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INFINITESIMAL GENERATORS FOR A FAMILY OF POLYNOMIAL PROCESSES – AN ALGEBRAIC APPROACH*

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Abstract. Quadratic harnesses are time-inhomogeneous Markov polynomial processes with linear conditional expectations and quadratic conditional variances with respect to the past-future filtrations. Typically they are determined by five numerical constants η, θ, τ, σ and q hidden in the form of conditional variances. In this paper we derive infinitesimal generators of such processes in the case $\sigma = 0$, extending previously known results. The infinitesimal generators are identified through a solution of a q-commutation equation in the algebra $\mathcal{Q}$ of infinite sequences of polynomials in one variable. The solution is a special element in $\mathcal{Q}$, whose coordinates satisfy a three-term recurrence and thus define a system of orthogonal polynomials. It turns out that the corresponding orthogonality measure $\nu_{x,t}$ uniquely determines the infinitesimal generator (acting on polynomials or bounded functions with bounded continuous second derivative) as an integro-differential operator with an explicit kernel, where integration is with respect to the measure $\nu_{x,t}$.

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1. INTRODUCTION AND PRELIMINARIES

Let us consider a Markov process $(X_t)_{t\geqslant 0}$ with transition probabilities $\mathbb{P}_{s,t}(x, \mathrm{d}y)$ for $(s,t) \in \Gamma := \{(r,u) : 0 \leqslant r \leqslant u\}$ and $x \in \operatorname{supp}(X_s) \subseteq \mathbb{R}$. Following [1, 2], we say that $(X_t)_{t\geqslant 0}$ is an *m-polynomial process*, $m \in \mathbb{N} \cup \{0\}$, if for all $k \in \{0, 1, \ldots, m\}$ and any polynomial f of degree at most k the following two

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conditions hold:

$$\mathbb{E}(f(X_t) \mid X_s = x) = \int_{\mathbb{R}} f(y)\, \mathbb{P}_{s,t}(x, \mathrm{d}y) \tag{1.1}$$

is a polynomial in x of degree at most k, and

$$(s,t) \mapsto \mathbb{E}(f(X_t) \mid X_s = x) \tag{1.2}$$

is in $\mathcal{C}^1(\Gamma)$, i.e., it is a continuously differentiable function in the interior of Γ, and there exist continuous extensions of this function and its derivatives to the boundary.

DEFINITION 1.1. If $(X_t)_{t\geqslant 0}$ is an m-polynomial process for all $m \geqslant 0$, then it is called a *polynomial process*.

Polynomial processes were introduced by Cuchiero [31] in the time-homogeneous case (then it is enough to assume that (1.2) is in $\mathcal{C}(\Gamma)$ instead of $\mathcal{C}^1(\Gamma)$). Applications of polynomial processes in financial and insurance mathematics (see e.g. [32]) triggered intensive studies of these processes in recent ten years. For recent extensions of polynomial processes to more abstract settings, see [33, 34, 7, 6].

In the above-mentioned references, much effort has been devoted to the properties of infinitesimal generators of polynomial processes, to relate them to martingale problems in order to simplify the calculation of some expectations. In this context, giving explicit formulas for infinitesimal generators for a broadest possible class of polynomial processes is of considerable interest.

In this paper, we study a wide subclass of polynomial processes called quadratic harnesses, introduced in [17]. Quadratic harnesses are square-integrable Markov processes with conditional expectations and conditional variances with respect to past-future sigma fields in a linear and quadratic form, respectively. More precisely, we assume that $(X_t)_{t\geqslant 0}$ is a square-integrable stochastic process such that

$$\mathbb{E}X_s = 0 \quad \text{and} \quad \mathbb{E}X_tX_s = \min\{s,t\} \qquad \text{for } s,t \geqslant 0. \tag{1.3}$$

Let $\mathcal{F}_{s,u} := \sigma\{X_t : t \in [0,s] \cup [u,\infty)\}$, $(s,u) \in \Gamma$, be the natural past-future filtration generated by the process. We say that $(X_t)_{t\geqslant 0}$ is a *quadratic harness* if for all $0 \leqslant s < t < u$ we have

$$\mathbb{E}(X_t|\mathcal{F}_{s,u}) = a_{tsu}X_s + b_{tsu}X_u \tag{1.4}$$

and

$$\mathbb{E}(X_t^2|\mathcal{F}_{s,u}) = A_{tsu}X_s^2 + B_{tsu}X_sX_u + C_{tsu}X_u^2 + D_{tsu}X_s + E_{tsu}X_u + F_{tsu}, \tag{1.5}$$

where $a_{tsu}, b_{tsu}, A_{tsu}, \ldots, F_{tsu}$ are deterministic functions depending only on $0 \leqslant s < t < u$. In view of (1.3), it is easy to see that $a_{tsu} = \frac{u-t}{u-s}$ and $b_{tsu} = \frac{t-s}{u-s}$.

It is well-known (see [17, Theorem 2.2]) that under mild technical assumptions, $A_{tsu}, \dots, F_{tsu}$ are explicitly identified in terms of five numerical constants

$$\eta, \theta \in \mathbb{R}, \quad \sigma \geqslant 0, \quad \tau \geqslant 0 \quad \text{and} \quad q \leqslant 1 + 2\sqrt{\sigma\tau} \tag{1.6}$$

in such a way that

$$\mathbb{V}\mathrm{ar}(X_t|\mathcal{F}_{s,u}) = \frac{(u-t)(t-s)}{\sigma su + u - qs + \tau} K\left(\frac{X_u - X_s}{u-s}, \frac{uX_s - sX_u}{u-s}\right), \tag{1.7}$$

where

$$K(x,y) = 1 + \eta y + \theta x + \sigma y^2 + \tau x^2 - (1-q)xy.$$

Typically, the distribution of a quadratic harness process is uniquely determined by conditions (1.3)–(1.5), and thus by the five constants in (1.6) (see [17, discussion after Theorem 2.4]); moreover, the construction of quadratic harnesses [21] shows that their supports are indeed compact in most cases. Therefore, we will write $(X_t)_{t\geqslant 0} \sim QH(\eta, \theta; \sigma, \tau; q)$ referring to the quadratic harness with the corresponding parameters. Well-known examples of quadratic harnesses are Wiener, Poisson, or Gamma processes. This class also includes classical versions of the free Brownian motion (see [10]), q-Gaussian processes (see [14]), q-Lévy–Meixner processes (see [3]), and the radial part of the quantum Bessel process (see [11]). It also contains a wide family of Askey–Wilson processes introduced in [21]. The latter is of special interest due to its relation to the ASEP (asymmetric simple exclusion process) with open boundaries through the representation of the generating function of the stationary law of the ASEP through joint moments of the Askey–Wilson (or quadratic harness) processes derived in [25]. The representation was one of the major tools recently used in [29] (see also [28]) to identify the multi-point Laplace transform of the stationary measure for the open KPZ equation, as well as to identify Brownian excursions and Brownian meanders as limiting processes in ASEPs of increasing sizes in [19]. Moreover, in [25], the representation combined with the form of the infinitesimal generator of the relevant quadratic harness allowed us to derive a formula for the so-called profile density for the ASEP. For additional references on (quadratic) harnesses and their applications, the reader may consult e.g. [8, 16, 35, 36, 37, 40, 41, 42, 43]. Let us emphasize that while for a large family of parameters $\eta, \theta, \sigma, \tau, q$ satisfying (1.6), the corresponding quadratic harnesses were constructed and they are Markov processes (with a wide spectrum of examples given above), it is not known whether quadratic harnesses exist (or are Markov) for the full range of parameters $\eta, \theta, \sigma, \tau, q$ defined through (1.6). Therefore when we write $(X_t)_{t\geqslant 0} \sim QH(\eta, \theta; \sigma, \tau; q)$, we tacitly assume that the process $(X_t)_{t\geqslant 0}$ exists and is Markov (then (1.6) necessarily holds true).

In this paper we will consider quadratic harnesses with $\sigma = 0$ (the case when $\sigma \neq 0$ looks technically much more involved and needs a separate investigation).

Then $\mathbb{E}|X_t|^r < \infty$ for all $r, t > 0$ by [17, Theorem 2.5] and therefore $(X_t)_{t\geqslant 0}$ is a uniquely determined Markov process satisfying conditions (1.3), (1.4) and (1.7). Since $\lim_{u\to\infty} X_u/u = 0$ a.s. (see [17, (2.9)]), from (1.4) and (1.7) we have

$$\mathbb{E}(X_t|\mathcal{F}_s) = X_s \quad \text{(martingale property)} \tag{1.8}$$

and

$$\mathbb{E}(X_t^2|\mathcal{F}_s) = X_s^2 + \eta(t-s)X_s + (t-s), \tag{1.9}$$

where $(\mathcal{F}_s)_{s\geqslant 0}$ is a natural filtration associated to this process. Moreover, one can calculate all conditional moments $\mathbb{E}(X_t^n|\mathcal{F}_s)$ and show that they are polynomials in X_s of degree at most n, for all $n \in \mathbb{N}$. As a result, $(X_t)_{t\geqslant 0}$ is a time-inhomogeneous polynomial process according to Definition 1.1.

There is an analogous situation regarding conditioning with respect to the future of the process, i.e., $\mathbb{E}(X_t^n|\mathcal{F}_{0,u})$ are also polynomials in X_u of degree at most n, so quadratic harnesses are polynomial processes not only with respect to the past but also with respect to the future of the process.

Our main goal is to derive explicit formulas for the infinitesimal generators of $(X_t)_{t\geqslant 0} \sim QH(\eta, \theta; 0, \tau; q)$. It turns out that the infinitesimal generators of quadratic harnesses acting on polynomials (or on bounded continuous functions with bounded continuous second derivatives) can be represented as integro-differential operators, where the integrals are taken with respect to the orthogonality measure for a concrete system of orthogonal polynomials, which is identified through its Jacobi matrix. Following the referee's remark we point out that while integro-differential operators may look non-standard as infinitesimal generators of Markov processes, already in the 1960's, Courrège [30] and von Waldenfels [44] derived such operator representations for generators of Feller semigroups satisfying appropriate assumptions.

Since quadratic harnesses are non-homogeneous Markov processes, their infinitesimal generators are indexed by the time variable $t \geqslant 0$. To recall the general definition, denote by $\{\mathbb{P}_{s,t}(x, \mathrm{d}y) : x \in \mathbb{R},\ 0 \leqslant s < t\}$ the transition probabilities of a non-homogeneous Markov process. The *weak left infinitesimal generator* is defined by

$$\mathbf{A}_t^- f(x) := \lim_{h\to 0^+} \int_{\mathbb{R}} \frac{f(y) - f(x)}{h}\, \mathbb{P}_{t-h,t}(x, \mathrm{d}y)$$

for $t > 0$ and f such that this pointwise limit exists. Analogously, the *weak right infinitesimal generator* is given by

$$\mathbf{A}_t^+ f(x) := \lim_{h\to 0^+} \int_{\mathbb{R}} \frac{f(y) - f(x)}{h}\, \mathbb{P}_{t,t+h}(x, \mathrm{d}y). \tag{1.10}$$

Typically, for a quadratic harness $(X_t)_{t\geqslant 0}$ there exists a family of martingale orthogonal polynomials $(p_n(\cdot, t))_{n\geqslant 0}$, $t \geqslant 0$, i.e., for any $n \geqslant 0$,

$$\mathbb{E}(p_n(X_t, t)|\mathcal{F}_s) = p_n(X_s, s), \quad 0 \leqslant s < t,$$

and for any $t \geqslant 0$,

$$\mathbb{E}\, p_n(X_t, t) p_m(X_t, t) = \kappa_n \, \delta_{n=m}, \quad n, m \geqslant 0.$$

An explicit three-term recurrence for the polynomials $(p_n(\cdot, t))_{n \geqslant 0}$ is given in [17, Theorem 4.5]. Moreover, $\mathbf{A}_t^{\pm}(p_n(x,t)) = -\frac{\partial p_n(x,t)}{\partial t}$, which in particular means that $\mathbf{A}_t^+$ and $\mathbf{A}_t^-$ coincide on polynomials – for details see [24, Section 1.4]. Therefore we will use the same symbol $\mathbf{A}_t$ for both. Furthermore, considering the Banach space of polynomials up to degree $m \in \mathbb{N} \cup \{0\}$ with a proper norm, [1, Proposition 2.2.10] says that the pointwise convergence in (1.10) for polynomial f implies convergence in norm.

Agoitia-Hurtado [1, Lemma 2.2.8] proved that for polynomial processes the infinitesimal generator (1.10) has the form

$$\mathbf{A}_t^+ f(x) = \sum_{l=0}^{k} \frac{\partial}{\partial t} \alpha_l^f(s,t) x^l \Big|_{s=t},$$

where $x \in \operatorname{supp}(X_t)$, f is a polynomial of degree at most $k \geqslant 0$ and $\alpha_0^f, \ldots, \alpha_k^f$ are the coefficients occurring on the right-hand side of (1.1), which assumes the form

$$\mathbb{E}(f(X_t) \mid X_s = x) = \sum_{l=0}^{k} \alpha_l^f(s,t) x^l.$$

We seek for a more explicit formula for the infinitesimal generator of quadratic harnesses. Over the years, there have been several different approaches to deriving explicit formulas for infinitesimal generators of quadratic harnesses with different restrictions on the parameters $\eta, \theta, \sigma, \tau, q$; see [12, 4, 15], [23] (generalized later in [25]) and [24]. They all lead to the representation of the infinitesimal generator $\mathbf{A}_t$ as an integro-differential operator of the form

$$\mathbf{A}_t \, f\,(x) = \int_{\mathbb{R}} \frac{\partial}{\partial x} \left(\frac{f(y) - f(x)}{y - x} \right) \nu_{x,t}(\mathrm{d}y), \tag{1.11}$$

where $\nu_{x,t}$ is some measure. Consequently, to determine $\mathbf{A}_t$ one has to identify the measure $\nu_{x,t}$. In particular, it is easy to check that for the Wiener process the above representation holds with $\nu_{x,t} = \delta_x$.

The methodology we propose in this paper has been inspired by quite distinct approaches from [23, 24]. We now briefly discuss them.

In [23], the authors introduced a system of so-called associated (orthogonal) polynomials to the system of polynomials orthogonal with respect to a measure that is a limiting version of transition probabilities of the quadratic harness. Knowing how the infinitesimal generator acts on martingale polynomials allowed deducing a formula for the infinitesimal generator in terms of the orthogonality measure for the system of the associated polynomials. In this way the measure $\nu_{x,t}$

in (1.11) for q-Meixner–Lévy processes, $QH(0,\theta;0,\tau;q)$, was identified. In [25], a similar approach was used to derive $\nu_{x,t}$ in (1.11) for the bi-Poisson process $QH(\eta,\theta;0,0;q)$. In both cases, $\nu_{x,t}$ was expressed in terms of special transition probabilities of the process.

The approach of [24] refers directly to the fact that quadratic harnesses are polynomial processes. It turns out that if $QH(\eta,\theta;\sigma,\tau;q)$ has all moments, then the infinitesimal generator $\mathbf{A}_t$ can be expressed in terms of a special element of a certain non-commutative algebra $\mathcal{Q}$ of infinite polynomial sequences. To identify this element, i.e., to identify the generator, one needs to solve a q-commutation equation in $\mathcal{Q}$. In [24] this equation was solved when $q=-\sqrt{\sigma\tau}$. Consequently, the infinitesimal generator for the free quadratic harness $X\sim QH(\eta,\theta;\sigma,\tau;-\sqrt{\sigma\tau})$ was identified in the form (1.11) with the explicit measure $\nu_{x,t}$, related to the transition probabilities of the process X. In particular, in this case the authors were able to postulate a special parametric form of the generator.

We expect that such an approach is not possible in general. Instead, we develop a more universal algebraic approach incorporating associated polynomials which were used more systematically in [23]. As a consequence, the approach we propose not only covers all already known cases when $\sigma=0$, but also allows us to derive the infinitesimal generators in new cases of $\tau>0$ and $\eta\neq 0$. We expect that this method extends to the much harder case of $\sigma>0$, but it is beyond the scope of the present investigations.

Here is our main result.

THEOREM 1.1. *Let* $(\mathbf{A}_t)_{t\geqslant 0}$ *be a family of infinitesimal generators of the process* $(X_t)_{t\geqslant 0}\sim QH(\eta,\theta;0,\tau;q)$. *Then for every polynomial* f, $t\geqslant 0$ *and* $x\in\operatorname{supp}(X_t)$,

$$\mathbf{A}_t f(x)=(1+\eta x)\int_{\mathbb{R}}\frac{\partial}{\partial x}\left(\frac{f(y)-f(x)}{y-x}\right)\nu_{x,t,\eta,\theta,\tau,q}(\mathrm{d}y), \tag{1.12}$$

where $\nu_{x,t,\eta,\theta,\tau,q}$ *is a probabilistic orthogonality measure of the polynomials* $(B_n(\cdot;x,t))_{n\geqslant 0}$ *defined through a three-step recurrence:*

(1.13)
$$\begin{aligned}
&B_{-1}(y;x,t)=0,\quad B_0(y;x,t)=1,\\
&yB_n(y;x,t)=B_{n+1}(y;x,t)\\
&\qquad+[(\gamma_t+\beta_t([n+1]_q+[n]_q))[n+1]_q+xq^{n+1}]B_n(y;x,t)\\
&\qquad+\alpha_t(1+\eta\gamma_t[n]_q+\eta\beta_t[n]_q^2+x\eta q^n)[n+1]_q[n]_qB_{n-1}(y;x,t),\quad n\geqslant 0,
\end{aligned}$$

with

$$\alpha_t:=\tau+(1-q)t,\quad \beta_t:=\eta\alpha_t,\quad \gamma_t:=\theta-\eta t. \tag{1.14}$$

Here $[n]_q$ denotes the q-natural number defined by

$$[n]_q=1+q+\dots+q^{n-1}\quad\text{for } n\geqslant 1\quad\text{and}\quad [0]_q=0. \tag{1.15}$$

The paper is organized as follows. In the next section we give an overview of an algebra $\mathcal{Q}$ of polynomial sequences, introduce some special elements of $\mathcal{Q}$ and analyze their properties. In Section 3 we attempt to solve a q-commutation equation in $\mathcal{Q}$, which is crucial for the identification of the measure $\nu_{x,t}$ in (1.11). To do this, we introduce and carefully examine a $\mathcal{Q}$-valued function $\mathbb{B}$ of real arguments. In Section 4 we give a proof of Theorem 1.1 and discuss its conclusions. In Section 5, we analyze the properties of $\nu_{x,t}$ by establishing its connection to an orthogonality measure for certain Askey–Wilson polynomials. Section 6 is an appendix which gives more details on the Askey–Wilson and related families of orthogonal polynomials.

2. ALGEBRA $\mathcal{Q}$ OF POLYNOMIAL SEQUENCES

In this section we derive several properties of an algebra of polynomial sequences which will be essential for the construction of infinitesimal generators of quadratic harnesses.

The algebra $\mathcal{Q}$ of polynomial sequences was introduced in [24] in order to study the properties of polynomial processes. It is defined as the linear space over $\mathbb{R}$ of all infinite sequences of polynomials in a real variable x with a non-commutative multiplication $\mathbb{R} = \mathbb{P}\mathbb{Q}$ for $\mathbb{P} = (P_0, P_1, \ldots)$, $\mathbb{Q} = (Q_0, Q_1, \ldots)$, with $\mathbb{R} = (R_0, R_1, \ldots) \in \mathcal{Q}$ given by

$$R_k(x) = \sum_{j=0}^{\deg(Q_k)} [Q_k]_j P_j(x), \quad k \geqslant 0, \tag{2.1}$$

where $[Q_k]_j$ is the coefficient of x^j in the polynomial Q_k. Note that the multiplication is associative and $\mathcal{Q}$ is an algebra with identity

$$\mathbb{E} = (1, x, x^2, x^3, \ldots).$$

For $a \in \mathbb{R}$ we denote $\mathbb{E}_a = (1, a, a^2, \ldots)$, i.e. all coordinate polynomials (as functions of the generic variable x) are of degree zero. Note that for $\mathbb{P} = (P_0, P_1, \ldots) \in \mathcal{Q}$ we have

$$\mathbb{E}_a \mathbb{P} = \mathbb{P}|_{x:=a} = (P_0(a), P_1(a), \ldots). \tag{2.2}$$

If $\deg P_n = n$ for all $n \geqslant 0$, then $\mathbb{P}$ is invertible in $\mathcal{Q}$ (see [24, Proposition 1.2]). In a remark below, we give an explicit formula for the inverse of $\mathbb{E} + \mathbb{X}$, where $\mathbb{X} \in \mathcal{Q}$ satisfies some special conditions.

REMARK 2.1. Let $\mathbb{X} \in \mathcal{Q}$ be such that its nth coordinate polynomial is of degree at most $n-1$ for $n \geqslant 0$ (where 0 is a polynomial of degree -1). Then $\mathbb{E} + \mathbb{X} \in \mathcal{Q}$ is invertible and

$$(\mathbb{E} + \mathbb{X})^{-1} = \sum_{k=0}^{\infty} (-\mathbb{X})^k. \tag{2.3}$$

Proof. Note that the infinite sum on the right-hand side of (2.3) is a well-defined element of $\mathcal{Q}$ due to the assumption about the degrees of the coordinate polynomials of $\mathbb{X}$. Indeed, (2.1) implies that the nth coordinate polynomial of $\mathbb{X}^k$ has a degree at most $n-k$ if $0 \leqslant k < n$ or -1 otherwise. Hence the sum is finite coordinatewise. Furthermore,

$$(\mathbb{E}+\mathbb{X})\sum_{k=0}^{\infty}(-\mathbb{X})^k = \sum_{k=0}^{\infty}(-\mathbb{X})^k - \sum_{k=0}^{\infty}(-\mathbb{X})^{k+1} = \mathbb{E},$$
$$\sum_{k=0}^{\infty}(-\mathbb{X})^k(\mathbb{E}+\mathbb{X}) = \sum_{k=0}^{\infty}(-\mathbb{X})^k - \sum_{k=0}^{\infty}(-\mathbb{X})^{k+1} = \mathbb{E}. \quad \blacksquare$$

We single out two elements of $\mathcal{Q}$:

$$\mathbb{D} := (0,1,x,x^2,\ldots) \quad \text{and} \quad \mathbb{F} := (x,x^2,x^3,\ldots),$$

which will play a basic role in what follows. It is easy to verify that

$$\mathbb{D}\mathbb{F} = \mathbb{E}, \tag{2.4}$$

but $\mathbb{E} - \mathbb{F}\mathbb{D} = (1,0,0,\ldots)$, so $\mathbb{D}$ and $\mathbb{F}$ do not commute.

Furthermore, for $q \in \mathbb{R}$ denote

$$\mathbb{D}_q := \sum_{k=0}^{\infty} q^k \mathbb{F}^k \mathbb{D}^{k+1}, \tag{2.5}$$

an element of $\mathcal{Q}$, which is important in the analysis below. Clearly, $\mathbb{D}_0 = \mathbb{D}$. Note that $\mathbb{D}_q$ can be written in terms of q-notation as

$$\mathbb{D}_q = ([0]_q, [1]_q, [2]_q x, [3]_q x^2, \ldots)$$

(recall (1.15)). When applied from the left to $\mathbb{P} \in \mathcal{Q}$ it acts coordinatewise as the q-derivative. In particular, for $q = 1$ we have

$$\mathbb{D}_1 := \sum_{k=0}^{\infty} \mathbb{F}^k \mathbb{D}^{k+1}, \tag{2.6}$$

and $\mathbb{D}_1$ represents the classical derivative. Moreover, $\mathbb{D}_q$ satisfies the following identities:

$$\mathbb{D}_q(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{0}, \tag{2.7}$$
$$\mathbb{D}_q\mathbb{F} = q\mathbb{F}\mathbb{D}_q + \mathbb{E}. \tag{2.8}$$

Here (2.7) is due to (2.5) and (2.4), while (2.8) follows from

$$\mathbb{D}_q\mathbb{F} - q\mathbb{F}\mathbb{D}_q = \sum_{k=0}^{\infty} q^k \mathbb{F}^k \mathbb{D}^k - \sum_{k=0}^{\infty} q^{k+1} \mathbb{F}^{k+1} \mathbb{D}^{k+1} = \mathbb{E}.$$

For future use, it will be convenient to introduce some identities involving additional special elements of $\mathcal{Q}$ that will be important in the derivation of the Jacobi matrix of the orthogonal polynomials related to the measure building up the infinitesimal generators we search for.

LEMMA 2.1. *For $\beta \in \mathbb{R}$ and*

$$\mathbb{W}_1 := \mathbb{E} + \beta\mathbb{D}_q\mathbb{F}\mathbb{D}_q, \quad \mathbb{W}_2 := \mathbb{E} + \beta\mathbb{F}\mathbb{D}_q^2,$$
$$\mathbb{W}_3 := \mathbb{E} + q\beta\mathbb{F}\mathbb{D}_q^2, \quad \mathbb{W}_4 := \mathbb{E} + \beta\mathbb{D}_q^2\mathbb{F},$$

the following identities hold:

$$\text{(i)}\ \mathbb{W}_1\mathbb{D}_q = \mathbb{D}_q\mathbb{W}_2, \quad \text{(iv)}\ \mathbb{W}_1 = \mathbb{W}_3 + \beta\mathbb{D}_q,$$
$$\text{(ii)}\ \mathbb{D}_q\mathbb{W}_1 = \mathbb{W}_4\mathbb{D}_q, \quad \text{(v)}\ \mathbb{D}_q\mathbb{F}\mathbb{W}_3 = \mathbb{E} + q\mathbb{W}_2\mathbb{F}\mathbb{D}_q,$$
$$\text{(iii)}\ \mathbb{D}_q^2\mathbb{W}_2 = \mathbb{W}_4\mathbb{D}_q^2, \quad \text{(vi)}\ \mathbb{W}_2\mathbb{W}_3 = \mathbb{W}_1(\mathbb{W}_2 - \beta\mathbb{D}_q).$$

Proof. Identities (i) and (ii) follow directly from the definitions of $\mathbb{W}_1$, $\mathbb{W}_2$ and $\mathbb{W}_4$, while (iii) is a combination of (i) and (ii). Identity (iv) is an immediate consequence of (2.8). To see (v), note that from (2.8) we have $\mathbb{D}_q\mathbb{F}^2\mathbb{D}_q = \mathbb{F}\mathbb{D}_q(q\mathbb{F}\mathbb{D}_q) + \mathbb{F}\mathbb{D}_q = \mathbb{F}\mathbb{D}_q^2\mathbb{F}$ and

$$\mathbb{D}_q\mathbb{F}\mathbb{W}_3 = \mathbb{D}_q\mathbb{F} + q\beta\mathbb{D}_q\mathbb{F}^2\mathbb{D}_q^2 = \mathbb{E} + q\mathbb{F}\mathbb{D}_q + q\beta\mathbb{F}\mathbb{D}_q^2\mathbb{F}\mathbb{D}_q = \mathbb{E} + q\mathbb{W}_2\mathbb{F}\mathbb{D}_q.$$

Finally, we use (iv) and (i) to show (vi) as follows:

$$\mathbb{W}_2\mathbb{W}_3 = \mathbb{W}_3\mathbb{W}_2 = (\mathbb{W}_1 - \beta\mathbb{D}_q)\mathbb{W}_2 = \mathbb{W}_1\mathbb{W}_2 - \beta\mathbb{D}_q\mathbb{W}_2 = \mathbb{W}_1(\mathbb{W}_2 - \beta\mathbb{D}_q). \quad \blacksquare$$

REMARK 2.2. The elements $\mathbb{W}_i$, $i = 1, 2, 3, 4$, are invertible and their inverses can be expressed by (2.3), where $\mathbb{X}$ is equal $\beta\mathbb{D}_q\mathbb{F}\mathbb{D}_q$, $\beta\mathbb{F}\mathbb{D}_q^2$, $q\beta\mathbb{F}\mathbb{D}_q^2$, $\beta\mathbb{D}_q^2\mathbb{F}$, respectively. This follows from Remark 2.1, since the nth coordinate polynomials of $\beta\mathbb{D}_q\mathbb{F}\mathbb{D}_q$, $\beta\mathbb{F}\mathbb{D}_q^2$, $q\beta\mathbb{F}\mathbb{D}_q^2$, $\beta\mathbb{D}_q^2\mathbb{F}$ is equal to $\beta[n]_q^2x^{n-1}$, $\beta[n]_q[n-1]_qx^{n-1}$, $q\beta[n]_q[n-1]_qx^{n-1}$, $\beta[n+1]_q[n]_qx^{n-1}$, respectively, for $n \geqslant 0$ (recall that $[0]_q = 0$).

The elements $\mathbb{W}_1$, $\mathbb{W}_2$, and $\mathbb{D}_q$ are the basic building blocks of more complicated elements of $\mathcal{Q}$, which are important for our considerations. For real coefficients α, β, γ let us define

$$\mathbb{S}(z) := \mathbb{R} + z(\mathbb{D} - \mathbb{Q}), \quad z \in \mathbb{R}, \tag{2.9}$$

where

$$\mathbb{R} := (\mathbb{W}_1 + \gamma\mathbb{D}_q)\mathbb{W}_2 + \alpha\mathbb{D}_q^2, \tag{2.10}$$
$$\mathbb{Q} := (1-q)\mathbb{D}_q\mathbb{W}_2 - \beta\mathbb{D}_q^2. \tag{2.11}$$

It is easy to see that (2.7) and the definition of $\mathbb{S}(z)$ imply that

$$\mathbb{S}(z)(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{E} - \mathbb{F}\mathbb{D}. \tag{2.12}$$

Moreover, as we will see later, $\mathbb{S}(z)$ plays the role of the Jacobi matrix of a system of orthogonal polynomials that will allow us to identify the orthogonality measure that is the basic ingredient of the infinitesimal generator of the process. In the lemma and its corollary we present two identities satisfied by $\mathbb{S}(z)$, $\mathbb{R}$, and $\mathbb{Q}$.

LEMMA 2.2. *For $\mathbb{X} \in \{\mathbb{R}, \mathbb{Q}\}$ we have*

$$\mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{X}\mathbb{F}\mathbb{W}_2\mathbb{W}_3 = \mathbb{X}(\mathbb{E} + q\mathbb{F}\mathbb{D}_q\mathbb{W}_2). \tag{2.13}$$

Proof. We first note that the identity

$$\mathbb{W}_4\mathbb{Y}\mathbb{W}_2^{-1}\mathbb{D}_q = \mathbb{D}_q\mathbb{Y} \tag{2.14}$$

holds for (1) $\mathbb{Y} := \mathbb{W}_1\mathbb{W}_2$, (2) $\mathbb{Y} := \mathbb{D}_q\mathbb{W}_2$ and (3) $\mathbb{Y} := \mathbb{D}_q^2$. Case (1) holds due to (i) and (ii), and cases (2) and (3) due to (iii) (see Lemma 2.1).

Since $\mathbb{R}$ and $\mathbb{Q}$ are linear in $(\mathbb{W}_1\mathbb{W}_2, \mathbb{D}_q\mathbb{W}_2, \mathbb{D}_q^2)$, we conclude that (2.14) also holds for $\mathbb{Y} := \mathbb{X}$. Consequently,

$$\mathbb{D}_q\mathbb{X}\mathbb{F}\mathbb{W}_2\mathbb{W}_3 = \mathbb{W}_4\mathbb{X}\mathbb{W}_2^{-1}\mathbb{D}_q\mathbb{F}\mathbb{W}_2\mathbb{W}_3,$$

but Lemma 2.1(v) implies

$$\mathbb{W}_2^{-1}\mathbb{D}_q\mathbb{F}\mathbb{W}_2\mathbb{W}_3 = \mathbb{E} + q\mathbb{F}\mathbb{D}_q\mathbb{W}_2.$$

Thus we obtain

$$\mathbb{D}_q\mathbb{X}\mathbb{F}\mathbb{W}_2\mathbb{W}_3 = \mathbb{W}_4\mathbb{X}(\mathbb{E} + q\mathbb{F}\mathbb{D}_q\mathbb{W}_2),$$

which, due to Lemma 2.1(ii) and Remark 2.2, is equivalent to (2.13). ■

COROLLARY 2.1. *For $z \in \mathbb{R}$ we have*

$$\mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{S}(z)\mathbb{F}\mathbb{W}_2\mathbb{W}_3 = \mathbb{R} + q\mathbb{S}(z)\mathbb{F}\mathbb{D}_q\mathbb{W}_2.$$

Proof. Writing $\mathbb{S}(z) = (\mathbb{R} - z\mathbb{Q}) + z\mathbb{D}$ and using Lemma 2.2 we see that the left-hand side above is

$$\mathbb{R} + q\mathbb{S}(z)\mathbb{F}\mathbb{D}_q\mathbb{W}_2 - z(\mathbb{Q} + q\mathbb{D}_q\mathbb{W}_2 - \mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{W}_2\mathbb{W}_3).$$

Due to Lemma 2.1(vi), the last term in the bracket is $\mathbb{D}_q\mathbb{W}_2 - \beta\mathbb{D}_q^2$. Thus, recalling the definition of $\mathbb{Q}$, we see that the whole bracket vanishes. ■

Recall that X_n denotes the nth coordinate polynomial of $\mathbb{X} \in \mathcal{Q}$, $n \geqslant 0$. If $\mathbb{X} \in \mathcal{Q}$ additionally depends on a parameter $z \in \mathbb{R}$, we write $\mathbb{X}(z)$, like for $\mathbb{S}(z)$ above. Then its nth coordinate, denoted by $X_n(z)$, is a polynomial in the generic variable x with coefficients depending on z, $n \geqslant 0$. In what follows, we need to evaluate the product $\mathbb{X}(z)\mathbb{Y}(z)$ at $z := x$. It is easy to see that even when $\mathbb{X}(z)|_{z:=x}$, $\mathbb{Y}(z)|_{z:=x} \in \mathcal{Q}$, the identity $(\mathbb{X}(z)\mathbb{Y}(z))|_{z:=x} = \mathbb{X}(z)|_{z:=x}\,\mathbb{Y}(z)|_{z:=x}$ may not hold.

LEMMA 2.3. *Let* $\mathbb{X}(z), \mathbb{Y}(z) \in \mathcal{Q}$ *for all* $z \in \mathbb{R}$. *Assume that for each* $n \geqslant 0$ *all coefficients of the coordinate polynomials* $X_n(z)$ *(in the generic variable* x*) are polynomials in* z. *Then* $\widetilde{\mathbb{X}} := \mathbb{X}(z)|_{z:=x} \in \mathcal{Q}$ *and*

$$(\mathbb{X}(z)\mathbb{Y}(z))|_{z:=x} = (\widetilde{\mathbb{X}}\mathbb{Y}(z))|_{z:=x}.$$

Proof. By the assumption on the coefficients of the coordinate polynomials of $\mathbb{X}(z)$, we conclude that $X_n(z)|_{z:=x}$, which is the nth coordinate of $\mathbb{X}(z)$ evaluated at $z := x$, is a polynomial in x, i.e., $\mathbb{X}(z)|_{z:=x} \in \mathcal{Q}$.

Note that $Y_n(z)$, the nth coordinate polynomial of $\mathbb{Y}(z)$, can be written as $Y_n(z) = \sum_{k=0}^{M_n} y_{k,n}(z)x^k$ for some $M_n \in \mathbb{N} \cup \{0\}$ and $(y_{k,n}(z))_{k=0,\dots,M_n} \in \mathbb{R}^{M_n+1}$, $n \geqslant 0$. Then, by (2.1), the nth coordinate of $\mathbb{X}(z)\mathbb{Y}(z)$ is $\sum_{k=0}^{M_n} y_{k,n}(z)X_k(z)$, whereas the nth coordinate of $\mathbb{X}(x)\mathbb{Y}(z)$ equals $\sum_{k=0}^{M_n} y_{k,n}(z)X_k(x)$. Therefore, considering these two objects as functions of z and inserting $z := x$ yields the desired equality coordinatewise. ■

3. INFINITESIMAL GENERATOR AS AN ELEMENT OF THE ALGEBRA

As already mentioned, $\mathbf{A}_t$ evaluated on polynomials also gives polynomials. Thus, $\mathbf{A}_t$ has a unique representation $\mathbb{A}_t \in \mathcal{Q}$ with the nth coordinate equal to $\mathbf{A}_t(x^n)$ for $n \geqslant 0$ (see [24, Section 1.4]). It was also explained in [24, Theorem 2.1] that the infinitesimal generator $\mathbb{A}_t$ of the quadratic harness $X \sim QH(\eta, \theta; \sigma, \tau; q)$ can be identified through an auxiliary element

$$\mathbb{H}_t := \mathbb{A}_t\mathbb{F} - \mathbb{F}\mathbb{A}_t \in \mathcal{Q}, \tag{3.1}$$

which is a solution of the following q-commutation equation for $t \geqslant 0$:

$$\mathbb{H}_t\mathbb{T}_t - q\mathbb{T}_t\mathbb{H}_t = \mathbb{E} + \theta\mathbb{H}_t + \eta\mathbb{T}_t + \tau\mathbb{H}_t^2 + \sigma\mathbb{T}_t^2, \tag{3.2}$$

where $\mathbb{T}_t := \mathbb{F} - t\mathbb{H}_t$, with the initial condition $\mathbb{H}_t(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{0}$. It has been proved in [24, Proposition 2.4] that (3.2) has the unique solution $\mathbb{H}_t$ when $\sigma\tau \neq 1$. Having identified $\mathbb{H}_t$, the generator $\mathbb{A}_t$ is recovered from (3.1). This plan was successfully realized in [24, Theorem 2.5] for the free quadratic harness, i.e., for $X \sim QH(\eta, \theta; \sigma, \tau; -\tau\sigma)$ and in [24, Section 5.2] for the classical version of the quantum Bessel process (see [9]), i.e. for $X \sim QH(\eta, \theta; 0, 0; 1)$. In the former case, the solution of (3.2) is given as

$$\mathbb{H}_t = \frac{1}{1 + \sigma t}(\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\,\phi_t(\mathbb{D})\mathbb{D},$$

where ϕ_t is a function satisfying certain quadratic equation (see [24, Lemma 3.1]). In the latter case, the solution of (3.2) was shown to be

$$\mathbb{H}_t = \frac{1}{\theta - t\eta}\,(\mathbb{E} + \eta\mathbb{F})(e^{(\theta - t\eta)\mathbb{D}_1} - \mathbb{E}),$$

where $\mathbb{D}_1$, defined by (2.5) with $q = 1$, acts as the classical derivative.

We will extend this algebraic method to all quadratic harnesses with $\sigma = 0$. The case of $\sigma \neq 0$ is more delicate due to the term $\sigma\mathbb{T}_t^2$ in the q-commutation equation (3.2). Indeed, in the argument presented below, one can see that $\sigma \neq 0$ adds the term $\sigma\mathbb{F}^2$ to the equations we are dealing with, and it triggers additional difficulties that we do not know how to overcome. Therefore we focus on $\sigma = 0$, which is already quite complicated.

Recall that our primary goal is to solve equation (3.2). Consider $\widetilde{\mathbb{H}}_t$ satisfying a similar equation,

$$\widetilde{\mathbb{H}}_t\mathbb{F} - q\mathbb{F}\widetilde{\mathbb{H}}_t = \mathbb{E} + \gamma_t\widetilde{\mathbb{H}}_t + \alpha_t\widetilde{\mathbb{H}}_t(\mathbb{E} + \eta\mathbb{F})\widetilde{\mathbb{H}}_t \tag{3.3}$$

with the initial condition

$$\widetilde{\mathbb{H}}_t(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{0}, \tag{3.4}$$

where α_t and γ_t are given in (1.14). The following observation justifies that it suffices to find a solution of (3.3).

LEMMA 3.1. *If* $\widetilde{\mathbb{H}}_t$ *satisfies* (3.3) *and* (3.4)*, then*

$$\mathbb{H}_t = (\mathbb{E} + \eta\mathbb{F})\widetilde{\mathbb{H}}_t$$

is a solution of (3.2) *with the initial condition* $\mathbb{H}_t(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{0}$.

Proof. Obviously, $\mathbb{H}_t(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{0}$. Upon multiplication (3.3) from the left by $\mathbb{E} + \eta\mathbb{F}$, we conclude that $\mathbb{H}_t$ satisfies (3.2). ∎

Fix $t > 0$ and let $R(\mathbb{X}) := \mathbb{E} + \gamma_t\mathbb{X} + \alpha_t\mathbb{X}^2 + \beta_t\mathbb{X}\mathbb{F}\mathbb{X}$. We rewrite (3.3) in the equivalent form

$$\widetilde{\mathbb{H}}_t\mathbb{F} = q\mathbb{F}\widetilde{\mathbb{H}}_t + R(\widetilde{\mathbb{H}}_t). \tag{3.5}$$

Let $\mathbb{S}(z)$ be as defined in (2.9)–(2.11) with $(\alpha, \beta, \gamma) = (\alpha_t, \beta_t, \gamma_t)$ (see (1.14)). From now on, since t is fixed, we suppress it in subscripts. Consider a function $\mathbb{B} : \mathbb{R} \to \mathcal{Q}$ satisfying

$$\mathbb{F}\mathbb{B}(z) = \mathbb{B}(z)\mathbb{S}(z)\mathbb{F} \tag{3.6}$$

with the initial condition

$$\mathbb{B}(z)(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{E} - \mathbb{F}\mathbb{D}. \tag{3.7}$$

The above identities for $\mathbb{B}(z)$ yield an explicit connection between coordinates $B_n(z)$, $n \geqslant 0$, of $\mathbb{B}(z)$ and orthogonal polynomials defined in (1.13). This connection is detailed in the proof of the next lemma.

LEMMA 3.2. *Fix* $z \in \mathbb{R}$. *Assume that* $\mathbb{B}(z) \in \mathcal{Q}$ *satisfies* (3.6) *and* (3.7). *Then* $\mathbb{B}(z)$ *is unique and invertible.*

Proof. Clearly, (3.6) and (3.7) uniquely determine $\mathbb{B}(z)$ in terms of α, β, γ. Indeed, coordinatewise (after simplification) (3.6) is equivalent to the three-term recurrence (1.13) with (y, x, t) replaced by (x, z, t), i.e. for polynomials $B_n(x; z, t)$ instead of $B_n(y; x, t)$ (alternatively, the nth coordinate of $(\mathbb{E}_y\mathbb{B}(z))|_{z=x}$ is $B_n(y; x, t)$) for $n \geqslant 0$. Consequently, $B_n(z)$, the nth entry of $\mathbb{B}(z)$, evaluated on the generic variable x, coincides with $B_n(x; z, t)$. Therefore, $B_n(z)$ is a monic polynomial of degree $n \geqslant 0$, whence $\mathbb{B}(z)$ is invertible. ∎

Another immediate consequence of the representation of $\mathbb{B}(z)$ through polynomials of the three-term recurrence (1.13) mentioned in the proof above, is the following useful observation.

REMARK 3.1. The coordinates $B_n(z)$, $n \geqslant 0$, of $\mathbb{B}(z)$ (which are polynomials in the generic variable $x \in \mathbb{R}$) are also polynomials in $z \in \mathbb{R}$. Alternatively, $\mathbb{E}_y\mathbb{B}(z)|_{z=x} \in \mathcal{Q}$.

Next, we give a useful representation of $\mathbb{B}(z)$.

LEMMA 3.3. *The identities* (3.6) *and* (3.7) *are equivalent to*

$$\mathbb{F}\mathbb{B}(z)\mathbb{D} + \mathbb{E} - \mathbb{F}\mathbb{D} = \mathbb{B}(z)\mathbb{S}(z). \tag{3.8}$$

Proof. If we multiply (3.6) on the right by $\mathbb{D}$ and use (2.12), we obtain (3.8) as follows:

$$\begin{aligned}\mathbb{F}\mathbb{B}(z)\mathbb{D} &= \mathbb{B}(z)\mathbb{S}(z)\mathbb{F}\mathbb{D} = \mathbb{B}(z)\mathbb{S}(z) - \mathbb{B}(z)\mathbb{S}(z)(\mathbb{E} - \mathbb{F}\mathbb{D})\\ &= \mathbb{B}(z)\mathbb{S}(z) - \mathbb{B}(z)(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{B}(z)\mathbb{S}(z) - (\mathbb{E} - \mathbb{F}\mathbb{D}).\end{aligned}$$

Conversely, in view of (2.4) the identity (3.8) multiplied on the right by $\mathbb{F}$ gives (3.6) directly. Similarly, in view of (2.12), multiplication of (3.8) on the right by $\mathbb{E} - \mathbb{F}\mathbb{D}$ gives (3.7). ∎

The next result brings important relations between $\mathbb{B}(z)$ and $\widetilde{\mathbb{H}}$. Recall that $\mathbb{W}_i$, $i = 1, 2, 3$, are defined in Lemma 2.1 with $\beta = \beta_t$ (see (1.14)).

THEOREM 3.1. *Assume that* $\widetilde{\mathbb{H}}$ *satisfies* (3.5) *and* $\widetilde{\mathbb{H}}(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{0}$. *Then for all* $z \in \mathbb{R}$ *we have*

$$\widetilde{\mathbb{H}} = \mathbb{B}(z)\mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{B}(z)^{-1}. \tag{3.9}$$

Moreover, for $\widetilde{\mathbb{M}} := \widetilde{\mathbb{H}}\mathbb{F} - \mathbb{F}\widetilde{\mathbb{H}}$ *we get*

$$\widetilde{\mathbb{M}}\mathbb{B}(z)\mathbb{W}_2\mathbb{W}_3 = \mathbb{E} - \mathbb{F}\mathbb{D} + (\mathbb{F} - z\mathbb{E})\mathbb{B}(z)(\mathbb{D} - \mathbb{Q}). \tag{3.10}$$

Proof. In the proof we fix $z \in \mathbb{R}$ and we suppress (z) in $\mathbb{B}(z)$, $\mathbb{X}(z)$, and $\mathbb{S}(z)$ for simplicity.

Proof of (3.9). Since $\mathbb{W}_1$ (see Remark 2.2) and $\mathbb{B}$ (see Lemma 3.2) are invertible,

$$\mathbb{X} := \mathbb{B}\mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{B}^{-1} \tag{3.11}$$

is a well-defined element of $\mathcal{Q}$. Due to (3.7) we have $\mathbb{B}^{-1}(\mathbb{E}-\mathbb{F}\mathbb{D}) = \mathbb{E}-\mathbb{F}\mathbb{D}$ and according to (2.7) we get $\mathbb{W}_1(\mathbb{E}-\mathbb{F}\mathbb{D}) = \mathbb{E}-\mathbb{F}\mathbb{D}$, thus we finally obtain

$$\begin{aligned}\mathbb{X}(\mathbb{E}-\mathbb{F}\mathbb{D}) &= \mathbb{B}\mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{B}^{-1}(\mathbb{E}-\mathbb{F}\mathbb{D}) = \mathbb{B}\mathbb{D}_q\mathbb{W}_1^{-1}(\mathbb{E}-\mathbb{F}\mathbb{D})\\ &= \mathbb{B}\mathbb{D}_q(\mathbb{E}-\mathbb{F}\mathbb{D}) = \mathbb{0}.\end{aligned}$$

Since equation (3.5) with the initial condition determines $\widetilde{\mathbb{H}}$ uniquely, it suffices to show that $\mathbb{X}$ defined by (3.11) satisfies

$$\text{LHS} := \mathbb{X}\mathbb{F} - q\mathbb{F}\mathbb{X} - \mathbb{E} - \gamma\mathbb{X} - \alpha\mathbb{X}^2 - \beta\mathbb{X}\mathbb{F}\mathbb{X} = \mathbb{0}, \tag{3.12}$$

because we have already checked that $\mathbb{X}(\mathbb{E}-\mathbb{F}\mathbb{D}) = \mathbb{0}$. Since (3.6) implies $\mathbb{B}^{-1}\mathbb{F} = \mathbb{S}\mathbb{F}\mathbb{B}^{-1}$, after simplification LHS above takes the form

$$\begin{aligned}\text{LHS} &= \mathbb{B}\mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{S}\mathbb{F}\mathbb{B}^{-1} - q\mathbb{B}\mathbb{S}\mathbb{F}\mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{B}^{-1} - \mathbb{E} - \gamma\mathbb{B}\mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{B}^{-1}\\ &\quad - \alpha\mathbb{B}(\mathbb{D}_q\mathbb{W}_1^{-1})^2\mathbb{B}^{-1} - \beta\mathbb{B}\mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{S}\mathbb{F}\mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{B}^{-1}\\ &= \mathbb{B}\mathbb{W}\mathbb{B}^{-1},\end{aligned}$$

where

$$\begin{aligned}\mathbb{W} &:= \mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{S}\mathbb{F} - q\mathbb{S}\mathbb{F}\mathbb{D}_q\mathbb{W}_1^{-1} - \mathbb{E} - \gamma\mathbb{D}_q\mathbb{W}_1^{-1}\\ &\quad - \alpha(\mathbb{D}_q\mathbb{W}_1^{-1})^2 - \beta\mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{S}\mathbb{F}\mathbb{D}_q\mathbb{W}_1^{-1}.\end{aligned}$$

Consequently, applying Lemma 2.1(i), we obtain

$$\begin{aligned}\mathbb{W}\mathbb{W}_1\mathbb{W}_2 &= \mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{S}\mathbb{F}\mathbb{W}_1\mathbb{W}_2 - q\mathbb{S}\mathbb{F}\mathbb{D}_q\mathbb{W}_2 - \mathbb{W}_1\mathbb{W}_2 - \gamma\mathbb{D}_q\mathbb{W}_2\\ &\quad - \alpha\mathbb{D}_q^2 - \beta\mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{S}\mathbb{F}\mathbb{D}_q\mathbb{W}_2\\ &= \mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{S}\mathbb{F}(\mathbb{W}_1 - \beta\mathbb{D}_q)\mathbb{W}_2 - q\mathbb{S}\mathbb{F}\mathbb{D}_q\mathbb{W}_2 - \mathbb{R},\end{aligned}$$

where the last equality holds due to (2.10). From Lemma 2.1(iv) and Corollary 2.1 we have

$$\mathbb{W}\mathbb{W}_1\mathbb{W}_2 = \mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{S}\mathbb{F}\mathbb{W}_3\mathbb{W}_2 - q\mathbb{S}\mathbb{F}\mathbb{D}_q\mathbb{W}_2 - \mathbb{R} = \mathbb{0}.$$

Because $\mathbb{W}_1$ and $\mathbb{W}_2$ are invertible, we have $\mathbb{W} = \mathbb{0}$ and consequently (3.12) holds true.

Proof of (3.10). Now we consider $\widetilde{\mathbb{M}}$. Using (3.6) and (3.9) we get

$$\begin{aligned}\widetilde{\mathbb{M}} = \widetilde{\mathbb{H}}\mathbb{F} - \mathbb{F}\widetilde{\mathbb{H}} &= \mathbb{B}\mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{B}^{-1}\mathbb{F} - \mathbb{F}\mathbb{B}\mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{B}^{-1}\\ &= \mathbb{B}(\mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{S}\mathbb{F} - \mathbb{S}\mathbb{F}\mathbb{D}_q\mathbb{E}_1^{-1})\mathbb{B}^{-1}.\end{aligned}$$

Therefore by Lemma 2.1(vi) and Corollary 2.1 we get

$$\begin{aligned}\widetilde{\mathbb{M}}\mathbb{B}\mathbb{W}_2\mathbb{W}_3 &= \mathbb{B}(\mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{S}\mathbb{F}\mathbb{W}_2\mathbb{W}_3 - \mathbb{S}\mathbb{F}\mathbb{D}_q\mathbb{W}_1^{-1}\mathbb{W}_2\mathbb{W}_3)\\ &= \mathbb{B}(\mathbb{R} + q\mathbb{S}\mathbb{F}\mathbb{D}_q\mathbb{W}_2 - \mathbb{S}\mathbb{F}\mathbb{D}_q(\mathbb{W}_2 - \beta\mathbb{D}_q)).\end{aligned}$$

Using the fact that $\mathbb{R} = \mathbb{S} - z(\mathbb{D} - \mathbb{Q})$ and recalling (2.11), we can rewrite the above as

$$\widetilde{\mathbb{M}}\mathbb{B}\mathbb{W}_2\mathbb{W}_3 = \mathbb{B}(\mathbb{R} - \mathbb{S}\mathbb{F}((1-q)\mathbb{D}_q\mathbb{W}_2 - \beta\mathbb{D}_q^2)) = \mathbb{B}\mathbb{S} - z\mathbb{B}(\mathbb{D} - \mathbb{Q}) - \mathbb{B}\mathbb{S}\mathbb{F}\mathbb{Q}.$$

According to (3.8) and (3.6) we finally obtain

$$\begin{aligned}\widetilde{\mathbb{M}}\mathbb{B}\mathbb{W}_2\mathbb{W}_3 &= \mathbb{F}\mathbb{B}\mathbb{D} + \mathbb{E} - \mathbb{F}\mathbb{D} - z\mathbb{B}(\mathbb{D} - \mathbb{Q}) - \mathbb{F}\mathbb{B}\mathbb{Q}\\ &= \mathbb{E} - \mathbb{F}\mathbb{D} + (\mathbb{F} - z\mathbb{E})\mathbb{B}(\mathbb{D} - \mathbb{Q}). \quad \blacksquare\end{aligned}$$

The following result, a consequence of Theorem 3.1, will be crucial in deriving the explicit form of $\mathbf{A}_t$ in the next section.

COROLLARY 3.1. *For* $\widetilde{\mathbb{M}}$ *and* $\mathbb{B}(z)$ *defined above,*

$$\widetilde{\mathbb{M}} = ((\mathbb{E} - \mathbb{F}\mathbb{D})\mathbb{B}(z)^{-1})|_{z:=x}. \tag{3.13}$$

Proof. Directly from (2.4) we deduce that

$$(\mathbb{E} - \mathbb{F}\mathbb{D})\mathbb{W}_2\mathbb{W}_3 = \mathbb{E} - \mathbb{F}\mathbb{D}.$$

Then (3.10) multiplied on the right by $\mathbb{W}_3^{-1}\mathbb{W}_2^{-1}\mathbb{B}(z)^{-1}$ (recall that in view of Lemma 3.2 the element $\mathbb{B}(z)$ is invertible) yields

$$\widetilde{\mathbb{M}} = \big(\mathbb{E} - \mathbb{F}\mathbb{D} + (\mathbb{F} - z\mathbb{E})\mathbb{B}(z)(\mathbb{D} - \mathbb{Q})\mathbb{W}_3^{-1}\mathbb{W}_2^{-1}\big)\mathbb{B}(z)^{-1} = \mathbb{X}(z)\mathbb{Y}(z),$$

where $\mathbb{X}(z) = \mathbb{E} - \mathbb{F}\mathbb{D} + (\mathbb{F} - z\mathbb{E})\mathbb{B}(z)(\mathbb{D} - \mathbb{Q})\mathbb{W}_3^{-1}\mathbb{W}_2^{-1}$ and $\mathbb{Y}(z) = \mathbb{B}(z)^{-1}$. Since $\widetilde{\mathbb{M}}$ does not depend on z, we can write

$$\widetilde{\mathbb{M}} = (\mathbb{X}(z)\mathbb{Y}(z))|_{z:=x}. \tag{3.14}$$

Note that

$$\mathbb{X}(z)|_{z:=x} = \mathbb{E} - \mathbb{F}\mathbb{D} + (\widetilde{\mathbb{X}}(z)\widetilde{\mathbb{Y}}(z))|_{z:=x},$$

where $\widetilde{\mathbb{X}}(z) = \mathbb{F} - z\mathbb{E}$ and $\widetilde{\mathbb{Y}}(z) = \mathbb{B}(z)(\mathbb{D} - \mathbb{Q})\mathbb{W}_3^{-1}\mathbb{W}_2^{-1}$. Since

$$\widetilde{\mathbb{X}}(z)|_{z:=x} = (\mathbb{F} - z\mathbb{E})|_{z:=x} = \mathbb{F} - \mathbb{F} = \mathbb{0},$$

in view of Lemma 2.3 applied to $\widetilde{\mathbb{X}}(z)$ and $\widetilde{\mathbb{Y}}(z)$, we conclude that $\mathbb{X}(z)|_{z:=x} = \mathbb{E} - \mathbb{F}\mathbb{D}$. Applying Lemma 2.3 again, this time to $\mathbb{X}(z)$ and $\mathbb{Y}(z)$, the right-hand side of (3.14) simplifies to (3.13). $\blacksquare$

Since $\widetilde{\mathbb{H}}\mathbb{F}\mathbb{D} = \widetilde{\mathbb{H}}$, it follows from the definition of $\widetilde{\mathbb{M}}$ that $\widetilde{\mathbb{H}} = \widetilde{\mathbb{M}}\mathbb{D} + \mathbb{F}\widetilde{\mathbb{H}}\mathbb{D}$. Iterating this equality we get

$$\widetilde{\mathbb{H}} = \sum_{k\geqslant 0} \mathbb{F}^k \widetilde{\mathbb{M}} \mathbb{D}^{k+1}, \tag{3.15}$$

where, coordinatewise, all sums have a finite number of non-zero summands. Note that as intended, (3.15) is the solution $\widetilde{\mathbb{H}}$ of equation (3.3) given in terms of $\mathbb{B}(z)$. Consequently, we have also found a solution to the initial q-commutation equation (3.2).

4. INFINITESIMAL GENERATOR OF THE QUADRATIC HARNESS $QH(\eta, \theta; 0, \tau; q)$

Equipped with the results obtained in the previous section, we are ready to prove our main result which gives an explicit integral representation of the infinitesimal generator of $QH(\eta, \theta; 0, \tau; q)$.

Proof of Theorem 1.1. Fix $t \geqslant 0$ and $z \in \mathbb{R}$. By Favard's theorem (see [27, Theorem 4.4]), the polynomials $\{B_n(\cdot; z, t) : n \geqslant 0\}$ are "orthogonal" with respect to the unique moment functional $\mathcal{L}_{z,t,\theta,\tau,\eta,q}$, which acts on polynomials in $y \in \mathbb{R}$, i.e.,

$$\mathcal{L}_{z,t,\theta,\tau,\eta,q}(B_n(y; z, t)B_k(y; z, t)) = \kappa_n \mathbb{1}(n = k),$$

where $\kappa_0 \neq 0$. Without any loss of generality, we assume that $\mathcal{L}_{z,t,\theta,\tau,\eta,q}$ is normalized, i.e., $\kappa_0 = 1$. Then for $\mathbb{E}_y \in \mathcal{Q}$, $y \in \mathbb{R}$ (see (2.2)), we obtain

$$\mathcal{L}_{z,t,\theta,\tau,\eta,q}(\mathbb{E}_y\mathbb{B}(z)) = \mathbb{E} - \mathbb{F}\mathbb{D}, \tag{4.1}$$

where $\mathcal{L}_{z,t,\theta,\tau,\eta,q}$ on the left-hand side of (4.1) acts coordinatewise on $\mathbb{E}_y\mathbb{B}(z)$. Moreover, let us consider

$$\mathbb{Z}(z) := \big(\mathcal{L}_{z,t,\theta,\tau,\eta,q}(1), \mathcal{L}_{z,t,\theta,\tau,\eta,q}(y), \mathcal{L}_{z,t,\theta,\tau,\eta,q}(y^2), \ldots\big) = \mathcal{L}_{z,t,\theta,\tau,\eta,q}(\mathbb{E}_y).$$

Note that $\mathbb{Z}(z)$ is a well-defined element of the algebra $\mathcal{Q}$ with all coordinates being polynomials of degree zero (in x). By the linearity of the moment functional, we obtain, in view of (4.1),

$$\mathbb{Z}(z)\mathbb{B}(z) = \mathcal{L}_{z,t,\theta,\tau,\eta,q}(\mathbb{E}_y\mathbb{B}(z)) = \mathbb{E} - \mathbb{F}\mathbb{D}.$$

Hence,

$$\mathbb{Z}(z) = (\mathbb{E} - \mathbb{F}\mathbb{D})\mathbb{B}(z)^{-1}.$$

Since the above equality holds for all fixed $z \in \mathbb{R}$, a comparison with (3.13) gives $\mathbb{Z}(z)|_{z=x} \in \mathcal{Q}$ and

$$\widetilde{\mathbb{M}} = \mathbb{Z}(z)|_{z=x} = \mathcal{L}_{z,t,\theta,\tau,\eta,q}(\mathbb{E}_y)|_{z=x} = \mathcal{L}_{x,t,\theta,\tau,\eta,q}(\mathbb{E}_y). \tag{4.2}$$

Thus, inserting (4.2) into (3.15), linearity of $\mathcal{L}_{x,t,\theta,\tau,\eta,q}$ implies

$$\widetilde{\mathbb{H}} = \mathcal{L}_{x,t,\theta,\tau,\eta,q}(\mathbb{Q}_y), \tag{4.3}$$

where

$$\mathbb{Q}_y := \sum_{k=0}^{\infty} \mathbb{F}^k \mathbb{E}_y \mathbb{D}^{k+1}.$$

Recall that (see [24, (3.8)])

$$\mathbb{A} = \sum_{j=0}^{\infty} \mathbb{F}^j \mathbb{H} \mathbb{D}^{j+1}. \tag{4.4}$$

Since $\mathbb{H} = (\mathbb{E} + \eta\mathbb{F})\widetilde{\mathbb{H}}$, plugging this together with (4.3) into (4.4) we obtain

$$\mathbb{A} = \sum_{j=0}^{\infty} \mathbb{F}^j (\mathbb{E} + \eta\mathbb{F})\widetilde{\mathbb{H}}\mathbb{D}^{j+1} = (\mathbb{E} + \eta\mathbb{F})\mathcal{L}_{x,t,\theta,\tau,\eta,q}\Big(\sum_{j=0}^{\infty} \mathbb{F}^j \mathbb{Q}_y \mathbb{D}^{j+1}\Big).$$

Since $\mathbb{D}\mathbb{E}_y = \mathbb{0}$, according to (2.6) we have

$$\begin{aligned}\mathbb{D}_1\mathbb{Q}_y &= \Big(\sum_{j=0}^{\infty} \mathbb{F}^j \mathbb{D}^{j+1}\Big)\Big(\sum_{k=0}^{\infty} \mathbb{F}^k \mathbb{E}_y \mathbb{D}^{k+1}\Big) = \sum_{j=0}^{\infty} \mathbb{F}^j \Big(\sum_{k\geqslant j+1} \mathbb{F}^{k-j-1} \mathbb{E}_y \mathbb{D}^{k+1}\Big) \\ &= \sum_{j=0}^{\infty} \mathbb{F}^j \Big(\sum_{k\geqslant j+1} \mathbb{F}^{k-j-1} \mathbb{E}_y \mathbb{D}^{k-j}\Big)\mathbb{D}^{j+1} = \sum_{j=0}^{\infty} \mathbb{F}^j \mathbb{Q}_y \mathbb{D}^{j+1}.\end{aligned}$$

Thus,

$$\mathbb{A} = (\mathbb{E} + \eta\mathbb{F})\mathcal{L}_{x,t,\theta,\tau,\eta,q}(\mathbb{D}_1\mathbb{Q}_y).$$

Since $\mathbb{E}_y\mathbb{D}^{k+1}$ has nth coordinate zero for $n \leqslant k$, and y^{n-k-1} for $n \geqslant k+1$, the nth coordinate of $\mathbb{Q}_y$ is

$$y^{n-1} + y^{n-2}x + \cdots + yx^{n-2} + x^{n-1} = \frac{y^n - x^n}{y - x}.$$

Therefore, the nth coordinate of $\mathbb{D}_1\mathbb{Q}_y$ is $\frac{\partial}{\partial x}\frac{y^n - x^n}{y - x}$. Consequently, $A_n(x)$, the nth coordinate of $\mathbb{A}$, is

$$A_n(x) = (1 + \eta x)\mathcal{L}_{x,t,\theta,\tau,\eta,q}\left(\frac{\partial}{\partial x}\frac{y^n - x^n}{y - x}\right).$$

Recall that $A_n(x) = \mathbf{A}_t(x^n)$. By linearity, for any polynomial f we get

$$\mathbf{A}_t f(x) = (1 + \eta x)\mathcal{L}_{x,t,\theta,\tau,\eta,q}\left(\frac{\partial}{\partial x}\frac{f(y) - f(x)}{y - x}\right). \tag{4.5}$$

Now are going to show that the moment functional $\mathcal{L}_{x,t,\theta,\tau,\eta,q}$ is non-negative. Consider $\mathbb{G} := (G_0, G_1, G_2, \ldots) \in \mathcal{Q}$ with

$$G_k(x) := \lim_{h\to 0^+} \int_{\mathbb{R}} y^k \frac{(y - x)^2}{h}\, \mathbb{P}_{t,t+h}(x, \mathrm{d}y), \quad k = 0, 1, \ldots.$$

For any $k \in \mathbb{N} \cup \{0\}$,

$$\int_{\mathbb{R}} y^k \frac{(y-x)^2}{h} \mathbb{P}_{t,t+h}(x, \mathrm{d}y) = \int_{\mathbb{R}} \frac{y^{k+2} - x^{k+2}}{h} \mathbb{P}_{t,t+h}(x, \mathrm{d}y) - 2x \int_{\mathbb{R}} \frac{y^{k+1} - x^{k+1}}{h} \mathbb{P}_{t,t+h}(x, \mathrm{d}y) + x^2 \int_{\mathbb{R}} \frac{y^k - x^k}{h} \mathbb{P}_{t,t+h}(x, \mathrm{d}y),$$

whence

$$G_k(x) = \mathbf{A}_t(x^{k+2}) - 2x\mathbf{A}_t(x^{k+1}) + x^2\mathbf{A}_t(x^k), \quad k = 0, 1, \ldots,$$

In view of the fact that $\mathbb{H}_t = \mathbb{A}_t\mathbb{F} - \mathbb{F}\mathbb{A}_t$, the above equality implies

$$\mathbb{G} = \mathbb{A}_t\mathbb{F}^2 - 2\mathbb{F}\mathbb{A}_t\mathbb{F} + \mathbb{F}^2\mathbb{A}_t = \mathbb{H}_t\mathbb{F} - \mathbb{F}\mathbb{H}_t.$$

Consequently, since $\mathbb{H} = (\mathbb{E} + \eta\mathbb{F})\widetilde{\mathbb{H}}$ and $\widetilde{\mathbb{H}}\mathbb{F} - \mathbb{F}\widetilde{\mathbb{H}} = \widetilde{\mathbb{M}}$ (see (4.2)), we obtain

$$\mathbb{G} = (\mathbb{E} + \eta\mathbb{F})(\widetilde{\mathbb{H}}\mathbb{F} - \mathbb{F}\widetilde{\mathbb{H}}) = (\mathbb{E} + \eta\mathbb{F})\mathcal{L}_{x,t,\theta,\tau,\eta,q}(\mathbb{E}_y).$$

Looking coordinatewise at the above identity, we find that for all $k \geqslant 0$,

$$\lim_{h\to 0^+} \int_{\mathbb{R}} y^k \frac{(y-x)^2}{h} \mathbb{P}_{t,t+h}(x, \mathrm{d}y) = (1 + \eta x)\mathcal{L}_{x,t,\theta,\tau,\eta,q}(y^k) \tag{4.6}$$

and as a result for any polynomial $f \geqslant 0$ (i.e., $f(y) \geqslant 0$ for all $y \in \mathbb{R}$) we have

$$0 \leqslant \lim_{h\to 0^+} \int_{\mathbb{R}} f(y) \frac{(y-x)^2}{h} \mathbb{P}_{t,t+h}(x, \mathrm{d}y) = (1 + \eta x)\mathcal{L}_{x,t,\theta,\tau,\eta,q}(f(y)). \tag{4.7}$$

From (1.8) and (1.9) the conditional variance for quadratic harnesses is given by

$$\mathbb{V}\mathrm{ar}(X_t|\mathcal{F}_s) = (t-s)(1 + \eta X_s). \tag{4.8}$$

If $x \in \mathrm{supp}(X_s)$ is such that $1 + \eta x = 0$, then on the set $\{X_s = x\}$ we have $\mathbb{V}\mathrm{ar}(X_t|\mathcal{F}_s) = 0$. Consequently, x is an absorbing state, and the infinitesimal generator is zero according to (1.10). Consequently, Theorem 1.1 holds in this case.

Otherwise, if still $x \in \mathrm{supp}(X_s)$ then (4.8) yields $1 + \eta x > 0$ since the conditional variance is non-negative. Consequently, (4.7) implies that $\mathcal{L}_{x,t,\theta,\tau,\eta,q}$ is non-negative definite, i.e., $\mathcal{L}_{x,t,\theta,\tau,\eta,q}f \geqslant 0$ for all polynomials $f \geqslant 0$. The proof of [27, Theorem 4.4] implies that the product of consecutive coefficients of B_{n-1} from recurrence (1.13) is non-negative. Therefore by [21, Theorem A.1] there exists a probability measure $\nu_{x,t,\eta,\theta,\tau,q}$ such that for all polynomials f we have

$$\mathcal{L}_{x,t,\theta,\tau,\eta,q}f = \int_{\mathbb{R}} f(y)\, \nu_{x,t,\eta,\theta,\tau,q}(\mathrm{d}y). \tag{4.9}$$

Putting (4.5) and (4.9) together ends the proof. ■

The measure $\nu_{x,t,\eta,\theta,\tau,q}$ that appears on the right-hand side of (1.12) may not be unique. The problem of uniqueness of the orthogonal measure is equivalent to the Hamburger moment problem with moments $\mathcal{L}_{x,t,\theta,\tau,\eta,q}(y^n)$, $n \geqslant 0$ (see [27, p. 71]).

In addition to the integral representation for the infinitesimal generator, we have shown in the proof above (see (4.6) and (4.9)) that all moments of the measure $\frac{(y-x)^2}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y)$ converge to the respective moments of $(1+\eta x)\nu_{x,t,\eta,\theta,\tau,q}(\mathrm{d}y)$ as $h \to 0^+$. Analogously, we can show the same for $\frac{(y-x)^2}{h}\mathbb{P}_{t-h,t}(x,\mathrm{d}y)$. Moreover, we can normalize the measures $\frac{(y-x)^2}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y)$ and $\frac{(y-x)^2}{h}\mathbb{P}_{t-h,t}(x,\mathrm{d}y)$ in order to make them probability measures.

REMARK 4.1. Let $1+\eta x > 0$. The measures

$$\frac{(y-x)^2}{h(1+\eta x)}\,\mathbb{P}_{t,t+h}(x,\mathrm{d}y), \quad t \geqslant 0,\ h > 0,$$

and

$$\frac{(y-x)^2}{h(1+\eta x)}\,\mathbb{P}_{t-h,t}(x,\mathrm{d}y), \quad t \geqslant h > 0,$$

are probability measures.

Proof. Note that, except for the trivial case in (4.8), we have $1+\eta x > 0$ for all $x \in \operatorname{supp}(X_t)$. Then $\frac{(y-x)^2}{h(1+\eta x)}\mathbb{P}_{t,t+h}(x,\mathrm{d}y)$ is non-negative and

$$\int_{\mathbb{R}} \frac{(y-x)^2}{h(1+\eta x)}\,\mathbb{P}_{t,t+h}(x,\mathrm{d}y) = \frac{\mathbb{V}\mathrm{ar}(X_{t+h} \mid X_t = x)}{h(1+\eta x)} = 1.$$

The first formula, applied to $t-h$ instead of t, gives the second one. ■

From now on, we restrict the domain of q to $q \in [-1,1)$. Then the coefficients in the recurrence (1.13) are bounded for fixed t and x. Consequently, polynomials $(B_n(\cdot;x,t))_{n\geqslant 0}$ are orthogonal with respect to a measure with bounded support (see [38, Theorems 2.5.4 and 2.5.5]). As a result, $\nu_{x,t,\eta,\theta,\tau,q}$ is uniquely determined by its moments, so the convergence of the moments of the measures $\frac{(y-x)^2}{h(1+\eta x)}\mathbb{P}_{t,t+h}(x,\mathrm{d}y)$ and $\frac{(y-x)^2}{h(1+\eta x)}\mathbb{P}_{t-h,t}(x,\mathrm{d}y)$ to the moments of $\nu_{x,t,\eta,\theta,\tau,q}$ implies the weak convergence of these measures (see [13, Theorem 30.2]). As a result, when $q \in [-1,1)$, we can extend the domain of the infinitesimal generator $\mathbf{A}_t$ to the space of bounded continuous functions with bounded continuous second derivative.

COROLLARY 4.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function with a bounded continuous second derivative. Then for $x \in \operatorname{supp}(X_t)$ we have*

$$\text{(4.10)} \qquad \mathbf{A}_t g(x) = \frac{1+\eta x}{2} g''(x)\nu_{x,t,\eta,\theta,\tau,q}(\{x\}) + (1+\eta x) \int_{\mathbb{R}\setminus\{x\}} \frac{\partial}{\partial x}\left(\frac{g(y)-g(x)}{y-x}\right) \nu_{x,t,\eta,\theta,\tau,q}(\mathrm{d}y),$$

where $\nu_{x,t,\eta,\theta,\tau,q}$ is the probability measure defined in Theorem 1.1.

Proof. As already observed, if $1+\eta x = 0$, then x is an absorbing state and hence (4.10) is trivially satisfied (see (1.10)). So assume that $1+\eta x > 0$. We will prove (4.10) only for $\mathbf{A}_t^+$, because repeating the same argument for $\mathbb{P}_{t-h,h}(x, \mathrm{d}y)$ yields the same for $\mathbf{A}_t^-$.

Fix $x \in \operatorname{supp}(X_t)$. Define $\phi_x : \mathbb{R} \to \mathbb{R}$ by

$$\phi_x(y) := \begin{cases} \dfrac{\partial}{\partial x} \dfrac{g(y)-g(x)}{y-x} & \text{for } y \neq x, \\ \frac{1}{2} g''(x) & \text{for } y = x. \end{cases}$$

By Taylor's theorem, ϕ_x is a bounded continuous function since

$$\frac{\partial}{\partial x}\frac{g(y)-g(x)}{y-x} = \frac{1}{(y-x)^2}\int_x^y g''(z)(y-z)\,\mathrm{d}z \xrightarrow{y\to x} \tfrac{1}{2} g''(x)$$

and

$$\left|\frac{\partial}{\partial x}\frac{g(y)-g(x)}{y-x}\right| = \frac{1}{(y-x)^2}\left|\int_x^y g''(z)(y-z)\,\mathrm{d}z\right| \leqslant \tfrac{1}{2}\sup_{y\in\mathbb{R}} |g''(y)|.$$

Because $g(y) - g(x) = (y-x)g'(x) + \int_x^y (y-z)g''(z)\,\mathrm{d}z$ we can write

$$\text{(4.11)} \qquad \int_{\mathbb{R}} \frac{g(y)-g(x)}{h}\,\mathbb{P}_{t,t+h}(x,\mathrm{d}y) = g'(x)\int_{\mathbb{R}} \frac{y-x}{h}\,\mathbb{P}_{t,t+h}(x,\mathrm{d}y) + J(h,x),$$

where

$$\begin{aligned} J(h,x) &= \int_{\mathbb{R}} \frac{\int_x^y g''(z)(y-z)\,\mathrm{d}z}{h}\,\mathbb{P}_{t,t+h}(x,\mathrm{d}y) \\ &= \int_{\mathbb{R}\setminus\{x\}} \frac{\int_x^y g''(z)(y-z)\,\mathrm{d}z}{(y-x)^2}\cdot\frac{(y-x)^2}{h}\,\mathbb{P}_{t,t+h}(x,\mathrm{d}y). \end{aligned}$$

Since $\phi_x(y) = \frac{\int_x^y g''(z)(y-z)\,\mathrm{d}z}{(y-x)^2}$ when $y \neq x$, we get

$$J(h,x) = \int\limits_{\mathbb{R}\setminus\{x\}} \phi_x(y)\frac{(y-x)^2}{h}\,\mathbb{P}_{t,t+h}(x,\mathrm{d}y) = \int\limits_{\mathbb{R}} \phi_x(y)\frac{(y-x)^2}{h}\,\mathbb{P}_{t,t+h}(x,\mathrm{d}y).$$

Since $\frac{(y-x)^2}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y)$ converges weakly to $(1+\eta x)\nu_{x,t,\eta,\theta,\tau,q}(\mathrm{d}y)$, we conclude that

$$\lim_{h\to 0^+} J(h,x) = (1+\eta x)\int\limits_{\mathbb{R}} \phi_x(y)\nu_{x,t,\eta,\theta,\tau,q}(\mathrm{d}y).$$

Consequently, letting $h \to 0^+$ in (4.11) and using the fact that

$$\lim_{h\to 0^+} \int\limits_{\mathbb{R}} \frac{y-x}{h}\,\mathbb{P}_{t,t+h}(x,\mathrm{d}y) = \mathbf{A}_t x = 0,$$

which follows from Theorem 1.1, we obtain the desired result. ■

It is worth mentioning that in all known cases of $QH(\eta,\theta;0,\tau;q)$ processes with $q \in [-1,1)$, the supports of $\mathbb{P}_{t,t+h}(x,\mathrm{d}y)$ and $\mathbb{P}_{t-h,t}(x,\mathrm{d}y)$ are bounded for any fixed $t,h>0$ and $x\in\mathbb{R}$ (see [21]). Hence, it is sufficient to assume in Corollary 4.1 that g is a C^2 function (we do not need to assume boundedness).

5. MORE ON THE MEASURE $\nu_{x,t,\theta,\tau,\eta,q}$

Note that Theorem 1.1 and Corollary 4.1 give explicit formulas for the infinitesimal generators in terms of a measure $\nu_{x,t} := \nu_{x,t,\eta,\theta,\tau,q}$. In this section, we embed the polynomials $(B_n(y;x,t))_{n\geqslant 0}$ in the Askey–Wilson scheme, which allows us to describe $\nu_{x,t}$ in terms of some measures well-known from the literature.

5.1. Case $|q|<1$. In this section we assume (1.6) with

$$\sigma = 0 \quad\text{and}\quad |q|<1. \tag{5.1}$$

As discussed in Section 4, the probability measures $\frac{(y-x)^2}{h(1+\eta x)}\mathbb{P}_{t,t+h}(x,\mathrm{d}y)$ and $\frac{(y-x)^2}{h(1+\eta x)}\mathbb{P}_{t-h,t}(x,\mathrm{d}y)$ converge weakly to the probability measure $\nu_{x,t}$ for all $t>0$. Moreover, the coefficient of B_{n-1} in the recurrence (1.13) is bounded, so $\nu_{x,t}$ is uniquely determined. Consequently, for all fixed $t>0$ and x in the support of X_t, Favard's theorem implies that the coefficient of B_{n-1} in (1.13) satisfies

$$\prod_{n=1}^{N}\left\{\alpha_t(1+\eta\gamma_t[n]_q+\eta\beta_t[n]_q^2+x\eta q^n)[n+1]_q[n]_q\right\}\geqslant 0$$

for all $N\geqslant 1$. Because $\alpha_t>0$ for $t>0$, the above condition is equivalent to

$$\prod_{n=1}^{N}(1+\eta\gamma_t[n]_q+\eta\beta_t[n]_q^2+x\eta q^n)\geqslant 0 \quad\text{for all } N\geqslant 1; \tag{5.2}$$

note that $q \in (-1, 1)$ implies $[n]_q > 0$ for all $n \geqslant 1$. According to (5.2), only the following two situations are possible:

• If there is $n \geqslant 1$ such that

$$1 + \eta\gamma_t[n]_q + \eta\beta_t[n]_q^2 + x\eta q^n = 0, \tag{5.3}$$

then $\nu_{x,t}$ is supported on distinct zeros of the polynomial $B_{N_0}(\cdot; x, t)$, where N_0 is the smallest $n \geqslant 1$ satisfying (5.3) (see [21, Theorem A.1]). Moreover, for all $1 \leqslant n < N_0$ we have

$$1 + \eta\gamma_t[n]_q + \eta\beta_t[n]_q^2 + x\eta q^n > 0,$$

which imposes additional constraints on the parameters η, θ, τ, q and on the support of X_t for fixed $t > 0$.

• If for all $n \geqslant 1$,

$$1 + \eta\gamma_t[n]_q + \eta\beta_t[n]_q^2 + x\eta q^n > 0,$$

then letting $n \to \infty$ we get

$$0 \leqslant 1 + \frac{\eta\gamma_t}{1-q} + \frac{\eta\beta_t}{(1-q)^2} = 1 + \frac{\eta\widetilde{\theta}}{1-q}, \tag{5.4}$$

where

$$\widetilde{\theta} := \theta + \frac{\eta\tau}{1-q}. \tag{5.5}$$

Consequently, not all combinations of the parameters of the quadratic harness are possible. Moreover, the identity (5.2) for $N = 1$ implies

$$1 + \eta\theta + \eta^2\tau \geqslant 0. \tag{5.6}$$

Indeed, letting $t \searrow 0$ and considering a version of the process with càdlàg trajectories (the quadratic harness has the martingale property, see (1.8), so such a version exists), we can also let x tend to zero. Thus, (5.6) follows.

As a result, if $\tau = 0$, then (5.4) and (5.6) imply that $QH(\eta, \theta; 0, 0; q)$ exists only when

$$1 + \eta\theta \geqslant \max\{0, q\}.$$

Under the above condition, the bi-Poisson process is constructed in [18].

In general, the measure $\nu_{x,t}$ can be conveniently described as a linear transformation of a measure occurring in the Askey–Wilson scheme.

THEOREM 5.1. *Assume* (5.4) *holds. Then for* $x \in \mathbb{R}$ *we have*

$$\nu_{x,t}(A) = \mu(\{(y - w)/u : y \in A\}), \quad A \in \mathcal{B}(\mathbb{R}), \tag{5.7}$$

where

(i) *for* $1-q+\eta\widetilde{\theta}>0$,

$$u=-\frac{\sqrt{(1-q)^3}}{2\sqrt{\alpha_t}\sqrt{1-q+\eta\widetilde{\theta}}},\quad w=\frac{(1-q)\widetilde{\theta}+\beta_t}{2\sqrt{\alpha_t}\sqrt{1-q}\sqrt{1-q+\eta\widetilde{\theta}}},$$

and μ *is the orthogonality measure for Askey–Wilson polynomials* (*see* (6.2)) *with parameters*

(5.8)
$$a=\frac{\eta\sqrt{\alpha_t}}{\sqrt{(1-q)(1-q+\eta\widetilde{\theta})}},$$

(5.9)
$$b,c=\frac{1}{2\sqrt{\alpha_t}}\left(q\frac{(1-q)\widetilde{\theta}+\beta_t-(1-q)^2x}{\sqrt{(1-q)(1-q+\eta\widetilde{\theta})}}\pm|q|\sqrt{\frac{((1-q)\widetilde{\theta}+\beta_t-(1-q)^2x)^2}{(1-q)(1-q+\eta\widetilde{\theta})}-4\alpha_t}\right)$$

and $d=0$*;*

(ii) *for* $1-q+\eta\widetilde{\theta}=0$ *and*

(a) $q=0$*:* $u=1$, $w=-\widetilde{\theta}$ *and* $\mu=\delta_0$,

(b) $q\neq 0$ *and* $\frac{\eta\beta_t}{(1-q)^2}\neq 1+x\eta$*:*

$$u=-\frac{q\eta(1-q)^2}{\eta\beta_t-(1+x\eta)(1-q)^2}\quad\text{and}\quad w=-u\frac{\eta\beta_t-(1-q)^2}{\eta(1-q)^2},$$

and μ *is the orthogonality measure for big* q*-Jacobi polynomials* (*see* (6.5)) *with parameters*

$$a=q,\quad b=0\quad\text{and}\quad c=\frac{\eta\beta_t}{\eta\beta_t-(1+\eta x)(1-q)^2};\tag{5.10}$$

(c) $q\neq 0$ *and* $\frac{\eta\beta_t}{(1-q)^2}=1+x\eta$*:*

$$u=\frac{(1-q)^2}{\beta_t}\quad\text{and}\quad w=\frac{u}{\eta},$$

and μ *is the orthogonality measure for little* q*-Jacobi polynomials with* $a=q$ *and* $b=0$ (*see* (6.7)).

Proof. Consider the polynomials $(p_n(y))_{n\geqslant 0}$ given by

$$p_n(y):=u^nB_n((y-w)/u;x,t),\quad n\geqslant 0,\tag{5.11}$$

where the parameters u and w are given in the formulation of the theorem and, in general, depend on x and t.

In view of (5.4), we restrict our considerations to two cases:

(i) $1-q+\eta\widetilde{\theta}>0$. Then $(p_n(y))_{n\geqslant 0}$ in (5.11) are the Askey–Wilson polynomials with parameters (5.8). Under the assumed range of parameters (see (1.6) and (5.1)), b and c are either a complex conjugate pair or both real.

(ii) $1-q+\eta\widetilde{\theta}=0$. As a result, $\eta\neq 0$.

  (a) $q=0$: in this case we have $p_n(y)=y(y-\beta_t)^{n-1}$, $n\geqslant 1$.
  (b) $q\neq 0$ and $\frac{\eta\beta_t}{(1-q)^2}\neq 1+x\eta$: it is easy to verify that $(p_n(y))_{n\geqslant 0}$ in (5.11) are the big q-Jacobi polynomials with parameters (5.10).
  (c) $q\neq 0$ and $\frac{\eta\beta_t}{(1-q)^2}=1+x\eta$: then the polynomials $(p_n(y))_{n\geqslant 0}$ are the little q-Jacobi polynomials with $a=q$ and $b=0$ (see (6.7)).

In the simplest case (ii)(a) we clearly have $\mu=\delta_0$ (see [21, Thereom A.1]). In all other cases, the polynomials $(p_n(y))_{n\geqslant 0}$ as well as their orthogonality measures μ are well-known in the literature. Consequently, in view of (5.11), the measure $\nu_{x,t}$ is identified through (5.7). ■

To conclude this subsection, we derive an exact formula for the infinitesimal generator of a free quadratic harness.

EXAMPLE 5.1. When $q=0$, the recurrence (1.13) assumes the form

$$\begin{aligned} B_0(y;x,t)&=1,\quad B_1(y;x,t)=y-\theta-\eta\tau,\\ yB_n(y;x,t)&=B_{n+1}(y;x,t)+(\theta+2\eta\tau+\eta t)B_n(y;x,t)\\ &\qquad+(\tau+t)(1+\eta\theta+\eta^2\tau)B_{n-1}(y;x,t)\quad \text{for } n\geqslant 1. \end{aligned}$$

It is easy to check that the polynomials $(B_n(y;x,t))_{n\geqslant 0}$ are orthogonal with respect to a probability measure if and only if $1+\eta\theta+\eta^2\tau\geqslant 0$. Therefore, if $1+\eta\theta+\eta^2\tau<0$, then the quadratic harness $QH(\eta,\theta;0,\tau;0)$ does not exist.

If $1+\eta\theta+\eta^2\tau=0$, then $(B_n(y;x,t))_{n\geqslant 0}$ are orthogonal with respect to the Dirac measure concentrated at $\theta+\eta\tau=-\frac{1}{\eta}$. The polynomials $(B_n(y;x,t))_{n\geqslant 0}$ do not depend on x, so the measure does not depend on x either. Moreover, $\mathbf{A}_t$ does not depend on t (since $\nu_{x,t}$ does not depend on t) and (4.10) takes the form

$$\mathbf{A}_t g(x)=\begin{cases}\eta g'(x)+\eta^2\dfrac{g(-1/\eta)-g(x)}{1+\eta x}, & x\neq -1/\eta,\\ 0, & x=-1/\eta.\end{cases} \tag{5.12}$$

Note that this formula does not depend on the values of τ and θ (even though we only assume that $1+\eta\theta+\eta^2\tau=0$).

It is not difficult to verify that (5.12) is the infinitesimal generator for the following process: Consider a Markov process $(X_t)_{t\geqslant 0}$ starting from zero, with the support at time $t>0$ containing only two points:

$$x_-(t)=-1/\eta\quad\text{and}\quad x_+(t)=\eta t.$$

For $i, j \in \{-, +\}$ its transition probabilities are

$$\mathbb{P}(X_t = x_i(t) \mid X_s = x_j(s)) = p_{ji}(s,t), \quad 0 < s < t,$$

where

$$\begin{aligned} p_{--}(s,t) &= 1, & p_{-+}(s,t) &= 0, \\ p_{+-}(s,t) &= \frac{\eta^2(t-s)}{1+\eta^2 t}, & p_{++}(s,t) &= \frac{1+\eta^2 s}{1+\eta^2 t}, \end{aligned}$$

and its univariate distributions are

$$\mathbb{P}(X_t = x_-(t)) = p_{+-}(0,t), \quad \mathbb{P}(X_t = x_+(t)) = p_{++}(0,t), \quad t \geqslant 0.$$

It can be easily checked that

$$(X_t)_{t\geqslant 0} \sim QH(\eta, -1/\eta - \eta\tau; 0, \tau; 0) = QH(\eta, -1/\eta; 0, 0; 0),$$

that is, $(X_t)_{t\geqslant 0}$ satisfies (1.3), (1.4) and (1.7) with appropriate parameters. Note that the value of τ is, in fact, irrelevant for this process.

When $1 + \eta\theta + \eta^2\tau > 0$, comparing respective recurrences, we conclude that

$$B_n(y; x, t) = (2\sigma_t)^n P_n^*\left(\frac{y - m_t}{2\sigma_t}, -\frac{\eta t + \eta\tau}{2\sigma_t}\right), \quad n \geqslant 0,$$

where $m_t := \theta + 2\eta\tau + \eta t$, $\sigma_t^2 := (\tau + t)(1 + \eta\theta + \eta^2\tau)$ and $(P_n^*(y, c))_{n\geqslant 0}$ are the polynomials from [26, Section 5, Example (a)]. Consequently, if $|\eta(t + \tau)| \leqslant \sigma_t$, then $(B_n(y; x, t))_{n\geqslant 0}$ are orthogonal with respect to

$$\begin{aligned} \nu_{x,t}(\mathrm{d}y) &= \frac{1}{2\pi} \frac{\sqrt{4\sigma_t^2 - (y - m_t)^2}}{\eta^2(t+\tau)^2 + \sigma_t^2 + \eta(t+\tau)(y - m_t)} \mathbb{1}_{(m_t - 2\sigma_t, m_t + 2\sigma_t)}(y)\,\mathrm{d}y \\ &= \frac{1}{2\pi} \frac{\sqrt{4\sigma_t^2 - (y - m_t)^2}}{(t+\tau)(1+\eta y)} \mathbb{1}_{(m_t - 2\sigma_t, m_t + 2\sigma_t)}(y)\,\mathrm{d}y. \end{aligned}$$

Therefore,

$$\mathbf{A}_t f(x) = \frac{1 + \eta x}{2\pi} \int_{m_t - 2\sigma_t}^{m_t + 2\sigma_t} \frac{\partial}{\partial x} \frac{f(y) - f(x)}{y - x} \frac{\sqrt{4\sigma_t^2 - (y - m_t)^2}}{(t+\tau)(1+\eta y)}\,\mathrm{d}y$$

for f being a polynomial or a continuous function with a continuous second derivative (in this case, the supports of the transition probabilities are compact, so f is bounded on the support).

In particular, when $\eta = 0$, the measure $\nu_{x,t}$ is Wigner's semicircle law with mean θ and variance $\tau + t$.

If $|\eta(t+\tau)| > \sigma_t$, then $(B_n(y;x,t))_{n\geqslant 0}$ are orthogonal with respect to

$$\nu_{x,t}(\mathrm{d}y) = \frac{1}{2\pi}\frac{\sqrt{4\sigma_t^2-(y-m_t)^2}}{(t+\tau)(1+\eta y)}\mathbb{1}_{(m_t-2\sigma_t,m_t+2\sigma_t)}(y)\,\mathrm{d}y + \left(1-\frac{\sigma_t^2}{\eta^2(t+\tau)^2}\right)\delta_{-1/\eta}(\mathrm{d}y).$$

Thus, in this case, we need to include an additional summand in the formula for the infinitesimal generator arising from the atom of $\nu_{x,t}$.

5.2. Case $q = 1$ and $\eta = 0$. It is noteworthy that quadratic harnesses with $q = 1$ and $\eta = 0$ are Lévy processes (the family is sometimes referred to as *Lévy–Meixner processes*) [20, Remark 3.2]. The subsequent theorem describes measures $\nu_{x,t}$ for such processes.

THEOREM 5.2. *Assume $\eta = 0$ and $q = 1$. Then for $x \in \mathbb{R}$ we have*

$$\nu_{x,t}(A) = \mu(\{(y-w)/u : y \in A\}), \quad A \in \mathcal{B}(\mathbb{R}),$$

where

(i) *for $\theta^2 < 4\tau$,*

$$u = \sqrt{4\tau-\theta^2}, \quad w = x,$$

and μ is the orthogonality measure for Meixner–Pollaczek polynomials (see [39, *formula* (9.7.4)]) *with parameters $\lambda = 1$ and $\phi = \arccos(-\frac{\theta}{2\sqrt{\tau}})$;*

(ii) *for $\theta^2 = 4\tau > 0$,*

$$u = \theta/2, \quad w = x,$$

and μ is the orthogonality measure for Laguerre polynomials (see [39, *formula* (9.12.4)]) *with parameter $\alpha = 1$;*

(iii) *for $\theta^2 > 4\tau > 0$,*

$$u = \sqrt{\theta^2-4\tau}, \quad w = x+u,$$

and μ is the orthogonality measure for Meixner polynomials (see [39, *formula* (9.10.4)]) *with parameters $\beta = 2$ and*

$$c = \frac{\theta^2-2\tau-\sqrt{\theta^4-4\theta^2\tau}}{2\tau} \in (0,1);$$

(iv) *for $\theta^2 \geqslant 4\tau = 0$,*

$$u = 1, \quad w = x,$$

and μ is the Dirac measure concentrated at θ.

Proof. As before in Theorem 5.1, consideration of polynomials $(p_n(y))_{n\geqslant 0}$ given by

$$p_n(y) := u^n B_n((y-w)/u;x,t), \quad n \geqslant 0,$$

with parameters u and w gives the desired results. ∎

In the illustrative examples below we assume that g is a polynomial (note that Corollary 4.1 formally does not cover the case $q = 1$).

EXAMPLE 5.2 (Wiener process). As the Wiener process is $QH(0, 0; 0, 0; 1)$, in view of Theorem 5.2(iv) we conclude that $\nu_{x,t} = \delta_x$ and thus

$$\mathbf{A}_t g(x) = \tfrac{1}{2} g''(x).$$

EXAMPLE 5.3 (Poisson type process). If $(N_t)_{t \geqslant 0}$ is a Poisson process with rate $\lambda > 0$ then $Y_t := \frac{N_t - \lambda t}{\sqrt{\lambda}}$, $t \geqslant 0$, is $QH(0, 1/\sqrt{\lambda}; 0, 0; 1)$. In view of Theorem 5.2(iv) we conclude that $\nu_{x,t} = \delta_{x+1/\sqrt{\lambda}}$ and thus

$$\mathbf{A}_t g(x) = \lambda(g(x + 1/\sqrt{\lambda}) - g(x)) - \sqrt{\lambda}\, g'(x).$$

EXAMPLE 5.4 (Gamma type process). Let us consider $QH(0, \theta; 0, \theta^2/4; 1)$, $\theta \neq 0$, which is a standardized version of the Gamma process. According to [39, formula (9.12.2)] and Theorem 5.2(ii), we see that the polynomials $(B_n(\cdot; x, t))_{n \geqslant 0}$ are orthogonal with respect to the measure $\nu_{x,t}$ whose density $f(y)$ is proportional to (we do not have to assume that $\theta > 0$)

$$\frac{y - x}{\theta/2} \exp\left(-\frac{y - x}{\theta/2}\right) \mathbb{1}\left(\frac{y - x}{\theta/2} > 0\right).$$

For quadratic harnesses that are Lévy processes we introduce an alternative algebraic representation for the infinitesimal generator, which does not refer explicitly to the integro-differential form of Theorem 1.1. To this end, for a function f which has a power series expansion $f(x) = \sum_{k=0}^{\infty} d_k x^k$ in a neighbourhood of zero, we define

$$f(\mathbb{D}_1) := \sum_{k=0}^{\infty} d_k \mathbb{D}_1^k$$

and note that $f(\mathbb{D}_1) \in \mathcal{Q}$.

THEOREM 5.3. *Assume $q = 1$ and $\eta = 0$ and denote $a := \theta^2 - 4\tau$. Then $\mathbb{A}_t$ can be represented as*

$$\mathbb{A}_t = f(\mathbb{D}_1), \tag{5.13}$$

where f takes the following forms:

a) *if $\theta = \tau = 0$ (Wiener process):*

$$f(z) = \tfrac{1}{2} z^2;$$

b) *if $\theta^2 = 4\tau > 0$ (Gamma type process):*

$$f(z) = -\frac{2}{\theta} z - \frac{4}{\theta^2} \ln(1 - \theta z/2);$$

c) *if* $|\theta| > \tau = 0$ (*Poisson type process*)*:*

$$f(z) = \frac{1}{\theta^2} e^{\theta z} - \frac{1}{\theta} z - \frac{1}{\theta^2};$$

d) *if* $\theta^2 > 4\tau > 0$ (*Pascal type process*)*:*

$$f(z) = -\frac{1}{\tau} \ln\left(\frac{1}{2}\left(1+\frac{\theta}{\sqrt{a}}\right)\exp\left(\frac{\theta-\sqrt{a}}{2}\right)+\frac{1}{2}\left(1-\frac{\theta}{\sqrt{a}}\right)\exp\left(\frac{\theta+\sqrt{a}}{2}z\right)\right);$$

e) *if* $\theta^2 < 4\tau$ (*Meixner type process*)*:*

$$f(z) = -\frac{\theta}{2\tau} z - \frac{1}{\tau} \ln(\cos(\sqrt{-a}z/2) - \frac{\theta}{\sqrt{-a}} \sin(\sqrt{-a}z/2)).$$

Proof. It can be inductively shown that

$$\mathbb{D}_1^{m+1}\mathbb{F} - \mathbb{F}\mathbb{D}_1^{m+1} = (m+1)\mathbb{D}_1^m, \qquad m \geqslant 0. \tag{5.14}$$

Moreover, let us consider a real sequence $(c_k)_{k\geqslant 1}$ defined recursively as follows:

$$c_k = \begin{cases} 1, & k = 1, \\ \frac{\theta}{2} c_1 & k = 2, \\ \frac{1}{k}\Big(\theta c_{k-1} + \tau \sum_{m=1}^{k-2} c_m c_{k-1-m}\Big), & k \geqslant 3. \end{cases}$$

By simple algebra the recurrence (5.14) shows that $\sum_{k=1}^{\infty} c_k \mathbb{D}_1^k$ satisfies (3.2) when $\sigma = \eta = 0$ and $q = 1$. Furthermore, the proper initial condition is also satisfied. The solution of the q-commutation equation is unique, so

$$\mathbb{H}_t = \sum_{k=1}^{\infty} c_k \mathbb{D}_1^k.$$

In view of [24, (5.2)] the proof is finished after relating the coefficients $(\frac{c_k}{k+1})_{k\geqslant 1}$ to the power series expansion of f's. ■

On the other hand, infinitesimal generators for Lévy process are well-known (and do not depend on t). For Lévy–Meixner processes (but not only), the infinitesimal generator can be written as

$$\mathbf{A}_t = \mathbf{A} = \psi(-i\partial_x), \tag{5.15}$$

where ψ is the cumulant generating function of the Lévy process; see e.g. [4, Section 3]. The right-hand side of the above expression should be understood as

$$\sum_{k=1}^{\infty} c_k \partial_x^k,$$

where $\sum_{k=1}^{\infty} c_k(iz)^k$ is a power series expansion of ψ around 0 and ∂_x^k is the kth derivative with respect to x. Consequently, the function f of Theorem 5.3 necessarily satisfies

$$f(z) = \psi(-iz). \tag{5.16}$$

Since the formulas for ψ for quadratic harnesses $QH(0, \theta; 0, \tau; 1)$ are well-known (e.g. one can take logarithms of the expressions in [20, Theorem 4.2] and set $t = 1$ to match the setting of [4]), one can check (5.16) by a direct calculation.

Furthermore, using the definition (2.1) of multiplication, it can be easily verified that the nth coordinate of $f(\mathbb{D}_1)$ is indeed $\mathbf{A}_t(x^n)$ with $\mathbf{A}_t$ given in (5.15).

5.3. Case $q = -1$. In this case the polynomials $(B_n(y; x, t))_{n\geqslant 0}$ are orthogonal with respect to the Dirac measure $\delta_{\theta+\eta(t+\tau)-x}$ concentrated at $\theta + \eta(t + \tau) - x$, because the coefficient of B_{n-2} in the recurrence (1.13) vanishes ($[2n]_q = 0$ for all $n \geqslant 0$). Therefore, Theorem 1.1 and Corollary 4.1 imply that the domain of the infinitesimal generator contains all polynomials and bounded continuous functions g with bounded continuous second derivative, and it takes the form

(5.17)
$$\mathbf{A}_t(g)(x) = \begin{cases} \dfrac{1+\eta x}{2} g''(x) & \text{when } \theta + \eta(t+\tau) = 2x, \\ \dfrac{1+\eta x}{\theta + \eta(t+\tau) - 2x}\left(\dfrac{g(\theta+\eta(t+\tau)-x) - g(x)}{\theta+\eta(t+\tau)-2x} - g'(x)\right) & \\ & \text{when } \theta + \eta(t+\tau) \neq 2x. \end{cases}$$

Moreover, the construction of a bi-Poisson process $QH(\eta, \theta; 0, 0; -1)$ was presented when $1 + \eta\theta \geqslant 0$ in [18, Section 3.2]. In particular, there exists $QH(\eta, \theta + \eta\tau; 0, 0; -1)$ with parameters satisfying (5.6). Surprisingly, tedious calculations show that this process also satisfies (1.4) and (1.7). Therefore, $QH(\eta, \theta + \eta\tau; 0, 0; -1)$ is also $QH(\eta, \theta; 0, \tau; -1)$. Then, by reading off the transition probabilities (see [18, Section 3.2]), it is easy to compute the infinitesimal generator directly and obtain exactly the formula from the second line in (5.17).

6. APPENDIX

6.1. Askey–Wilson polynomials. For $a, b, c, d \in \mathbb{C}$ such that

$$abcd, qabcd \notin [1, \infty) \tag{6.1}$$

and $|q| < 1$, let us define polynomials $(P_n(y))_{n\geqslant 0}$ by the recurrence

$$2yP_n(y) = \widetilde{A}_n P_{n+1}(y) + B_n P_n(y) + \widetilde{C}_n P_{n-1}(y), \quad n \geqslant 0,$$

with the initial conditions $P_{-1} \equiv 0$ and $P_0 \equiv 1$, where for $n \geqslant 0$ we have

$$\begin{aligned}
\widetilde{A}_n &:= \frac{A_n}{(1-abq^n)(1-acq^n)(1-adq^n)},\\
B_n &:= a + \frac{1}{a} - \frac{A_n}{a} - aC_n,\\
\widetilde{C}_n &:= C_n(1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1}),\\
A_n &:= \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{(1-abcdq^{2n-1})(1-abcdq^{2n})},\\
C_n &:= \frac{(1-q^n)(1-bcq^n)(1-bdq^n)(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}.
\end{aligned}$$

Here A_0 and C_0 should be interpreted as $\frac{(1-ab)(1-ac)(1-ad)}{1-abcd}$ and 0, respectively. Then the polynomials $(P_n(y))_{n\geqslant 0}$ are *Askey–Wilson polynomials* (see [39, Section 14.1]). Because the coefficients $\widetilde{A}_n$, B_n, $\widetilde{C}_n$ do not depend on the order of the parameters a, b, c, d, these polynomials are well-defined also for $a = 0$.

Moreover, we can normalize the polynomials $(P_n(y))_{n\geqslant 0}$ by

$$p_n(y) := 2^n \prod_{k=0}^{n-1} \widetilde{A}_k P_n(y), \quad n \geqslant 0,$$

with the convention that $\widetilde{A}_{-1} := 1$, to find that the polynomials $(p_n(y))_{n\geqslant 0}$ satisfy

$$yp_n(y) = p_{n+1}(y) + \tfrac{1}{2}B_n p_n(y) + \tfrac{1}{4}A_{n-1}C_n p_{n-1}(y), \quad n \geqslant 0, \tag{6.2}$$

with $A_{-1} = 1$.

The polynomials $(p_n(y))_{n\geqslant 0}$ satisfy a three-step recurrence, so there exists a moment functional that makes them orthogonal. However, it is difficult to give explicit conditions in terms of a, b, c, and d guaranteeing that the orthogonality measure $\mu_{a,b,c,d}(\mathrm{d}y)$ for the moment functional exists. It is known only in some special cases, so let us present results covering all the problems discussed in Section 5.

Denote

$$\begin{aligned}
m_1 &:= \sharp\big(\{ab, ac, ad, bc, bd, cd\} \cap [1, \infty)\big),\\
m_2 &:= \sharp\big(\{qab, qac, qad, qbc, qbd, qcd\} \cap [1, \infty)\big).
\end{aligned} \tag{6.3}$$

According to [21, Lemma 3.1], if a, b, c, d are either real or come in complex conjugate pairs and satisfy (6.1), then the orthogonality measure $\mu_{a,b,c,d}$ exists only in the following cases:

1. If $q \geqslant 0$ and $m_1 = 0$, then $\mu_{a,b,c,d}$ only has a continuous component.
2. If $q < 0$ and $m_1 = m_2 = 0$, then $\mu_{a,b,c,d}$ only has a continuous component.

3. If $q \geqslant 0$ and $m_1 = 2$, then $\mu_{a,b,c,d}$ is well-defined if either $q = 0$ or the smaller of the two products that fall into $[1, \infty)$ is of the form $1/q^N$, and in the latter case $\mu_{a,b,c,d}$ is a purely discrete measure with $N + 1$ atoms.
4. If $q < 0$ and $m_1 = 2$, $m_2 = 0$, then $\mu_{a,b,c,d}$ is well-defined if the smaller of the two products in $[1, \infty)$ equals $1/q^N$ with N even. Then $\mu_{a,b,c,d}$ is a purely discrete measure with $N + 1$ atoms.
5. If $q < 0$, $m_1 = 0$ and $m_2 = 2$, then $\mu_{a,b,c,d}$ is well-defined if the smaller of the two products in $[1, \infty)$ equals $1/q^N$ with N even. Then $\mu_{a,b,c,d}$ is a purely discrete measure with $N + 2$ atoms.

Introducing for $w, w_1, \ldots, w_k \in \mathbb{C}$ the following notation:

$$(w; q)_n := \begin{cases} 1 & \text{when } n = 0, \\ \prod_{j=0}^{n-1} (1 - wq^j) & \text{when } n = 1, 2, \ldots, \\ \prod_{j=0}^{\infty} (1 - wq^j) & \text{when } n = \infty, \end{cases} \tag{6.4}$$

and

$$(w_1, \ldots, w_k; q)_n := (w_1; q)_n \cdot \ldots \cdot (w_k; q)_n,$$

the probability measure $\mu_{a,b,c,d}$ can be written explicitly as

$$\mu_{a,b,c,d}(\mathrm{d}y) = f_{a,b,c,d}(y) \mathbb{1}_{\{|y|<1\}} \, \mathrm{d}y + \sum_{x \in F_{a,b,c,d}} \rho(x) \delta_x(\mathrm{d}y),$$

where for θ such that $y = \cos(\theta)$ we have

$$f_{a,b,c,d}(y) := \frac{(q, ab, ac, ad, bc, bd, cd; q)_\infty}{2\pi (abcd; q)_\infty \sqrt{1 - y^2}} \left| \frac{(e^{2i\theta}; q)_\infty}{(ae^{2i\theta}, be^{2i\theta}, ce^{2i\theta}, de^{2i\theta}; q)_\infty} \right|^2,$$

and $F_{a,b,c,d}$ is an empty or finite set of atoms that arise from the respective parameters a, b, c, d with an absolute value larger than 1. For example, if $a \in (-\infty, -1) \cup (1, \infty)$, then the corresponding atoms are

$$x_k = \frac{aq^k + (aq^k)^{-1}}{2}$$

for $k = 0, 1, \ldots$ such that $|aq^k| > 1$. The probabilities of x_k are then equal to

$$\rho(x_0) := \frac{(1/a^2, bc, bd, cd; q)_\infty}{(b/a, c/a, d/a, abcd; q)_\infty},$$

$$\rho(x_k) := \rho(x_0) \frac{(a^2, ab, ac, ad; q)_k (1 - a^2 q^{2k})}{(q, qa/b, qa/c, qa/d; q)_k (1 - a^2)} \left(\frac{q}{abcd} \right)^k, \quad k \geqslant 1.$$

The above formula has to be rewritten when $abcd = 0$. Especially, when $d = 0$ we have

$$\rho(x_k) = \rho(x_0)\frac{(a^2, ab, ac; q)_k(1 - a^2q^{2k})}{(q, qa/b, qa/c; q)_k(1 - a^2)}(-1)^k q^{-\binom{k}{2}}\left(\frac{1}{a^2bc}\right)^k, \quad k \geqslant 1,$$

where, by convention, we put $\binom{1}{2} = 0$.

6.2. Big q-Jacobi polynomials. Let us consider polynomials $(p_n(x))_{n\geqslant 0}$ given by a recurrence

$$xp_n(x) = p_{n+1}(x) + (1-(A_n+C_n))p_n(x) + A_{n-1}C_np_{n-1}(x), \quad n \geqslant 0, \tag{6.5}$$

with the initial conditions $p_{-1} \equiv 0$ and $p_0 \equiv 1$, where for $n \geqslant 0$ we have

$$A_n := \frac{(1 - aq^{n+1})(1 - abq^{n+1})(1 - cq^{n+1})}{(1 - abq^{2n+1})(1 - abq^{2n+2})},$$
$$C_n := -acq^{n+1}\frac{(1 - q^n)(1 - abc^{-1}q^n)(1 - bq^n)}{(1 - abq^{2n})(1 - abq^{2n+1})}.$$

Then the polynomials $(p_n(y))_{n\geqslant 0}$ are normalized *big q-Jacobi polynomials* (see [39, Section 14.5]).

The orthogonality relation for $0 < aq < 1$, $0 \leqslant bq < 1$ and $c < 0$ is

$$\int_{cq}^{aq} w(x)p_m(x)p_n(x)\,\mathrm{d}_q(x) = h_n\delta_{mn},$$

where

$$w(x) := \frac{(a^{-1}x, c^{-1}x; q)_\infty}{(x, bc^{-1}x; q)_\infty},$$
$$h_n = aq(1 - q)\frac{(q, abq^2, a^{-1}c, ac^{-1}q; q)_\infty}{(aq, bq, cq, abc^{-1}q; q)_\infty}\frac{(1 - abq)}{(1 - abq^{2n+1})}$$
$$\times \frac{(q, aq, bq, cq, abc^{-1}q; q)_n}{(abq, abq^{n+1}, abq^{n+1}; q)_n}(-acq^2)^n q^{\binom{n}{2}}$$

and

$$\int_{cq}^{aq} f(x)\,\mathrm{d}_q(x) := aq(1 - q)\sum_{k=0}^{\infty} f(aq^{k+1})q^k - cq(1 - q)\sum_{k=0}^{\infty} f(cq^{k+1})q^k. \tag{6.6}$$

Above, we have used the notation introduced in (6.4). More information about big q-Jacobi polynomials can be found in [39, Section 14.5]. In particular, it turns out that they can be obtained as a limit of specially reparameterized Askey–Wilson polynomials.

6.3. Little q-Jacobi polynomials. Substituting cqx for x in $(p_n(x))_{n\geqslant 0}$ given in (6.5) and letting $c \to -\infty$ leads to the *little q-Jacobi polynomials* which, after normalization, satisfy the following recurrence:

$$xw_n(x) = w_{n+1}(x) + (\widetilde{A}_n + \widetilde{C}_n)w_n(x) + \widetilde{A}_{n-1}\widetilde{C}_n w_{n-1}(x), \qquad n \geqslant 0, \tag{6.7}$$

with $w_{-1} \equiv 0$ and $w_0 \equiv 1$, where

$$\widetilde{A}_n := q^n \frac{(1-aq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})},$$
$$\widetilde{C}_n := aq^n \frac{(1-q^n)(1-bq^n)}{(1-abq^{2n})(1-abq^{2n+1})}.$$

In this case, the orthogonality relation takes the form

$$\sum_{k=0}^{\infty} \frac{(bq;q)_k}{(q,q)_k}(aq)^k p_n(q^k)p_m(q^k)$$
$$= \frac{(abq^2;q)_\infty}{(ab;q)_\infty} \frac{(1-abq)(aq^n)^n}{(1-abq^{2n+1})} \frac{(q,aq,bq;q)_n}{(abq,abq^{n+1},abq^{n+1};q)_n}\delta_{nm}$$

for $0 < aq < 1$ and $bq < 1$. For more information on the little q-Jacobi polynomials, see [39, Section 14.12].

6.4. Al-Salam–Carlitz I polynomials. Let $a \in \mathbb{R}$. We consider the polynomials $(p_n(y))_{n\geqslant 0}$ given by the following three-step recurrence:

$$xp_n(x) = p_{n+1}(x) + (a+1)q^n p_n(x) - aq^{n-1}(1-q^n)p_{n-1}(x), \quad n \geqslant 0, \tag{6.8}$$

with $p_{-1} \equiv 0$ and $p_0 \equiv 1$. The polynomials $(p_n(y))_{n\geqslant 0}$ are called *Al-Salam–Carlitz I polynomials* (see [39, Section 14.24]).

For $a < 0$, these polynomials are orthogonal and satisfy

$$\int_a^1 (qx, a^{-1}qx;q)_\infty p_n(x)p_m(x)\,\mathrm{d}_q(x)$$
$$= (-a)^n(1-q)(q;q)_n(q,a,a^{-1}q;q)_\infty q^{\binom{n}{2}}\delta_{mn},$$

where we use the notation introduced in (6.4) and (6.6).

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