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ON THE ASYMPTOTIC DISTRIBUTION OF THE MAXIMA FROM GAUSSIAN FUNCTIONS SUBJECT TO MISSING OBSERVATIONS*

BY

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Abstract. In this paper, we derive the asymptotic distributions for the maxima of two types of Gaussian functions, including a χ-random sequence and a Gaussian order statistics sequence subject to missing observations, where the Gaussian functions are generated by stationary Gaussian sequences with covariance functions r_n satisfying $r_n \log n \to \gamma \in [0, \infty)$ as $n \to \infty$.

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1. INTRODUCTION

Let $\{X_n,\, n \geqslant 1\}$ be a sequence of independent and identically distributed random variables, and let $M_n(X) = \max\{X_j : 1 \leqslant j \leqslant n\}$. Suppose there exist constants $c_n > 0, d_n \in \mathbb{R}, n \geqslant 1$, and a non-degenerate distribution $G(x)$ such that

$$\lim_{n\to\infty} P(c_n(M_n - d_n) \leqslant x) = G(x). \tag{1.1}$$

Then G must be one of three types of extreme value distributions (see, e.g., Leadbetter et al. [16] and Piterbarg [25]). The dependent case of the above classical result can also be found in these monographs.

Missing observations may occur randomly in practical applications, but the model (1.1) cannot be applied directly in such situations. In extreme value theory, it is important to investigate the asymptotic relation between the maxima of complete samples and incomplete samples. Consider a sequence $\{X_n,\, n \geqslant 1\}$ of stationary random variables, and let $\{\varepsilon_n,\, n \geqslant 1\}$ denote a sequence of Bernoulli random variables indicating whether the variables $\{X_n,\, n \geqslant 1\}$ are observed. Furthermore,

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suppose that $\{\varepsilon_n,\, n \geqslant 1\}$ is independent of $\{X_n,\, n \geqslant 1\}$. Let $S_n = \sum_{i=1}^{n} \varepsilon_i$ satisfy

$$\frac{S_n}{n} \xrightarrow{P} \lambda \quad \text{as } n \to \infty, \tag{1.2}$$

where λ is a random or non-random variable.

If λ is a constant, under a type of long dependence condition $D(u_n, v_n)$ (see [21] for the definition) and a local dependence condition $D'(u_n)$ (see [16] for the definition), Mladenović and Piterbarg [21] first derived the following result for all real $x < y$:

$$\lim_{n\to\infty} P(M_n(X,\boldsymbol{\varepsilon}) \leqslant \vartheta_n(x),\, M_n(X) \leqslant \vartheta_n(y)) = G^{\lambda}(x)G^{1-\lambda}(y), \tag{1.3}$$

where $M_n(X,\boldsymbol{\varepsilon}) = \max\{X_j : 1 \leqslant j \leqslant n,\, \varepsilon_j = 1\}$ denotes the maxima of samples observed and $\vartheta_n(x) = c_n^{-1}x + d_n$.

Peng et al. [23] and Cao and Peng [4] extended the above result to Gaussian cases. Tong and Peng [35] and Tan and Wang [34] considered the almost sure version of complete maxima and incomplete maxima, while Glavaš et al. [10] and Glavaš and Mladenović [9] studied similar problems for autoregressive processes and linear processes, respectively. Some related studies including the cases of maxima and minima, sums and maxima, non-stationary random fields and exceedance point processes can be found in Hashorva and Weng [12], Krajka and Rychlik [15], Panga and Pereira [22], Peng et al. [24], Li and Tan [17] and Zheng and Tan [39]. Some similar studies for continuous time stochastic processes can be found in Piterbarg [26], Tan and Tang [33], Xu et al. [37], Ling et al. [19], Lu and Peng [20] and the references therein.

When λ is a random variable, under the same conditions of $D(u_n, v_n)$ and $D'(u_n)$, Krajka [14] obtained the following result for any $x < y \in \mathbb{R}$:

$$\lim_{n\to\infty} P(M_n(X,\boldsymbol{\varepsilon}) \leqslant \vartheta_n(x), M_n(X) \leqslant \vartheta_n(y)) = E\big(G^{\lambda}(x)G^{1-\lambda}(y)\big). \tag{1.4}$$

Note that the condition $D(u_n, v_n)$ is weakly dependent and (1.4) indicates that the complete maxima and the incomplete maxima are asymptotic independent conditional on λ. Since the strongly dependent phenomenon is very common in financial time series (see, e.g., Baillie and Kapetanios [3]), it is interesting to study the model (1.4) in the strongly dependent setting. Hashorva et al. [11] extended (1.4) to strongly dependent Gaussian sequences and found that asymptotic conditional independence is destroyed by strong dependence. A naturally arising problem is whether the asymptotic conditional independence between complete maxima and incomplete maxima can be preserved for strongly dependent non-Gaussian random sequences.

In this paper, we extend the model (1.4) to some non-Gaussian strongly dependent random sequences. In what follows, $\{X_n,\, n \geqslant 1\}$ will be a sequence of stationary standard Gaussian random variables with covariance functions $r_n = \mathrm{Cov}(X_1, X_{n+1})$ satisfying

$$r_n \log n \to \gamma \in [0, +\infty) \quad \text{as } n \to \infty. \tag{1.5}$$

The Gaussian sequence $\{X_n,\, n \geqslant 1\}$ is said to be *strongly dependent* when (1.5) holds with $\gamma > 0$ (see, e.g., Leadbetter et al. [16, Chapter 5]). Let $\{X_{ij},\, j \geqslant 1,\, i = 1, \ldots, d\}$, $d \geqslant 1$, be independent copies of $\{X_n,\, n \geqslant 1\}$. Define two types of Gaussian functions generated by the Gaussian sequence:

$$\chi_{dj}(X) = \Big(\sum_{i=1}^{d} X_{ij}^2\Big)^{1/2}, \quad j \geqslant 1, \tag{1.6}$$

and for $r \in \{1, \ldots, d\}$,

(1.7)
$$O_{dj}^{(d)}(X) = \min_{i=1}^{d} X_{ij} \leqslant \cdots \leqslant O_{dj}^{(r)}(X) \leqslant \cdots \leqslant O_{dj}^{(1)}(X) = \max_{i=1}^{d} X_{ij}, j \geqslant 1.$$

Note that the sequences $\{\chi_{dj}(X),\, j \geqslant 1\}$ and $\{O_{dj}^{(r)}(X),\, j \geqslant 1\}$ are a χ-random sequence and a Gaussian rth order statistics sequence, respectively, and both are strongly dependent when (1.5) holds with $\gamma > 0$. The χ-random sequence has been extensively studied in theoretical fields (see, e.g., Sharpe [27], Tan and Hashorva [31, 32], Sun and Tan [29]), and in applied fields (see, e.g., Albin and Jarušková [1], Chareka et al. [5], Jarušková and Piterbarg [13]). The Gaussian rth order statistics random variables play an important role in many applied fields such as in models concerned with the analysis of surface roughness during machinery processes and functional magnetic resonance imaging data. For related studies on Gaussian rth order statistics, we refer to Alodat [2], Dębicki et al. [6, 7, 8], Worsley and Friston [36], Zhao [38] and the references therein.

The studies on the asymptotic relation between the maxima of complete samples and incomplete samples are far from complete. In this paper, we extend the model (1.4) to a stationary χ-random sequence and a Gaussian rth order statistics sequence with missing observations when (1.2) holds with λ a random variable. Section 2 gives the main results and their proofs appear in Section 3.

2. MAIN RESULTS

Now we state our main results. The first result is about the asymptotic distribution for the maxima of χ-random sequences with missing observations.

THEOREM 2.1. *Let $\{X_n,\, n \geqslant 1\}$ be a sequence of stationary Gaussian variables with covariance functions r_n satisfying* (1.5) *and $\{\chi_{dn}(X), n \geqslant 1\}$ be a sequence defined as in* (1.6). *Assume that $\{\varepsilon_n,\, n \geqslant 1\}$ is a sequence of indicators which is independent of $\{\chi_{dn}(X), n \geqslant 1\}$ and such that* (1.2) *holds with λ*

a random variable. If moreover

$$a_n = (2\log n)^{1/2}, \qquad b_n = a_n + \frac{\log\left(2^{1-d/2}(\Gamma(d/2))^{-1}a_n^{d-2}\right)}{a_n}, \tag{2.1}$$

then for any real $x < y$,

$$\begin{aligned}\lim_{n\to\infty} P\big(M_n(\chi_d(X),\boldsymbol{\varepsilon}) \leqslant u_n(x),\, M_n(\chi_d(X)) \leqslant u_n(y)\big) \\ = E\Big(\int_{\mathbb{R}^d} \exp\big(-\lambda e^{-(x+\gamma-\sqrt{2\gamma}\|\mathbf{z}\|)} - (1-\lambda)e^{-(y+\gamma-\sqrt{2\gamma}\|\mathbf{z}\|)}\big)\, d\Phi(\mathbf{z})\Big),\end{aligned} \tag{2.2}$$

where $\Gamma(\cdot)$ is the Euler gamma function,

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt,$$

$\mathbf{z} = (z_1, \ldots, z_d)$, $\|\mathbf{z}\| = \sqrt{z_1^2 + \cdots + z_d^2}$, $u_n(x) = a_n^{-1}x + b_n$, *and* $d\Phi(\mathbf{z})$ *stands for* $d\Phi(z_1)\cdots d\Phi(z_d)$ *for simplicity.*

REMARK 2.1. (i) The weakly dependent case $\gamma = 0$ has been proved by Zheng and Tan [39, Theorem 3.2]; the case $\gamma > 0$ is strongly dependent.

(ii) A similar study for continuous time χ-processes can be found in Ling and Tan [18].

For the maxima of Gaussian order statistics random variables with missing observations, we have the following result.

THEOREM 2.2. *Let $\{X_n,\, n \geqslant 1\}$ be a sequence of stationary Gaussian variables with covariance functions r_n satisfying* (1.5) *and $\{O_{dn}^{(r)}(X),\, n \geqslant 1\}$ be a sequence defined as in* (1.7). *Assume that $\{\varepsilon_n,\, n \geqslant 1\}$ is a sequence of indicators which is independent of $\{O_{dn}^{(r)}(X),\, n \geqslant 1\}$ and such that* (1.2) *holds with λ a random variable. If moreover*

$$\alpha_n = (2r\log n)^{1/2}, \qquad \beta_n = \frac{1}{r}\alpha_n - \frac{r(\log 4\pi - \log r + \log\log n)}{2\alpha_n}, \tag{2.3}$$

then for any real $x < y$,

$$\begin{aligned}\lim_{n\to\infty} P\big(M_n(O_d^{(1)}(X),\boldsymbol{\varepsilon}) \leqslant v_n(x),\, M_n(O_d^{(1)}(X)) \leqslant v_n(y)\big) \\ = E\Big(\int_{\mathbb{R}^d} \exp\Big(-\lambda \sum_{i=1}^{d} e^{-(x+\gamma-\sqrt{2\gamma}z_i)} - (1-\lambda)\sum_{i=1}^{d} e^{-(y+\gamma-\sqrt{2\gamma}z_i)}\Big)\, d\Phi(\mathbf{z})\Big)\end{aligned} \tag{2.4}$$

and

$$\text{(2.5)}\qquad \lim_{n\to\infty} P\big(M_n(O_d^{(d)}(X),\boldsymbol{\varepsilon}) \leqslant v_n(x),\ M_n(O_d^{(d)}(X)) \leqslant v_n(y)\big)$$
$$= E\Big(\int_{\mathbb{R}^d} \exp\big(-\lambda e^{-(x+\gamma-\sqrt{2\gamma}\overline{\mathbf{z}})} - (1-\lambda)e^{-(y+\gamma-\sqrt{2\gamma}\overline{\mathbf{z}})}\big)\, d\Phi(\mathbf{z})\Big),$$

where $v_n(x) = \alpha_n^{-1}x + \beta_n$ *and* $\overline{\mathbf{z}} = \frac{1}{d}\sum_{i=1}^d z_i$.

REMARK 2.2. (i) The case $\gamma = 0$ has been proved by Zheng and Tan [39, Theorem 3.3].

(ii) Note that the cases $r \in \{2, 3, \ldots, d-1\}$ are not covered by our method, since the transformations (3.12) and (3.13) below are only valid for $r = 1, d$.

(iii) A similar study for continuous time Gaussian order statistics processes can be found in Tan [30], where there are some mistakes in the strongly dependent case, but they can be rectified by the method of the present paper.

3. PROOFS OF THE MAIN RESULTS

Let Y_{ij}, U_i be i.i.d. $N(0,1)$ random variables, where $j \geqslant 1$, $i = 1, \ldots, d$ with an integer $d \geqslant 1$. Let $\rho_n = \gamma/\log n$ and define

$$\text{(3.1)}\qquad Z_{ij} = (1-\rho_n)^{1/2} Y_{ij} + \rho_n^{1/2} U_i, \quad j \geqslant 1,\ i = 1, \ldots, d.$$

It is easy to see that the covariance function of $\{Z_{ij},\ j \geqslant 1,\ i = 1, \ldots, d\}$ is

$$\mathrm{Cov}(Z_{ij}, Z_{lk}) = \begin{cases} \rho_n, & i = l, \\ 0, & i \neq l. \end{cases}$$

Define χ-random sequences and rth order statistics sequences generated by the random sequences $\{Y_{ij},\ j \geqslant 1,\ i = 1, \ldots, d\}$ as follows:

$$\text{(3.2)}\qquad \chi_{dj}(Y) = \Big(\sum_{i=1}^d Y_{ij}^2\Big)^{1/2}, \quad j \geqslant 1,$$

and for $r \in \{1, \ldots, d\}$,

(3.3)

$$O_{dj}^{(d)}(Y) = \min_{i=1}^d Y_{ij} \leqslant \cdots \leqslant O_{dj}^{(r)}(Y) \leqslant \cdots \leqslant O_{dj}^{(1)}(Y) = \max_{i=1}^d Y_{ij},\ j \geqslant 1.$$

Similarly, we can define random sequences $\{\chi_{dj}(Z),\ j \geqslant 1\}$ and $\{O_{dj}^{(r)}(Z),\ j \geqslant 1\}$. For fixed k, let $K_s = \{j : (s-1)m + 1 \leqslant j \leqslant sm\}$ for $1 \leqslant s \leqslant k$, where

$m = [n/k]$ denotes the integer part of n/k. Let $\boldsymbol{\theta} = \{\theta_n,\ n \geqslant 1\}$ be a non-random sequence taking values in $\{0, 1\}$. For any random sequence $\{X_n,\ n \geqslant 1\}$, let

$$M_n(X, K_s, \boldsymbol{\theta}) = \begin{cases} \max\limits_{j:\, j \in K_s,\, \theta_j = 1} X_j & \text{if } \sum_{j \in K_s} \theta_j > 0, \\ -\infty & \text{otherwise,} \end{cases}$$

and

$$M_n(X, \boldsymbol{\theta}) = \begin{cases} \max\limits_{j:\, 1 \leqslant j \leqslant n,\, \theta_j = 1} X_j & \text{if } \sum_{j=1}^{n} \theta_j > 0, \\ -\infty & \text{otherwise.} \end{cases}$$

Further, for a random variable λ such that $0 \leqslant \lambda \leqslant 1$ a.s., set

$$B_{t,k} = \left\{ \omega : \lambda(\omega) \in \begin{cases} [0, 1/2^k], & t = 0, \\ (t/2^k, (t+1)/2^k], & 0 < t \leqslant 2^k - 1 \end{cases} \right\},$$

and define

$$B_{t,k,\boldsymbol{\theta},n} = \{\omega : \varepsilon_j(\omega) = \theta_j, 1 \leqslant j \leqslant n\} \cap B_{t,k}.$$

In the following, C will denote a constant whose values may change from line to line, and $\mathbf{1}(\cdot)$ will stand for the indicator function.

To prove Theorems 2.1 and 2.2, we first prove the following lemmas.

LEMMA 3.1. *Suppose that the conditions of Theorem* 2.1 *are satisfied. Assume that, for large* n*, the positive integers* l *are such that* $k < l < m = [n/k]$ *and* $l = o(n)$*. Then, for any real* $x < y$,

$$\Big| P\big(M_n(\chi_d(Y), \boldsymbol{\theta}) \leqslant u_n(x),\ M_n(\chi_d(Y)) \leqslant u_n(y)\big) - \prod_{s=1}^{k} P\big(M_n(\chi_d(Y), K_s, \boldsymbol{\theta}) \leqslant u_n(x),\ M_n(\chi_d(Y), K_s) \leqslant u_n(y)\big) \Big| \leqslant (4k+2)l(1 - F_\chi(u_n(x)))$$

uniformly for all $\boldsymbol{\theta} \in \{0, 1\}^n$*, where* $F_\chi(\cdot)$ *denotes the distribution function of a* χ*-random variable with* d *degrees of freedom.*

Proof. The proof is similar to that in Hashorva et al. [11, Lemma 1], so it is omitted. ■

LEMMA 3.2 *Suppose that the conditions of Theorem* 2.2 *are satisfied. Assume that, for large* n*, the positive integers* l *are such that* $k < l < m = [n/k]$ *and*

$l = o(n)$. Let $v_n(x, z_i) = (1-\rho_n)^{-1/2}(v_n(x) - \rho_n^{1/2} z_i)$ with $z_i \in \mathbb{R}$ and define

$$\mathcal{A}_r = \begin{cases} \bigcap_{\substack{j:\,\theta_j=1\\ 1\leqslant j\leqslant n}} \bigcap_{i=1}^{d} \{Y_{ij} \leqslant v_n(x, z_i)\} \cap \bigcap_{j=1}^{n} \bigcap_{i=1}^{d} \{Y_{ij} \leqslant v_n(y, z_i)\}, & r = 1, \\ \Big\{ \bigcap_{\substack{j:\,\theta_j=1\\ 1\leqslant j\leqslant n}} \bigcup_{i=1}^{d} \{Y_{ij} \leqslant v_n(x, z_i)\} \Big\} \cap \Big\{ \bigcap_{j=1}^{n} \bigcup_{i=1}^{d} \{Y_{ij} \leqslant v_n(y, z_i)\} \Big\}, & r = d, \end{cases}$$

and

$$\mathcal{B}_{r,K_s} = \begin{cases} \bigcap_{\substack{j:\,\theta_j=1\\ j\in K_s}} \bigcap_{i=1}^{d} \{Y_{ij} \leqslant v_n(x, z_i)\} \cap \bigcap_{j\in K_s} \bigcap_{i=1}^{d} \{Y_{ij} \leqslant v_n(y, z_i)\}, & r = 1, \\ \Big\{ \bigcap_{\substack{j:\,\theta_j=1\\ j\in K_s}} \bigcup_{i=1}^{d} \{Y_{ij} \leqslant v_n(x, z_i)\} \Big\} \cap \Big\{ \bigcap_{j\in K_s} \bigcup_{i=1}^{d} \{Y_{ij} \leqslant v_n(y, z_i)\} \Big\}, & r = d. \end{cases}$$

Then, for all real $x < y$, we have

$$\text{(3.4)} \qquad \Big| P(\mathcal{A}_r) - \prod_{s=1}^{k} P(\mathcal{B}_{r,K_s}) \Big| \leqslant \begin{cases} (4k+2) l \sum_{i=1}^{d} (1 - \Phi(v_n(x, z_i))), & r = 1, \\ (4k+2) l \prod_{i=1}^{d} (1 - \Phi(v_n(x, z_i))), & r = d, \end{cases}$$

uniformly for all $\boldsymbol{\theta} \in \{0,1\}^n$.

Proof. First we divide the km integers into $2k$ consecutive intervals as follows. For large n, let l be integers such that $k < l < m$ and $l = o(n)$. Write $I_s = \{(s-1)m + 1, \ldots, sm - l\}$ and $J_s = \{sm - l + 1, \ldots, sm\}$ for $s = 1, \ldots, k$, and set $I_{k+1} = \{(k-1)m + l + 1, \ldots, km\}$ and $J_{k+1} = \{km + 1, \ldots, km + l\}$. For $r = 1$, using the triangular inequality, we get

$$\begin{aligned} \Big| P(\mathcal{A}_1) &- \prod_{s=1}^{k} P(\mathcal{B}_{1,K_s}) \Big| \\ &= \Big| P\Big(\bigcap_{\substack{j:\,\theta_j=1\\ 1\leqslant j\leqslant n}} \bigcap_{i=1}^{d} \{Y_{ij} \leqslant v_n(x, z_i)\} \cap \bigcap_{j=1}^{n} \bigcap_{i=1}^{d} \{Y_{ij} \leqslant v_n(y, z_i)\} \Big) \\ &\quad - \prod_{s=1}^{k} P\Big(\bigcap_{\substack{j:\,\theta_j=1\\ j\in K_s}} \bigcap_{i=1}^{d} \{Y_{ij} \leqslant v_n(x, z_i)\} \cap \bigcap_{j\in K_s} \bigcap_{i=1}^{d} \{Y_{ij} \leqslant v_n(y, z_i)\} \Big) \Big| \\ &\leqslant \Sigma_1 + \Sigma_2 + \Sigma_3, \end{aligned}$$

where

$$\Sigma_1 = \Big|P\Big(\bigcap_{\substack{j:\,\theta_j=1\\1\leqslant j\leqslant n}}\bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(x,z_i)\}\cap\bigcap_{j=1}^{n}\bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(y,z_i)\}\Big)$$
$$-P\Big(\bigcap_{s=1}^{k}\Big\{\bigcap_{\substack{j:\,\theta_j=1\\j\in I_s}}\bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(x,z_i)\}\cap\bigcap_{j\in I_s}\bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(y,z_i)\}\Big\}\Big)\Big|,$$

$$\Sigma_2 = \Big|P\Big(\bigcap_{s=1}^{k}\Big\{\bigcap_{\substack{j:\,\theta_j=1\\j\in I_s}}\bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(x,z_i)\}\cap\bigcap_{j\in I_s}\bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(y,z_i)\}\Big\}\Big)$$
$$-\prod_{s=1}^{k}P\Big(\bigcap_{\substack{j:\,\theta_j=1\\j\in I_s}}\bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(x,z_i)\}\cap\bigcap_{j\in I_s}\bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(y,z_i)\}\Big)\Big|,$$

$$\Sigma_3 = \Big|\prod_{s=1}^{k}P\Big(\bigcap_{\substack{j:\,\theta_j=1\\j\in I_s}}\bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(x,z_i)\}\cap\bigcap_{j\in I_s}\bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(y,z_i)\}\Big)$$
$$-\prod_{s=1}^{k}P\Big(\bigcap_{\substack{j:\,\theta_j=1\\j\in K_s}}\bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(x,z_i)\}\cap\bigcap_{j\in K_s}\bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(y,z_i)\}\Big)\Big|.$$

Since

$$\bigcap_{s=1}^{k}\Big\{\bigcap_{\substack{j:\,\theta_j=1\\j\in I_s}}\bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(x,z_i)\}\cap\bigcap_{j\in I_s}\bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(y,z_i)\}\Big\}$$
$$-\bigcap_{\substack{j:\,\theta_j=1\\1\leqslant j\leqslant n}}\bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(x,z_i)\}\cap\bigcap_{j=1}^{n}\bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(y,z_i)\}$$
$$\subset\bigcup_{s=1}^{k+1}\Big\{\bigcup_{\substack{j:\,\theta_j=1\\j\in J_s}}\bigcup_{i=1}^{d}\{Y_{ij}> v_n(x,z_i)\}\cup\bigcup_{j\in J_s}\bigcup_{i=1}^{d}\{Y_{ij}> v_n(x,z_i)\}\Big\},$$

it follows that

(3.5)

$$\begin{aligned}\Sigma_1 &\leqslant P\Big(\bigcup_{s=1}^{k+1}\Big\{\bigcup_{\substack{j:\,\theta_j=1\\j\in J_s}}\bigcup_{i=1}^{d}\{Y_{ij}> v_n(x,z_i)\}\cup\bigcup_{j\in J_s}\bigcup_{i=1}^{d}\{Y_{ij}> v_n(x,z_i)\}\Big\}\Big)\\ &\leqslant 2\sum_{s=1}^{k+1}P\Big(\bigcup_{j\in J_s}\bigcup_{i=1}^{d}\{Y_{ij}>v_n(x,z_i)\}\Big)\\ &\leqslant 2l\sum_{s=1}^{k+1}\sum_{i=1}^{d}(1-\Phi(v_n(x,z_i)))\\ &= 2(k+1)l\sum_{i=1}^{d}(1-\Phi(v_n(x,z_i))).\end{aligned}$$

Since $\{Y_{ij},\, j \geqslant 1,\, i = 1, \ldots, d\}$ is a sequence of independent standard normal random variables, we obtain

$$\Sigma_2 = 0. \tag{3.6}$$

By using the fact that

$$\Big|\prod_{s=1}^{k} a_s - \prod_{s=1}^{k} b_s\Big| \leqslant \sum_{s=1}^{k} |a_s - b_s| \tag{3.7}$$

for all $a_s, b_s \in [0,1]$, we get

(3.8)

$$\begin{aligned}\Sigma_3 \leqslant \sum_{s=1}^{k} \Big| P\Big(\bigcap_{\substack{j:\,\theta_j=1\\ j\in I_s}} \bigcap_{i=1}^{d}\{Y_{ij} \leqslant v_n(x,z_i)\} \cap \bigcap_{j\in I_s}\bigcap_{i=1}^{d}\{Y_{ij} \leqslant v_n(y,z_i)\}\Big) \\ - P\Big(\bigcap_{\substack{j:\,\theta_j=1\\ j\in K_s}} \bigcap_{i=1}^{d}\{Y_{ij} \leqslant v_n(x,z_i)\} \cap \bigcap_{j\in K_s}\bigcap_{i=1}^{d}\{Y_{ij} \leqslant v_n(y,z_i)\}\Big)\Big| \\ \leqslant 2l \sum_{s=1}^{k}\sum_{i=1}^{d}(1-\Phi(v_n(x,z_i))) = 2kl\sum_{i=1}^{d}(1-\Phi(v_n(x,z_i))).\end{aligned}$$

Combining (3.5), (3.6) with (3.8), the case $r = 1$ of (3.4) follows.

The proof for $r = d$ is similar, hence we omit it. ■

LEMMA 3.3. *Under the conditions of Theorem* 2.1, *for any real* $x < y$ *we have*

$$\begin{aligned}\Big| P\big(M_n(\chi_d(X),\boldsymbol{\theta}) \leqslant u_n(x),\, M_n(\chi_d(X)) \leqslant u_n(y)\big) \\ - \int_{\mathbb{R}^d} P\big(M_n(\chi_d(Y),\boldsymbol{\theta}) \leqslant u_n(x,\|\mathbf{z}\|),\, M_n(\chi_d(Y)) \leqslant u_n(y,\|\mathbf{z}\|)\big)\, d\Phi(\mathbf{z})\Big| \\ \leqslant Cnd\sum_{k=1}^{n}\frac{|r_k-\rho_n|(u_n(x)u_n(y))^{d-1}}{(1-\omega_k^2)^{1/2d}}\exp\Big(-\frac{u_n^2(x)+u_n^2(y)}{2(1+\omega_k)}\Big),\end{aligned}$$

where $u_n(x,\|\mathbf{z}\|) = (1-\rho_n)^{-1/2}(u_n(x)-\rho_n^{1/2}\|\mathbf{z}\|)$ *and* $\omega_k = \max(|r_k|, |\rho_n|)$.

Proof. By using the comparison inequality for χ-variables (see, e.g., Song et al. [28, Lemma 3.2]), we get

$$\begin{aligned}\big| P\big(M_n(\chi_d(X),\boldsymbol{\theta}) \leqslant u_n(x),\, M_n(\chi_d(X)) \leqslant u_n(y)\big) \\ - P\big(M_n(\chi_d(Z),\boldsymbol{\theta}) \leqslant u_n(x),\, M_n(\chi_d(Z)) \leqslant u_n(y)\big)\big| \\ \leqslant Cnd\sum_{k=1}^{n}\frac{|r_k-\rho_n|(u_n(x)u_n(y))^{d-1}}{(1-\omega_k^2)^{1/2d}}\exp\Big(-\frac{u_n^2(x)+u_n^2(y)}{2(1+\omega_k)}\Big).\end{aligned} \tag{3.9}$$

Define a Gaussian random field by

$$Z_{(j,\boldsymbol{\nu})} = Z_{1j}\nu_1 + Z_{2j}\nu_2 + \cdots + Z_{dj}\nu_d, \quad j \geqslant 1,$$

with

$$\boldsymbol{\nu} = (\nu_1, \ldots, \nu_d) \in \mathbb{S}_{d-1},$$

where $\mathbb{S}_{d-1}$ stands for the unit sphere in $\mathbb{R}^d$ and the sequences $\{Z_{ij},\, j \geqslant 1,\, i = 1, \ldots, d\}$ are defined in (3.1). We have (see, e.g., Piterbarg [25])

$$\chi_{dj}(Z) = \Big(\sum_{i=1}^{d} Z_{ij}^2\Big)^{1/2} = \sup_{\boldsymbol{\nu}\in\mathbb{S}_{d-1}} Z_{(j,\boldsymbol{\nu})} = \sup_{\boldsymbol{\nu}\in\mathbb{S}_{d-1}} \sum_{i=1}^{d} Z_{ij}\nu_i, \quad j \geqslant 1.$$

Hence

$$\begin{aligned}
&(3.10)\quad P\big(M_n(\chi_d(Z),\boldsymbol{\theta}) \leqslant u_n(x),\, M_n(\chi_d(Z)) \leqslant u_n(y)\big)\\
&= P\Big(\max_{\substack{(j,\boldsymbol{\nu})\in[1,n]\times\mathbb{S}_{d-1}\\ j:\,\theta_j=1}} \sum_{i=1}^{d}\big((1-\rho_n)^{1/2}Y_{ij} + \rho_n^{1/2}U_i\big)\nu_i \leqslant u_n(x),\\
&\qquad\qquad \max_{(j,\boldsymbol{\nu})\in[1,n]\times\mathbb{S}_{d-1}} \sum_{i=1}^{d}\big((1-\rho_n)^{1/2}Y_{ij} + \rho_n^{1/2}U_i\big)\nu_i \leqslant u_n(y)\Big)\\
&= P\Big(\bigcap_{\substack{(j,\boldsymbol{\nu})\in[1,n]\times\mathbb{S}_{d-1}\\ j:\,\theta_j=1}} \Big\{\sum_{i=1}^{d} Y_{ij}\nu_i \leqslant (1-\rho_n)^{-1/2}\Big(u_n(x) - \rho_n^{1/2}\sum_{i=1}^{d} U_i\nu_i\Big)\Big\}\\
&\qquad \cap \bigcap_{(j,\boldsymbol{\nu})\in[1,n]\times\mathbb{S}_{d-1}} \Big\{\sum_{i=1}^{d} Y_{ij}\nu_i \leqslant (1-\rho_n)^{-1/2}(u_n(y) - \rho_n^{1/2}\sum_{i=1}^{d} U_i\nu_i)\Big\}\Big)\\
&= \int_{\mathbb{R}^d} P\Big(\bigcap_{\substack{(j,\boldsymbol{\nu})\in[1,n]\times\mathbb{S}_{d-1}\\ j:\,\theta_j=1}} \Big\{\sum_{i=1}^{d} Y_{ij}\nu_i \leqslant (1-\rho_n)^{-1/2}\Big(u_n(x) - \rho_n^{1/2}\sum_{i=1}^{d} z_i\nu_i\Big)\Big\}\\
&\qquad \cap \bigcap_{(j,\boldsymbol{\nu})\in[1,n]\times\mathbb{S}_{d-1}} \Big\{\sum_{i=1}^{d} Y_{ij}\nu_i \leqslant (1-\rho_n)^{-1/2}(u_n(y) - \rho_n^{1/2}\sum_{i=1}^{d} z_i\nu_i)\Big\}\Big)\, d\Phi(\mathbf{z})\\
&= \int_{\mathbb{R}^d} P\big(M_n(\chi_d(Y),\boldsymbol{\theta}) \leqslant u_n(x, \|\mathbf{z}\|),\, M_n(\chi_d(Y)) \leqslant u_n(y, \|\mathbf{z}\|)\big)\, d\Phi(\mathbf{z}).
\end{aligned}$$

Plugging (3.10) into (3.9), we complete the proof of Lemma 3.3. ∎

LEMMA 3.4. *Under the conditions of Theorem* 2.2, *we have, for* $r = 1, d$ *and any real* $x < y$,

$$\Big|P\big(M_n(O_d^{(r)}(X),\boldsymbol{\theta}) \leqslant v_n(x),\, M_n(O_d^{(r)}(X)) \leqslant v_n(y)\big) - \int_{\mathbb{R}^d} P(\mathcal{A}_r)\, d\Phi(\mathbf{z})\Big|$$
$$\leqslant Cnd\sum_{k=1}^{n} \frac{|r_k - \rho_n|}{(v_n(x)v_n(y))^{r-1}} \exp\Big(-\frac{r(v_n^2(x) + v_n^2(y))}{2(1+\omega_k)}\Big),$$

where $\mathcal{A}_r$ *is defined in Lemma* 3.2 *and* $\omega_k = \max(|r_k|, |\rho_n|)$.

Proof. By using the comparison inequality for order statistics variables (see e.g. Song et al. [28, Lemma 3.2]), we get

$$
\begin{aligned}
(3.11)\quad & \Big|P\big(M_n(O_d^{(r)}(X),\boldsymbol{\theta}) \leqslant v_n(x),\, M_n(O_d^{(r)}(X)) \leqslant v_n(y)\big) \\
& \qquad - P\big(M_n(O_d^{(r)}(Z),\boldsymbol{\theta}) \leqslant v_n(x),\, M_n(O_d^{(r)}(Z)) \leqslant v_n(y)\big)\Big| \\
& \leqslant Cnd\sum_{k=1}^{n} \frac{|r_k-\rho_n|}{(v_n(x)v_n(y))^{r-1}} \exp\Big(-\frac{r(v_n^2(x)+v_n^2(y))}{2(1+\omega_k)}\Big), \qquad r=1,d.
\end{aligned}
$$

For $r=1$, we have

$$
\begin{aligned}
(3.12)\quad & P\big(M_n(O_d^{(1)}(Z),\boldsymbol{\theta}) \leqslant v_n(x),\, M_n(O_d^{(1)}(Z)) \leqslant v_n(y)\big) \\
&= P\Big(\max_{\substack{j:\,\theta_j=1\\ 1\leqslant j\leqslant n}} \max_{1\leqslant i\leqslant d} Z_{ij} \leqslant v_n(x),\, \max_{1\leqslant j\leqslant n}\max_{1\leqslant i\leqslant d} Z_{ij} \leqslant v_n(y)\Big) \\
&= P\Big(\max_{\substack{j:\,\theta_j=1\\ 1\leqslant j\leqslant n}} \max_{1\leqslant i\leqslant d}\{(1-\rho_n)^{1/2}Y_{ij}+\rho_n^{1/2}U_i\} \leqslant v_n(x), \\
&\qquad\qquad \max_{1\leqslant j\leqslant n}\max_{1\leqslant i\leqslant d}\{(1-\rho_n)^{1/2}Y_{ij}+\rho_n^{1/2}U_i\} \leqslant v_n(y)\Big) \\
&= P\Big(\bigcap_{\substack{j:\,\theta_j=1\\ 1\leqslant j\leqslant n}} \bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(x,U_i)\} \cap \bigcap_{1\leqslant j\leqslant n}\bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(y,U_i)\}\Big) \\
&= \int_{\mathbb{R}^d} P\Big(\bigcap_{\substack{j:\,\theta_j=1\\ 1\leqslant j\leqslant n}} \bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(x,z_i)\} \cap \bigcap_{1\leqslant j\leqslant n}\bigcap_{i=1}^{d}\{Y_{ij}\leqslant v_n(y,z_i)\}\Big)\, d\Phi(\mathbf{z}).
\end{aligned}
$$

By the same arguments, for $r=d$ we have

$$
\begin{aligned}
(3.13)\quad & P\big(M_n(O_d^{(d)}(Z),\boldsymbol{\theta}) \leqslant v_n(x),\, M_n(O_d^{(d)}(Z)) \leqslant v_n(y)\big) \\
&= \int_{\mathbb{R}^d} P\Big(\bigcap_{\substack{j:\,\theta_j=1\\ 1\leqslant j\leqslant n}} \bigcup_{i=1}^{d}\{Y_{ij}\leqslant v_n(x,z_i)\} \cap \bigcap_{1\leqslant j\leqslant n}\bigcup_{i=1}^{d}\{Y_{ij}\leqslant v_n(y,z_i)\}\Big)\, d\Phi(\mathbf{z}).
\end{aligned}
$$

Recall the definition of $\mathcal{A}_r$ in Lemma 3.2. Combining (3.12) with (3.13), we get

$$
\begin{aligned}
(3.14)\quad & P\big(M_n(O_d^{(r)}(Z),\boldsymbol{\theta}) \leqslant v_n(x),\, M_n(O_d^{(r)}(Z)) \leqslant v_n(y)\big) \\
&\qquad = \int_{\mathbb{R}^d} P(\mathcal{A}_r)\, d\Phi(\mathbf{z}), \qquad r=1,d.
\end{aligned}
$$

Plugging (3.14) into (3.11) completes the proof of Lemma 3.4. ∎

LEMMA 3.5. *Under the conditions of Theorem* 2.1*, we have, as* $n\to\infty$*,*

$$
Cnd\sum_{k=1}^{n} \frac{|r_k-\rho_n|(u_n(x)u_n(y))^{d-1}}{(1-\omega_k^2)^{1/2d}} \exp\Big(-\frac{u_n^2(x)+u_n^2(y)}{2(1+\omega_k)}\Big) \to 0,
$$

where $\omega_k=\max(|r_k|,|\rho_n|)$.

Proof. From (2.1), it is easy to see that for any $x \in \mathbb{R}$,

$$\exp\biggl(-\frac{u_n^2(x)}{2}\biggr) \sim \frac{C(u_n(x))^{-(d-2)}}{n}, \qquad u_n(x) \sim (2\log n)^{1/2}. \tag{3.15}$$

Let $\delta(k) = \sup_{k<m\leqslant n} \omega_m$. Clearly ω_k, and hence $\delta(k)$, also depend on n, but we do not make this dependence explicit in the notation. In view of Leadbetter et al. [16, p. 86], we have $\sup_{n\geqslant 1} |r_n| < 1$ and $\sup_{n\geqslant 1} |\rho_n| < 1$. Furthermore, it is easy to see that $\sup_{n\geqslant 1} |\omega_n| < 1$ and $\delta(0) < 1$. Further, let τ be such that $0 < \tau < (1-\delta(0))/(1+\delta(0))$ for all sufficiently large n, and let $p = [n^\tau]$. Then

$$\begin{aligned}
Cnd \sum_{k=1}^{n} &\frac{|r_k-\rho_n|(u_n(x)u_n(y))^{d-1}}{(1-\omega_k^2)^{1/2d}} \exp\biggl(-\frac{u_n^2(x)+u_n^2(y)}{2(1+\omega_k)}\biggr)\\
&\leqslant Cnd \sum_{k=1}^{n} |r_k-\rho_n|(u_n(y))^{2(d-1)} \exp\biggl(-\frac{u_n^2(x)}{1+\omega_k}\biggr)\\
&= Cnd \sum_{k=1}^{p} |r_k-\rho_n|(u_n(y))^{2(d-1)} \exp\biggl(-\frac{u_n^2(x)}{1+\omega_k}\biggr)\\
&\quad + Cnd \sum_{k=p+1}^{n} |r_k-\rho_n|(u_n(y))^{2(d-1)} \exp\biggl(-\frac{u_n^2(x)}{1+\omega_k}\biggr)\\
&=: B_{n,1}+B_{n,2}.
\end{aligned}$$

Using (3.15) and $\omega_k \leqslant \delta(0)$, we obtain

$$\begin{aligned}
B_{n,1} &\leqslant Cnd \sum_{k=1}^{p} (u_n(y))^{2(d-1)} \exp\biggl(-\frac{u_n^2(x)}{1+\delta(0)}\biggr)\\
&\leqslant Cdn^{1+\tau}(u_n(y))^{2(d-1)} \exp\biggl(-\frac{u_n^2(x)}{2}\biggr)^{\frac{2}{1+\delta(0)}}\\
&\sim Cdn^{\tau-\frac{1-\delta(0)}{1+\delta(0)}}(\log n)^{d-1-\frac{d-2}{1+\delta(0)}}.
\end{aligned}$$

Since $0 < \tau < \frac{1-\delta(0)}{1+\delta(0)}$, we get $B_{n,1} \to 0$ as $n \to \infty$.

For the second part $B_{n,2}$, we have

$$\begin{aligned}
B_{n,2} &\leqslant Cnd \sum_{k=p+1}^{n} |r_k-\rho_n|(u_n(y))^{2(d-1)} \exp\biggl(-\frac{u_n^2(x)}{1+\delta(p)}\biggr)\\
&= Cd\frac{n^2}{\log n}(u_n(y))^{2(d-1)} \exp\biggl(-\frac{u_n^2(x)}{1+\delta(p)}\biggr)\frac{\log n}{n}\sum_{k=p+1}^{n} |r_k-\rho_n|.
\end{aligned}$$

Since $r_n \log n \to \gamma$, there is a constant C such that $r_n \log n \leqslant C$ for all $n \geqslant 1$. Hence also

$$\delta(p) \log p \leqslant C, \tag{3.16}$$

so by (3.15) and (3.16) we have

$$\begin{aligned} Cd\frac{n^2}{\log n}(u_n(y))^{2(d-1)} \exp\Big(-\frac{u_n^2(x)}{1+\delta(p)}\Big) &\leqslant Cd\frac{n^2}{\log n}(u_n(y))^{2(d-1)} \exp\Big(-\frac{u_n^2(x)}{1+C/\log n^\tau}\Big) \\ &\leqslant Cdn^{\frac{2C}{\tau \log n + C}} (\log n)^{\frac{C(d-2)}{\tau \log n + C}} = O(1). \end{aligned} \tag{3.17}$$

According to Leadbetter et al. [16, p. 135], we have

$$\lim_{n\to\infty} \frac{\log n}{n} \sum_{k=p+1}^{n} |r_k - \rho_n| = 0. \tag{3.18}$$

Combining (3.17) with (3.18), we have $B_{n,2} \to 0$ as $n \to \infty$. Therefore, the proof of Lemma 3.5 is complete. ∎

LEMMA 3.6. *Under the conditions of Theorem* 2.2 *we have, as* $n \to \infty$,

$$Cnd \sum_{k=1}^{n} \frac{|r_k - \rho_n|}{(v_n(x)v_n(y))^{r-1}} \exp\Big(-\frac{r(v_n^2(x)+v_n^2(y))}{2(1+\omega_k)}\Big) \to 0, \tag{3.19}$$

where $\omega_k = \max(|r_k|, |\rho_n|)$.

Proof. We use the notations of Lemma 3.5. From (2.3), it is easy to see that

$$\exp\Big(-\frac{rv_n^2(x)}{2}\Big) \sim \frac{Cv_n^r(x)}{n}, \qquad v_n(x) \sim \Big(\frac{2\log n}{r}\Big)^{1/2}. \tag{3.20}$$

As in the proof of Lemma 3.5, split the sum in (3.19) into two parts:

$$\begin{aligned} Cnd \sum_{k=1}^{n} \frac{|r_k - \rho_n|}{(v_n(x)v_n(y))^{r-1}} \exp\Big(-\frac{r(v_n^2(x)+v_n^2(y))}{2(1+\omega_k)}\Big) &\leqslant Cnd \sum_{k=1}^{p} \frac{|r_k - \rho_n|}{(v_n(x))^{2(r-1)}} \exp\Big(-\frac{rv_n^2(x)}{1+\omega_k}\Big) \\ &\quad + Cnd \sum_{k=p+1}^{n} \frac{|r_k - \rho_n|}{(v_n(x))^{2(r-1)}} \exp\Big(-\frac{rv_n^2(x)}{1+\omega_k}\Big) \\ &=: B_{n,3} + B_{n,4}. \end{aligned}$$

Using (3.20) and $0 < \tau < \frac{1-\delta(0)}{1+\delta(0)}$, we have

$$\begin{aligned} B_{n,3} &\leqslant Cnd \sum_{k=1}^{p} \frac{|r_k - \rho_n|}{(v_n(x))^{2(r-1)}} \exp\left(-\frac{rv_n^2(x)}{1+\delta(0)}\right) \\ &\leqslant Cdn^{1+\tau} \frac{1}{(v_n(x))^{2(r-1)}} \exp\left(-\frac{rv_n^2(x)}{2}\right)^{\frac{2}{1+\delta(0)}} \\ &\sim Cdn^{\tau - \frac{1-\delta(0)}{1+\delta(0)}} (\log n)^{1-r+\frac{r}{1+\delta(0)}} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

For $B_{n,4}$, we have

$$\begin{aligned} B_{n,4} &\leqslant Cnd \sum_{k=p+1}^{n} \frac{|r_k - \rho_n|}{(v_n(x))^{2(r-1)}} \exp\left(-\frac{rv_n^2(x)}{1+\delta(p)}\right) \\ &= Cd \frac{n^2}{(\log n)(v_n(x))^{2(r-1)}} \exp\left(-\frac{rv_n^2(x)}{1+\delta(p)}\right) \frac{\log n}{n} \sum_{k=p+1}^{n} |r_k - \rho_n|. \end{aligned}$$

In view of (3.16) and (3.20),

$$\begin{aligned} &Cd \frac{n^2}{(\log n)(v_n(x))^{2(r-1)}} \exp\left(-\frac{rv_n^2(x)}{1+\delta(p)}\right) \\ &\qquad \leqslant Cd \frac{n^2}{(\log n)(v_n(x))^{2(r-1)}} \exp\left(-\frac{rv_n^2(x)}{1+C/\log n^{\tau}}\right) \\ &\qquad \leqslant Cd \frac{n^2}{(\log n)^r} \frac{(\log n)^{\frac{r}{1+C/\log n^{\tau}}}}{n^{\frac{2}{1+C/\log n^{\tau}}}} \\ &\qquad = Cdn^{\frac{2C}{C+\tau \log n}} (\log n)^{-\frac{rC}{C+\tau \log n}} = O(1). \end{aligned}$$

So by using (3.18), we obtain $\lim_{n\to\infty} B_{n,4} = 0$. Hence the proof of Lemma 3.6 is complete. ∎

Proof of Theorem 2.1. Let $\Psi(n, x, \|\mathbf{z}\|) = n(1 - F_{\chi}(u_n(x, \|\mathbf{z}\|)))$. We use the techniques of Krajka [14]. By the total probability rule and the triangle inequality, we get

$$\begin{aligned} &\Bigg| P\big(M_n(\chi_d(X), \boldsymbol{\varepsilon}) \leqslant u_n(x),\ M_n(\chi_d(X)) \leqslant u_n(y)\big) \\ &\qquad - E\left(\int_{\mathbb{R}^d} \prod_{s=1}^{k} \left(1 - \frac{\lambda\Psi(n, x, \|\mathbf{z}\|) + (1-\lambda)\Psi(n, y, \|\mathbf{z}\|)}{k}\right) d\Phi(\mathbf{z})\right)\Bigg| \\ &\leqslant \sum_{t=0}^{2^k-1} \sum_{\boldsymbol{\theta}\in\{0,1\}^n} E\Bigg| P\big(M_n(\chi_d(X), \boldsymbol{\theta}) \leqslant u_n(x),\ M_n(\chi_d(X)) \leqslant u_n(y)\big) \\ &\qquad - \int_{\mathbb{R}^d} \prod_{s=1}^{k} \left(1 - \frac{\lambda\Psi(n, x, \|\mathbf{z}\|) + (1-\lambda)\Psi(n, y, \|\mathbf{z}\|)}{k}\right) d\Phi(\mathbf{z})\Bigg| \mathbf{1}(B_{t,k,\boldsymbol{\theta},n}) \\ &\leqslant \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4, \end{aligned}$$

where

$$\Delta_1 = \sum_{t=0}^{2^k-1} \sum_{\boldsymbol{\theta}\in\{0,1\}^n} E\Big(\Big|P\big(M_n(\chi_d(X),\boldsymbol{\theta}) \leqslant u_n(x),\ M_n(\chi_d(X)) \leqslant u_n(y)\big) - \int_{\mathbb{R}^d} P(\mathcal{C}_{\boldsymbol{\theta},\mathbf{z}})\, d\Phi(\mathbf{z})\Big|\, \mathbf{1}(B_{t,k,\boldsymbol{\theta},n})\Big),$$

$$\Delta_2 = \sum_{t=0}^{2^k-1} \sum_{\boldsymbol{\theta}\in\{0,1\}^n} E\Big(\int_{\mathbb{R}^d} \Big|P(\mathcal{C}_{\boldsymbol{\theta},\mathbf{z}}) - \prod_{s=1}^{k} P\big(\mathcal{C}_{\boldsymbol{\theta},K_s,\mathbf{z}}\big)\Big|\, d\Phi(\mathbf{z})\, \mathbf{1}(B_{t,k,\boldsymbol{\theta},n})\Big),$$

$$\Delta_3 = \sum_{t=0}^{2^k-1} \sum_{\boldsymbol{\theta}\in\{0,1\}^n} E\Big(\int_{\mathbb{R}^d} \Big| \prod_{s=1}^{k} P(\mathcal{C}_{\boldsymbol{\theta},K_s,\mathbf{z}}) - \prod_{s=1}^{k}\Big(1 - \frac{\frac{t}{2^k}\Psi(n,x,\|\mathbf{z}\|) + \big(1-\frac{t}{2^k}\big)\Psi(n,y,\|\mathbf{z}\|)}{k}\Big)\Big|\, d\Phi(\mathbf{z})\, \mathbf{1}(B_{t,k,\boldsymbol{\theta},n})\Big),$$

$$\Delta_4 = \sum_{t=0}^{2^k-1} \sum_{\boldsymbol{\theta}\in\{0,1\}^n} E\Big(\int_{\mathbb{R}^d} \Big| \prod_{s=1}^{k}\Big(1 - \frac{\frac{t}{2^k}\Psi(n,x,\|\mathbf{z}\|) + \big(1-\frac{t}{2^k}\big)\Psi(n,y,\|\mathbf{z}\|)}{k}\Big) - \prod_{s=1}^{k}\Big(1 - \frac{\lambda\Psi(n,x,\|\mathbf{z}\|) + (1-\lambda)\Psi(n,y,\|\mathbf{z}\|)}{k}\Big)\Big|\, d\Phi(\mathbf{z})\, \mathbf{1}(B_{t,k,\boldsymbol{\theta},n})\Big)$$

with

$$\mathcal{C}_{\boldsymbol{\theta},\mathbf{z}} = \{M_n(\chi_d(Y),\boldsymbol{\theta}) \leqslant u_n(x,\|\mathbf{z}\|)\} \cap \{M_n(\chi_d(Y)) \leqslant u_n(y,\|\mathbf{z}\|)\}$$

and

$$\mathcal{C}_{\boldsymbol{\theta},K_s,\mathbf{z}} = \{M_n(\chi_d(Y),K_s,\boldsymbol{\theta}) \leqslant u_n(x,\|\mathbf{z}\|)\} \cap \{M_n(\chi_d(Y),K_s) \leqslant u_n(y,\|\mathbf{z}\|)\}.$$

Using Lemmas 3.3 and 3.5, we have

$$\lim_{n\to\infty} \Delta_1 = 0. \tag{3.21}$$

For Δ_2, according to Lemma 3.1, we have

$$\Delta_2 \leqslant (4k+2)\frac{l}{n} \int_{\mathbb{R}^d} \Psi(n,x,\|\mathbf{z}\|)\, d\Phi(\mathbf{z}).$$

It follows from Leadbetter et al. [16, proof of Theorem 6.5.1] that, as $n \to \infty$,

$$u_n(x,\|\mathbf{z}\|) = u_n(x + \gamma - \sqrt{2\gamma}\,\|\mathbf{z}\|) + o(a_n^{-1}), \tag{3.22}$$

and taking into account the tail asymptotic of χ-variables (see, e.g., Piterbarg [25]), we have

$$\lim_{n\to\infty} n(1 - F_\chi(u_n(x))) = \exp(-x). \tag{3.23}$$

Combining (3.22) with (3.23), we have

$$\lim_{n\to\infty} \Psi(n,x,\|\mathbf{z}\|) = \exp\bigl(-(x+\gamma-\sqrt{2\gamma}\,\|\mathbf{z}\|)\bigr) =: g(x,\gamma,\|\mathbf{z}\|).$$

Noting that $l = o(n)$ as $n \to \infty$, the dominated convergence theorem yields

$$\lim_{n\to\infty} \Delta_2 = 0. \tag{3.24}$$

For Δ_3, according to (3.7) and using the same arguments as in Krajka [14, (7)], we have

$$\begin{aligned}
\Delta_3 \leqslant{}& \sum_{t=0}^{2^k-1} \sum_{\boldsymbol{\theta}\in\{0,1\}^n} E\biggl(\int_{\mathbb{R}^d} \sum_{s=1}^{k} \biggl| P(\mathcal{C}_{\boldsymbol{\theta},K_s,\mathbf{z}}) \\
&\quad - \biggl(1 - \frac{\frac{t}{2^k}\Psi(n,x,\|\mathbf{z}\|) + \bigl(1-\frac{t}{2^k}\bigr)\Psi(n,y,\|\mathbf{z}\|)}{k}\biggr)\biggr| \, d\Phi(\mathbf{z})\, \mathbf{1}(B_{t,k,\boldsymbol{\theta},n})\biggr) \\
\leqslant{}& \sum_{t=0}^{2^k-1} \sum_{\boldsymbol{\theta}\in\{0,1\}^n} E\biggl(\int_{\mathbb{R}^d} \sum_{s=1}^{k} \biggl| \frac{\sum_{j\in K_s}\theta_j}{m} - \frac{t}{2^k} \biggr| \\
&\quad \times \frac{\Psi(n,x,\|\mathbf{z}\|) - \Psi(n,y,\|\mathbf{z}\|)}{k}\, d\Phi(\mathbf{z})\, \mathbf{1}(B_{t,k,\boldsymbol{\theta},n})\biggr) \\
&\quad + \frac{1}{k}\int_{\mathbb{R}^d} (\Psi(n,x,\|\mathbf{z}\|))^2\, d\Phi(\mathbf{z}) \\
={}& \sum_{t=0}^{2^k-1} \sum_{s=1}^{k} E\biggl(\biggl| \frac{\sum_{j\in K_s}\varepsilon_j}{m} - \frac{t}{2^k} \biggr| \mathbf{1}(B_{t,k})\biggr) \\
&\quad \times \int_{\mathbb{R}^d} \frac{\Psi(n,x,\|\mathbf{z}\|) - \Psi(n,y,\|\mathbf{z}\|)}{k}\, d\Phi(\mathbf{z}) + \frac{1}{k}\int_{\mathbb{R}^d} (\Psi(n,x,\|\mathbf{z}\|))^2\, d\Phi(\mathbf{z}) \\
\leqslant{}& \sum_{s=1}^{k} \biggl(E\biggl| \frac{\sum_{j\in K_s}\varepsilon_j}{m} - \lambda \biggr| + \frac{1}{2^k}\biggr) \int_{\mathbb{R}^d} \frac{\Psi(n,x,\|\mathbf{z}\|) - \Psi(n,y,\|\mathbf{z}\|)}{k} d\Phi(\mathbf{z}) \\
&\quad + \frac{1}{k}\int_{\mathbb{R}^d} (\Psi(n,x,\|\mathbf{z}\|))^2\, d\Phi(\mathbf{z}) \\
={}& \sum_{s=1}^{k} \biggl(E\biggl| \frac{S_{sm}}{sm}s - \frac{S_{(s-1)m}}{(s-1)m}(s-1) - \lambda \biggr| + \frac{1}{2^k}\biggr) \\
&\quad \times \int_{\mathbb{R}^d} \frac{\Psi(n,x,\|\mathbf{z}\|) - \Psi(n,y,\|\mathbf{z}\|)}{k}\, d\Phi(\mathbf{z}) + \frac{1}{k}\int_{\mathbb{R}^d} (\Psi(n,x,\|\mathbf{z}\|))^2\, d\Phi(\mathbf{z}).
\end{aligned}$$

Since $\frac{S_n}{n} \xrightarrow{P} \lambda$ as $n \to \infty$, we have $\frac{S_{sm}}{sm} \xrightarrow{P} \lambda$ as $m \to \infty$. Furthermore,

$$\frac{S_{sm}}{sm}s - \frac{S_{(s-1)m}}{(s-1)m}(s-1) - \lambda \xrightarrow{P} 0 \quad \text{as } m \to \infty. \tag{3.25}$$

Thus, we get

$$\limsup_{n\to\infty} \Delta_3 \leqslant \frac{1}{2^k} \int_{\mathbb{R}^d} (g(x,\gamma,\|\mathbf{z}\|) - g(y,\gamma,\|\mathbf{z}\|))\, d\Phi(\mathbf{z}) + \frac{1}{k} \int_{\mathbb{R}^d} (g(x,\gamma,\|\mathbf{z}\|))^2\, d\Phi(\mathbf{z}). \tag{3.26}$$

For Δ_4, we have

$$\begin{aligned}
\Delta_4 &\leqslant \sum_{t=0}^{2^k-1} \sum_{\boldsymbol{\theta}\in\{0,1\}^n} E\Big(\int_{\mathbb{R}^d} \sum_{s=1}^{k} \Big|\lambda - \frac{t}{2^k}\Big| \\
&\qquad \times \frac{\Psi(n,x,\|\mathbf{z}\|) + \Psi(n,y,\|\mathbf{z}\|)}{k}\, d\Phi(\mathbf{z}) \mathbf{1}(B_{t,k,\boldsymbol{\theta},n})\Big) \\
&= \int_{\mathbb{R}^d} (\Psi(n,x,\|\mathbf{z}\|) + \Psi(n,y,\|\mathbf{z}\|))\, d\Phi(\mathbf{z}) \\
&\quad \times \sum_{t=0}^{2^k-1} E\Big(\Big|\lambda - \frac{t}{2^k}\Big| \mathbf{1}(B_{t,k})\Big) \\
&\leqslant \int_{\mathbb{R}^d} \frac{\Psi(n,x,\|\mathbf{z}\|) + \Psi(n,y,\|\mathbf{z}\|))}{2^k}\, d\Phi(\mathbf{z}) \\
&\to \frac{1}{2^k} \int_{\mathbb{R}^d} (g(x,\gamma,\|\mathbf{z}\|) + g(y,\gamma,\|\mathbf{z}\|))\, d\Phi(\mathbf{z}) \quad \text{as } n\to\infty.
\end{aligned} \tag{3.27}$$

Hence, combining (3.21), (3.24), (3.26) with (3.27), we have

$$\begin{aligned}
\limsup_{n\to\infty} \Big| P\big(M_n(\chi_d(X),\boldsymbol{\varepsilon}) \leqslant u_n(x),\ M_n(\chi_d(X)) \leqslant u_n(y)\big) \\
- E\Big(\int_{\mathbb{R}^d} \Big(1 - \frac{\lambda g(x,\gamma,\|\mathbf{z}\|) + (1-\lambda) g(y,\gamma,\|\mathbf{z}\|)}{k}\Big)^k d\Phi(\mathbf{z})\Big)\Big| \\
\leqslant \frac{1}{2^{k-1}} \int_{\mathbb{R}^d} g(x,\gamma,\|\mathbf{z}\|)\, d\Phi(\mathbf{z}) + \frac{1}{k} \int_{\mathbb{R}^d} (g(x,\gamma,\|\mathbf{z}\|))^2\, d\Phi(\mathbf{z}).
\end{aligned}$$

The claimed result follows by letting $k\to\infty$.

Proof of Theorem 2.2. Let

$$\varphi_r(n,x,\mathbf{z}) = \begin{cases} n\sum_{i=1}^{d} (1 - \Phi(v_n(x,z_i))), & r = 1, \\ n\prod_{i=1}^{d} (1 - \Phi(v_n(x,z_i))), & r = d. \end{cases}$$

By the total probability rule and the triangle inequality, we get

$$\begin{aligned}&\Big|P\big(M_n(O_d^{(r)}(X),\boldsymbol{\varepsilon}\big)\leqslant v_n(x),\,M_n(O_d^{(r)}(X))\leqslant v_n(y))\\&\qquad-E\bigg(\int\limits_{\mathbb{R}^d}\prod_{s=1}^{k}\bigg(1-\frac{\lambda\varphi_r(n,x,\mathbf{z})+(1-\lambda)\varphi_r(n,y,\mathbf{z})}{k}\bigg)\,d\Phi(\mathbf{z})\bigg)\Big|\\&\leqslant\sum_{t=0}^{2^k-1}\sum_{\boldsymbol{\theta}\in\{0,1\}^n}E\Big|P\big(M_n(O_d^{(r)}(X),\boldsymbol{\theta})\leqslant v_n(x),\,M_n(O_d^{(r)}(X))\leqslant v_n(y)\big)\\&\quad-\int\limits_{\mathbb{R}^d}\prod_{s=1}^{k}\bigg(1-\frac{\lambda\varphi_r(n,x,\mathbf{z})+(1-\lambda)\varphi_r(n,y,\mathbf{z})}{k}\bigg)\,d\Phi(\mathbf{z})\Big|\,\mathbf{1}(B_{t,k,\boldsymbol{\theta},n})\\&\leqslant\Delta_5+\Delta_6+\Delta_7+\Delta_8,\end{aligned}$$

where

$$\begin{aligned}\Delta_5&=\sum_{t=0}^{2^k-1}\sum_{\boldsymbol{\theta}\in\{0,1\}^n}E\Big(\Big|P\big(M_n(O_d^{(r)}(X),\boldsymbol{\theta})\leqslant v_n(x),\,M_n(O_d^{(r)}(X))\leqslant v_n(y)\big)\\&\qquad-\int\limits_{\mathbb{R}^d}P(\mathcal{A}_r)\,d\Phi(\mathbf{z})\Big|\mathbf{1}(B_{t,k,\boldsymbol{\theta},n})\Big),\\\Delta_6&=\sum_{t=0}^{2^k-1}\sum_{\boldsymbol{\theta}\in\{0,1\}^n}E\Big(\int\limits_{\mathbb{R}^d}\Big|P(\mathcal{A}_r)\,d\Phi(\mathbf{z})-\prod_{s=1}^{k}P(\mathcal{B}_{r,K_s}\Big)\Big|\,d\Phi(\mathbf{z})\mathbf{1}(B_{t,k,\boldsymbol{\theta},n})),\\\Delta_7&=\sum_{t=0}^{2^k-1}\sum_{\boldsymbol{\theta}\in\{0,1\}^n}E\bigg(\int\limits_{\mathbb{R}^d}\bigg|\prod_{s=1}^{k}\bigg(1-\frac{\frac{t}{2^k}\varphi_r(n,x,\mathbf{z})+\big(1-\frac{t}{2^k}\big)\varphi_r(n,y,\mathbf{z})}{k}\bigg)\\&\qquad-\prod_{s=1}^{k}P(\mathcal{B}_{r,K_s})\bigg|\,d\Phi(\mathbf{z})\mathbf{1}(B_{t,k,\boldsymbol{\theta},n})\bigg),\\\Delta_8&=\sum_{t=0}^{2^k-1}\sum_{\boldsymbol{\theta}\in\{0,1\}^n}E\bigg(\int\limits_{\mathbb{R}^d}\bigg|\prod_{s=1}^{k}\bigg(1-\frac{\lambda\varphi_r(n,x,\mathbf{z})+(1-\lambda)\varphi_r(n,y,\mathbf{z})}{k}\bigg)\\&\qquad-\prod_{s=1}^{k}\bigg(1-\frac{\frac{t}{2^k}\varphi_r(n,x,\mathbf{z})+\big(1-\frac{t}{2^k}\big)\varphi_r(n,y,\mathbf{z})}{k}\bigg)\bigg|\,d\Phi(\mathbf{z})\mathbf{1}(B_{t,k,\boldsymbol{\theta},n})\bigg).\end{aligned}$$

By Lemmas 3.4 and 3.6,

$$\lim_{n\to\infty}\Delta_5=0. \tag{3.28}$$

By Lemma 3.2,

$$\Delta_6\leqslant(4k+2)\frac{l}{n}\int\limits_{\mathbb{R}^d}\varphi_r(n,x,\mathbf{z})\,d\Phi(\mathbf{z}).$$

Again according to Leadbetter et al. [16, proof of Theorem 6.5.1], we have

$$v_n(x, z_i) = v_n(x + \gamma - \sqrt{2\gamma}\, z_i) + o(\alpha_n^{-1}).$$

By using (2.3), we obtain

$$\lim_{n\to\infty} \varphi_r(n, x, \mathbf{z}) = \begin{cases} \sum_{i=1}^d \exp(-(x + \gamma - \sqrt{2\gamma}\, z_i)), & r = 1, \\ \exp(-(x + \gamma - \sqrt{2\gamma}\, \overline{\mathbf{z}})), & r = d, \end{cases}$$
$$=: f_r(x, \gamma, \mathbf{z}).$$

The dominated convergence theorem and the fact that $l = o(n)$ as $n \to \infty$ imply

$$\lim_{n\to\infty} \Delta_6 = 0. \tag{3.29}$$

For Δ_7, by the same arguments as for Δ_3, we have

$$\begin{aligned}
\Delta_7 &\leqslant \sum_{t=0}^{2^k-1} \sum_{\boldsymbol{\theta}\in\{0,1\}^n} E\Bigg(\int\limits_{\mathbb{R}^d} \sum_{s=1}^k \Bigg| \Bigg(1 - \frac{\frac{t}{2^k}\varphi_r(n, x, \mathbf{z}) + \big(1 - \frac{t}{2^k}\big)\varphi_r(n, y, \mathbf{z})}{k}\Bigg) \\
&\qquad\qquad - P(\mathcal{B}_{r,K_s}) \Bigg| \, d\Phi(\mathbf{z})\, \mathbf{1}(B_{t,k,\boldsymbol{\theta},n}) \Bigg) \\
&\leqslant \sum_{t=0}^{2^k-1} \sum_{\boldsymbol{\theta}\in\{0,1\}^n} E\Bigg(\int\limits_{\mathbb{R}^d} \sum_{s=1}^k \Bigg| \frac{\sum_{j\in K_s} \theta_j}{m} - \frac{t}{2^k} \Bigg| \\
&\quad \times \frac{\varphi_r(n, x, \mathbf{z}) - \varphi_r(n, y, \mathbf{z})}{k}\, d\Phi(\mathbf{z})\, \mathbf{1}(B_{t,k,\boldsymbol{\theta},n}) \Bigg) + \frac{1}{k} \int\limits_{\mathbb{R}^d} (\varphi_r(n, x, \mathbf{z}))^2 \, d\Phi(\mathbf{z}) \\
&= \sum_{t=0}^{2^k-1} \sum_{s=1}^k E\Bigg(\Bigg| \frac{\sum_{j\in K_s} \varepsilon_j}{m} - \frac{t}{2^k} \Bigg| \mathbf{1}(B_{t,k}) \Bigg) \int\limits_{\mathbb{R}^d} \frac{\varphi_r(n, x, \mathbf{z}) - \varphi_r(n, y, \mathbf{z})}{k}\, d\Phi(\mathbf{z}) \\
&\qquad\qquad + \frac{1}{k} \int\limits_{\mathbb{R}^d} (\varphi_r(n, x, \mathbf{z}))^2 \, d\Phi(\mathbf{z}) \\
&\leqslant \sum_{s=1}^k \Bigg(E\Bigg| \frac{\sum_{j\in K_s} \varepsilon_j}{m} - \lambda \Bigg| + \frac{1}{2^k} \Bigg) \int\limits_{\mathbb{R}^d} \frac{\varphi_r(n, x, \mathbf{z}) - \varphi_r(n, y, \mathbf{z})}{k}\, d\Phi(\mathbf{z}) \\
&\qquad\qquad + \frac{1}{k} \int\limits_{\mathbb{R}^d} (\varphi_r(n, x, \mathbf{z}))^2 \, d\Phi(\mathbf{z}) \\
&= \sum_{s=1}^k \Bigg(E\Bigg(\frac{S_{sm}}{sm} s - \frac{S_{(s-1)m}}{(s-1)m}(s-1) - \lambda \Bigg) + \frac{1}{2^k} \Bigg) \\
&\qquad \times \int\limits_{\mathbb{R}^d} \frac{\varphi_r(n, x, \mathbf{z}) - \varphi_r(n, y, \mathbf{z})}{k}\, d\Phi(\mathbf{z}) + \frac{1}{k} \int\limits_{\mathbb{R}^d} (\varphi_r(n, x, \mathbf{z}))^2 \, d\Phi(\mathbf{z}).
\end{aligned}$$

Using (3.25), we get

$$\limsup_{n\to\infty} \Delta_7 \leqslant \frac{1}{2^k} \int_{\mathbb{R}^d} (f_r(x,\gamma,\mathbf{z}) - f_r(y,\gamma,\mathbf{z}))\, d\Phi(\mathbf{z}) + \frac{1}{k} \int_{\mathbb{R}^d} (f_r(x,\gamma,\mathbf{z}))^2\, d\Phi(\mathbf{z}). \tag{3.30}$$

For Δ_8, as $n \to \infty$ we have

$$\begin{aligned} \Delta_8 &\leqslant \sum_{t=0}^{2^k-1} \sum_{\boldsymbol{\theta}\in\{0,1\}^n} E\bigg(\int_{\mathbb{R}^d} \sum_{s=1}^{k} \left|\lambda - \frac{t}{2^k}\right| \\ &\qquad \times \frac{\varphi_r(n,x,\mathbf{z}) + \varphi_r(n,y,\mathbf{z})}{k}\, d\Phi(\mathbf{z})\, \mathbf{1}(B_{t,k,\boldsymbol{\theta},n})\bigg) \\ &= \int_{\mathbb{R}^d} (\varphi_r(n,x,\mathbf{z}) + \varphi_r(n,y,\mathbf{z}))\, d\Phi(\mathbf{z}) \\ &\quad \times \sum_{t=0}^{2^k-1} E\left(\left|\lambda - \frac{t}{2^k}\right| \mathbf{1}(B_{t,k})\right) \\ &\leqslant \int_{\mathbb{R}^d} \frac{\varphi_r(n,x,\mathbf{z}) + \varphi_r(n,y,\mathbf{z})}{2^k}\, d\Phi(\mathbf{z}) \\ &\to \frac{1}{2^k} \int_{\mathbb{R}^d} (f_r(x,\gamma,\mathbf{z}) + f_r(y,\gamma,\mathbf{z}))\, d\Phi(\mathbf{z}). \end{aligned} \tag{3.31}$$

Hence, combining (3.28) – (3.31), we have

$$\begin{aligned} \limsup_{n\to\infty} \bigg| & P\big(M_n(O_d^{(r)}(X),\boldsymbol{\varepsilon}) \leqslant v_n(x),\ M_n(O_d^{(r)}(X)) \leqslant v_n(y)\big) \\ &- E\bigg(\int_{\mathbb{R}^d} \left(1 - \frac{\lambda f_r(x,\gamma,\mathbf{z}) + (1-\lambda) f_r(y,\gamma,\mathbf{z})}{k}\right)^k d\Phi(\mathbf{z})\bigg)\bigg| \\ &\leqslant \frac{1}{2^{k-1}} \int_{\mathbb{R}^d} f_r(x,\gamma,\mathbf{z})\, d\Phi(\mathbf{z}) + \frac{1}{k} \int_{\mathbb{R}^d} (f_r(x,\gamma,\mathbf{z}))^2\, d\Phi(\mathbf{z}). \end{aligned}$$

Letting $k \to \infty$, we complete the proof.

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