_{3,n}x + B_{3,n})} \sim \frac{\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{-\sqrt{2n}(\sqrt{1+1/\sqrt{n}}-1)x-\sqrt{2n}} e^{-t^2/2}\, dt}{\frac{1}{\sqrt{2\pi}} \int\limits_{\sqrt{2n}(\sqrt{1+1/\sqrt{n}}-1)x+\sqrt{2n}}^{\infty} e^{-t^2/2}\, dt} = 1.$$

In view of Theorem 4.1(3), we have $\frac{F_X(-A_{3,n}x-B_{3,n})}{\overline{F}_X(A_{3,n}x+B_{3,n})} \underset{n}{\rightarrow} 1$. It follows that $F_{|X|}(A_{3,n}x + B_{3,n}) \in \mathcal{D}_{\text{urec}}(\Psi_3(x))$.

EXAMPLE 4.3 (Weibull distribution). The Weibull distribution is defined as $F_X(x) = 1-e^{-x^c}, 0 < x < \infty, c > 0$. It is known (see [2]) that $F_X(A_{3,n}x+B_{3,n}) \in \mathcal{D}_{\text{urec}}(\Psi_3)$ with normalizing constants

$$A_{3,n} = n^{1/c}((1 + 1/\sqrt{n})^{1/c} - 1) \quad \text{and} \quad B_{3,n} = n^{1/c}.$$

Then it is easy to check that $F_X(-A_{3,n}x - B_{3,n}) \underset{n}{\rightarrow} 0$ and

$$\frac{F_X(-A_{3,n}x - B_{3,n})}{\overline{F}_X(A_{3,n}x + B_{3,n})} \underset{n}{\rightarrow} \begin{cases} 0, \text{ or} \\ L > 0. \end{cases}$$

Therefore,

$$F_X(-A_{3,n}x - B_{3,n}) = 1 - e^{-(-n^{1/c}((1+1/\sqrt{n})^{1/c}-1)x-n^{1/c})^c}$$
$$= 1 - e^{-n(-((1+1/\sqrt{n})^{1/c}-1)x-1)^c} \underset{n}{\rightarrow} \begin{cases} -\infty & \text{if } c \text{ is even,} \\ 1 & \text{if } c \text{ is odd.} \end{cases}$$

Hence, $F_{|X|}(A_{3,n}x + B_{3,n}) \notin \mathcal{D}_{\text{urec}}(\Psi_3)$.

5. CONCLUSION

This study examined the asymptotic behavior of random record values and intermediate OSs arising from a folded distribution by comparing the asymptotic behavior of those statistics arising from the unfolded distribution itself. It was observed that the limit behavior of such statistics is significantly influenced by the distribution's right and left endpoints. Furthermore, it was demonstrated that the upper record value's weak convergence based on a folded DF is identical to that based on the original DF (the unfolded distribution). Lastly, a few examples were provided for illustration.

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