PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 44, Fasc. 1 (2024), pp. 87–118
Published online 7.8.2024
doi:10.37190/0208-4147.00174

COMPLETE f-MOMENT CONVERGENCE FOR ARRAYS OF RANDOM VARIABLES AND ITS APPLICATIONS IN SEMIPARAMETRIC AND EV REGRESSION MODELS*

BY

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Abstract. We study the complete f-moment convergence for arrays of rowwise random variables satisfying a Rosenthal type moment inequality, and then establish general results on the complete moment convergence and complete convergence for partial sums and weighted sums of arrays of rowwise random variables. As applications, we further describe the statistical properties of complete f-moment convergence in both semiparametric regression models and simple linear errors-in-variables models. The asymptotic properties for estimators are established. We also provide some simulations to verify the validity of the theoretical results.

2020 Mathematics Subject Classification: Primary 60F15; Secondary 62F12, 62G20.

Key words and phrases: complete convergence, complete f-moment convergence, strong consistency, semiparametric regression models, simple linear errors-in-variables models.

1. INTRODUCTION

Limit theory is an important research direction of probability theory and mathematical statistics, which mainly concerns the convergence of random variable sequences and sequences of distribution functions. In this work, we study a new type of convergence called complete f-moment convergence, which is more general and much stronger than both complete convergence and complete moment convergence. First of all, we recall some classical convergence concepts.

1.1. Classical convergence concepts. The concept of complete convergence was introduced by Hsu and Robbins (1947) as follows:

* Supported by the National Social Science Foundation of China (22BTJ059).

DEFINITION 1.1. A sequence $\{X_n, n \geqslant 1\}$ of random variables *converges completely* to the constant C if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - C| > \varepsilon) < \infty.$$

By the Borel–Cantelli lemma, this implies that $X_n \to C$ almost surely and thus complete convergence is stronger than almost sure convergence. As complete convergence is an important tool in establishing almost sure convergence of random variables, this result has been extended by many authors. For example, Hsu and Robbins (1947) proved that if $\{X, X_n, n \geqslant 1\}$ is a sequence of independent and identically distributed random variables with $EX_1 = \mu$ and $EX^2 < \infty$, then $\frac{1}{n}\sum_{k=1}^{n} X_k \to \mu$ completely; Erdős (1949) gave the corresponding converse statement; Baum and Katz (1965) studied conditions equivalent to complete convergence; Sung (2010) obtained a complete convergence result for weighted sums of identically distributed ρ^*-mixing random variables; Wang et al. (2014) investigated complete convergence for arrays of rowwise negatively superadditive dependent (NSD, for short) random variables; Miao et al. (2022) and Chang and Miao (2023) provided some interesting results on complete convergence for dependent random variables with general moment conditions, and so forth.

Chow (1988) introduced a more general concept named complete moment convergence:

DEFINITION 1.2. Suppose that $\{X_n, n \geqslant 1\}$ is a sequence of random variables and $a_n > 0, b_n > 0, q > 0$. We say that $\{X_n, n \geqslant 1\}$ *converges moment completely* if

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|X_n| - \varepsilon\}_+^q < \infty \quad \text{for all } \varepsilon > 0,$$

where $a_+ = \max\{0, a\}$.

Sung (2009) proved that complete moment convergence implies complete convergence. Properties of complete moment convergence have been obtained by many scholars. For instance, Li and Zhang (2004) investigated the complete moment convergence of moving average processes under the condition of negative association (NA); Qiu and Chen (2014) established complete convergence and complete moment convergence for weighted sums of widely orthant dependent (WOD) random variables. More results can be found e.g. in Yang et al. (2013), Qiu et al. (2014), Wang et al. (2014), Wu et al. (2014), Qiu et al. (2017), Ko (2017), Tang et al. (2017), and Deng and Wang (2017).

Inspired by the concept of complete moment convergence, Wu et al. (2019) introduced the concept of complete f-moment convergence:

DEFINITION 1.3. Let $\{S_n, n \geqslant 1\}$ be a sequence of random variables, $\{c_n, n \geqslant 1\}$ be a sequence of positive constants and $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing continuous function with $f(0) = 0$. We say that $\{S_n, n \geqslant 1\}$ *converges* *f-moment completely* if

$$\sum_{n=1}^{\infty} c_n Ef(\{|S_n| - \varepsilon\}_+) < \infty \quad \text{for all } \varepsilon > 0.$$

A particular case of complete f-moment convergence with special choices of c_n, $f(t) = t$, $t \geqslant 0$, and $S_n = \frac{1}{n^\alpha}\sum_{k=1}^n X_k$, where $\{X_n,\, n \geqslant 1\}$ is a sequence of independent and identically distributed random variables, has been considered by Chow (1988, Theorem 2.5).

It is easy to check that complete f-moment convergence implies complete convergence if $c_n = 1$ for all $n \geqslant 1$, which was proved by Wu et al. (2019). Thus, complete f-moment convergence is much more general than complete convergence. Understanding the complete f-moment convergence behavior is crucial for applications in nonparametric regression models, semiparametric models, and other complex statistical frameworks. Numerous authors, including Lang et al. (2023), Wang et al. (2023a), Wang et al. (2023b), Zhou et al. (2023a), Zhou et al. (2023b), Li et al. (2024), and Zheng et al. (2024), investigated complete f-moment convergence for specific classes of random variables or regression models.

In this paper, we extend these results to suitable sequences of random variables under milder conditions, providing a more general approach to complete f-moment convergence. Additionally, we verify through simulations that the specific sequences of random variables considered in these studies also satisfy our theoretical results.

1.2. Brief review. Recently, our interest has been attracted by the results of Silva (2020) about convergence of series of moments for rowwise sums of random variables. The convergence was established by assuming that for any $t > 0$, the array of random variables satisfies a Rosenthal type inequality which can be found in Petrov (1995): there exist sequences $\{\beta_n,\, n \geqslant 1\}$ and $\{\xi_n,\, n \geqslant 1\}$ of positive numbers such that for some $q > 2$,

$$(1.1)\qquad E\Big|\sum_{k=1}^{n}[h_t(X_{n,k}) - Eh_t(X_{n,k})]\Big|^q \leqslant \beta_n \sum_{k=1}^{n} E|h_t(X_{n,k})|^q + \xi_n\Big[\sum_{k=1}^{n} E|h_t(X_{n,k})|^2\Big]^{q/2}$$

for any $n \geqslant 1$ and $t > 0$, where

$$h_t(X_{n,k}) = -tI(X_{n,k} < -t) + X_{n,k}I(|X_{n,k}| \leqslant t) + tI(X_{n,k} > t).$$

Based on this inequality, Silva (2020) established the following two results on complete moment convergence for rowwise sums of random variables.

THEOREM A. *Let $p > 1$, and let $\{X_{n,k},\, 1 \leqslant k \leqslant n,\, n \geqslant 1\}$ be an array of random variables with $E|X_{n,k}|^p < \infty$ for each $1 \leqslant k \leqslant n$ and $n \geqslant 1$, and satisfying* (1.1) *for $q > \max\{p, 2\}$ and some sequences $\{\beta_n,\, n \geqslant 1\}$, $\{\xi_n,\, n \geqslant 1\}$*

of positive numbers. If $\{b_n, n \geqslant 1\}$ *and* $\{c_n, n \geqslant 1\}$ *are real sequences of positive numbers such that*

(a) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} \beta_n c_n b_n^{-q} \int_0^{b_n^q} P(|X_{n,k}|^q > t)\, dt < \infty,$

(b) $\sum_{n=1}^{\infty} \xi_n c_n b_n^{-p} \int_0^{b_n^{p-q}} \Big(\sum_{k=1}^{n} \int_0^{t^{2/(p-q)}} P(X_{n,k}^2 > s)\, ds\Big)^{q/2} dt < \infty,$

(c) $\sum_{n=1}^{\infty} \xi_n c_n b_n^{-q} \Big(\sum_{k=1}^{n} \int_0^{b_n^2} P(X_{n,k}^2 > t)\, dt\Big)^{q/2} < \infty,$

(d) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{c_n}{b_n} E|X_{n,k}| I(|X_{n,k}| > b_n) < \infty,$

(e) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} (1+\beta_n) c_n b_n^{-p} \int_{b_n^p}^{\infty} P(|X_{n,k}|^p > t)\, dt < \infty,$

then

$$\sum_{n=1}^{\infty} c_n E\bigg[\frac{|\sum_{k=1}^{n}(X_{n,k} - EX_{n,k})|}{b_n} - \varepsilon\bigg]_+^p < \infty \quad \textit{for all } \varepsilon > 0.$$

THEOREM B. *Let* $\{X_{n,k}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ *be an array of random variables with* $E|X_{n,k}| < \infty$ *for each* $1 \leqslant k \leqslant n$ *and* $n \geqslant 1$*, and satisfying* (1.1) *for* $q > 2$ *and some sequences* $\{\beta_n,\ n \geqslant 1\}$, $\{\xi_n,\ n \geqslant 1\}$ *of positive numbers. If* $\{b_n, n \geqslant 1\}$ *and* $\{c_n, n \geqslant 1\}$ *are sequences of positive numbers such that*

(a) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} \beta_n c_n b_n^{-q} \int_0^{b_n^q} P(|X_{n,k}|^q > t)\, dt < \infty,$

(b) $\sum_{n=1}^{\infty} \xi_n c_n b_n^{-1} \int_0^{b_n^{1-q}} \Big(\sum_{k=1}^{n} \int_0^{t^{2/(1-q)}} P(X_{n,k}^2 > s)\, ds\Big)^{q/2} dt < \infty,$

(c) $\sum_{n=1}^{\infty} \xi_n c_n b_n^{-q} \Big(\sum_{k=1}^{n} \int_0^{b_n^2} P(X_{n,k}^2 > t)\, dt\Big)^{q/2} < \infty,$

(d) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{c_n}{b_n} E|X_{n,k}| I(|X_{n,k}| > b_n) < \infty,$

(e) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} \beta_n c_n b_n^{-1} \int_{b_n}^{\infty} P(|X_{n,k}| > t)\, dt < \infty,$

then

$$\sum_{n=1}^{\infty} c_n E\bigg[\frac{|\sum_{k=1}^{n}(X_{n,k} - EX_{n,k})|}{b_n} - \varepsilon\bigg]_+ < \infty \quad \textit{for all } \varepsilon > 0.$$

As we can see, in Theorems A and B only the cases $f(t) = t^p$, $t \geqslant 0$, and $f(t) = t$ were considered in view of the definition of complete f-moment convergence. The main purpose of the present investigation is to extend the complete moment convergence to complete f-moment convergence for arrays of random variables under some more general conditions and give some corollaries and statistical applications to semiparametric models and errors-in-variables regression models.

1.3. Stochastic domination

DEFINITION 1.4. An array $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ of random variables is said to be *stochastically dominated* by a random variable X if there exists a positive constant C such that

$$P(|X_{n,k}| > x) \leqslant CP(|X| > x)$$

for all $x \geqslant 0$, $1 \leqslant k \leqslant n$ and $n \geqslant 1$.

The following lemma is an important property of stochastic domination. The first inequality in the lemma is due to Adler and Rosalsky (1987) and the second one can be found in Adler et al. (1989).

LEMMA 1.1. *Let* $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ *be an array of random variables which is stochastically dominated by a random variable* X*. For any* $\alpha > 0$ *and* $b > 0$*, the following two statements hold:*

$$E|X_{n,k}|^\alpha I(|X_{n,k}| \leqslant b) \leqslant C_1[E|X|^\alpha I(|X| \leqslant b) + b^\alpha P(|X| > b)],$$
$$E|X_{n,k}|^\alpha I(|X_{n,k}| > b) \leqslant C_2 E|X|^\alpha I(|X| > b),$$

where C_1 *and* C_2 *are two positive constants.*

Throughout this paper, the symbols C and C_1 represent positive constants which may vary in different places. Let $I(A)$ be the indicator function of the set A. Denote $a_+ = \max\{0, a\}$ and $\log x = \ln \max\{x, e\}$, where $\ln x$ is the natural logarithm.

This work is organized as follows: In Section 1, we recall some classical concepts of convergence and the target of the work is determined. The main results and their proofs are stated in Section 2. Some applications are presented in Section 3.

2. THE MAIN RESULTS AND THEIR PROOFS

At first, we introduce a function

$$f : \mathbb{R}^+ \to \mathbb{R}^+,$$

which is increasing and continuous with $f(0) = 0$.

Let

$$g : \mathbb{R}^+ \to \mathbb{R}^+$$

be the inverse function of $f(t)$, that is, $g(f(t)) = t$ for $t \geqslant 0$. Assume that for some positive constants p and δ, the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the following condition:

$$\int_{f(\delta)}^{\infty} g^{-p}(t)\, dt < \infty. \tag{2.1}$$

Using the above functions f and g, we present our main results.

THEOREM 2.1. *Let $p > 1$, $\{X_{n,k},\, 1 \leqslant k \leqslant n,\, n \geqslant 1\}$ be an array of random variables with $E|X_{n,k}|^p < \infty$ for each $1 \leqslant k \leqslant n$ and $n \geqslant 1$, and satisfies* (1.1) *and* (2.1) *for $q > \max\{p, 2\}$ and some sequences $\{\beta_n,\, n \geqslant 1\}$, $\{\xi_n,\, n \geqslant 1\}$ of positive numbers. If $\{b_n,\, n \geqslant 1\}$ and $\{c_n,\, n \geqslant 1\}$ are sequences of positive numbers such that conditions* (a)–(e) *in Theorem A hold, then for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} c_n E f([|S_n| - \varepsilon]_+) < \infty,$$

that is, the sequence $\{S_n,\, n \geqslant 1\}$ converges f-moment completely, where $S_n = \sum_{k=1}^{n}(X_{n,k} - EX_{n,k})/b_n$ for $n \geqslant 1$.

Proof. For any $\varepsilon > 0$, we can easily see by Markov's inequality and (2.1) that

$$\begin{aligned}
\sum_{n=1}^{\infty} c_n E f([|S_n| - \varepsilon]_+) &= \sum_{n=1}^{\infty} c_n \int_0^{\infty} P(|S_n| > \varepsilon + g(t))\, dt \\
&= \sum_{n=1}^{\infty} c_n \int_0^{f(\delta)} P(|S_n| > \varepsilon + g(t))\, dt + \sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} P(|S_n| > \varepsilon + g(t))\, dt \\
&\leqslant \sum_{n=1}^{\infty} c_n \int_0^{f(\delta)} P(|S_n| > \varepsilon)\, dt + \sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} \frac{E[|S_n| - \varepsilon]_+^p}{g^p(t)}\, dt \\
&\leqslant f(\delta) \sum_{n=1}^{\infty} c_n P(|S_n| > \varepsilon) + \sum_{n=1}^{\infty} c_n E[|S_n| - \varepsilon]_+^p \Big(\int_{f(\delta)}^{\infty} g^{-p}(t)\, dt \Big) \\
&\leqslant C \sum_{n=1}^{\infty} c_n P(|S_n| > \varepsilon) + C \sum_{n=1}^{\infty} c_n E[|S_n| - \varepsilon]_+^p =: I_1 + I_2.
\end{aligned}$$

Noting that $S_n = \sum_{k=1}^{n}(X_{n,k} - EX_{n,k})/b_n$, we can get $I_2 < \infty$ immediately by Theorem A. We can also get $I_1 < \infty$ because

$$\begin{aligned}
\sum_{n=1}^{\infty} c_n E[|S_n| - \varepsilon]_+^p &= \sum_{n=1}^{\infty} c_n \int_0^{\infty} P(|S_n| > \varepsilon + t^{1/p})\, dt \\
&= \sum_{n=1}^{\infty} c_n \int_0^{\varepsilon^p} P(|S_n| > \varepsilon + t^{1/p})\, dt + \sum_{n=1}^{\infty} c_n \int_{\varepsilon^p}^{\infty} P(|S_n| > \varepsilon + t^{1/p})\, dt
\end{aligned}$$

$$\geqslant \varepsilon^p \sum_{n=1}^{\infty} c_n P(|S_n| > 2\varepsilon) + \sum_{n=1}^{\infty} c_n \int_{\varepsilon^p}^{\infty} P(|S_n| > \varepsilon + t^{1/p})\, dt$$
$$\geqslant C \sum_{n=1}^{\infty} c_n P(|S_n| > 2\varepsilon).$$

Hence, the proof is complete. ■

REMARK 2.1. If we take $f(t) = t^s$, $t \geqslant 0$, $1 < s < p$ in Theorem 2.1, then we can get a series of results which are similar to Theorem A. This means that Theorem A is a special case of our results.

REMARK 2.2. The function f can be chosen much more generally such as $f(t)$ satisfying $f^{-1}(t) = g(t) \geqslant Ct^{1/p}(\log t)^{\mu/p}$ for some $\mu > 1$. Moreover, it can be found that many sequences of random variables satisfy (1.1), including extended negatively dependent (END) random variables, negatively orthant dependent (NOD) random variables, WOD random variables, NSD random variables, and NA random variables.

If we take $c_n = 1$, $b_n = n^{\alpha}$, $1/2 < \alpha \leqslant 1$, $1/\alpha < p < 2/\alpha$, $f(t) = t^s$, $1 < s < p$ in Theorem 2.1, and further assume that $\{\beta_n,\, n \geqslant 1\}$ and $\{\xi_n,\, n \geqslant 1\}$ are constant sequences, and $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ is an array of random variables stochastically dominated by a random variable X, then we can get the following corollary.

COROLLARY 2.1. *Let* $1/2 < \alpha \leqslant 1$, $1 < \alpha p < 2$, $1 < s < p$, *and* $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ *be an array of random variables which is stochastically dominated by a random variable* X *with* $E|X|^{2/\alpha} < \infty$, *and satisfies* (1.1) *for* $q > \max\{p, \frac{2}{2\alpha-1}\}$ *and constant sequences* $\{\beta_n,\, n \geqslant 1\}$ *and* $\{\xi_n,\, n \geqslant 1\}$. *Then*

$$\sum_{n=1}^{\infty} E\Big[n^{-\alpha}\Big|\sum_{k=1}^{n}(X_{n,k} - EX_{n,k})\Big| - \varepsilon\Big]_+^s < \infty \quad \text{for all } \varepsilon > 0. \tag{2.2}$$

In addition,

$$\sum_{n=1}^{\infty} P\Big(n^{-\alpha}\Big|\sum_{k=1}^{n}(X_{n,k} - EX_{n,k})\Big| > \varepsilon\Big) < \infty \quad \text{for all } \varepsilon > 0, \tag{2.3}$$

and thus

$$n^{-\alpha}\sum_{k=1}^{n}(X_{n,k} - EX_{n,k}) \xrightarrow[n\to\infty]{\text{a.s.}} 0. \tag{2.4}$$

Proof. We only need to prove (2.2), since (2.3) follows from (2.2), and (2.4) follows from (2.3) immediately. To prove (2.2) we only need to verify that conditions (a)–(e) of Theorem A are satisfied, where $b_n = n^{\alpha}$, $c_n = 1$, $\beta_n \equiv C$, $\xi_n \equiv C$.

First, note that

$$
\begin{aligned}
&E|X_{n,k}|^q I(|X_{n,k}| \leqslant n^\alpha) \\
&\quad = \int_0^{n^{\alpha q}} P(|X_{n,k}|^q > t, |X_{n,k}| \leqslant n^\alpha)\,dt + \int_{n^{\alpha q}}^{\infty} P(|X_{n,k}|^q > t, |X_{n,k}| \leqslant n^\alpha)\,dt \\
&\quad = \int_0^{n^{\alpha q}} P(|X_{n,k}|^q > t)\,dt - n^{\alpha q} P(|X_{n,k}| > n^\alpha).
\end{aligned}
$$

Hence, by Lemma 1.1 we have

$$
\begin{aligned}
(2.5)\qquad &\sum_{n=1}^{\infty}\sum_{k=1}^{n} n^{-\alpha q}\int_0^{n^{\alpha q}} P(|X_{n,k}|^q > t)\,dt \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{n} n^{-\alpha q}[E|X_{n,k}|^q I(|X_{n.k}| \leqslant n^\alpha) + n^{\alpha q} P(|X_{n,k}| > n^\alpha)] \\
&\leqslant C\sum_{n=1}^{\infty} n^{1-\alpha q}[E|X|^q I(|X| \leqslant n^\alpha) + n^{\alpha q} P(|X| > n^\alpha)] \\
&\leqslant C\sum_{n=1}^{\infty} n^{1-\alpha q}\sum_{j=1}^{n} E|X|^q I((j-1)^\alpha < |X| \leqslant j^\alpha) \\
&\quad + C\sum_{n=1}^{\infty} n\sum_{j=n}^{\infty} P(j^\alpha < |X| \leqslant (j+1)^\alpha) \\
&\leqslant C\sum_{j=1}^{\infty} E|X|^q I((j-1)^\alpha < |X| \leqslant j^\alpha)\sum_{n=j}^{\infty} n^{1-\alpha q} \\
&\quad + C\sum_{j=1}^{\infty} EI(j^\alpha < |X| \leqslant (j+1)^\alpha)\sum_{n=1}^{j} n \\
&\leqslant C\sum_{j=1}^{\infty} E|X|^{2/\alpha} I((j-1)^\alpha < |X| \leqslant j^\alpha) \\
&\quad + C\sum_{j=1}^{\infty} E|X|^{2/\alpha} I(j^\alpha < |X| \leqslant (j+1)^\alpha) \\
&\leqslant CE|X|^{2/\alpha} < \infty,
\end{aligned}
$$

which implies the validity of condition (a).

Next, for condition (b), by Lemma 1.1 and $E|X|^{2/\alpha} < \infty$ we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{-\alpha p}\int_0^{n^{\alpha(p-q)}}\Big(\sum_{k=1}^{n}\int_0^{t^{2/(p-q)}} P(X_{n,k}^2 > s)\,ds\Big)^{q/2} dt \\
&\qquad \leqslant C\sum_{n=1}^{\infty} n^{q/2-\alpha p}\int_0^{n^{\alpha(p-q)}}\Big(\int_0^{t^{2/(p-q)}} P(X^2 > s)\,ds\Big)^{q/2} dt \\
&\qquad \leqslant C\sum_{n=1}^{\infty} n^{q/2-\alpha p}\cdot n^{\alpha(p-q)}(EX^2)^{q/2} \leqslant C\sum_{n=1}^{\infty} n^{(1/2-\alpha)q} < \infty.
\end{aligned}
$$

For condition (c), in the same manner we have

$$\sum_{n=1}^{\infty} n^{-\alpha q}\Big(\sum_{k=1}^{n}\int_{0}^{n^{2\alpha}} P(X_{n,k}^2 > t)\,dt\Big)^{q/2} \leqslant C\sum_{n=1}^{\infty} n^{(1/2-\alpha)q}\Big(\int_{0}^{n^{2\alpha}} P(X^2 > t)\,dt\Big)^{q/2}$$
$$\leqslant C\sum_{n=1}^{\infty} n^{(1/2-\alpha)q} < \infty.$$

From Lemma 1.1 and $E|X|^{2/\alpha} < \infty$ again, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{k=1}^{n} n^{-\alpha} E|X_{n,k}|I(|X_{n,k}| > n^{\alpha})\\
&\qquad\leqslant C\sum_{n=1}^{\infty} n^{1-\alpha}\sum_{j=n}^{\infty} E|X|I(j^{\alpha} < |X| \leqslant (j+1)^{\alpha})\\
&\qquad\leqslant C\sum_{j=1}^{\infty} j^{2-\alpha} E|X|I(j^{\alpha} < |X| \leqslant (j+1)^{\alpha})\\
&\qquad\leqslant C\sum_{j=1}^{\infty} j^{\alpha(2/\alpha-1)} E|X|I(j^{\alpha} < |X| \leqslant (j+1)^{\alpha})\\
&\qquad\leqslant C\sum_{j=1}^{\infty} E|X|^{2/\alpha} I(j^{\alpha} < |X| \leqslant (j+1)^{\alpha})\\
&\qquad\leqslant CE|X|^{2/\alpha} < \infty,
\end{aligned}$$

which implies the validity of condition (d).

Finally, we will check (e). Under our assumptions, it is easy to find that

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{k=1}^{n} n^{-\alpha p}\int_{n^{\alpha p}}^{\infty} P(|X_{n,k}|^p > t)\,dt &\leqslant C\sum_{n=1}^{\infty} n^{1-\alpha p}\int_{n^{\alpha p}}^{\infty} P(|X|^p > t)\,dt\\
&\leqslant C\sum_{n=1}^{\infty} n^{1-\alpha p} E|X|^p I(|X| > n^{\alpha})\\
&= C\sum_{n=1}^{\infty} n^{1-\alpha p}\sum_{j=n}^{\infty} E|X|^p I(j^{\alpha} < |X| \leqslant (j+1)^{\alpha})\\
&= C\sum_{j=1}^{\infty} E|X|^p I(j^{\alpha} < |X| \leqslant (j+1)^{\alpha})\sum_{n=1}^{j} n^{1-\alpha p}\\
&\leqslant CE|X|^{2/\alpha} < \infty.
\end{aligned}$$

Hence, all the conditions of Theorem A have been verified. The desired result (2.2) follows from Theorem 2.1 immediately. ∎

If we further consider a constant sequence $\{c_{n,k}, 1 \leqslant k \leqslant n,\, n \geqslant 1\}$ in Corollary 2.1 as the weight coefficients, we get the following corollary.

COROLLARY 2.2. *Let* $1/2 < \alpha \leqslant 1$, $1 < \alpha p < 2$, $1 < s < p$, $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ *be an array of random variables which is stochastically dominated by a random variable* X *with* $E|X|^{2/\alpha} < \infty$ *and satisfies* (1.1) *for* $q > \max\{p, \frac{2}{2\alpha-1}\}$ *and constant sequences* $\{\beta_n,\ n \geqslant 1\}$ *and* $\{\xi_n,\ n \geqslant 1\}$. *Let* $\{c_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ *be an array of constants satisfying*

$$\max_{1\leqslant k\leqslant n} |c_{n,k}| = O(1).$$

Then

$$\sum_{n=1}^{\infty} E\Big[n^{-\alpha}\Big|\sum_{k=1}^{n} c_{n,k}(X_{n,k} - EX_{n,k})\Big| - \varepsilon\Big]_+^s < \infty \quad \textit{for all } \varepsilon > 0.$$

In addition,

$$\sum_{n=1}^{\infty} P\Big(n^{-\alpha}\Big|\sum_{k=1}^{n} c_{n,k}(X_{n,k} - EX_{n,k})\Big| > \varepsilon\Big) < \infty \quad \textit{for all } \varepsilon > 0,$$

and thus

$$n^{-\alpha}\sum_{k=1}^{n} c_{n,k}(X_{n,k} - EX_{n,k}) \xrightarrow[n\to\infty]{\text{a.s.}} 0.$$

Proof. The proof follows that of Corollary 2.1; all we need to do is replace $X_{n,k}$ by $c_{n,k}X_{n,k}$. Without loss of generality, we assume that $0 < \max_{1\leqslant k\leqslant n}|c_{n,k}| \leqslant C_1$.

First, for condition (a), noting that $\max_{1\leqslant k\leqslant n}|c_{n,k}| \leqslant C_1$, we can easily get

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{k=1}^{n} n^{-\alpha q}\int_0^{n^{\alpha q}} P(|c_{n,k}X_{n,k}|^q > t)\,dt \leqslant \sum_{n=1}^{\infty}\sum_{k=1}^{n} n^{-\alpha q}\int_0^{n^{\alpha q}} P(|C_1X_{n,k}|^q > t)\,dt\\
&\quad= \sum_{n=1}^{\infty}\sum_{k=1}^{n} n^{-\alpha q}[E|C_1X_{n,k}|^q I(|C_1X_{n,k}| \leqslant n^{\alpha}) + n^{\alpha q}P(|C_1X_{n,k}| > n^{\alpha})]\\
&\quad= \sum_{n=1}^{\infty}\sum_{k=1}^{n} n^{-\alpha q}E|C_1X_{n,k}|^q I(|C_1X_{n,k}| \leqslant n^{\alpha}) + \sum_{n=1}^{\infty}\sum_{k=1}^{n} P(|C_1X_{n,k}| > n^{\alpha})\\
&\quad=: I_1 + I_2.
\end{aligned}$$

For I_1, similar to the proof of (2.5), by Lemma 1.1 and $E|X|^{2/\alpha} < \infty$ we have

$$\begin{aligned}
I_1 &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} n^{-\alpha q}E|C_1X_{n,k}|^q I(|C_1X_{n,k}| \leqslant n^{\alpha})\\
&\leqslant \sum_{n=1}^{\infty} n^{1-\alpha q}[E|C_1X|^q I(|C_1X| \leqslant n^{\alpha}) + n^{\alpha q}P(|X| > n^{\alpha})]\\
&\leqslant \sum_{n=1}^{\infty} n^{1-\alpha q}E|C_1X|^q I(|C_1X| \leqslant n^{\alpha}) + \sum_{n=1}^{\infty} nP(|X| > n^{\alpha})\\
&\leqslant CE|X|^{2/\alpha} < \infty.
\end{aligned}$$

Next, we will show $I_2 < \infty$. Noting that $\max_{1\leqslant k\leqslant n}|c_{n,k}| \leqslant C_1$, we have

$$\begin{aligned} I_2 &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} P\Big(|X_{n,k}| > \frac{n^{\alpha}}{C_1}\Big) \leqslant C\sum_{n=1}^{\infty} nP\Big(|X| > \frac{n^{\alpha}}{C_1}\Big) \\ &= C\sum_{n=1}^{\infty} n\sum_{j=n}^{\infty} P\Big(\frac{j^{\alpha}}{C_1} < |X| \leqslant \frac{(j+1)^{\alpha}}{C_1}\Big) \\ &= C\sum_{j=1}^{\infty} P\Big(\frac{j^{\alpha}}{C_1} < |X| \leqslant \frac{(j+1)^{\alpha}}{C_1}\Big)\sum_{n=1}^{j} n \\ &= C\sum_{j=1}^{\infty} j^2 EI\Big(\frac{j^{\alpha}}{C_1} < |X| \leqslant \frac{(j+1)^{\alpha}}{C_1}\Big) \leqslant CE|X|^{2/\alpha} < \infty. \end{aligned}$$

For condition (b), noting that $q > \frac{2}{2\alpha-1}$ means $(1/2-\alpha)q < -1$, by Lemma 1.1, $\max_{1\leqslant k\leqslant n}|c_{n,k}| \leqslant C_1$ and $E|X|^{2/\alpha} < \infty$ we have

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{-\alpha p}\int_0^{n^{\alpha(p-q)}}\Big(\sum_{k=1}^{n}\int_0^{t^{2/(p-q)}} P(c_{n,k}^2X_{n,k}^2 > s)\,ds\Big)^{q/2} dt \\ &\qquad \leqslant C\sum_{n=1}^{\infty} n^{q/2-\alpha p}\int_0^{n^{\alpha(p-q)}}\Big(\int_0^{t^{2/(p-q)}} P(X^2 > s/C_1^2)\,ds\Big)^{q/2} dt \\ &\qquad \leqslant C\sum_{n=1}^{\infty} n^{q/2-\alpha p}\cdot n^{\alpha(p-q)}(EX^2)^{q/2} \leqslant C\sum_{n=1}^{\infty} n^{(1/2-\alpha)q} < \infty. \end{aligned}$$

For condition (c), in the same manner we have

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{-\alpha q}\Big(\sum_{k=1}^{n}\int_0^{n^{2\alpha}} P(c_{n,k}^2X_{n,k}^2 > t)\,dt\Big)^{q/2} \\ &\qquad\qquad \leqslant C\sum_{n=1}^{\infty} n^{(1/2-\alpha)q}\Big(\int_0^{n^{2\alpha}} P(X^2 > t/C_1^2)\,dt\Big)^{q/2} \\ &\qquad\qquad \leqslant C\sum_{n=1}^{\infty} n^{(1/2-\alpha)q}(EX^2)^{q/2} \leqslant C\sum_{n=1}^{\infty} n^{(1/2-\alpha)q} < \infty. \end{aligned}$$

From Lemma 1.1 and $E|X|^{2/\alpha} < \infty$ again, it is obvious that

$$\begin{aligned} &\sum_{n=1}^{\infty}\sum_{k=1}^{n} n^{-\alpha}E|c_{n,k}X_{n,k}|I(|c_{n,k}X_{n,k}| > n^{\alpha}) \\ &\leqslant C\sum_{n=1}^{\infty}\sum_{k=1}^{n} n^{-\alpha}E|X_{n,k}|I(|X_{n,k}| > n^{\alpha}/C_1) \\ &\leqslant C\sum_{n=1}^{\infty} n^{1-\alpha}\sum_{j=n}^{\infty} E|X|I(j^{\alpha}/C_1 < |X| \leqslant (j+1)^{\alpha}/C_1) \leqslant CE|X|^{2/\alpha} < \infty, \end{aligned}$$

which implies the validity of condition (d).

Finally, we will check (e). Under our assumptions, it is easy to get

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} n^{-\alpha p}\int_{n^{\alpha p}}^{\infty} P(|c_{n,k}X_{n,k}|^p > t)\,dt \leqslant C\sum_{n=1}^{\infty} n^{1-\alpha p}\int_{n^{\alpha p}}^{\infty} P(|X|^p > t/C_1^p)\,dt$$
$$\leqslant C\sum_{n=1}^{\infty} n^{1-\alpha p} E|X|^p I(|X| > n^{\alpha}/C_1) \leqslant CE|X|^{2/\alpha} < \infty.$$

Hence, all the conditions of Theorem A have been verified. This completes the proof of the corollary. □

3. STATISTICAL APPLICATIONS

In this section, we will provide some applications to statistical models of the complete convergence results that we established in Section 2.

3.1. Application to semiparametric regression models. Semiparametric regression was introduced by Engle et al. (1986) as a generalization of parametric regression and nonparametric regression. Later, this classical model has been extended by many authors. For example, Baek et al. (2006) studied heteroscedastic semiparametric regression models with NA random errors, and established strong consistency and asymptotic normality for least squares estimators and weighted least squares estimators of β and g; Johnson et al. (2008) proposed a general strategy for selection in semiparametric regression models by penalizing appropriate estimating functions and gave some applications to semiparametric linear regression with censored responses and missing predictors, respectively. Duran et al. (2012) considered the difference between a ridge regression estimator and a Liu type estimator of the regression parameters in partial linear semiparametric regression models, and extended the results to account for heteroscedasticity and autocovariance in the error terms; Deng et al. (2019) established a general result on complete convergence for weighted sums of linear processes and gave its application to semiparametric regression models.

Hu (2006) gave the following specific form of the semiparametric regression model:

$$y_i^{(n)} = x_i^{(n)}\beta + g(t_i^{(n)}) + \varepsilon_i^{(n)}, \quad i = 1, \ldots, n,\ n \geqslant 1,$$

where g is an unknown function defined on a compact set A in $\mathbb{R}^p$, and β is an unknown parameter in $\mathbb{R}$, $x_i^{(n)}$ and $t_i^{(n)}$ are known to be nonrandom, $y_i^{(n)}$ represents the ith response which is observable at $x_i^{(n)}$, and $t_i^{(n)}$, $\varepsilon_i^{(n)}$ are random errors.

By the least squares and weight functions method, the following estimators of β and $g(t)$ were first introduced by Pan et al. (2003):

$$\hat{\beta}_n = S_n^{-2}\sum_{i=1}^{n}\tilde{x}_i^{(n)}\tilde{y}_i^{(n)}, \quad \hat{g}_n(t) = \sum_{i=1}^{n} W_{ni}(t)(y_i^{(n)} - x_i^{(n)}\hat{\beta}_n),$$

where $W_{ni}(t) = W_{ni}(t; t_1^{(n)}, \ldots, t_n^{(n)})$ are measurable weight functions,

$$\tilde{x}_i^{(n)} = x_i^{(n)} - \sum_{k=1}^{n} W_{nk}(t_i^{(n)}) x_k^{(n)}, \quad \tilde{y}_i^{(n)} = y_i^{(n)} - \sum_{k=1}^{n} W_{nk}(t_i^{(n)}) y_k^{(n)},$$
$$S_n^2 = \sum_{i=1}^{n} (\tilde{x}_i^{(n)})^2.$$

Now we list some basic assumptions on the weight functions $W_{ni}(t)$:

(A1) $g(\cdot)$ satisfies the Lipschitz condition.

(A2) $\liminf_{n\to\infty} S_n^2/n \geqslant D$, where D is a positive constant.

(A3) $\left|\sum_{k=1}^{n} W_{nk}(t) - 1\right| = O(k_n/n) \quad$ for any $t \in A$.

(A4) $\sum_{k=1}^{n} |W_{nk}(t)| = O(1) \quad$ for any $t \in A$.

(A5) $\max_{1\leqslant k\leqslant n} \sum_{i=1}^{n} |W_{nk}(t_i^{(n)})| = O(1)$.

(A6) There exist $1 \leqslant k_n \leqslant n$ such that

$$\lim_{n\to\infty} k_n = \infty,$$
$$\sum_{k=1}^{n} |W_{nk}(t)| I\left(\|t - t_k^{(n)}\| > \frac{k_n}{n}\right) = O\left(\frac{k_n}{n}\right) \quad \text{for any } t \in A.$$

(A7) There exist $1 \leqslant k_n \leqslant n$ such that

(i) $\lim_{n\to\infty} \frac{k_n}{n} = 0, \quad \max_{1\leqslant i\leqslant n} |x_i^{(n)}| = O(n^{1-\alpha}), \quad 1/2 < \alpha \leqslant 1;$

(ii) $\lim_{n\to\infty} \frac{k_n}{n^{(1+\alpha)/2}} = 0, \quad \max_{1\leqslant i\leqslant n} |x_i^{(n)}| = O(n^{(1-\alpha)/2}), \quad 1/2 < \alpha \leqslant 1.$

(A8) $\max_{1\leqslant k\leqslant n} |W_{nk}(t)| = O(n^{-\alpha}), \quad 1/2 < \alpha \leqslant 1.$

REMARK 3.1. Assumptions similar to (A1), (A2) and (A4)–(A6) for weight functions can be found in Hu (2006), and in Section 3 there, it is shown that the assumptions are satisfied for the nearest neighbour weights. This is the basis on which we will carry out numerical simulations later. Assumptions (A3), (A7) and (A8) can also be easily satisfied, e.g. for the nearest neighbour weights mentioned above.

THEOREM 3.1. *Let $1/2 < \alpha \leqslant 1$, and $\{\varepsilon_i^{(n)}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of zero mean random variables which is stochastically dominated by a random variable X with $E|X|^{2/\alpha} < \infty$, and satisfies* (1.1) *for $q > \frac{2}{2\alpha-1}$ and constant sequences $\{\beta_n, n \geqslant 1\}$ and $\{\xi_n, n \geqslant 1\}$. If conditions* (A1)–(A6) *and* (A7)(i) *hold, then*

$$\hat{\beta}_n \xrightarrow[n\to\infty]{} \beta \quad \textit{completely}. \tag{3.1}$$

Proof. Without loss of generality, we can assume that $W_{ni}(t) \geqslant 0$ for $1 \leqslant i \leqslant n$, $n \geqslant 1$ and any $t \in A$. It is easily seen that

$$\hat{\beta}_n - \beta = S_n^{-2}\Big\{\sum_{i=1}^n \tilde{x}_i^{(n)} \tilde{g}(t_i^{(n)}) + \sum_{i=1}^n \tilde{x}_i^{(n)} \varepsilon_i^{(n)} - \sum_{i=1}^n \tilde{x}_i^{(n)} \sum_{k=1}^n W_{nk}(t_i^{(n)})\varepsilon_k^{(n)}\Big\}.$$

In order to prove (3.1), we only need to show that

$$\begin{aligned}
I_1 &= S_n^{-2} \sum_{i=1}^n \tilde{x}_i^{(n)} \tilde{g}(t_i^{(n)}) \xrightarrow[n\to\infty]{} 0, \\
I_2 &= \sum_{n=1}^\infty P\bigg(\bigg|\frac{1}{n}\sum_{i=1}^n \tilde{x}_i^{(n)}\varepsilon_i^{(n)}\bigg| > \varepsilon\bigg) < \infty \quad \text{for any } \varepsilon > 0, \\
I_3 &= \sum_{n=1}^\infty P\bigg(\bigg|\frac{1}{n}\sum_{i=1}^n \tilde{x}_i^{(n)} \sum_{k=1}^n W_{nk}(t_i^{(n)})\varepsilon_k^{(n)}\bigg| > \varepsilon\bigg) < \infty \quad \text{for any } \varepsilon > 0,
\end{aligned}$$

by (A2). It is easily checked by (A1)–(A6) and (A7)(i) that

$$\begin{aligned}
S_n^{-2}\Big|\sum_{i=1}^n \tilde{x}_i^{(n)} \tilde{g}(t_i^{(n)})\Big| &\leqslant \Big(\max_{1\leqslant i\leqslant n} |\tilde{g}(t_i^{(n)})|\Big)\Big(S_n^{-2}\sum_{i=1}^n |\tilde{x}_i^{(n)}|\Big), \\
S_n^{-2}\sum_{i=1}^n |\tilde{x}_i^{(n)}| &\leqslant n^{1/2} S_n^{-2}\Big(\sum_{i=1}^n (\tilde{x}_i^{(n)})^2\Big)^{1/2} = \bigg(\frac{n}{S_n^2}\bigg)^{1/2} \leqslant C
\end{aligned}$$

and

$$\begin{aligned}
\max_{1\leqslant i\leqslant n} |\tilde{g}(t_i^{(n)})| &= \max_{1\leqslant i\leqslant n} \Big|g(t_i^{(n)}) - \sum_{j=1}^n W_{nj}(t_i^{(n)}) g(t_j^{(n)})\Big| \\
&\leqslant \max_{1\leqslant i\leqslant n} |g(t_i^{(n)})| \Big|\sum_{j=1}^n W_{nj}(t_i^{(n)}) - 1\Big| \\
&\quad + \max_{1\leqslant i\leqslant n} \sum_{j=1}^n |W_{nj}(t_i^{(n)})| \cdot |g(t_i^{(n)}) - g(t_j^{(n)})| I(\|t_i^{(n)} - t_j^{(n)}\| > k_n/n) \\
&\quad + \max_{1\leqslant i\leqslant n} \sum_{j=1}^n |W_{nj}(t_i^{(n)})| \cdot |g(t_i^{(n)}) - g(t_j^{(n)})| I(\|t_i^{(n)} - t_j^{(n)}\| \leqslant k_n/n) \\
&\leqslant C\frac{k_n}{n} \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

Thus, $I_1 \xrightarrow[n\to\infty]{} 0$. It is obvious by (A4) and (A7)(i) that

$$|\tilde{x}_i^{(n)}| \leqslant |x_i^{(n)}| + \max_{1\leqslant k\leqslant n} |x_k^{(n)}| \sum_{k=1}^{n} |W_{nk}(t_i^{(n)})| \leqslant Cn^{1-\alpha} \quad \text{for each } 1 \leqslant i \leqslant n,$$

and thus

$$\max_{1\leqslant i\leqslant n} \frac{|\tilde{x}_i^{(n)}|}{n^{1-\alpha}} = O(1).$$

Applying Corollary 2.2 with $X_{n,i} = \varepsilon_i^{(n)}$, $c_{n,i} = \tilde{x}_i^{(n)}/n^{1-\alpha}$, $p = \frac{3}{2\alpha}$, $s = \frac{3}{2\alpha} - \frac{1}{4}$, and noting that $p < \frac{2}{2\alpha-1}$, we get

$$I_2 = \sum_{n=1}^{\infty} P\left(n^{-\alpha}\left|\sum_{i=1}^{n} \frac{\tilde{x}_i^{(n)}}{n^{1-\alpha}}\varepsilon_i^{(n)}\right| > \varepsilon\right) < \infty.$$

It follows from (A5) and (A7)(i) that

$$\left|\sum_{i=1}^{n} W_{nk}(t_i^{(n)})\tilde{x}_i^{(n)}\right| \leqslant \max_{1\leqslant i\leqslant n} |\tilde{x}_i^{(n)}| \cdot \max_{1\leqslant k\leqslant n} \sum_{i=1}^{n} |W_{nk}(t_i^{(n)})| \leqslant Cn^{1-\alpha},$$

which means

$$\max_{1\leqslant k\leqslant n} \frac{|\sum_{i=1}^{n} W_{nk}(t_i^{(n)})\tilde{x}_i^{(n)}|}{n^{1-\alpha}} = O(1).$$

Therefore, $I_3 < \infty$ follows from Corollary 2.2 in the same way. Thus the proof is complete. ■

THEOREM 3.2. *Let* $1/2 < \alpha \leqslant 1$, *and* $\{\varepsilon_i^{(n)},\, 1 \leqslant i \leqslant n,\, n \geqslant 1\}$ *be an array of zero mean random variables which is stochastically dominated by a random variable* X *with* $E|X|^{2/\alpha} < \infty$, *and satisfies* (1.1) *for* $q > \frac{2}{2\alpha-1}$ *and constant sequences* $\{\beta_n,\, n \geqslant 1\}$ *and* $\{\xi_n,\, n \geqslant 1\}$. *If conditions* (A1)–(A6), (A7)(ii) *and* (A8) *hold, then for any* $t \in A$,

$$\hat{g}_n(t) \xrightarrow[n\to\infty]{} g(t) \quad \textit{completely}.$$

Proof. It is easy to see that condition (A7)(ii) implies (A7)(i). Hence we can get $\hat{\beta}_n \to \beta$ completely by Theorem 3.1 immediately. Note that

$$\hat{g}_n(t) - g(t) = \sum_{k=1}^{n} W_{nk}\varepsilon_k^{(n)} - (\hat{\beta}_n - \beta) \sum_{k=1}^{n} W_{nk}(t)x_k^{(n)} - \tilde{g}(t),$$

where $\tilde{g}(t) = g(t) - \sum_{k=1}^{n} W_{nk}(t)g(t_k^{(n)})$. For any $t \in A$ and $\varepsilon > 0$, it is obvious that

$$\sum_{n=1}^{\infty} P(|\hat{g}_n(t) - g(t)| > \varepsilon) \leqslant \sum_{n=1}^{\infty} P\Big(\Big|\sum_{k=1}^{n} W_{nk}(t)\varepsilon_k^{(n)}\Big| > \varepsilon/3\Big) + \sum_{n=1}^{\infty} P\Big(\Big|(\hat{\beta}_n - \beta)\sum_{k=1}^{n} W_{nk}(t)x_k^{(n)}\Big| > \varepsilon/3\Big) + \sum_{n=1}^{\infty} P(|\tilde{g}(t)| > \varepsilon/3) =: J_1 + J_2 + J_3.$$

Obviously, $J_3 < \infty$ follows from (A2), (A3) and (A6). For J_1, applying Corollary 2.2 with $X_{n,k} = \varepsilon_k^{(n)}$, $|c_{nk}| = n^{\alpha} \cdot |W_{nk}(t)|$, $p = \frac{3}{2\alpha}$, $s = \frac{3}{2\alpha} - \frac{1}{4}$, we can obtain $J_1 < \infty$ immediately by (A8).

For J_2, by (A4) and (A7)(ii) we have

$$\sum_{n=1}^{\infty} P\Big(\Big|(\hat{\beta}_n - \beta)\sum_{k=1}^{n} W_{nk}(t)x_k^{(n)}\Big| > \varepsilon/3\Big) \leqslant \sum_{n=1}^{\infty} P\Big(\max_{1\leqslant k\leqslant n}|x_k^{(n)}|\sum_{k=1}^{n}|W_{nk}(t)|\cdot|\hat{\beta}_n - \beta| > \varepsilon/3\Big) \leqslant \sum_{n=1}^{\infty} P(Cn^{(1-\alpha)/2}|\hat{\beta}_n - \beta| > \varepsilon/3).$$

Hence, to prove $J_2 < \infty$, we only need to show that

$$I_4 = n^{(1-\alpha)/2}\cdot S_n^{-2}\sum_{i=1}^{n}\tilde{x}_i^{(n)}\tilde{g}(t_i^{(n)}) \xrightarrow[n\to\infty]{} 0,$$

$$I_5 = \sum_{n=1}^{\infty} P\bigg(\bigg|n^{(1-\alpha)/2}\cdot\frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i^{(n)}\varepsilon_i^{(n)}\bigg| > \varepsilon\bigg) = \sum_{n=1}^{\infty} P\Big(n^{-\alpha}\Big|\sum_{i=1}^{n}(n^{(\alpha-1)/2}\tilde{x}_i^{(n)})\varepsilon_i^{(n)}\Big| > \varepsilon\Big) < \infty \quad \text{for any } \varepsilon > 0,$$

and

$$I_6 = \sum_{n=1}^{\infty} P\bigg(\bigg|n^{(1-\alpha)/2}\cdot\frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i^{(n)}\sum_{k=1}^{n}W_{nk}(t_i^{(n)})\varepsilon_k^{(n)}\bigg| > \varepsilon\bigg) = \sum_{n=1}^{\infty} P\Big(n^{-\alpha}\Big|\sum_{k=1}^{n}\Big[n^{(\alpha-1)/2}\sum_{i=1}^{n}W_{nk}(t_i^{(n)})\tilde{x}_i^{(n)}\Big]\varepsilon_k^{(n)}\Big| > \varepsilon\Big) < \infty \quad \text{for any } \varepsilon > 0.$$

Similar to the proof of I_1, by condition (A7)(ii) we again have

$$|I_4| \leqslant Cn^{(1-\alpha)/2}\cdot\frac{k_n}{n} = C\frac{k_n}{n^{(\alpha+1)/2}} \xrightarrow[n\to\infty]{} 0,$$

which implies $I_4 \xrightarrow[n\to\infty]{} 0$. It is easy to check by (A5) and (A7)(ii) that

$$\max_{1\leqslant i\leqslant n} |n^{(\alpha-1)/2}\tilde{x}_i^{(n)}| = n^{(\alpha-1)/2}\cdot \max_{1\leqslant i\leqslant n} |\tilde{x}_i^{(n)}| \leqslant Cn^{(\alpha-1)/2}\cdot n^{(1-\alpha)/2} = O(1)$$

and

$$\max_{1\leqslant k\leqslant n} \Big| n^{(\alpha-1)/2} \sum_{i=1}^{n} W_{nk}(t_i^{(n)})\tilde{x}_i^{(n)} \Big| \leqslant n^{(\alpha-1)/2}\cdot \max_{1\leqslant i\leqslant n} |\tilde{x}_i^{(n)}| \cdot \max_{1\leqslant k\leqslant n} \sum_{i=1}^{n} |W_{nk}(t_i^{(n)})| = O(1).$$

So applying Corollary 2.2 with $p = \frac{3}{2\alpha}$ and $s = \frac{3}{2\alpha} - \frac{1}{4}$ again, we get $I_5 < \infty$ and $I_6 < \infty$ immediately. Hence $J_2 < \infty$. This completes the proof of the theorem. ■

3.2. Application to simple linear errors-in-variables models. Consider the following simple linear errors-in-variables (EV) model:

$$\eta_i = \theta + \beta x_i + \varepsilon_i, \quad \xi_i = x_i + \delta_i, \quad 1 \leqslant i \leqslant n, \tag{3.2}$$

where θ and β are unknown parameters, $x_1, x_2, \ldots$ are unknown constants, $(\varepsilon_1, \delta_1)$, $(\varepsilon_2, \delta_2), \ldots$ are random vectors and ξ_i, η_i, $i = 1, 2, \ldots,$ are observable variables. From (3.4) we have

$$\eta_i = \theta + \beta\xi_i + \nu_i, \quad \nu_i = \varepsilon_i - \beta\delta_i, \quad 1 \leqslant i \leqslant n. \tag{3.3}$$

Considering formula (3.3) as a usual regression model of η_i on ξ_i, we get the least squares (LS) estimators of θ and β as follows:

$$\hat{\beta} = \frac{\sum_{i=1}^{n}(\xi - \overline{\xi}_n)(\eta_i - \overline{\eta}_n)}{\sum_{i=1}^{n}(\xi - \overline{\xi}_n)^2}, \quad \hat{\theta} = \overline{\eta}_n - \hat{\beta}\overline{\xi}_n,$$

where $\overline{\xi}_n = n^{-1}\sum_{i=1}^{n}\xi_i$. Other similar notations, such as $\overline{\eta}_n$, $\overline{\delta}_n$ and $\overline{x}_n$, are defined in the same way. Denote $S_n = \sum_{i=1}^{n}(x_i - \overline{x}_n)^2$ for each $n \geqslant 1$. With the notations above, we get

$$\hat{\beta} - \beta = \frac{\sum_{i=1}^{n}(\delta_i - \overline{\delta}_n)\varepsilon_i + \sum_{i=1}^{n}(x_i - \overline{x}_n)(\varepsilon_i - \beta\delta_i) - \beta\sum_{i=1}^{n}(\delta_i - \overline{\delta}_n)^2}{\sum_{i=1}^{n}(\xi_i - \overline{\xi}_n)^2} \tag{3.4}$$

and

$$\hat{\theta} - \theta = (\beta - \hat{\beta})\overline{x}_n + (\beta - \hat{\beta})\overline{\delta}_n + \overline{\varepsilon}_n - \beta\overline{\delta}_n. \tag{3.5}$$

The EV regression model was proposed by Deaton (1985) to model noisy observations and is more practical than the ordinary regression models. Due to its simple form and wide applicability, the EV model has attracted much attention. For more details, we refer the readers to Liu and Chen (2005), Fan et al. (2010) and Miao et al. (2011) among others for consistency and asymptotic normality of $\hat{\beta}$ and $\hat{\theta}$. In particular, Liu and Chen (2005) discussed necessary and sufficient conditions for the strong consistency of β and the weak consistency of θ. Hu et al. (2017) extended the results of Liu and Chen (2015) to the case of ψ-mixing random variables and further obtained necessary and sufficient conditions for the strong consistency of $\hat{\beta}$ and $\hat{\theta}$. Chen et al. (2020) obtained a necessary and sufficient condition for the convergence rate in the strong consistency of the least squares estimators of β and θ.

To present our results, the following assumption is indispensable:

(B1) $\{\varepsilon_i,\, i \geqslant 1\}$ and $\{\delta_i,\, i \geqslant 1\}$ are sequences of zero mean random variables satisfying (1.1) with $q > 2$ and stochastically dominated by ε and δ, respectively, with $0 < E\varepsilon^4 < \infty$ and $0 < E\delta^4 < \infty$. In addition, $\{(\varepsilon_i)_+^2,\, i \geqslant 1\}$, $\{(\varepsilon_i)_-^2,\, i \geqslant 1\}$, $\{(\delta_i)_+^2,\, i \geqslant 1\}$ and $\{(\delta_i)_-^2,\, i \geqslant 1\}$ are sequences of random variables satisfying (1.1) with $q > 2$.

We point out that condition (B1) is very general, because many dependent sequences possess this property, including END sequences, NOD sequences, WOD sequences, NSD sequences, NA sequences, and mixing sequences.

Based on Corollaries 2.1 and 2.2, we obtain the strong consistency for the LS estimators $\hat{\beta}$ and $\hat{\theta}$:

THEOREM 3.3. *Under the model* (3.2), *assume that condition* (B1) *holds. If*

$$\frac{S_n}{n} \to \infty, \qquad \frac{n}{S_n^{1/2}} = O(1),$$

then

$$\hat{\beta} \xrightarrow[n\to\infty]{\text{a.s.}} 0. \tag{3.6}$$

Proof. According to (3.4), it suffices to prove

$$S_n^{-1} \sum_{i=1}^{n} (\delta_i - \overline{\delta}_n)\varepsilon_i \xrightarrow[n\to\infty]{\text{a.s.}} 0, \tag{3.7}$$

$$S_n^{-1} \sum_{i=1}^{n} (x_i - \overline{x}_n)(\varepsilon_i - \beta\delta_i) \xrightarrow[n\to\infty]{\text{a.s.}} 0, \tag{3.8}$$

$$S_n^{-1} \sum_{i=1}^{n} (\delta_i - \overline{\delta}_n)^2 \xrightarrow[n\to\infty]{\text{a.s.}} 0, \tag{3.9}$$

$$S_n^{-1} \sum_{i=1}^{n} (\xi_i - \overline{\xi}_n)^2 \xrightarrow[n\to\infty]{\text{a.s.}} 1. \tag{3.10}$$

From Lemma 1.1 and $S_n/n \to \infty$, we can derive

$$S_n^{-1}\Big|\sum_{i=1}^n E\delta_i\varepsilon_i\Big| \leqslant S_n^{-1}\sum_{i=1}^n E|\delta_i\varepsilon_i| \leqslant S_n^{-1}\sum_{i=1}^n (E\delta_i^2)^{1/2}(E\varepsilon_i^2)^{1/2}$$
$$\leqslant C\frac{n}{S_n}(E\delta^2)^{1/2}(E\varepsilon^2)^{1/2} \xrightarrow[n\to\infty]{} 0.$$

Applying Corollary 2.1 with $\alpha = 1$, $p = 3/2$ and $s = 5/4$, we obtain

$$S_n^{-1}\sum_{i=1}^n(\delta_i-\overline{\delta}_n)^2 \leqslant \frac{n}{S_n}\left|\frac{1}{n}\sum_{i=1}^n(\delta_i^2 - E\delta_i^2)\right| + \frac{n}{S_n}\overline{\delta}_n^2 + S_n^{-1}\sum_{i=1}^n E\delta_i^2 \xrightarrow[n\to\infty]{\text{a.s.}} 0,$$

which implies (3.10). Similarly, we get $S_n^{-1}\sum_{i=1}^n(\varepsilon_i-\overline{\varepsilon}_n)^2 \xrightarrow[n\to\infty]{} 0$. Hence, by Hölder's inequality,

$$S_n^{-1}\Big|\sum_{i=1}^n(\delta_i-\overline{\delta}_n)\varepsilon_i\Big| = S_n^{-1}\Big|\sum_{i=1}^n(\delta_i-\overline{\delta}_n)(\varepsilon_i-\overline{\varepsilon}_n)\Big|$$
$$\leqslant \left[S_n^{-1}\sum_{i=1}^n(\delta_i-\overline{\delta}_n)^2\cdot S_n^{-1}\sum_{i=1}^n(\varepsilon_i-\overline{\varepsilon}_n)^2\right]^{1/2} \xrightarrow[n\to\infty]{\text{a.s.}} 0,$$

which yields (3.7). To prove (3.8), let $c_{n,i} = n(x_i - \overline{x}_n)/S_n$ for $1 \leqslant i \leqslant n$, $n \geqslant 1$. Then

$$\max_{1\leqslant i\leqslant n}|c_{n,i}| = \frac{n}{S_n}\cdot\max_{1\leqslant i\leqslant n}|x_i-\overline{x}_n| \leqslant \frac{n}{S_n}\cdot\Big(\sum_{i=1}^n(x_i-\overline{x}_n)^2\Big)^{1/2} = \frac{n}{S_n^{1/2}} = O(1).$$

So applying Corollary 2.2 with $\alpha = 1$, $p = \frac{3}{2}$ and $s = \frac{5}{4}$ again, we obtain

$$S_n^{-1}\sum_{i=1}^n(x_i-\overline{x}_n)\varepsilon_i = n^{-1}\sum_{i=1}^n c_{ni}\varepsilon_i \xrightarrow[n\to\infty]{\text{a.s.}} 0,$$
$$S_n^{-1}\sum_{i=1}^n(x_i-\overline{x}_n)\delta_i = n^{-1}\sum_{i=1}^n c_{ni}\delta_i \xrightarrow[n\to\infty]{\text{a.s.}} 0, \tag{3.11}$$

from which we can derive (3.8). For (3.10), noting that

$$S_n^{-1}\sum_{i=1}^n(\xi_i-\overline{\xi}_n)^2 = 1 + 2S_n^{-1}\sum_{i=1}^n(x_i-\overline{x}_n)\delta_i + S_n^{-1}\sum_{i=1}^n(\delta_i-\overline{\delta}_n)^2,$$

we obtain (3.10) immediately from (3.9) and (3.11). The proof is complete. ■

THEOREM 3.4. *Suppose that the conditions of Theorem* 3.3 *are satisfied. If*

$$\frac{n|\overline{x}_n|}{S_n} \to 0, \qquad \max_{1\leqslant i\leqslant n}\frac{n|\overline{x}_n|\,|x_i-\overline{x}_n|}{S_n} = O(1), \tag{3.12}$$

then

$$\hat{\theta} \xrightarrow[n\to\infty]{\text{a.s.}} \theta. \tag{3.13}$$

Proof. According to (3.5), to prove (3.13), it suffices to show that

$$\overline{\varepsilon}_n - \beta\overline{\delta}_n \xrightarrow[n\to\infty]{\text{a.s.}} 0, \tag{3.14}$$

$$(\beta - \hat{\beta})\overline{x}_n \xrightarrow[n\to\infty]{\text{a.s.}} 0, \tag{3.15}$$

$$(\beta - \hat{\beta})\overline{\delta}_n \xrightarrow[n\to\infty]{\text{a.s.}} 0. \tag{3.16}$$

Applying Corollary 2.1 with $\alpha = 1$, we have

$$\begin{aligned} \overline{\varepsilon}_n &= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i \xrightarrow[n\to\infty]{\text{a.s.}} 0, \\ \overline{\delta}_n &= \frac{1}{n}\sum_{i=1}^{n}\delta_i \xrightarrow[n\to\infty]{\text{a.s.}} 0, \end{aligned} \tag{3.17}$$

which yield (3.14).

By Theorem 3.3, we obtain $\beta - \hat{\beta} \xrightarrow[n\to\infty]{\text{a.s.}} 0$, which combined with (3.17) yields (3.16).

Finally, we will show (3.15). Combining (3.4) and (3.10), we need to get the following result:

$$\frac{|\overline{x}_n|}{S_n}\Big(\sum_{i=1}^{n}(\delta_i - \overline{\delta}_n)\varepsilon_i + \sum_{i=1}^{n}(x_i - \overline{x}_n)(\varepsilon_i - \beta\delta_i) - \beta\sum_{i=1}^{n}(\delta_i - \overline{\delta}_n)^2\Big) \xrightarrow[n\to\infty]{\text{a.s.}} 0. \tag{3.18}$$

On the one hand, similar to the proofs of (3.7) and (3.9), by (3.12) we have

$$\begin{aligned} &\frac{|\overline{x}_n|}{S_n}\sum_{i=1}^{n}(\delta_i - \overline{\delta}_n)^2 \\ &\quad\leqslant \frac{n|\overline{x}_n|}{S_n}\cdot\left|\frac{1}{n}\sum_{i=1}^{n}(\delta_i^2 - E\delta_i^2)\right| + \frac{n|\overline{x}_n|}{S_n}\overline{\delta}_n^2 + \frac{|\overline{x}_n|}{S_n}\sum_{i=1}^{n}E\delta_i^2 \xrightarrow[n\to\infty]{\text{a.s.}} 0, \end{aligned} \tag{3.19}$$

and

$$\frac{|\overline{x}_n|}{S_n}\sum_{i=1}^{n}(\delta_i - \overline{\delta}_n)\varepsilon_i \xrightarrow[n\to\infty]{\text{a.s.}} 0. \tag{3.20}$$

On the other hand, let $c_{n,i} = n|\overline{x}_n|(x_i - \overline{x}_n)/S_n$ for $1 \leqslant i \leqslant n$, $n \geqslant 1$. By (3.12),

$$\max_{1\leqslant i\leqslant n}|c_{n,i}| = \frac{n|\overline{x}_n|}{S_n}\max_{1\leqslant i\leqslant n}|x_i - \overline{x}_n| = O(1).$$

Applying Corollary 2.2 with $\alpha = 1$, we obtain

$$\begin{aligned}\frac{|\overline{x}_n|}{S_n}\sum_{i=1}^{n}(x_i-\overline{x}_n)(\varepsilon_i-\beta\delta_i) &= \frac{n|\overline{x}_n|}{S_n}\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x}_n)\varepsilon_i-\beta\frac{n|\overline{x}_n|}{S_n}\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x}_n)\delta_i \\ &= \frac{1}{n}\sum_{i=1}^{n}c_{n,i}\varepsilon_i-\beta\frac{1}{n}\sum_{i=1}^{n}c_{n,i}\delta_i \xrightarrow[n\to\infty]{\text{a.s.}} 0,\end{aligned}$$

which, together with (3.19) and (3.20), yields (3.18), and thus (3.15) holds. The proof is complete. ■

4. NUMERICAL SIMULATIONS

4.1. Semiparametric regression models. The observations are generated from the following model:

$$y_i^{(n)} = 3x_i^{(n)} + \frac{e^{(t_i^{(n)})}\cos t_i^{(n)}}{300} + \varepsilon_i^{(n)}, \qquad i = 1,\ldots,n,\ n \geqslant 1,$$

where $A = [0,1]$, $g(t) = (e^t\cos t)/300$, $t_i^{(n)} = i/n$, $x_i^{(n)} = (-1)^i \cdot i/n$, $i = 1,\ldots,n$, $(\varepsilon_1^{(n)},\ldots,\varepsilon_n^{(n)})$ has the same distribution as $(\varepsilon_1,\ldots,\varepsilon_n)$, and $(\varepsilon_1,\ldots,\varepsilon_n)^T \sim N(\mathbf{0},\mathbf{\Sigma})$, $\mathbf{0}$ is the zero column vector, and

$$\mathbf{\Sigma} = \begin{pmatrix} \frac{1}{2}+\rho^2 & -\rho & 0 & \cdots & 0 & 0 & 0 \\ -\rho & \frac{1}{2}+\rho^2 & -\rho & \cdots & 0 & 0 & 0 \\ 0 & -\rho & \frac{1}{2}+\rho^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2}+\rho^2 & -\rho & 0 \\ 0 & 0 & 0 & \cdots & -\rho & \frac{1}{2}+\rho^2 & -\rho \\ 0 & 0 & 0 & \cdots & 0 & -\rho & \frac{1}{2}+\rho^2 \end{pmatrix}_{n\times n}, \qquad \rho = 0.25.$$

It is easily proved that $\{\varepsilon_i,\ i \geqslant 1\}$ generated from the method above are NA by Joag-Dev and Proschan (1983). Here, we take the nearest neighbour weights to be the weight functions $W_{ni}(\cdot)$. For any $t \in A$, we rewrite $|t_1^{(n)} - t|, ,\ldots, |t_n^{(n)} - t|$ as

$$|t_{R_1(t)}^{(n)}| \leqslant \cdots \leqslant |t_{R_n(t)}^{(n)}|;$$

if $|t_i^{(n)} - t| = |t_j^{(n)} - t|$, then $|t_i^{(n)} - t|$ is moved before $|t_j^{(n)} - t|$ when $i < j$. Take $k_n = \lfloor n^a/\log n\rfloor$, $a = \frac{63}{64}$, $\alpha = 1$ and define the nearest neighbour weight functions as follows:

$$W_{ni}(t) = \begin{cases} \frac{1}{k_n} & \text{if } |t_i - t| \leqslant |t_{R_{k_n}(x)}^{(n)} - t|, \\ 0 & \text{otherwise.}\end{cases}$$

Based on Section 3 in Hu (2006), all the assumptions in Theorem 3.2 are satisfied. Next, by taking $t = 0.2, 0.5, 0.8$ and the sample sizes $n = 100, 500, 1000$, respectively, we compute $\hat{\beta}_n - \beta$ and $\hat{g}_n(t) - g(t)$ for 1000 times and get the corresponding boxplots in Figures 1–3. The corresponding values of Mean Absolute Error (MAE), Standard Deviation (SD) and Root Mean Squared Error (RMSE) for $\hat{\beta}_n$ and $\hat{g}_n(t)$ are listed in Table 1.

Figures 1–3 show that $\hat{\beta}_n - \beta$ and $\hat{g}_n(t) - g(t)$, regardless of $t = 0.2, 0.5$ or 0.8, fluctuate to zero and the variation ranges decrease substantially as the sample size n increases. From Table 1, we can see that MAE, SD and RMSE of the estimators of β and $g(t)$ decrease as n increases. Hence, the numerical results here confirm the consistency of $\hat{\beta}_n$ and $\hat{g}_n(t)$.

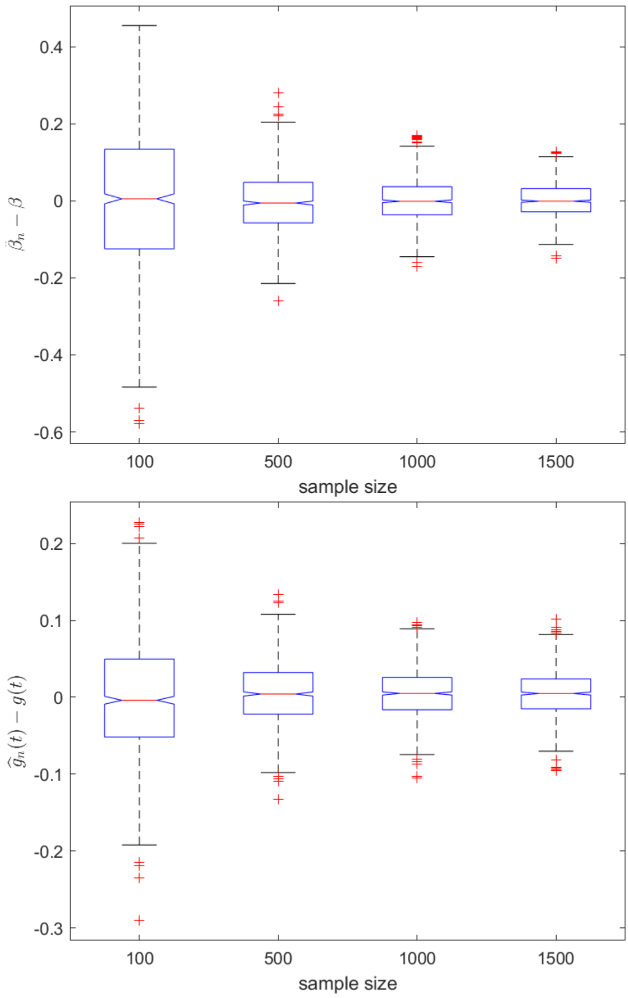

FIGURE 1. Boxplots of $\hat{\beta}_n - \beta$ and $\hat{g}_n(t) - g(t)$ with $g(t) = (e^t \cos t)/300$ and $t = 0.2$

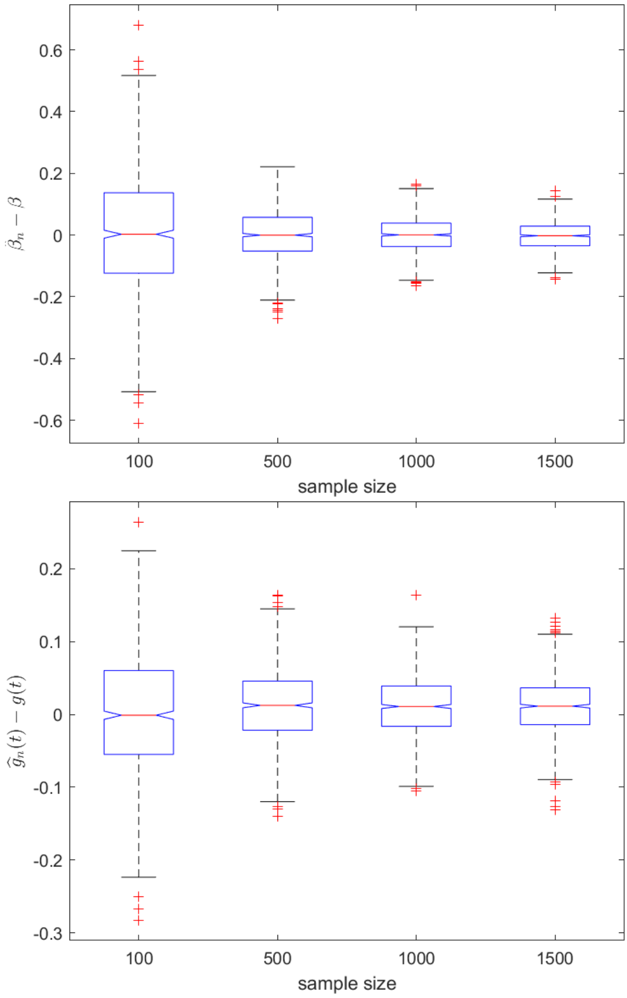

FIGURE 2. Boxplots of $\hat{\beta}_n - \beta$ and $\hat{g}_n(t) - g(t)$ with $g(t) = (e^t \cos t)/300$ and $t = 0.5$

4.2. EV regression models. In this subsection, we mainly evaluate the convergence behaviour of (3.6) and (3.13) and verify the consistency of $\hat{\beta}$ and $\hat{\theta}$. Firstly, we generate the data. Let $(\varepsilon_1, \ldots, \varepsilon_n)^T \sim N(\mathbf{0}, \mathbf{\Sigma})$ and $(\delta_1, \ldots, \delta_n)^T \sim N(\mathbf{0}, \mathbf{\Sigma})$, where $\mathbf{0}$ represents the zero column vector and

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & -\rho & 0 & \cdots & 0 & 0 & 0 \\ -\rho & 1 & -\rho & \cdots & 0 & 0 & 0 \\ 0 & -\rho & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\rho & 0 \\ 0 & 0 & 0 & \cdots & -\rho & 1 & -\rho \\ 0 & 0 & 0 & \cdots & 0 & -\rho & 1 \end{pmatrix}_{n\times n}, \quad \rho = 0.1.$$

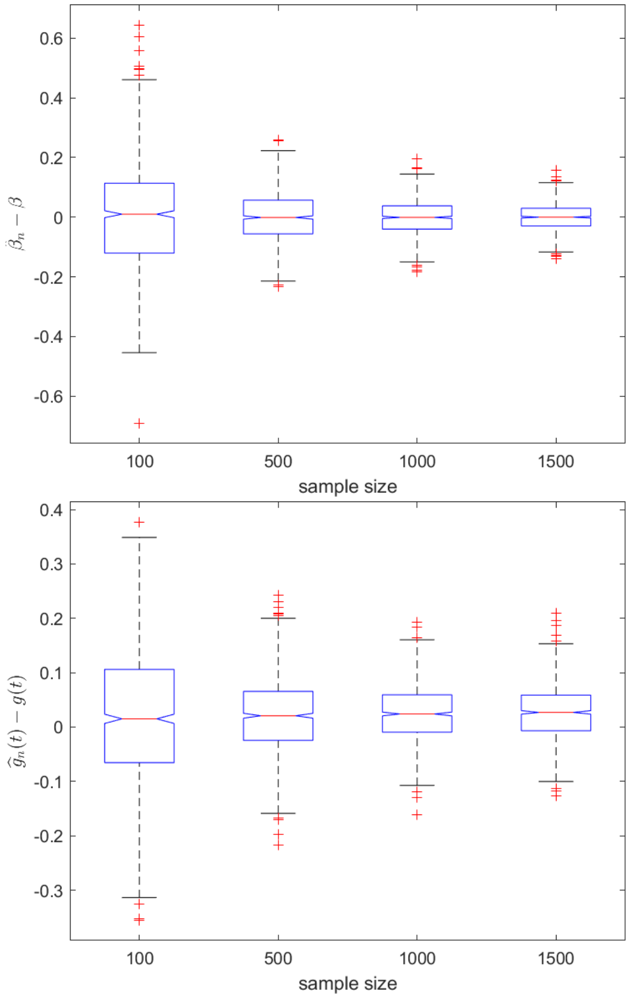

FIGURE 3. Boxplots of $\hat{\beta}_n - \beta$ and $\hat{g}_n(t) - g(t)$ with $g(t) = (e^t \cos t)/300$ and $t = 0.8$

It is obvious that $\{\varepsilon_i, i \geqslant 1\}$ and $\{\delta_i, i \geqslant 1\}$ generated above are both NA by Joag-Dev and Proschan (1983). Set $x_i = i/n^{0.2}$ for all $1 \leqslant i \leqslant n$. It is easy to verify that the conditions required in Theorems 3.3 and 3.4 are satisfied. For different values of β and θ, we consider the following two cases.

CASE 1: $\theta = 3.5$ and $\beta = 0.5$. By taking the sample sizes $n = 100, 500, 1000$ respectively, we compute the values of $\hat{\beta} - \beta$ and $\hat{\theta} - \theta$ for 1000 times and present the plots in Figures 4 and 5, obtained by MATLAB software. Moreover, we record the MAE and RMSE of $\hat{\beta}$ and $\hat{\theta}$ respectively in Table 2.

CASE 2: $\theta = 1$ and $\beta = 2$. We also compute the values of $\hat{\beta} - \beta$ and $\hat{\theta} - \theta$ for 1000 times under the sample sizes $n = 100, 500, 1000$ respectively and present

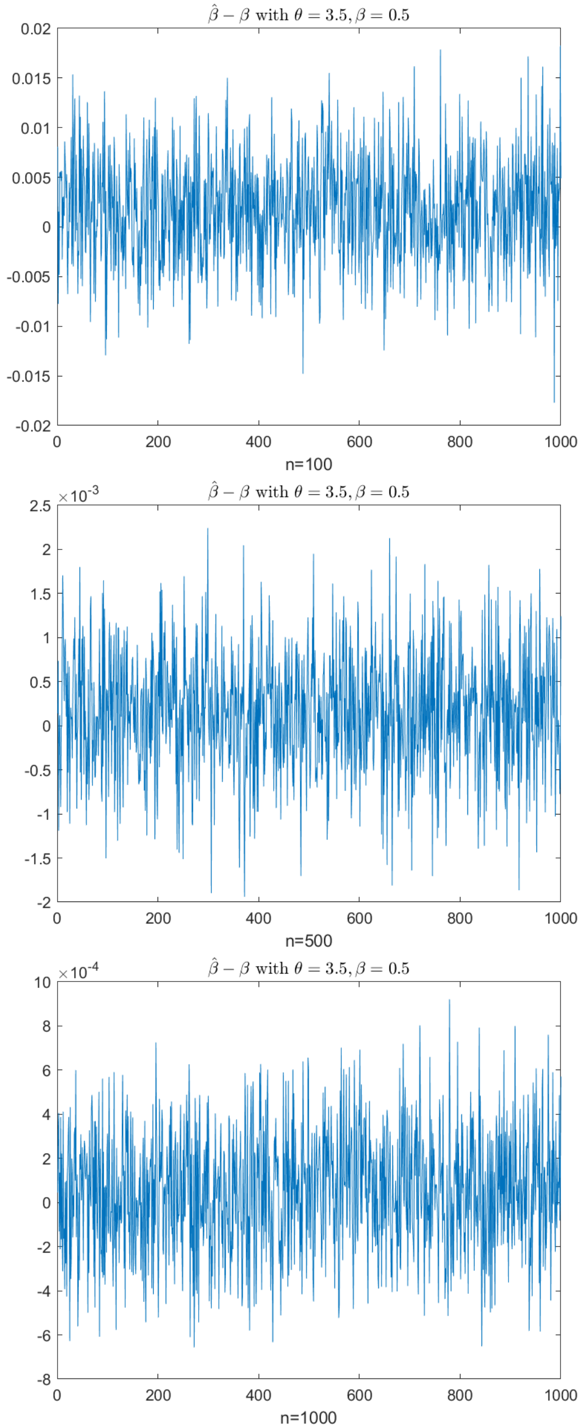

FIGURE 4. $\hat{\beta}-\beta$ with $\theta = 3.5$ and $\beta = 0.5$

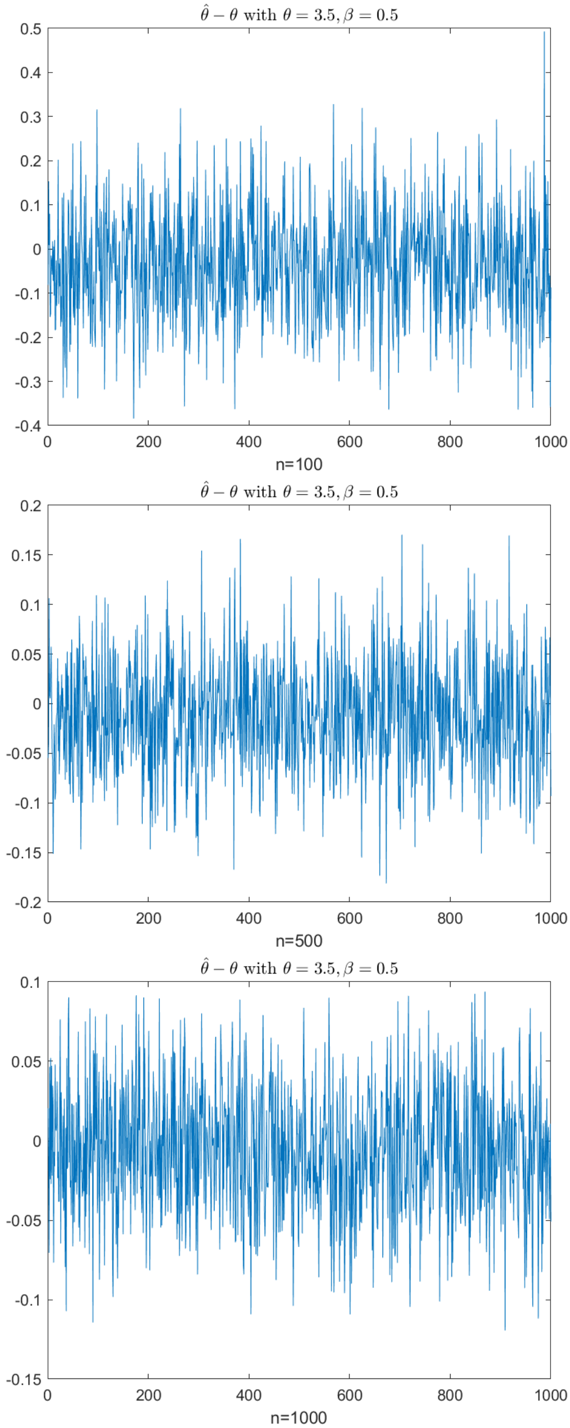

FIGURE 5. $\hat{\theta}-\theta$ with $\theta = 3.5$ and $\beta = 0.5$

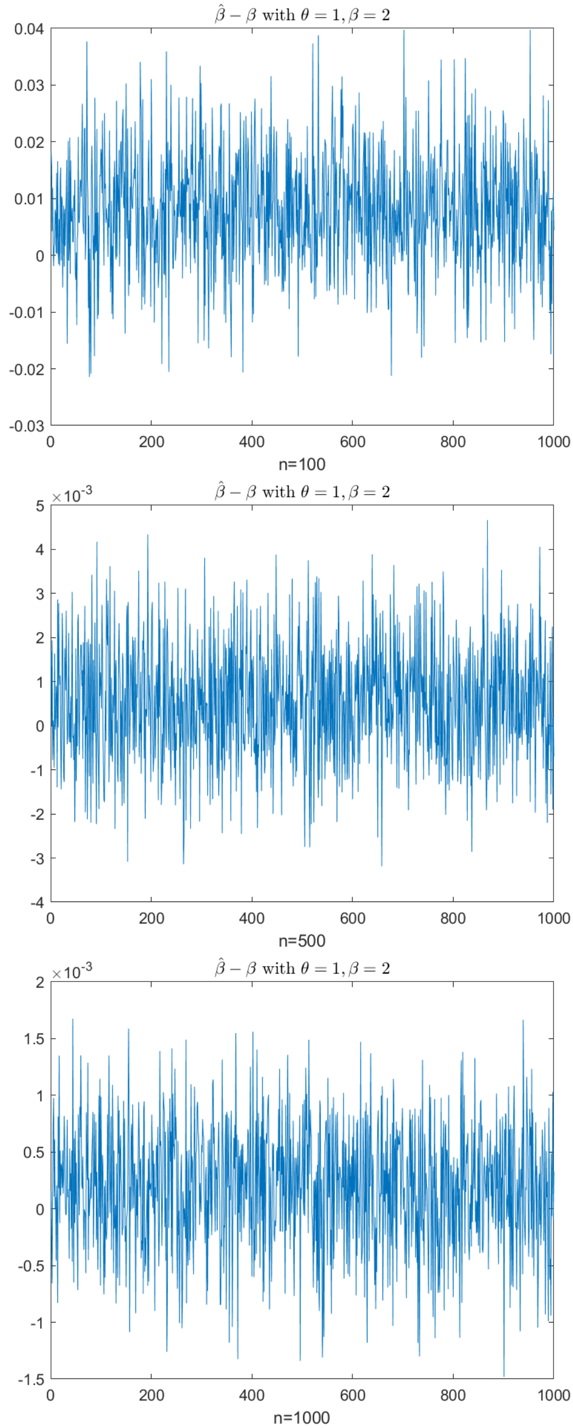

FIGURE 6. $\hat{\beta}-\beta$ with $\theta=1$ and $\beta=2$

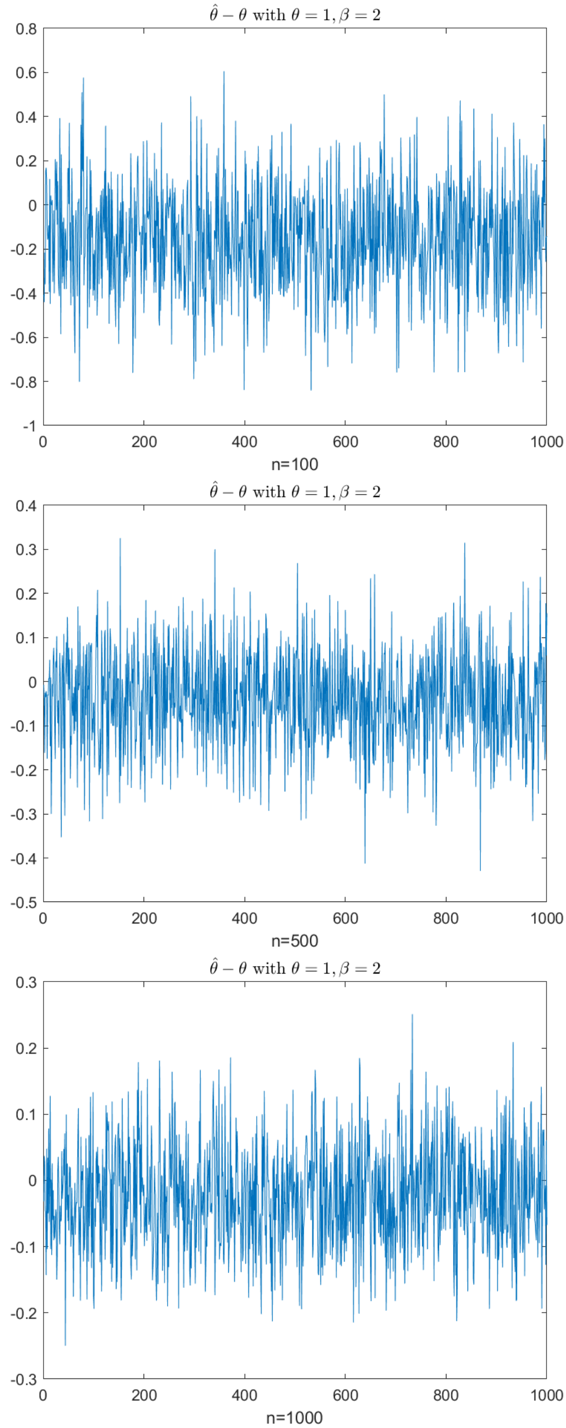

FIGURE 7. $\hat{\theta}-\theta$ with $\theta=1$ and $\beta=2$

TABLE 1. The MAE, SD and RMSE of $\hat{\beta}_n$ and $\hat{g}_n(t)$

		$\hat{\beta}_n$			$\hat{g}_n(t)$		
t	n	MAE	SD	RMSE	MAE	SD	RMSE
0.2	100	0.1435	0.1774	0.1773	0.0615	0.0772	0.0770
	500	0.0642	0.0805	0.0806	0.0321	0.0401	0.0404
	1000	0.0438	0.0553	0.0553	0.0259	0.0324	0.0328
	1500	0.0353	0.0436	0.0436	0.0233	0.0291	0.0294
0.5	100	0.1423	0.1751	0.1751	0.0675	0.0839	0.0839
	500	0.0658	0.0830	0.0829	0.0405	0.0492	0.0506
	1000	0.0443	0.0547	0.0547	0.0333	0.0397	0.0414
	1500	0.0371	0.0460	0.0461	0.0315	0.0380	0.0396
0.8	100	0.1425	0.1795	0.1794	0.0980	0.1222	0.1236
	500	0.0668	0.0827	0.0827	0.0557	0.0669	0.0700
	1000	0.0452	0.0569	0.0568	0.0464	0.0523	0.0576
	1500	0.0361	0.0455	0.0454	0.0451	0.0496	0.0565

TABLE 2. The MAE and RMSE of $\hat{\beta}$ and $\hat{\theta}$ with $\theta = 3.5$ and $\beta = 0.5$

n	$n = 100$	$n = 500$	$n = 1000$
MAE: $\hat{\beta}$	0.0045	5.6073e-04	2.2697e-04
RMSE: $\hat{\beta}$	0.0056	7.0232e-04	2.8109e-04
MAE: $\hat{\theta}$	0.1021	0.0465	0.0330
RMSE: $\hat{\theta}$	0.1288	0.0578	0.0408

TABLE 3. The MAE and RMSE of $\hat{\beta}$ and $\hat{\theta}$ with $\theta = 1$ and $\beta = 2$

n	$n = 100$	$n = 500$	$n = 1000$
MAE: $\hat{\beta}$	0.0105	0.0012	4.6698e-04
RMSE: $\hat{\beta}$	0.0130	0.0014	5.7516e-04
MAE: $\hat{\theta}$	0.2298	0.0926	0.0670
RMSE: $\hat{\theta}$	0.2861	0.1165	0.0829

the plots in Figures 6 and 7. Moreover, we record the MAE and RMSE of $\hat{\beta}$ and $\hat{\theta}$ respectively in Table 3.

Figures 4–7 lead to the conclusion that the values of $\hat{\beta} - \beta$ and $\hat{\theta} - \theta$ fluctuate around zero. From Tables 2 and 3, we can clearly find that MAE and RMSE of $\hat{\beta}$ and $\hat{\theta}$ fluctuate around zero and the fluctuation ranges decrease as n increases. These conclusions agree with the theoretical results.

Acknowledgments. The authors are grateful to the Editor and the anonymous referee for carefully reading the manuscript and for valuable suggestions which helped improve an earlier version of this paper.

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Received 5.3.2024;
accepted 3.7.2024