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POWER MEANS OF RANDOM VARIABLES AND CHARACTERIZATIONS OF DISTRIBUTIONS VIA FRACTIONAL CALCULUS

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Abstract. We investigate fractional moments and expectations of power means of complex-valued random variables by using fractional calculus. We deal with both negative and positive orders of the fractional derivatives. The one-dimensional distributions are characterized in terms of the fractional moments without any moment assumptions. We explicitly compute the expectations of the power means for both the univariate Cauchy distribution and the Poincaré distribution on the upper half-plane. We show that for these distributions the expectations are invariant with respect to the sample size and the value of the power.

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1. INTRODUCTION

The strong law of large numbers is fundamental to probability. This law states that the arithmetic mean of independent and identically distributed (i.i.d.) random variables converges almost surely to a constant. This constant is equal to the expectation of the respective random variable. For stochastic processes, once we establish the law of large numbers, we move on to more sophisticated limit theorems such as the central limit theorem. For non-integrable i.i.d. random variables such as Cauchy distributions, the law of large numbers fails for the arithmetic mean. In order to establish universality for non-integrable i.i.d. random variables, we need to consider an alternative statistic other than the arithmetic mean.

The framework of *quasi-arithmetic means* provides an alternative approach to universality. This notion was introduced independently by Kolmogorov [7], Nagumo [12], and de Finetti [6]. They proposed axioms of means and showed

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that if the axioms hold for an n-ary operation on an interval of real numbers, then the operation is a quasi-arithmetic mean. It is defined by

$$M_n^f = M_n^f(x_1, \dots, x_n) := f^{-1}\left(\frac{1}{n}\sum_{j=1}^{n} f(x_j)\right)$$

for a generator f; M_n^f is the arithmetic mean if $f(x) = x$. This framework includes geometric, harmonic, and power means. The random variable $M_n^f(X_1, \dots, X_n)$, where $X_1, \dots, X_n$ are i.i.d. random variables, can be integrable even if X_1 is non-integrable, and it plays a similar role to the arithmetic mean of i.i.d. random variables. Limit theorems such as the strong law of large numbers and the central limit theorem hold. The strong law of large numbers states that $\lim_{n\to\infty} M_n^f = f^{-1}(E[f(X_1)])$ almost surely and it follows directly from the usual strong law of large numbers if $f(X_1) \in L^1$. The central limit theorem is more complicated, but it follows from the delta method (Carvalho [5]). Recently, limit theorems of more general means of i.i.d. random variables have been considered by Barczy and Burai [2] and Barczy and Páles [3]. By considering *complex-valued* quasi-arithmetic means of i.i.d. random variables, we can deal with heavy-tailed random variables *supported on* $\mathbb{R}$. Akaoka and the present authors [1] considered integrability and asymptotic variances of quasi-arithmetic means and gave applications to quasi-arithmetic means of Cauchy distributions.

The expectation of the arithmetic mean of i.i.d. integrable random variables is equal to the expectation of the respective random variable. However, the expectation of M_n^f, $E[M_n^f]$, is difficult to compute explicitly, because M_n^f is defined by using f^{-1}, which can be non-linear. In this paper, we consider $E[M_n^f]$ for the case of complex-valued *power means*, more specifically, the generator is given by $f(z) = z^p$, $p \in [-1, 1], p \neq 0$. Each power mean is a homogeneous quasi-arithmetic mean. The class of power means for $p \in (-1, 0)$ interpolates between the harmonic mean and the geometric mean, and, for $p \in (0, 1)$ the class interpolates between the geometric mean and the arithmetic mean.

We deal with the fractional moment $E[Z^\lambda]$ of a complex-valued random variable Z with a complex power λ, and furthermore the expectation of the power mean $E\left[\left(\frac{1}{n}\sum_{j=1}^n Z_j^p\right)^{1/p}\right]$ of i.i.d. complex-valued random variables $(Z_j)_j$ for $p \in [-1, 1]$. We explain why we use *fractional calculus* to compute them. Let $\varphi_Z(t) := E[\exp(itZ)]$. If $E[|Z|^n] < +\infty$ for some $n \in \mathbb{N}$, then

$$\left.\frac{\partial^n}{\partial t^n}\varphi_Z(t)\right|_{t=0} = i^n E[Z^n].$$

We replace the natural number n, the order of the derivative, with a *fractional* number λ and then we formally obtain

$$\left.\frac{\partial^\lambda}{\partial t^\lambda}\varphi_Z(t)\right|_{t=0} = i^\lambda E[Z^\lambda].$$

Part of our paper is devoted to justifying this heuristic argument. There are several different frameworks for fractional derivatives (Oldham and Spanier [15]). Here we adopt the Riemann–Liouville integral for $\mathrm{Re}(\lambda) < 0$ and Marchaud's fractional derivative [10] for $\mathrm{Re}(\lambda) > 0$. Then we proceed to consider the case of $Z = \frac{1}{n}\sum_{j=1}^{n} Z_j^p$ and $\lambda = p \in [-1, 1]$. The relationship between fractional calculus and the fractional *absolute* moment $E[|X|^p]$ of a real-valued random variable X for $p \in \mathbb{R}$ has been considered in many papers. For more details, see Matsui and Pawlas [11] and references therein. Here we consider connections between fractional calculus and fractional (non-absolute) moments of complex-valued random variables.

By using our results for the fractional moment, we show that certain subfamilies of the expectations $\{E[(X+\alpha)^\lambda] : \alpha \in \overline{\mathbb{H}},\ \lambda \in \mathbb{C}\}$ characterize the distribution of a real-valued random variable X, where $\mathbb{H}$ is the upper half-plane and $\overline{\mathbb{H}}$ is its closure. Our results are partly similar to those of Lin [9], but with notable differences. We consider $X + \alpha$ instead of X, which makes the considerations for $\mathrm{Re}(\lambda) < 0$ much clearer, since with this modification we do not need to impose any integrability conditions on X. Furthermore, the notion of determining sets of holomorphic functions is involved.

Our framework is applicable to *Poincaré distributions*, which are a parametric family of distributions supported on $\mathbb{H}$ recently introduced by Tojo and Yoshino [18, 19]. If Z follows a Poincaré distribution, we can explicitly compute φ_Z. By using our results for the fractional moment, we can compute $E[Z_1^\lambda]$ and $E\big[\big(\frac{1}{n}\sum_{j=1}^{n} Z_j^p\big)^{1/p}\big]$ if Z_1 follows a Poincaré distribution. Similarly, we can deal with the case of $Z_j = X_j + \alpha$ where $\alpha \in \mathbb{H}$ and $(X_j)_j$ are i.i.d. real-valued random variables following a Cauchy distribution or a t-distribution with three degrees of freedom. For the Poincaré and Cauchy distributions, $E\big[\big(\frac{1}{n}\sum_{j=1}^{n} Z_j^p\big)^{1/p}\big]$ does not depend on the sample size n or the parameter p. However, this fails for the t-distribution with three degrees of freedom.

The paper is organized as follows. In Sections 2 and 3, we give integral expressions for the fractional moment $E[Z^\lambda]$ and the expectation of the power mean $E\big[\big(\frac{1}{n}\sum_{j=1}^{n} Z_j^p\big)^{1/p}\big]$ by using φ_Z. The cases of $\mathrm{Re}(\lambda) < 0$ and $\mathrm{Re}(\lambda) > 0$ are considered in Sections 2 and 3 respectively. In Section 4, by using the results of Sections 2 and 3, we show that certain subfamilies of $\{E[(X+\alpha)^\lambda]\}_{\alpha,\lambda}$ characterize the distribution of a real-valued random variable X. In Section 5, we compute the expectations of the complex-valued power means of Cauchy distributions, t-distributions with three degrees of freedom, and Poincaré distributions. In the Appendix, we compare the fractional absolute moment with the absolute value of the fractional moment.

Notation. For $z \in \mathbb{C}, z \neq 0$, we let $\log z := \log|z| + i\theta$, where $z = r\exp(i\theta)$, $-\pi < \theta \leqslant \pi$ and $r > 0$. For $\lambda \in \mathbb{C}$, let

$$z^\lambda := \begin{cases} \exp(\lambda \log z), & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Let $\overline{A}$ be the closure of a subset A of $\mathbb{C}$. Let $\mathbb{H} := \{x + yi : x \in \mathbb{R}, y > 0\}$, and $-\mathbb{H} := \{x + yi : x \in \mathbb{R}, y < 0\}$. Let $U := i\mathbb{H} = \{x + yi \mid x < 0, y \in \mathbb{R}\}$ and $V := -U = -i\mathbb{H} = \{x + yi \mid x > 0, y \in \mathbb{R}\}$.

Let the Gamma function be $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1} \exp(-t)\, dt$ for $\lambda \in V$. For a complex-valued random variable Z, we denote the distribution of Z by P^Z. For $r > 0$, we write $Z \in L^r$ if $|Z| \in L^r$. For every $\alpha \in \mathbb{C}$, $Z \in L^r$ if and only if $Z + \alpha \in L^r$.

2. FRACTIONAL DERIVATIVE OF NEGATIVE ORDER

We use the Riemann–Liouville integral as in Wolfe [21] and Cressie and Borkent [4, Definition 1].

DEFINITION 2.1. Let $\lambda \in U$. For a Borel measurable function $f : (-\infty, 0] \to \mathbb{C}$,

$$\frac{\partial^\lambda}{\partial t^\lambda} f(t)\bigg|_{t=0} := \frac{1}{\Gamma(-\lambda)} \int_0^\infty u^{-\lambda-1} f(-u)\, du$$

if the integral exists.

It suffices to define the left derivative at $t = 0$ only. As indicated in [21], it is natural to consider the fractional derivative of complex order. Here the negative order means that the real part of the complex order is negative.

LEMMA 2.2. *Let $\lambda \in U$. Then we have the following assertions:*

(i) *Let Z be an $\mathbb{H}$-valued random variable. Assume that $E[\mathrm{Im}(Z)^{\mathrm{Re}(\lambda)}] < +\infty$. Then $Z^\lambda \in L^1$ and*

$$\frac{\partial^\lambda}{\partial t^\lambda} E[\exp(-itZ)]\bigg|_{t=0} = (-i)^\lambda E[Z^\lambda]. \tag{2.1}$$

(ii) *Let Z be a $(-\mathbb{H})$-valued random variable. Assume that $E[(-\mathrm{Im}(Z))^{\mathrm{Re}(\lambda)}] < +\infty$. Then $Z^\lambda \in L^1$ and*

$$\frac{\partial^\lambda}{\partial t^\lambda} E[\exp(itZ)]\bigg|_{t=0} = i^\lambda E[Z^\lambda].$$

Proof. We remark that $\mathrm{Re}(\lambda) < 0$. We show (i); the proof of (ii) is similar. Let $z = re^{i\theta} \in \mathbb{H}$ be such that $0 < \theta < \pi$. Then

$$\begin{aligned}\int_0^\infty |u^{-\lambda-1} \exp(iuz)|\, du &= \int_0^\infty u^{-\mathrm{Re}(\lambda)-1} \exp(-u\,\mathrm{Im}(z))\, du \\ &= \mathrm{Im}(z)^{\mathrm{Re}(\lambda)} \Gamma(-\mathrm{Re}(\lambda)).\end{aligned}$$

Hence by the assumption,

$$\int_{\mathbb{H}}\int_0^\infty |u^{-\lambda-1}\exp(iuz)|\,du\,P^Z(dz) = E[\mathrm{Im}(Z)^{\mathrm{Re}(\lambda)}]\Gamma(-\mathrm{Re}(\lambda)) < +\infty.$$

Therefore we can use Fubini's theorem to obtain

$$\begin{aligned}\frac{\partial^\lambda}{\partial t^\lambda}E[\exp(-itZ)]\bigg|_{t=0} &= \frac{1}{\Gamma(-\lambda)}\int_0^\infty u^{-\lambda-1}E[\exp(iuZ)]\,du \\ &= \frac{1}{\Gamma(-\lambda)}E\Big[\int_0^\infty u^{-\lambda-1}\exp(iuZ)\,du\Big].\end{aligned}$$

By the Cauchy integral theorem,

$$\begin{aligned}\int_0^\infty u^{-\lambda-1}\exp(iuz)\,du &= z^\lambda\int_{C_z}\zeta^{-\lambda-1}\exp(i\zeta)\,d\zeta \\ &= z^\lambda\int_{C_i}\zeta^{-\lambda-1}\exp(i\zeta)\,d\zeta \\ &= z^\lambda i^{-\lambda}\int_0^\infty u^{-\lambda-1}\exp(-u)\,du = z^\lambda i^{-\lambda}\Gamma(-\lambda),\end{aligned}\tag{2.2}$$

where we let $C_w := \{tw : t \geqslant 0\}$ for $w \in \mathbb{H}$. Hence

$$|z^\lambda| \leqslant \frac{1}{|i^{-\lambda}\Gamma(-\lambda)|}\int_0^\infty |u^{-\lambda-1}\exp(iuz)|\,du = \frac{\Gamma(-\mathrm{Re}(\lambda))}{|i^{-\lambda}\Gamma(-\lambda)|}\,\mathrm{Im}(z)^{\mathrm{Re}(\lambda)}.\tag{2.3}$$

Now the assumption yields $Z^\lambda \in L^1$, and we get (2.1). ■

We can generalize (2.3) to the case where $z \in -\mathbb{H}$. For $\lambda \in U$, there exists a constant $C(\lambda)$ such that for $z \in \mathbb{C}\setminus\mathbb{R}$,

$$|z^\lambda| \leqslant C(\lambda)|\mathrm{Im}(z)|^{\mathrm{Re}(\lambda)}.\tag{2.4}$$

For example, we can put $C(\lambda) := \frac{\Gamma(-\mathrm{Re}(\lambda))\exp(\pi|\mathrm{Im}(\lambda)|/2)}{|\Gamma(-\lambda)|}$.

Now we give an application of Lemma 2.2.

PROPOSITION 2.3. *Let X be a real-valued random variable and φ_X be the characteristic function of X. Let $\lambda \in U$. Let $\alpha \in \mathbb{H}$. Then $(X+\alpha)^\lambda \in L^\infty$ and*

$$E[(X+\alpha)^\lambda] = \frac{i^\lambda}{\Gamma(-\lambda)}\int_0^\infty t^{-\lambda-1}\varphi_X(t)\exp(i\alpha t)\,dt.$$

Proof. By (2.4), $\sup_{x\in\mathbb{R}}|(x+\alpha)^\lambda| \leqslant C(\lambda)|\mathrm{Im}(\alpha)|^{\mathrm{Re}(\lambda)}$. We now apply Lemma 2.2(i) to $Z = X+\alpha$ to get the assertion. ■

REMARK 2.4. Let $\alpha \in \mathbb{H}$ and $\alpha \to 0$. Then, formally,

$$E[X^{\lambda}] = \frac{i^{\lambda}}{\Gamma(-\lambda)} \int_0^{\infty} t^{-\lambda-1} \varphi_X(t)\, dt.$$

This is justified if $E[|X|^{\mathrm{Re}(\lambda)}] < +\infty$ and $t^{-\mathrm{Re}(\lambda)-1}\varphi_X(t) \in L^1$. If $-1 < \mathrm{Re}(\lambda) < 0$ and $E[|X|^{\mathrm{Re}(\lambda)}] < +\infty$, then $\int_0^\infty t^{-\lambda-1}\varphi_X(t)dt$ exists as a Riemann improper integral (see [21, Section 5]). The integration contour in [21, Section 5] is different from the contour in (2.2).

We now give an application to the power mean of random variables.

THEOREM 2.5. *Let $-1 \leqslant p < 0$. Let $n \geqslant 2$. Let $Z_1, \dots, Z_n$ be i.i.d. $\mathbb{H}$-valued random variables such that $E[(\mathrm{Im}(Z_1)^{1/p}|Z_1|^{1-1/p})^{1/n}] < +\infty$. Then*

$$\frac{\partial^{1/p}}{\partial t^{1/p}} E\left[\exp\left(i\frac{t}{n} Z_1^p\right)\right]^n \Bigg|_{t=0} = i^{1/p} E\left[\left(\frac{1}{n}\sum_{j=1}^n Z_j^p\right)^{1/p}\right].$$

Proof. Let $Z := \frac{1}{n}\sum_{j=1}^n Z_j^p$. Since $-1 \leqslant p < 0$, Z is $(-\mathbb{H})$-valued. We argue as in [1, proof of Proposition 3.3(ii)]. By the geometric mean-arithmetic mean inequality,

$$E[(-\mathrm{Im}(Z))^{1/p}] \leqslant E[(-\mathrm{Im}(Z_1^p))^{1/(np)}]^n. \tag{2.5}$$

Let $r_1 > 0$ and $\theta_1 \in (0, \pi)$ be such that $Z_1 = r_1 \exp(i\theta_1)$. Then

$$(-\mathrm{Im}(Z_1^p))^{1/p} = \mathrm{Im}(Z_1)^{1/p}|Z_1|^{1-1/p}\left(\frac{\sin\theta_1}{\sin(-p\theta_1)}\right)^{-1/p}. \tag{2.6}$$

We see that $\sup_{\theta\in(0,\pi)}\left(\frac{\sin\theta}{\sin(-p\theta)}\right)^{-1/p} < +\infty$. From this, (2.6), and the assumption, $E[(-\mathrm{Im}(Z_1^p))^{1/(np)}] < +\infty$, and hence $E[(-\mathrm{Im}(Z))^{1/p}] < +\infty$. Therefore we can apply Lemma 2.2(ii) to get the assertion. ■

We now consider the continuity of the expectation of the power mean with respect to the parameter.

PROPOSITION 2.6. *Let $-1 < p_0 < 0$. Let $n \geqslant 2$. Let $Z_1, \dots, Z_n$ be i.i.d. $\mathbb{H}$-valued random variables such that*

$$\sup_{p\in(p_0-\varepsilon_0, p_0+\varepsilon_0)} E[(\mathrm{Im}(Z_1)^{1/p}|Z_1|^{1-1/p})^{1/n+\varepsilon_0}] < +\infty$$

for some $\varepsilon_0 > 0$. Then $E\left[\left(\frac{1}{n}\sum_{j=1}^n Z_j^p\right)^{1/p}\right]$ is continuous at $p = p_0$ as a function of p.

Proof. Let $-1 < p < 0$. Let $Z^{(p)} := \frac{1}{n}\sum_{j=1}^{n} Z_j^p$. Since

$$E\left[\left(\frac{1}{n}\sum_{j=1}^{n} Z_j^p\right)^{1/p}\right] = \frac{1}{\Gamma(-1/p)}\int_0^\infty u^{-1/p-1}E[\exp(-iuZ^{(p)})]\,du,$$

it suffices to show the continuity of $p \mapsto \int_0^\infty u^{-1/p-1}E[\exp(-iuZ^{(p)})]\,du$ at $p = p_0$. By Fubini's theorem,

$$\int_0^\infty u^{-1/p-1}E[\exp(u\,\mathrm{Im}(Z^{(p)}))]\,du = \Gamma\left(-\frac{1}{p}\right)E[(-\mathrm{Im}(Z^{(p)}))^{-1/p}].$$

By (2.5), (2.6), and the assumption, the functions $p \mapsto E[(-\mathrm{Im}(Z^{(p)}))^{-1/p}]$ and $p \mapsto \int_0^\infty u^{-1/p-1}E[\exp(u\,\mathrm{Im}(Z^{(p)}))]\,du$ are continuous at $p = p_0$. We remark that $|E[\exp(-iuZ^{(p)})]| \leqslant E[\exp(u\,\mathrm{Im}(Z^{(p)}))]$. By the Lebesgue convergence theorem, the functions $p \mapsto E[\exp(-iuZ^{(p)})]$ and $p \mapsto u^{-1/p-1}E[\exp(-iuZ^{(p)})]$ are continuous. Now we can apply a generalized Lebesgue convergence theorem (see [16, Chapter 4, Theorem 19] for example) to conclude that $p \mapsto \int_0^\infty u^{-1/p-1}E[\exp(-iuZ^{(p)})]\,du$ is continuous at $p = p_0$. ∎

We state a result applicable to general distributions on $\mathbb{R}$ satisfying an integrability condition.

COROLLARY 2.7. *Let $\alpha \in \mathbb{H}$. Then we have the following:*

(i) *Let $-1 \leqslant p < 0$. Let $X_1, \dots, X_n$ be i.i.d. real-valued random variables such that $X_1 \in L^{(p-1)/(np)}$. Then*

$$\frac{\partial^{1/p}}{\partial t^{1/p}}E\left[\exp\left(i\frac{t}{n}(X_1+\alpha)^p\right)\right]^n\Bigg|_{t=0} = i^{1/p}E\left[\left(\frac{1}{n}\sum_{j=1}^{n}(X_j+\alpha)^p\right)^{1/p}\right].$$

(ii) *Let $r > 2/n$. Let $X_1, \dots, X_n$ be i.i.d. real-valued random variables such that $X_1 \in L^r$. Then $E\big[\big(\frac{1}{n}\sum_{j=1}^{n}(X_j+\alpha)^p\big)^{1/p}\big]$ is continuous with respect to p on $\big(-1, -\frac{1}{nr-1}\big)$.*

Proof. The proof is the same as that of Theorem 2.5. If $Z_j := X_j + \alpha$, then $\mathrm{Im}(Z_1)^{1/p} = \mathrm{Im}(\alpha)^{1/p} < +\infty$. By substituting this into (2.6), we have assertion (i). We show (ii). Let $p \in \big(-1, -\frac{1}{nr-1}\big)$. Then, by the Hölder inequality, for sufficiently small $\varepsilon > 0$,

$$\sup_{p' \in (p-\varepsilon, p+\varepsilon)} E[|X_1+\alpha|^{(1-1/p')(1/n+\varepsilon)}] \leqslant \sup_{s\in[0,1]} E[|X_1+\alpha|^r]^s < +\infty.$$

Hence we can apply Proposition 2.6. ∎

REMARK 2.8. If $p \in U$ and $p \notin \mathbb{R}$, then it does not necessarily hold that $\mathrm{Im}(Z_1^p) < 0$ P-a.s., even if $Z_1 \in \mathbb{H}$. The above proof does not apply to this case. When we consider the power means, it is natural to consider $p \in \mathbb{R}$. One reason is that it does not satisfy [1, Assumption 2.1], more specifically, if we let $f(z) := z^p$, then $f(\mathbb{H})$ may be *non-convex*. Indeed, $0 \notin f(\mathbb{H})$, but on the other hand, there exist two points $x_1 < 0$ and $x_2 > 0$ such that $x_1, x_2 \in f(\mathbb{H})$. Due to the branch cut, $\log(\sum_{j=1}^n Z_j^p)$ is not continuous with respect to p. The power means of complex orders are interesting but much harder to deal with.

3. FRACTIONAL DERIVATIVE OF POSITIVE ORDER

We adopt the following definition as a fractional derivative of positive order. As in the case of negative order, we consider the fractional derivative of complex order. Here the *positive order* means that the real part of the complex order is positive. We adopt Marchaud's fractional derivative [10] instead of the Riemann–Liouville fractional derivative, because it involves the derivative after an integration in order to define the fractional operator. Recall that $V = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$.

DEFINITION 3.1. Let k be a non-negative integer. Let $\lambda \in V$ be such that $\lambda = k + \delta$ and $0 < \mathrm{Re}(\delta) < 1$. Let $f \in C^k((-\infty, 0])$. Then

$$\frac{\partial^{k+\delta}}{\partial t^{k+\delta}} f(t)\bigg|_{t=0} := \frac{\delta}{\Gamma(1-\delta)} \int_0^\infty \frac{f^{(k)}(0) - f^{(k)}(-u)}{u^{1+\delta}}\, du.$$

LEMMA 3.2. *Let $\lambda \in V$. Assume that $\mathrm{Re}(\lambda) \notin \mathbb{N}$. Then we have the following assertions:*

(i) *Let Z be an $\overline{\mathbb{H}}$-valued random variable. Assume that $E[|Z|^{\mathrm{Re}(\lambda)}] < +\infty$. Then*

$$\frac{\partial^\lambda}{\partial t^\lambda} E[\exp(-itZ)]\bigg|_{t=0} = (-i)^\lambda E[Z^\lambda].$$

(ii) *Let Z be an $\overline{-\mathbb{H}}$-valued random variable. Assume that $E[|Z|^{\mathrm{Re}(\lambda)}] < +\infty$. Then*

$$\frac{\partial^\lambda}{\partial t^\lambda} E[\exp(itZ)]\bigg|_{t=0} = i^\lambda E[Z^\lambda].$$

Proof. We show (i); the proof of (ii) is similar. Let $f(t) := E[\exp(-itZ)]$. Let k be the integer part of $\mathrm{Re}(\lambda)$ and let $\delta := \lambda - k$. Then $0 < \mathrm{Re}(\delta) < 1$. By the assumption that $E[|Z|^{\mathrm{Re}(\lambda)}] < +\infty$, we see that $f \in C^k((-\infty, 0])$ and furthermore $f^{(k)}(t) = (-i)^k E[Z^k \exp(-itZ)]$ for $t \leqslant 0$.

We have

$$\int_0^\infty \left| \frac{1 - \exp(iuz)}{u^{1+\delta}} \right| du = |z|^{\mathrm{Re}(\delta)} \int_0^\infty \frac{|1 - \exp(ite^{i\theta})|}{t^{1+\mathrm{Re}(\delta)}}\, dt, \qquad z = re^{i\theta},\ \theta \in [0, \pi].$$

If $0 \leqslant t \leqslant 1$, then $\frac{|1-\exp(ite^{i\theta})|}{t^{1+\mathrm{Re}(\delta)}} \leqslant \frac{e}{t^{\mathrm{Re}(\delta)}}$. If $t > 1$, then $\frac{|1-\exp(ite^{i\theta})|}{t^{1+\mathrm{Re}(\delta)}} \leqslant \frac{2}{t^{1+\mathrm{Re}(\delta)}}$. Hence, we see that

$$\int_{\overline{\mathbb{H}}}\int_0^\infty |z|^k \left|\frac{1-\exp(iuz)}{u^{1+\delta}}\right| du\, P^Z(dz) < +\infty.$$

Therefore we can use Fubini's theorem to obtain

$$\begin{aligned}\frac{\partial^\lambda}{\partial t^\lambda} E[\exp(-itZ)]\bigg|_{t=0} &= \frac{(-i)^k\delta}{\Gamma(1-\delta)}\int_0^\infty \frac{E[Z^k(1-\exp(iuZ))]}{u^{1+\lambda}}\, du \\ &= \frac{(-i)^k\delta}{\Gamma(1-\delta)} E\left[Z^k \int_0^\infty \frac{1-\exp(iuZ)}{u^{1+\delta}}\, du\right].\end{aligned}$$

By the Cauchy integral theorem,

$$\begin{aligned}\int_0^\infty \frac{1-\exp(iuz)}{u^{1+\delta}}\, du &= z^\delta \int_{C_z} \frac{1-\exp(i\zeta)}{\zeta^{1+\delta}}\, d\zeta \\ &= z^\delta i^{-\delta}\int_0^\infty \frac{1-e^{-t}}{t^{1+\delta}}\, dt = \frac{\Gamma(1-\delta)}{\delta} z^\delta i^{-\delta}.\end{aligned} \tag{3.1}$$

Thus we have the assertion. ∎

We now give applications of Lemma 3.2 to power means of random variables. By Lemma 3.2(i), we immediately deduce

PROPOSITION 3.3. *Let* $\alpha \in \overline{\mathbb{H}}$ *and* $\lambda \in V$. *Assume that* $\mathrm{Re}(\lambda) \notin \mathbb{N}$ *and* $X \in L^{\mathrm{Re}(\lambda)}$. *Let k be the integer part of* $\mathrm{Re}(\lambda)$ *and let* $\delta := \lambda - k$. *Then*

$$E[(X+\alpha)^\lambda] = \frac{i^\delta\delta}{\Gamma(1-\delta)}\int_0^\infty \frac{E[(X+\alpha)^k(1-\exp(iu(X+\alpha)))]}{u^{1+\delta}}\, du.$$

We now give an application to power means of random variables. For $p = 0$, we regard $\left(\frac{1}{n}\sum_{j=1}^n z_j^p\right)^{1/p}$ as the geometric mean $\prod_{j=1}^n z_j^{1/n}$ for $z_1, \ldots, z_n \in \overline{\mathbb{H}}$.

THEOREM 3.4. *Let* $0 < p \leqslant 1$. *Let* $Z_1, \ldots, Z_n$ *be i.i.d.* $\overline{\mathbb{H}}$*-valued random variables such that* $Z_1 \in L^1$. *Then*

$$\frac{\partial^{1/p}}{\partial t^{1/p}} E\left[\exp\left(-i\frac{t}{n}Z_1^p\right)\right]^n\bigg|_{t=0} = (-i)^{1/p} E\left[\left(\frac{1}{n}\sum_{j=1}^n Z_j^p\right)^{1/p}\right]. \tag{3.2}$$

Furthermore, $E\left[\left(\frac{1}{n}\sum_{j=1}^n Z_j^p\right)^{1/p}\right]$ *is continuous with respect to p on* $[0,1]$.

If $p = 1/m$ for some natural number m, then $\frac{\partial^{1/p}}{\partial t^{1/p}}$ denotes the ordinary mth derivative. The proof of continuity with respect to the parameter does not involve fractional calculus.

Proof. The case that $p = 1/m$ for some natural number m is just the mth derivative of the characteristic function and the result is well-known [20, Chapter 16]. Let $Z := \frac{1}{n}\sum_{j=1}^n Z_j^p$. Then, by noting $0 < p < 1$, $Z \in \overline{\mathbb{H}}$. By the convexity of $x \mapsto |x|^{1/p}$ and the assumption that $Z_1 \in L^1$, we have $E[|Z|^{1/p}] < +\infty$. We can apply Lemma 3.2(i) to obtain (3.2).

We remark that $|Z_j| = (|Z_j|^p)^{1/p}$ and $1/p > 1$. Then, by the Hölder inequality,

$$\left|\left(\frac{1}{n}\sum_{j=1}^n Z_j^p\right)^{1/p}\right| \leqslant \left(\frac{1}{n}\sum_{j=1}^n |Z_j|^p\right)^{1/p} \leqslant \frac{1}{n}\sum_{j=1}^n |Z_j|.$$

By the assumption, $\frac{1}{n}\sum_{j=1}^n |Z_j|$ is integrable. We remark that

$$\begin{aligned}\lim_{p\to+0}\left(\frac{1}{n}\sum_{j=1}^n Z_j^p\right)^{1/p} &= \lim_{p\to+0}\exp\left(\frac{1}{p}\log\left(\frac{1}{n}\sum_{j=1}^n Z_j^p\right)\right)\\ &= \exp\left(\frac{d}{dp}\log\left(\frac{1}{n}\sum_{j=1}^n Z_j^p\right)\bigg|_{p=0}\right) = \exp\left(\frac{1}{n}\sum_{j=1}^n \log Z_j\right).\end{aligned}$$

The continuity assertion follows from the Lebesgue convergence theorem. ∎

As in the negative case, we state a result applicable to general distributions on $\mathbb{R}$ satisfying an integrability condition.

COROLLARY 3.5. *Let $0 < p < 1$. Let $\alpha \in \overline{\mathbb{H}}$. Let $X_1, \dots, X_n$ be i.i.d. $\overline{\mathbb{H}}$-valued random variables such that $X_1 \in L^1$. Then $\left(\frac{1}{n}\sum_{j=1}^n (X_j+\alpha)^p\right)^{1/p} \in L^1$ and*

$$\frac{\partial^{1/p}}{\partial t^{1/p}} E\left[\exp\left(-i\frac{t}{n}(X_1+\alpha)^p\right)\right]^n\bigg|_{t=0} = (-i)^{1/p} E\left[\left(\frac{1}{n}\sum_{j=1}^n (X_j+\alpha)^p\right)^{1/p}\right].$$

Furthermore, $E\left[\left(\frac{1}{n}\sum_{j=1}^n (X_j+\alpha)^p\right)^{1/p}\right]$ is continuous with respect to p on $[0, 1]$.

If $X_1 \notin L^1$, then $\left(\frac{1}{n}\sum_{j=1}^n (X_j+\alpha)^p\right)^{1/p} \notin L^1$; see [1, Proposition 5.1(ii)].

4. CHARACTERIZATIONS OF DISTRIBUTIONS ON $\mathbb{R}$

Throughout this section, we use λ for exponents of powers of complex-valued random variables. We first consider the case that $\mathrm{Re}(\lambda) < 0$. Recall that $U = i\mathbb{H} = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\}$.

We consider the analyticity of the moment with respect to the exponent λ. Let $U_M := \{z \in U : \mathrm{Re}(z) > -M\}$ for $M > 0$.

LEMMA 4.1. *Let Z be a complex-valued random variable. Assume that $E[|\mathrm{Im}(Z)|^{-M}] < +\infty$ and $P(Z \in \mathbb{R}) = 0$. Then the map $\lambda \mapsto E[Z^\lambda]$ is holomorphic on U_M, and furthermore*

$$\frac{d}{d\lambda}E[Z^\lambda] = E[Z^\lambda \log Z].$$

Proof. Let C be a Jordan curve in U_M. By (2.4), $E[Z^\lambda]$ is continuous on U_M and $E[\sup_{\lambda\in C}|Z^\lambda|] \leqslant E[|\mathrm{Im}(Z)|^{-M}] < +\infty$. By Fubini's theorem and the Cauchy integral theorem, $\int_C E[Z^\lambda]\, d\lambda = E[\int_C Z^\lambda\, d\lambda] = 0$. By Morera's theorem, $E[Z^\lambda]$ is holomorphic on U_M.

By the Cauchy integral formula and Fubini's theorem, we see that

$$\frac{d}{d\lambda}E[Z^\lambda] = \frac{1}{2\pi i}\int_C \frac{E[Z^\zeta]}{(\zeta-\lambda)^2}\, d\zeta = E\left[\frac{1}{2\pi i}\int_C \frac{Z^\zeta}{(\zeta-\lambda)^2}\, d\zeta\right] = E[Z^\lambda \log Z],$$

where C is a circle with center λ contained in U_M. ∎

There are several different proofs of this lemma. An alternative way is to use the integral expression of $E[Z^\lambda]$ obtained from Lemma 2.2. By (2.4), we can also show the following assertion in the same manner as the above lemma.

LEMMA 4.2. *Let $\lambda \in U$. Let Z be an $\overline{\mathbb{H}}$-valued random variable. Then the map $\alpha \mapsto E[(Z+\alpha)^\lambda]$ is holomorphic on $\mathbb{H}$, and furthermore*

$$\frac{d}{d\alpha}E[(Z+\alpha)^\lambda] = \lambda E[(Z+\alpha)^{\lambda-1}].$$

Let $\mathcal{P}(\mathbb{R})$ be the set of Borel probability measures on $\mathbb{R}$. Let

$$F_\eta(\alpha,\lambda) := \int_{\mathbb{R}} (x+\alpha)^\lambda\, \eta(dx), \qquad \alpha,\lambda \in \mathbb{C},\ \ \eta \in \mathcal{P}(\mathbb{R}),$$

if the integral exists. This is a function of two variables. Hereafter we *fix* either of the two variables. Let

$$\mathcal{F}_{(\cdot,\lambda)} := \{F_\eta(\cdot,\lambda) : \eta \in \mathcal{P}(\mathbb{R})\} \quad \text{and} \quad \mathcal{F}_{(\alpha,\cdot)} := \{F_\eta(\alpha,\cdot) : \eta \in \mathcal{P}(\mathbb{R})\}$$

for $\lambda, \alpha \in \mathbb{C}$. Let the characteristic function of $\eta \in \mathcal{P}(\mathbb{R})$ be

$$\varphi_\eta(t) := \int_{\mathbb{R}} \exp(itx)\, \eta(dx), \qquad t \in \mathbb{R}.$$

We remark that for every $\eta \in \mathcal{P}(\mathbb{R})$, $\alpha \in \mathbb{H}$, and $\lambda \in U$, the integral $F_\eta(\alpha,\lambda)$ is well-defined. By Lemmas 4.1 and 4.2, all elements of $\mathcal{F}_{(\cdot,\lambda)}$ and $\mathcal{F}_{(\alpha,\cdot)}$ are holomorphic functions on $\mathbb{H}$ and U, respectively.

Let D be a set and $\mathcal{F}$ be a class of complex-valued functions on D. We say that a subset S of D is a *determining set* of $(D,\mathcal{F})$ if it has the property that if $f, g \in \mathcal{F}$ and $f(x) = g(x)$ for every $x \in S$ then $f(x) = g(x)$ for every $x \in D$.

EXAMPLE 4.3. (i) Let D be a domain and $\mathcal{F}$ be the set of holomorphic functions on D. If S is a subset of D which has an accumulating point in D, then S is a determining set of $(D, \mathcal{F})$, by the identity theorem for holomorphic functions.

(ii) Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. For $a > 0$, let $\mathbb{H}_a := \{z \in \mathbb{H} : \mathrm{Im}(z) > a\}$ and $\varphi_a(z) := \frac{z-(a+1)i}{z-(a-1)i}$, $z \in \mathbb{H}_a$. The map $\varphi_a : \mathbb{H}_a \to \mathbb{D}$ is bi-holomorphic. Let $(z_n)_{n\geqslant 1}$ be a sequence in $\mathbb{H}_a$ such that $\sum_{n=1}^{\infty}(1 - |\varphi_a(z_n)|) = +\infty$. This means that the *Blaschke condition* fails for $(\varphi_a(z_n))_n$. For $\lambda \in (-\infty, 0)$ and $P \in \mathcal{P}(\mathbb{R})$, the map $w \mapsto F_P(\varphi_a^{-1}(w), \lambda)$ is holomorphic and *bounded* on $\mathbb{D}$, since $a > 0$. By [17, Theorem 15.23], the set $\{z_n\}_n$ is a determining set of $(\mathbb{H}_a, \mathcal{F}_{(\cdot,\lambda)})$.

For example, if $|\varphi_1(z_n)| = 1 - 1/n$ for each n, then $\{z_n\}_n$ is an unbounded determining set of $(\mathbb{H}_1, \mathcal{F}_{(\cdot,\lambda)})$. Since $\mathbb{H}_1 \subset \mathbb{H}$ and $\mathcal{F}_{(\cdot,\lambda)}$ is a class of holomorphic functions on $\mathbb{H}$, by the identity theorem for holomorphic functions, $\{z_n\}_n$ is an unbounded determining set of $(\mathbb{H}, \mathcal{F}_{(\cdot,\lambda)})$.

(iii) Let $(\lambda_n)_{n\geqslant 1}$ be a sequence in U such that $\sum_{n\geqslant 1} 1/(-\mathrm{Re}(\lambda_n)) = +\infty$ and $\sup_{n\geqslant 1} |\mathrm{Im}(\lambda_n)| < +\infty$. Then, by following the proof[1] of the Müntz–Szasz theorem in [17, Theorem 15.26], we see that $\{\lambda_n\}_{n\geqslant 1}$ is a determining set of $(U, \mathcal{F}_{(\alpha,\cdot)})$ for every $\alpha \in \mathbb{H}$.

Our motivation for introducing the notion of determining set comes from Example 4.3(ii), in which the determining set is a *divergent* sequence in $\mathbb{H}$. We consider a characterization of distributions.

THEOREM 4.4. *Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$. Then $\mu = \nu$ if either of the following two conditions holds:*

(i) *There exist a point $\lambda \in U$ and a determining set D_λ of $(\mathbb{H}, \mathcal{F}_{(\cdot,\lambda)})$ such that $F_\mu(\alpha, \lambda) = F_\nu(\alpha, \lambda)$ for every $\alpha \in D_\lambda$.*

(ii) *There exist a point $\alpha \in \mathbb{H}$ and a determining set D_α of $(U, \mathcal{F}_{(\alpha,\cdot)})$ such that $F_\mu(\alpha, \lambda) = F_\nu(\alpha, \lambda)$ for every $\lambda \in D_\alpha$.*

We do not need any moment assumptions for μ and ν. It is, for example, applicable to the Cauchy distributions. This is an extension of [14, Theorem 2.1], which is specific to the Cauchy distribution. The following proof is completely different from the proof of [14, Theorem 2.1], which depends on the Riesz–Markov–Kakutani theorem.

Proof of Theorem 4.4. Assume that (i) holds. By the assumption and Lemma 4.2, $F_\mu(\alpha, \lambda) = F_\nu(\alpha, \lambda)$ for every $\alpha \in \mathbb{H}$, in particular on the imaginary axis in $\mathbb{H}$. By Proposition 2.3,

$$\int_0^\infty t^{-\lambda-1}\varphi_\mu(t)\exp(-st)\,dt = \int_0^\infty t^{-\lambda-1}\varphi_\nu(t)\exp(-st)\,dt, \qquad s > 0.$$

[1]The Blaschke condition is crucial in it.

By the inversion formula for the Laplace transform, $\varphi_\mu(t) = \varphi_\nu(t)$ for $t > 0$. Hence, $\varphi_\mu(t) = \varphi_\nu(t)$ for $t \in \mathbb{R}$. By the Lévy inversion formula, $\mu = \nu$.

Assume that (ii) holds. The proof is identical to the above one, except for using the inversion formula for the Mellin transform. By the assumption and Lemma 4.1, $F_\mu(\alpha, \lambda) = F_\nu(\alpha, \lambda)$ for $\lambda \in (-\infty, 0)$. By Proposition 2.3,

$$\int_0^\infty t^{-\lambda-1}\varphi_\mu(t)\exp(i\alpha t)\,dt = \int_0^\infty t^{-\lambda-1}\varphi_\nu(t)\exp(i\alpha t)\,dt, \qquad \lambda \in (-\infty, 0).$$

We remark that

$$|(\varphi_\mu(t) - \varphi_\nu(t))\exp(i\alpha t)| \leqslant 2\exp(-t\,\mathrm{Im}(\alpha)).$$

By the inversion formula for the Mellin transform, $\varphi_\mu(t) = \varphi_\nu(t)$ for $t > 0$. Hence, $\mu = \nu$. ■

We next consider the case where $\mathrm{Re}(\lambda) > 0$. Recall that $V = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$. Let $V_M := \{\lambda \in V : \mathrm{Re}(\lambda) < M\}$ for $M > 0$. We let $z^\lambda \log z := 0$ if $z = 0$ and $\lambda \in V$. We can also show the following in the same manner as in the proof of Lemma 4.1.

LEMMA 4.5. *Let Z be a complex-valued random variable. Assume that $E[|Z|^M] < +\infty$. Then the map $\lambda \mapsto E[Z^\lambda]$ is holomorphic on V_M, and furthermore*

$$\frac{d}{d\lambda}E[Z^\lambda] = E[Z^\lambda \log Z].$$

There are some differences from Lemma 4.1. In the above assertion, we need to assume the integrability condition for Z. On the other hand, Z can take real values with positive probability.

We also have the following assertion which corresponds to Lemma 4.2.

LEMMA 4.6. *Let $\lambda \in V$. Let Z be an $\overline{\mathbb{H}}$-valued random variable such that $E[|Z|^{\mathrm{Re}(\lambda)}] < +\infty$. Then the map $\alpha \mapsto E[(Z+\alpha)^\lambda]$ is holomorphic on $\mathbb{H}$, and furthermore*

$$\frac{d}{d\alpha}E[(Z+\alpha)^\lambda] = \lambda E[(Z+\alpha)^{\lambda-1}]. \tag{4.1}$$

The following results deal with the positive moment case. We assume a moment condition for distributions.

THEOREM 4.7. *Let $M > 0$. Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ be such that*

$$\int_{\mathbb{R}} |x|^M\,\mu(dx) + \int_{\mathbb{R}} |x|^M\,\nu(dx) < +\infty.$$

Then $\mu = \nu$ if either of the following two conditions holds:

(i) *There exist a point* $\lambda \in V_M$ *and a determining set* D_λ *of* $(\mathbb{H}, \mathcal{F}_{(\cdot,\lambda)})$ *such that* $\mathrm{Re}(\lambda) \notin \mathbb{N}$ *and* $F_\mu(\alpha, \lambda) = F_\nu(\alpha, \lambda)$ *for every* $\alpha \in D_\lambda$.

(ii) *There exist a point* $\alpha \in \overline{\mathbb{H}}$ *and a determining set* D_α *of* $(V_M, \mathcal{F}_{(\alpha,\cdot)})$*such that* $F_\mu(\alpha, \lambda) = F_\nu(\alpha, \lambda)$ *for every* $\lambda \in D_\alpha$.

This is an extension of [14, Theorem 3.1], which was specific to the Cauchy distribution. By Lemmas 4.5 and 4.6, we can apply Example 4.3 as examples of determining sets. The following proof is similar to that of Theorem 4.4, but is a little more involved and rather different from the strategy taken in [14, proof of Theorem 3.1].

Proof of Theorem 4.7. Assume that (i) holds. By the assumption and Lemma 4.6,

$$\int_{\mathbb{R}} (x+\alpha)^\lambda \, \mu(dx) = \int_{\mathbb{R}} (x+\alpha)^\lambda \, \nu(dx) \tag{4.2}$$

for every $\alpha \in \mathbb{H}$. Let k be the integer part of $\mathrm{Re}(\lambda)$ and $\delta = \mathrm{Re}(\lambda) - k$. By the assumption, $0 < \mathrm{Re}(\delta) < 1$. Recall (4.1). By differentiating the two expectations in (4.2) with respect to α k times, we obtain

$$\int_{\mathbb{R}} (x+\alpha)^\delta \, \mu(dx) = \int_{\mathbb{R}} (x+\alpha)^\delta \, \nu(dx), \qquad \alpha \in \mathbb{H}.$$

By Proposition 3.3,

$$\int_0^\infty \frac{1 - \varphi_\mu(t)\exp(it\alpha)}{t^{1+\delta}} \, dt = \int_0^\infty \frac{1 - \varphi_\nu(t)\exp(it\alpha)}{t^{1+\delta}} \, dt, \qquad \alpha \in \mathbb{H}.$$

By differentiating these two integrals with respect to α, we find that

$$\int_0^\infty \frac{\varphi_\mu(t)\exp(it\alpha)}{t^{\delta}} \, dt = \int_0^\infty \frac{\varphi_\nu(t)\exp(it\alpha)}{t^{\delta}} \, dt, \qquad \alpha \in \mathbb{H},$$

in particular on the imaginary axis in $\mathbb{H}$. Hence,

$$\int_0^\infty \frac{\varphi_\mu(t)}{t^{\delta}} \exp(-tx) \, dt = \int_0^\infty \frac{\varphi_\mu(t)}{t^{\delta}} \exp(-tx) \, dt, \qquad x > 0.$$

Now by the uniqueness of the Laplace transform, $\varphi_\mu(t) = \varphi_\nu(t)$ for every $t > 0$, and hence $\mu = \nu$.

Assume that (ii) holds. By the assumption and Lemma 4.5, we see that $F_\mu(\alpha, \lambda) = F_\nu(\alpha, \lambda)$ for every $\lambda \in V_M$.

By Proposition 3.3,

$$\int_0^\infty \frac{1 - \varphi_\mu(t)\exp(it\alpha)}{t^{1+\delta}} \, dt = \int_0^\infty \frac{1 - \varphi_\nu(t)\exp(it\alpha)}{t^{1+\delta}} \, dt$$

for every $\delta \in V$ such that $0 < \mathrm{Re}(\delta) < \min\{1, M\}$. By the inversion formula for the Mellin transform, $\varphi_\mu(t) = \varphi_\nu(t)$ for every $t > 0$. Hence, $\mu = \nu$. ■

REMARK 4.8. (i) In condition (i) of Theorem 4.7, we need the assumption that $\mathrm{Re}(\lambda) \notin \mathbb{N}$. If $\lambda = \mathrm{Re}(\lambda) = 1$, then $E[(X+\alpha)^\lambda] = E[X]+\alpha$, so even if we move the value of α, we only know about the value of $E[X]$ and we cannot identify the distribution of X.

(ii) In condition (ii) of Theorem 4.7, α can be a real number.

(iii) The approach taken in the proof of [14, Theorem 3.1] was similar to that of [9, Theorem 1]. However, we cannot take these approaches here.

5. APPLICATIONS

In this section, we compute the expectations of the power means of the Cauchy distribution, the t-distribution whose degree of freedom is 3, and the Poincaré distribution. We first give an informal and heuristic argument. Let $(Z_n)_n$ be i.i.d. $\mathbb{H}$-valued random variables. We will show that there exists a constant β and an interval I such that $E[\exp(itZ_1)] = \exp(i\beta t)$ for every $t \in I$. Then, by the fractional derivative or complex analysis, $E[Z_1^p] = \beta^p$ for every p. By Taylor expansion, $E[\exp(itZ_1^p)] = \exp(i\beta^p t)$. Finally, by the fractional derivative again, we see that $E\big[\big(\frac{1}{n}\sum_{j=1}^n Z_j^p\big)^{1/p}\big] = \beta$ for every n and p. We need some modifications in the case of Cauchy distributions.

5.1. Cauchy distributions. We deal with the Cauchy distribution with location μ and scale σ. For $\mu \in \mathbb{R}$ and $\sigma > 0$, its probability density function $p(x)$ is given by

$$p(x) = \frac{\sigma}{\pi}\frac{1}{(x-\mu)^2+\sigma^2}, \qquad x \in \mathbb{R}.$$

Let $\gamma := \mu + \sigma i$.

THEOREM 5.1 ([1, Theorem 7.1]). *Let $\alpha \in \mathbb{H}$ and $-1 \leqslant p < 0$. Let $n \geqslant 2$. Let $X_1, \ldots, X_n$ be i.i.d. random variables following the Cauchy distribution with location μ and scale σ. Then*

$$E\left[\left(\frac{1}{n}\sum_{j=1}^n (X_j+\alpha)^p\right)^{1/p}\right] = \gamma + \alpha.$$

In [1], this assertion is shown by repeated use of the Cauchy integral formula. Here, by Corollary 2.7, we can naturally anticipate that the assertion holds. However, we cannot apply Corollary 2.7 directly, because $(p-1)/(np) < 1$ fails for some n.

Proof of Theorem 5.1. Let $Z := \frac{1}{n}\sum_{j=1}^n (X_j+\alpha)^p$. Let $P_M := P(\cdot|X_1 \leqslant M)$ for $M > 0$. We remark that $\mathrm{Im}((x+\alpha)^p) \leqslant 0$ for every $x \in \mathbb{R}$. Then there exists a positive constant c_M such that if $X_1 \leqslant M$ then

$$\operatorname{Im}(Z) \leqslant \frac{\operatorname{Im}((X_1+\alpha)^p)}{n} \leqslant -c_M < 0.$$

Therefore we can apply Lemma 2.2(ii) to obtain

$$\frac{\partial^{1/p}}{\partial t^{1/p}} E_M[\exp(itZ)]\bigg|_{t=0} = i^{-1/p} E_M[Z^{1/p}],$$

where E_M is the expectation with respect to P_M. This means that

$$\frac{1}{\Gamma(-1/p)} \int_0^\infty t^{-1/p-1} E_M[\exp(-itZ)]\,dt = i^{-1/p} E_M[Z^{1/p}]. \tag{5.1}$$

By the Lebesgue convergence theorem,

$$\lim_{M\to+\infty} E_M[Z^{1/p}] = E[Z^{1/p}], \tag{5.2}$$

and

$$\lim_{M\to+\infty} E_M\left[\exp\left(i\frac{t}{n}(X_1+\alpha)^p\right)\right] = E\left[\exp\left(i\frac{t}{n}(X_1+\alpha)^p\right)\right]. \tag{5.3}$$

By the Cauchy integral formula[2], for every $t \leqslant 0$,

$$E[\exp(it(X_1+\alpha)^p)] = E[\exp(itZ)] = \exp(it(\gamma+\alpha)^p).$$

From this, (5.3) and

$$E_M[\exp(itZ)] = \exp\left(i\frac{n-1}{n}t(\gamma+\alpha)^p\right) E_M\left[\exp\left(i\frac{t}{n}(X_1+\alpha)^p\right)\right],$$

we see that

$$\lim_{M\to+\infty} E_M[\exp(itZ)] = \exp(it(\gamma+\alpha)^p). \tag{5.4}$$

Since $\left|E_M\left[\exp\left(i\frac{t}{n}(X_1+\alpha)^p\right)\right]\right| \leqslant 1$, we find that

$$|u^{-1/p-1} E_M[\exp(-itZ)]| \leqslant t^{-1/p-1}\exp(t\sin(p\theta_0)),$$

where $\arg(\gamma+\alpha) = \theta_0 \in (0,\pi)$.

---

[2]As an alternative proof, we can also show the equality by using Taylor expansion; see the proof of Theorem 5.4 below. The alternative proof uses $E[(X_1+\alpha)^r] = (\gamma+\alpha)^r$ for every $r < 0$.

In view of this and (5.4), we can apply the Lebesgue convergence theorem to obtain

$$\lim_{M\to\infty}\int_0^\infty t^{-1/p-1}E_M[\exp(-itZ)]\,dt = \int_0^\infty t^{-1/p-1}\exp(-it(\gamma+\alpha)^p)\,dt. \tag{5.5}$$

By (2.2),

$$\int_0^\infty t^{-1/p-1}\exp(-it(\gamma+\alpha)^p)\,dt = (\gamma+\alpha)i^{-1/p}\Gamma\left(-\frac{1}{p}\right). \tag{5.6}$$

By (5.1), (5.2), (5.5) and (5.6), we get the assertion. ■

REMARK 5.2. (i) Let φ be the characteristic function of X_1. Then $\varphi(t) = \exp(i\gamma t)$ for every $t \geqslant 0$. By Lemma 2.2(i), we have $E[(X_1+\alpha)^p]^{1/p} = \gamma+\alpha$ for $p \in (-1,0)$ and $\alpha \in \mathbb{H}$. This is an alternative derivation of [1, (6.1)].

(ii) The *positive* power mean is more difficult since $X_1 \notin L^1$. [8, Theorem 2.2 (a)] does not hold for an $\mathbb{H}$-valued random variable $Z = \frac{1}{n}\sum_{j=1}^n (X_j+\alpha)^p$. Let $p>1$. Assume that $p\notin\mathbb{N}$. Let k be the integer part of $1/p$. Then $(\sum_{j=1}^n (X_j+\alpha)^p)^k \in L^1$, and

$$\frac{\partial^{1/p}}{\partial t^{1/p}}E[\exp(-itZ)]\Big|_{t=0} = \frac{\partial^{1/p}}{\partial t^{1/p}}\exp(-it(\gamma+\alpha)^p)\Big|_{t=0} = (-i)^{1/p}(\gamma+\alpha).$$

However, $(\sum_{j=1}^n (X_j+\alpha)^p)^{1/p} \notin L^1$.

5.2. t-distributions. We consider the location-scale family of a slightly modified t-distribution whose degree of freedom is 3. Let $f(x) := \frac{2}{\pi}(1+x^2)^{-2}$ for $x\in\mathbb{R}$. Let $\mu\in\mathbb{R}$ and $\sigma>0$. Let $\gamma := \mu+\sigma i$. Let

$$p(x) := \frac{1}{\sigma}f\left(\frac{x-\mu}{\sigma}\right) = \frac{2\sigma^3}{\pi}\frac{1}{|x-\gamma|^4}, \quad x\in\mathbb{R}.$$

Let $X_1, X_2, \ldots$ be i.i.d. random variables such that X_1 has density function $p(x)$. Let $\alpha\in\mathbb{H}$ and $p<0$.

Since we can argue in the same manner as in the case of Cauchy distributions, we only give a sketch. By the Cauchy integral formula,

$$E[(X_1+\alpha)^p] = (\gamma+\alpha)^{p-1}(\gamma+\alpha-ip\sigma)$$

and

$$E[\exp(it(X_1+\alpha)^p)] = (1+p\sigma(\gamma+\alpha)^{p-1}t)\exp(i(\gamma+\alpha)^p t), \quad t\geqslant 0.$$

As before, let $Z := \frac{1}{n}\sum_{j=1}^n (X_j+\alpha)^p$. Then

$$E[\exp(itZ)] = \sum_{k=0}^n \binom{n}{k}\left(\frac{p(\gamma+\alpha)^{p-1}}{n}\right)^k t^k\exp(i(\gamma+\alpha)^p t).$$

From this and (2.2),

$$\int_0^\infty t^{-1/p-1} E[\exp(itZ)]dt = i^{-1/p} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{ip}{n(\gamma+\alpha)}\right)^k \Gamma\left(k - \frac{1}{p}\right).$$

Thus we see that

$$\begin{aligned} (5.7) \quad & E\left[\left(\frac{1}{n}\sum_{j=1}^{n}(X_j+\alpha)^p\right)^{1/p}\right] \\ & = \frac{\gamma+\alpha}{\Gamma(-1/p)} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{ip}{n(\gamma+\alpha)}\right)^k \Gamma\left(k - \frac{1}{p}\right). \end{aligned}$$

Since $\frac{p^k\Gamma(k-1/p)}{\Gamma(-1/p)} = \prod_{j=0}^{k-1}(jp-1)$, the right hand side of (5.7) is a polynomial in p of degree $n-1$ with complex coefficients. The coefficient of the highest degree is $\frac{(n-1)!i^n}{n^n(\gamma+\alpha)^{n-1}}$. Hence, $E\left[\left(\frac{1}{n}\sum_{j=1}^{n}(X_j+\alpha)^p\right)^{1/p}\right]$ is not a constant function with respect to p. The same arguments apply to the t-distribution whose degree of freedom is an odd number, but the expression of the expectation would be more complicated.

5.3. Poincaré distributions. The probability density function of a Poincaré distribution with a parameter $\theta = (a, b, c)$ is given by

$$p_\theta(x,y) := \frac{D\exp(2D)}{\pi} \exp\left(-\frac{a(x^2+y^2)+2bx+c}{y}\right)\frac{1}{y^2}, \quad x \in \mathbb{R},\ y > 0,$$

where θ belongs to the parameter space $\Theta := \{(a,b,c) \in \mathbb{R}^3 : a > 0,\ c > 0,\ ac - b^2 > 0$ and we let $D := \sqrt{ac-b^2}$.

This is the upper half-plane realization of two-dimensional hyperboloid distributions and was introduced by [18]. The family is compatible with the Poincaré metric on $\mathbb{H}$. Properties of the Poincaré distribution have also been investigated in [19, 13].

PROPOSITION 5.3. *Let Z be an $\mathbb{H}$-valued random variable following a Poincaré distribution with a parameter $(a, b, c) \in \Theta$. Then*

$$E[\exp(itZ)] = \exp\left(-i\frac{b}{a}t - \frac{D}{a}t\right) \quad \text{for } t \geqslant 0.$$

Proof. We have $E[\exp(itZ)] = \int_{\mathbb{H}} \exp(itx - ty)p_\theta(x,y)\,dx\,dy$. We first calculate the integral with respect to x and we obtain

$$\begin{aligned} & \int_{\mathbb{R}} \exp(itx)\exp\left(-\frac{a(x^2+y^2)+2bx+c}{y}\right)dx \\ & = \sqrt{\frac{\pi y}{a}}\exp\left(-i\frac{b}{a}t - \left(a + \frac{t^2}{4a}\right)y - \frac{D^2}{ay}\right). \end{aligned}$$

We then integrate the above function with respect to y to find that

$$\int_0^\infty \frac{1}{y^{3/2}} \exp\left(-\frac{(t+2a)^2}{4a} y - \frac{D^2}{ay}\right) dy = \frac{\sqrt{\pi a}}{D} \exp\left(-\frac{D(t+2a)}{a}\right).$$

The assertion follows from this. ■

THEOREM 5.4. *Let $n \geqslant 2$. Let $Z_1, \dots, Z_n$ be i.i.d. $\mathbb{H}$-valued random variables following a Poincaré distribution with a parameter $(a, b, c) \in \Theta$. Then the following hold:*

(i) *For every p with $p \neq 0$,*

$$E[Z_1^p] = \left(-\frac{b}{a} + \frac{D}{a} i\right)^p. \tag{5.8}$$

(ii) $E[\exp(t|Z_1|^p)] < +\infty$ *for every $t > 0$ and p with $0 < |p| < 1$.*

(iii) *For every p with $0 < |p| \leqslant 1$,*

$$E\left[\left(\frac{1}{n}\sum_{j=1}^n Z_j^p\right)^{1/p}\right] = -\frac{b}{a} + \frac{D}{a} i. \tag{5.9}$$

For ease of notation, we let

$$I(a,b,c) := \int_{\mathbb{H}} \exp\left(-\frac{a(x^2+y^2)+2bx+c}{y}\right) \frac{1}{y^2}\, dx\, dy, \qquad \theta = (a,b,c) \in \Theta.$$

We show these assertions by dividing into cases according to the sign of p.

Proof for $p < 0$. (i) We will show that $E[\mathrm{Im}(Z_1)^{-\lambda}] < +\infty$ for every $\lambda > 0$. We see that

$$\begin{aligned} E[\mathrm{Im}(Z_1)^{-\lambda}] &\leqslant \frac{D\exp(2D)}{\pi} \int_{\mathbb{H}} \frac{1}{y^{2+\lambda}} \exp\left(-\frac{a(x^2+y^2)+2bx+c}{y}\right) dx\, dy \\ &< +\infty, \end{aligned}$$

because $\sup_{y>0} y^{-2-\lambda} \exp(-\varepsilon/y) < +\infty$ for every $\varepsilon > 0$, and $I(a,b,c-\varepsilon) < +\infty$ for sufficiently small $\varepsilon > 0$.

Therefore we can apply Lemma 2.2(i). By recalling Proposition 5.3 and (2.2), we get (5.8).

(ii) For every $t > 0$,

$$\begin{aligned} &E[\exp(t|Z_1|^p)] \leqslant E[\exp(t\,\mathrm{Im}(Z_1)^p)] \\ &\quad = \frac{D\exp(2D)}{\pi} \int_{\mathbb{H}} \frac{\exp(ty^p)}{y^2} \exp\left(-\frac{a(x^2+y^2)+2bx+c}{y}\right) dx\, dy < +\infty, \end{aligned}$$

because $\sup_{y>0}\exp(ty^p-\varepsilon/y)<+\infty$ for every $\varepsilon>0$, and $I(a,b,c-\varepsilon)<+\infty$ for sufficiently small $\varepsilon>0$.

(iii) For $n\in\mathbb{N}$ and $p\in[-1,1]$, we have[3] $z^{np}=(z^p)^n$ for every $z\in\mathbb{C}$. By Taylor expansion, (i) and (ii), we see that for every $u\geqslant 0$,

$$E[\exp(iuZ_1^p)]=\sum_{k=0}^{\infty}\frac{(iu)^kE[Z_1^{pk}]}{k!}=\sum_{k=0}^{\infty}\frac{(iu)^k\left(-\frac{b}{a}+\frac{D}{a}i\right)^{pk}}{k!}=\exp\left(iu\left(-\frac{b}{a}+\frac{D}{a}i\right)^p\right). \tag{5.10}$$

We remark that

$$\int_{\mathbb{H}}\frac{(x^2+y^2)^{(1-1/p)/2}}{y^{2-1/p}}\exp\left(-\frac{a(x^2+y^2)+2bx+c}{y}\right)dx\,dy<+\infty,$$

because for every $\varepsilon>0$ we have $\sup_{x+iy\in\mathbb{H}}(x^2+y^2)^{(1-1/p)/2}\exp\left(-\varepsilon\frac{x^2+y^2}{y}\right)<+\infty$ and $\sup_{y>0}y^{1/p}\exp\left(-\varepsilon/y\right)<+\infty$, and for sufficiently small $\varepsilon>0$, $I(a-\varepsilon,b,c-\varepsilon)<+\infty$. Hence, $E[\mathrm{Im}(Z_1)^{1/p}|Z_1|^{1-1/p}]<+\infty$. Let $Z:=\frac{1}{n}\sum_{j=1}^{n}Z_j^p$. By Theorem 2.5, we see that

$$\frac{\partial^{1/p}}{\partial t^{1/p}}E\left[\exp\left(i\frac{t}{n}Z_1^p\right)\right]^n\bigg|_{t=0}=i^{-1/p}E[Z^{1/p}].$$

By (5.10),

$$E\left[\exp\left(i\frac{t}{n}Z_1^p\right)\right]^n=\exp\left(it\left(-\frac{b}{a}+\frac{D}{a}i\right)^p\right).$$

From this and (2.2), we deduce (5.9). ∎

Proof for $p>0$. We first show

LEMMA 5.5. *$E[\exp(t|Z_1|)]<+\infty$ for sufficiently small $t>0$.*

Proof. Let $\varepsilon>0$ be such that $I(a-\varepsilon,b,c)<+\infty$. Then

$$E\left[\exp\left(\frac{\varepsilon}{2}|Z_1|\right)\right]\leqslant\frac{D\exp(2D)}{\pi}I(a-\varepsilon,b,c)<+\infty,$$

because $\sup_{x+yi\in\mathbb{H}}\exp\left(\frac{\varepsilon}{2}\sqrt{x^2+y^2}-\varepsilon\frac{x^2+y^2}{y}\right)\leqslant 1$. ∎

(i) If p is an integer, then we have (5.8) by Proposition 5.3 and Lemma 5.5. Assume that p is not an integer. By Lemma 5.5, $E[\mathrm{Im}(Z_1)^p]\leqslant E[|Z_1|^p]<+\infty$. Therefore we can apply Lemma 3.2(i) to obtain

$$\frac{\partial^p}{\partial t^p}E[\exp(-itZ_1)]\bigg|_{t=0}=(-i)^pE[Z_1^p].$$

From this, Proposition 5.3, and (2.2), we get (5.8).

[3] However, it is *not* true that $z^{np}=(z^n)^p$.

(ii) follows from Lemma 5.5.

(iii) The case of $p = 1$ is easy. Assume that $0 < p < 1$. Let $Z := \frac{1}{n}\sum_{j=1}^{n} Z_j^p$. By Taylor expansion, (i) and (ii), we see that for every $u \geqslant 0$,

$$E[\exp(iuZ_1^p)] = \sum_{k=0}^{\infty} \frac{(iu)^k E[Z_1^{pk}]}{k!} = \exp\biggl(iu\Bigl(-\frac{b}{a} + \frac{D}{a}i\Bigr)^p\biggr). \tag{5.11}$$

By Hölder's inequality,

$$\begin{aligned} E[\mathrm{Im}(Z)^{1/p}] &\leqslant E[|Z|^{1/p}] \leqslant E\biggl[\biggl(\frac{1}{n}\sum_{j=1}^{n}|Z_j|^p\biggr)^{1/p}\biggr] \leqslant E\biggl[\frac{1}{n}\sum_{j=1}^{n}|Z_j|\biggr] \\ &= E[|Z_1|] < +\infty. \end{aligned}$$

Therefore we can apply Theorem 3.4 to obtain

$$\frac{\partial^{1/p}}{\partial t^{1/p}} E\biggl[\exp\Bigl(-i\frac{t}{n}Z_1^p\Bigr)\biggr]^n\bigg|_{t=0} = (-i)^p E[Z^{1/p}].$$

By (5.11),

$$E\biggl[\exp\Bigl(-i\frac{t}{n}Z_1^p\Bigr)\biggr]^n = \exp\biggl(-it\Bigl(-\frac{b}{a} + \frac{D}{a}i\Bigr)^p\biggr).$$

From this and (3.1), we get (5.9). ∎

REMARK 5.6. (i) The case $p = 0$ corresponds to the geometric mean $E[Z_1^{1/n}]^n$, and the value is $-\frac{b}{a} + \frac{D}{a}i$ by (5.8).

(ii) Assume that Z and W are $\mathbb{H}$-valued. It can happen that $E[\exp(itZ)] = E[\exp(itW)]$ for every $t \geqslant 0$, but the distributions of Z and W are different, as in the above case. If $E[\exp(itZ)] = E[\exp(itW)]$ for every $t \geqslant 0$, then, by Lemmas 2.2(i) and 3.2(i), $E[Z^p] = E[W^p]$ for every $p \in \mathbb{R}$.

(iii) Numerical computations show that (5.9) fails if $|p| > 1$.

6. APPENDIX: COMPARISON WITH FRACTIONAL ABSOLUTE MOMENTS

It is natural to compare the fractional absolute moment $E[|Z^p|] = E[|Z|^p]$ with the absolute value of the fractional moment $|E[Z^p]|$ for $p \in \mathbb{R}$.

PROPOSITION 6.1. *Assume that* p *is a real number with* $|p| \leqslant 1$. *Let* Z *be an* $\overline{\mathbb{H}}$*-valued random variable or a* $-\overline{\mathbb{H}}$*-valued random variable. Then*

$$E[|Z|^p] \leqslant \frac{|E[Z^p]|}{\cos(p\pi/2)}. \tag{6.1}$$

Proof. The case $p \in \{0, \pm 1\}$ is easy. Assume that Z is $\overline{\mathbb{H}}$-valued and $0 < p < 1$; the other cases are shown in the same manner.

Let $n \geqslant 1$ and $Z_1, \dots, Z_n$ be i.i.d. copies of Z. Let $\xi_i := Z_i/|Z_i|$ if $Z_i \neq 0$ and $\xi_i := 1$ if $Z_i = 0$. Let $s_i := |Z_i|/\sum_{i=1}^n |Z_i|$ if $\sum_{i=1}^n |Z_i| > 0$ and $s_i := 1/n$ if $\sum_{i=1}^n |Z_i| = 0$. Then

$$\sum_{i=1}^n Z_i^p = \Big(\sum_{i=1}^n |Z_i|^p\Big)\Big(\sum_{i=1}^n s_i \xi_i^p\Big).$$

We remark that $\sum_{i=1}^n s_i \xi_i^p$ is in the convex hull of $\{\xi_i^p\}_{i=1}^n$ and the convex hull is contained in the region surrounded by the arc $\{\exp(i\theta) : 0 \leqslant \theta \leqslant p\pi\}$ and the line segment connecting 1 and $(-1)^p = \exp(ip\pi)$. Hence, $|\sum_{i=1}^n s_i \xi_i^p| \geqslant \cos(p\pi/2)$ and $|\sum_{i=1}^n Z_i^p| \geqslant (\sum_{i=1}^n |Z_i|^p)\cos(p\pi/2)$. Now the assertion follows from the law of large numbers. ∎

In the same manner, we show

PROPOSITION 6.2. *Assume that p is a real number with $|p| < 1/2$. Let Z be a complex-valued random variable. Then*

$$E[|Z|^p] \leqslant \frac{|E[Z^p]|}{\cos(p\pi)}.$$

REMARK 6.3. (i) If $|p| \leqslant 1$ and $P(Z = 1) = P(Z = -1) = 1/2$, then $E[|Z|^p] = 1$ and $|E[Z^p]| = |1 + (-1)^p|/2 = \cos(p\pi/2)$. Hence the bound in (6.1) is sharp in general.

(ii) If $1/2 < p \leqslant 1$ and $P(Z = 1) = P(Z = \exp(i(1 - 1/p)\pi)) = 1/2$, then $E[Z^p] = 0$ and $E[|Z|^p] = 1$. The case $-1 \leqslant p < 1/2$ is the same.

We give an application of Proposition 6.1. Let $(Z_n)_n$ be $\overline{\mathbb{H}}$-valued i.i.d. random variables such that $Z_1 \in L^p$ for some $p > 0$. Let $\ell \in \mathbb{N}$ be such that $Z_1 \in L^{1/\ell}$. Let $Y_j := Z_j^{1/(j+\ell)}/E[Z_j^{1/(j+\ell)}]$ for $j \geqslant 1$, and $M_n := \prod_{j=1}^n Y_j$ for $n \geqslant 1$. Let $\mathcal{F}_n := \sigma(Z_1, \dots, Z_n)$. Then $(M_n, \mathcal{F}_n)_n$ is a martingale. By (6.1),

$$\sup_{n \geqslant 1} E[|Z_n|] \leqslant \prod_{j=1}^\infty \frac{1}{\cos\big(\frac{\pi}{2}(j+\ell)\big)} \leqslant C \sum_{j=1}^\infty \frac{1}{(j+\ell)^2} < +\infty.$$

By the martingale convergence theorem [20, Chapter 11], $(Z_n)_n$ converges almost surely as $n \to \infty$. By Kronecker's lemma,

$$\prod_{j=1}^n \frac{Z_j^{1/n}}{E[Z_j^{1/(j+\ell)}]^{(j+\ell)/n}} \to 1 \quad \text{a.s. as } n \to \infty.$$

Since $Z_1 \in L^{1/\ell}$, we get

$$\prod_{j=1}^{n} Z_j^{1/n} \to \exp(E[\log Z_1]) \quad \text{a.s. as } n \to \infty.$$

This gives a martingale proof of the strong law of large numbers *for the geometric means*.

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