PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 44, Fasc. 1 (2024), pp. 119–132
Published online 20.9.2024
doi:10.37190/0208-4147.00156

BETA DISTRIBUTION AND ASSOCIATED STIRLING NUMBERS OF THE SECOND KIND

BY

JAKUB GISMATULLIN* (WROCŁAW) AND
PATRICK TARDIVEL** (WROCŁAW AND DIJON)

Abstract. This article gives a formula for associated Stirling numbers of the second kind based on the moment of a sum of independent random variables having a beta distribution. From this formula we deduce lower and upper bounds for these numbers, using a probabilistic approach.

2020 Mathematics Subject Classification: Primary 05A40; Secondary 44A60.

Key words and phrases: beta distribution, associated Stirling numbers.

1. INTRODUCTION

The classical Stirling numbers of the second kind $S(p,m)$ count the number of partitions of $\{1,\dots,p\}$ into m nonempty subsets, for $p\in\mathbb{N}_{>0}$ and $m\in\mathbb{N}$. More generally, the r-associated Stirling number $S_r(p,m)$, with $r\in\mathbb{N}_{>0}$, is the number of partitions of $\{1,\dots,p\}$ into m subsets where each subset contains at least r elements [3, p. 221]. Obviously $S(p,m)=S_1(p,m)$. Some subsequences of the multi-sequence $\{S_r(p,m) : p,m,r\in\mathbb{N}_{>0},\ p\geqslant rm\}$ appear in the On-Line Encyclopedia of Integer Sequences (OEIS) [11]. Specifically, the arrays $\{S_1(p,m)\}_{\{p,m\}}$, $\{S_2(p,m)\}_{\{p,m\}}$, $\{S_3(p,m)\}_{\{p,m\}}$ appear as A008277, A008299, A059022. Moreover, the sequences $\{S_2(k{+}6,3)\}_{\{k\}}$, $\{S_2(k{+}8,4)\}_{\{k\}}$, representing the number of ways of placing $k+6$ or $k+8$ labelled balls into 3 or 4 indistinguishable boxes with at least 2 balls in each box appear in the OEIS as A000478, A058844.

There are well-known connections between Stirling numbers of the second kind and probability theory. For example, the sequences $S_1(p,m)$ and $S_2(p,m)$

---

* Jakub Gismatullin is supported by The National Science Centre, Poland, NCN grants no. 2014/13/D/ST1/03491 and 2017/27/B/ST1/01467.

** The Institut de Mathématiques de Bourgogne (IMB) receives support from the EIPHI Graduate School (contract ANR-17-EURE-0002). Patrick Tardivel receives support from the region Bourgogne-Franche-Comté (EPADM project).

are asymptotically normal when p tends to $+\infty$ [6, 4]. More precisely, when $r \in \{1, 2\}$, the following convergence in distribution holds:

$$\frac{Y_p - \mathbb{E}(Y_p)}{\sqrt{\text{var}(Y_p)}} \xrightarrow[p \to +\infty]{d} \mathcal{N}(0, 1)$$

where

$$\mathbb{P}(Y_p = m) = \frac{S_r(p, m)}{\sum_{k=1}^{p} S_r(p, k)} \quad \text{for all } m \in \mathbb{N}_{>0}.$$

Furthermore, according to Dobiński's formula, the moment of order p of a Poisson distribution with parameter $\lambda \geqslant 0$ is $\sum_{m=1}^{p} S_1(p, m)\lambda^m$ (see, e.g., [3, p. 211]). However, to our knowledge, there is no closed formula in the literature for $S_r(k, m)$ based on moments of a sum of independent and identically distributed (i.i.d.) random variables. The main result in this article is Theorem 2.1 providing the following new identity:

$$S_r(p, m) = \frac{p!}{m!(r!)^m (p - rm)!} \mathbb{E}[(X_1 + \cdots + X_m)^{p-rm}], \tag{1.1}$$

where $X_1, \ldots, X_p$ are i.i.d. random variables having a beta distribution with parameter $(1, r)$. Note that a beta distribution with parameter $(1, 1)$ is a uniform distribution on $[0, 1]$. Thus, when $r = 1$, the above formula is quite simple:

$$S_1(p, m) = \binom{p}{m} \mathbb{E}(Z^{p-m})$$

where $Z = \sum_{i=1}^{m} X_i$ has an Irwin–Hall distribution on $[0, m]$. Propositions 3.1–3.3 give upper and lower bounds for $\mathbb{E}[(X_1 + \cdots + X_m)^{p-rm}]$. These bounds are sharp when m, r, or $p - rm$ tends to $+\infty$, and thus provide accurate approximations of r-associated Stirling numbers.

2. CLOSED FORMULA FOR STIRLING NUMBERS AND MOMENTS OF RANDOM VARIABLES

The density g_r of a beta distribution with parameters $(1, r)$ where $r \in \mathbb{N}_{>0}$ is

$$g_r(x) = \begin{cases} r(1 - x)^{r-1} & \text{if } x \in [0, 1], \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Let $X_1, \ldots, X_m$ be independent random variables having the same beta $(1, r)$ distribution. Consider the moment of order $k \in \mathbb{N}$ of the sum of these variables:

$$\mathcal{M}_r(k, m) = \mathbb{E}[(X_1 + \cdots + X_m)^k]. \tag{2.2}$$

Theorem 2.1 provides a closed formula for the Stirling numbers of the second kind involving $\mathcal{M}_r(k, m)$.

THEOREM 2.1. *Let $m, r \in \mathbb{N}_{>0}$ and $p \in \mathbb{N}$ where $p \geqslant rm$. The Stirling numbers of the second kind satisfy the identity*

$$S_r(p,m) = \frac{p!}{m!(r!)^m(p-rm)!}\mathcal{M}_r(p-rm,m). \tag{2.3}$$

From Theorem 2.1, one may deduce that $\mathbb{E}(Z^k) = S_1(m+k,m)/\binom{m+k}{m}$ where Z has an Irwin–Hall distribution on $[0,m]$ [8, 7]. Note that the moment generating function of Z is $\sum_{k\geqslant 0}\mathbb{E}(Z^k)t^k/k! = ((\exp(t)-1)/t)^m$, and therefore we recover the well-known exponential generating function of the Stirling numbers of the second kind $\sum_{p\geqslant m} S_1(p,m)t^p/p! = (\exp(t)-1)^m/m!$ (see [9, Theorem 3.3, p. 52]). The above expression of $S_r(p,m)$ is explicit up to the computation of $\mathcal{M}_r(k,m)$. Whereas computing it explicitly might be technically complicated, lower bounds, upper bounds and approximations of $\mathcal{M}_r(k,m)$ are tractable, as illustrated in the following section.

3. UPPER AND LOWER BOUNDS

We will use a probabilistic approach to derive upper and lower bounds for the moment $\mathcal{M}_r(k,m)$.

3.1. Sharp upper and lower bounds when m is large. Let $\overline{X}_m = (X_1 + \cdots + X_m)/m$. Jensen's inequality provides the following lower bound:

$$\mathcal{M}_r(k,m) = m^k\mathbb{E}(\overline{X}_m^k) \geqslant m^k\mathbb{E}(\overline{X}_m)^k = \frac{m^k}{(r+1)^k}. \tag{3.1}$$

This inequality relies on the linearization of the function $q(x) = x^k$ at $x_0 = \mathbb{E}(\overline{X}_m) = \frac{1}{r+1}$. Specifically, the following inequality holds for all $x \in [0,1]$:

$$\begin{aligned} x^k &= q(x) \geqslant q(x_0) + q'(x_0)(x-x_0) \\ &= \frac{1}{(r+1)^k} + \frac{k}{(r+1)^{k-1}}\left(x - \frac{1}{r+1}\right). \end{aligned} \tag{3.2}$$

Moreover, one may choose $c \geqslant 0$ for which the following inequality is true for all $x \in [0,1]$ (see Lemma 5.2):

$$x^k \leqslant q(x_0) + q'(x_0)(x-x_0) + c(x-x_0)^2. \tag{3.3}$$

Proposition 3.1 below is a consequence of inequalities (3.2) and (3.3).

PROPOSITION 3.1. *Let $r, m \in \mathbb{N}_{>0}$ and $k \in \mathbb{N}$. Then*

$$\frac{m^k}{(r+1)^k} \leqslant \mathcal{M}_r(k,m) \leqslant \frac{m^k}{(r+1)^k} + m^{k-1}\frac{(r+1)^k - 1 - kr}{(r+1)^k r(r+2)}. \tag{3.4}$$

When m is large, the leading term in both the lower and upper bounds is $m^k/(r+1)^k$. Therefore, the lower and upper bounds are asymptotically equivalent when $k \in \mathbb{N}_{\geqslant 0}$ and $r \in \mathbb{N}_{>0}$ are fixed and m tends to $+\infty$. These bounds are sharp when m is large since $\overline{X}_m$ converges to $\mathbb{E}(\overline{X}_m)$ and both (3.2) and (3.3) are sharp on the neighbourhood of $\mathbb{E}(\overline{X}_m)$.

3.2. Sharp upper and lower bounds when k is large. The asymptotic behaviour of moments when k is large depends on the density of $X_1 + \cdots + X_m$ on the tail, i.e. on the neighbourhood of m. This motivates us to prove the following inequality in Corollary 5.1:

$$g_r^{*m}(x) \leqslant \frac{(r!)^m}{(mr-1)!}(m-x)^{mr-1} \quad \text{for all } x \in [0, m], \tag{3.5}$$

where g_r is given by (2.1) and g_r^{*m} is the mth convolution power of g_r. Moreover, this inequality is an equality for $x \in [m-1, m]$. From this fact we derive lower and upper bounds for $\mathcal{M}_r(k, m)$ in the proposition below.

PROPOSITION 3.2. *For any $r \in \mathbb{N}_{>0}$, any $k \in \mathbb{N}$ and any $m \in \mathbb{N}_{>0}$,*

$$\mathcal{M}_r(k, m) \leqslant \frac{(r!)^m}{(mr-1)!} \int_0^m x^k (m-x)^{mr-1}\, dx = \frac{k!(r!)^m m^{k+rm}}{(k+mr)!},$$

$$\mathcal{M}_r(k, m) \geqslant \frac{(r!)^m}{(mr-1)!} \int_{m-1}^m x^k (m-x)^{mr-1}\, dx$$

$$\geqslant \frac{k!(r!)^m m^{k+mr}}{(k+mr)!}\left(1 - \frac{(m-1)^k}{m^{k+mr}} \sum_{i=1}^{mr} \binom{k+mr}{k+i} (m-1)^i\right).$$

These bounds are sharp when k is large since

$$\lim_{k \to +\infty} \frac{(m-1)^k}{m^{k+mr}} \sum_{i=1}^{mr} \binom{k+mr}{k+i} (m-1)^i = 0.$$

As a consequence of Proposition 3.2, we observe that $S_r(p, m) \leqslant m^p/m!$. Moreover, the lower and upper bounds are asymptotically equivalent when k tends to $+\infty$, and therefore $S_r(p, m) \sim m^p/m!$ when p is large. This approximation, well known when $r = 1$ (see [2]), remains true for $r > 1$.

3.3. Sharp upper bound when r is large. Proposition 3.3 below proves that the moment $\mathcal{M}_r(k, m)$ is bounded, up to an explicit expression, by the moment of a sum of independent random variables having the same standard exponential distribution.

PROPOSITION 3.3. *Let $k \in \mathbb{N}$, $m \in \mathbb{N}_{>0}$, $r \in \mathbb{N}_{>1}$ and let $\mathcal{E}_1, \ldots, \mathcal{E}_m$ be i.i.d. random variables having the standard exponential distribution with density* $\exp(-x)$.

(i) *The following inequality holds:*

$$r^k \mathcal{M}_r(k,m) \leqslant \left(\frac{r}{r-1}\right)^{2k} \mathbb{E}(\mathcal{E}_1 + \cdots + \mathcal{E}_m)^k = \left(\frac{r}{r-1}\right)^{2k} \frac{(m-1+k)!}{(m-1)!}. \tag{3.6}$$

(ii) *The upper bound in* (i) *is sharp since*

$$\lim_{r\to+\infty} r^k \mathcal{M}_r(k,m) = \mathbb{E}(\mathcal{E}_1 + \cdots + \mathcal{E}_m)^k = \frac{(m-1+k)!}{(m-1)!}. \tag{3.7}$$

It seems difficult to find lower bounds which are sharp when r tends to $+\infty$. Finally, we recap the lower and upper bounds for r-associated Stirling numbers of the second kind:

- Proposition 3.1 provides the following lower and upper bounds:

$$\begin{cases} S_r(p,m) \geqslant \dfrac{p!m^{p-rm}}{m!(r!)^m(p-rm)!(r+1)^{p-rm}}, \\ S_r(p,m) \leqslant \dfrac{p!m^{p-rm}}{m!(r!)^m(p-rm)!(r+1)^{p-rm}} \\ \qquad \times \left(1 + \dfrac{(r+1)^{p-rm} - 1 - r(p-rm)}{mr(r+2)}\right). \end{cases} \tag{3.8}$$

  These bounds are equivalent when $p - rm$ and r are fixed and when m tends to $+\infty$.

- Proposition 3.2 provides the following lower and upper bounds:

$$\begin{cases} S_r(p,m) \geqslant \dfrac{m^p}{m!} - \dfrac{(m-1)^{p-rm}}{m!} \displaystyle\sum_{i=1}^{mr} \binom{p}{p-rm+i} (m-1)^i, \\ S_r(p,m) \leqslant \dfrac{m^p}{m!}. \end{cases} \tag{3.9}$$

  These bounds are equivalent when m, r are fixed and when p tends to $+\infty$.

- Proposition 3.3 provides the following upper bound when $r \geqslant 2$:

$$S_r(p,m) \leqslant \frac{p!r^{2(p-rm)}(m-1+p-rm)!}{m!(r!)^m(p-rm)!(r-1)^{2(p-rm)}(m-1)!}. \tag{3.10}$$

  This upper bound is equivalent to $S_r(p,m)$ when $p - rm$ and m are fixed and when r tends to $+\infty$.

4. NUMERICAL EXPERIMENTS

4.1. Upper and lower bounds of Stirling numbers of the second kind. According to Propositions 3.1 and 3.2, for all $m \in \mathbb{N}$ and all $p \in \mathbb{N}_{>0}$, Stirling numbers of the second kind satisfy the following inequalities:

$$S_1(p,m) \leqslant \underbrace{\min\left\{\frac{m^p}{m!}, \binom{p}{m}\left(\frac{m}{2}\right)^{p-m}\left(1+\frac{2^{p-m}+m-p-1}{3m}\right)\right\}}_{U(p,m)}$$

$$S_1(p,m) \geqslant \underbrace{\max\left\{\frac{m^p}{m!} - \frac{(m-1)^{p-m}}{m!}\sum_{i=1}^{m}\binom{p}{m-i}(m-1)^i, \binom{p}{m}\left(\frac{m}{2}\right)^{p-m}\right\}}_{L(p,m)}.$$

First of all we are going to compare the bounds $U(m,p)$ and $L(p,m)$ to the bounds given by Rennie and Dobson [10]:

$$\underbrace{\frac{1}{2}(m^2+m+2)m^{p-m-1}-1}_{L_{\mathrm{rd}}(p,m)} \leqslant S_1(p,m) \leqslant \underbrace{\frac{1}{2}\binom{p}{m}m^{p-m}}_{U_{\mathrm{rd}}(p,m)}. \tag{4.1}$$

Numerical comparison between $U(p,m)$ and $U_{\mathrm{rd}}(p,m)$ is not needed since clearly

$$\frac{1}{2}\binom{p}{m}m^{p-m} \geqslant \binom{p}{m}\left(\frac{m}{2}\right)^{p-m}\left(1+\frac{2^{p-m}+m-p-1}{3m}\right)$$

for $m < p$. Unlike the upper bound, the lower bound $L(p,m)$ is not uniformly larger than the one given by Rennie and Dobson; for instance, $31 = L_{\mathrm{rd}}(6,2) > L(6,2) = 28.5$. Numerical experiments in Figure 1 illustrate that for most integers p, m, $L(p,m)$ is a better approximation of $S_1(p,m)$ than $L_{\mathrm{rd}}(p,m)$.

Figure 2 provides a comparison between $L(p,m), U(p,m)$ and $S_1(p,m)$.

4.2. Upper bounds of Bell numbers. The Bell number $B(p)$, where $p \in \mathbb{N}_{>0}$, represents the number of partitions of $\{1,\ldots,p\}$. Since the Bell number is a sum of Stirling numbers of the second kind, $B(p) = \sum_{m=1}^{p} S_1(p,m)$, the following inequality holds:

$$B(p) \leqslant \underbrace{\sum_{m=0}^{p}\min\left\{\frac{m^p}{m!}, \binom{p}{m}\left(\frac{m}{2}\right)^{p-m}\left(1+\frac{2^{p-m}+m-p-1}{3m}\right)\right\}}_{=U(p)} \tag{4.2}$$

for all $p \in \mathbb{N}_{>0}$. In Figure 3 we compare $U(p)$ with the upper bound $B(p) \leqslant U_{\mathrm{bt}}(p) = \left(\frac{0.792p}{\ln(p+1)}\right)^p$ given by Berend and Tassa [1].

Note that $U(p) \leqslant \sum_{m=0}^{+\infty} m^p/m! = eB(p)$ (the last equality is due to the Dobiński formula). In Figure 4 we show that $U(p)/B(p)$ is very close to e when p is large.

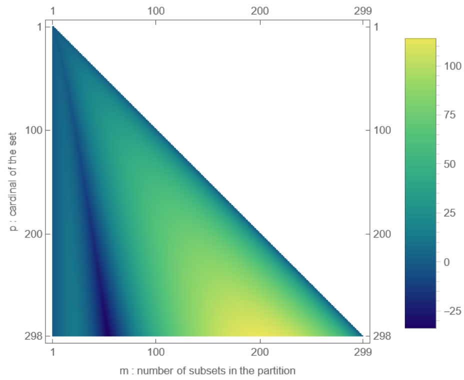

FIGURE 1. $\ln(L(p,m)) - \ln(L_{\rm rd}(p,m))$ as a function of m (on the x-axis) and p (on the y-axis). For most integers the lower bound $L(p,m)$ is a better approximation of $S_1(p,m)$ than $L_{\rm rd}(p,m)$ (as $\ln(L(p,m)) - \ln(L_{\rm rd}(p,m)) > 0$).

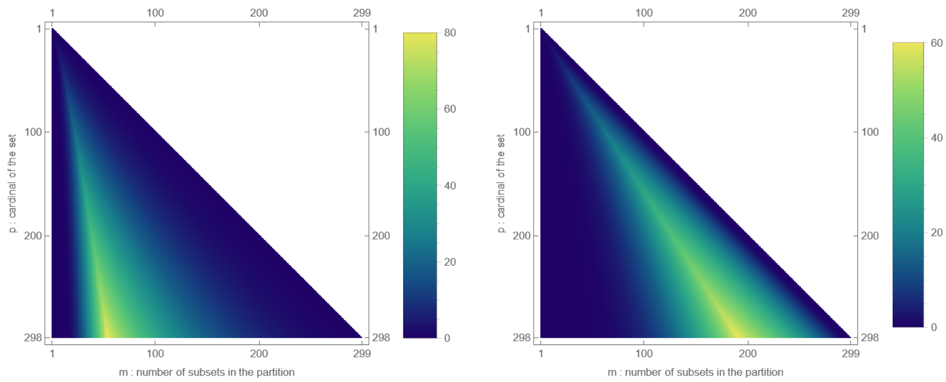

FIGURE 2. $\ln(S_1(p,m)) - \ln(L(p,m))$ (left) and $\ln(U(p,m)) - \ln(S_1(p,m))$ (right) as a function of m and p. These numerical experiments comply with Propositions 3.1 and 3.2 since both lower and upper bounds sharply approximate $S_1(p,m)$ when p is large and m is small or when m is large and $p - m$ is small.

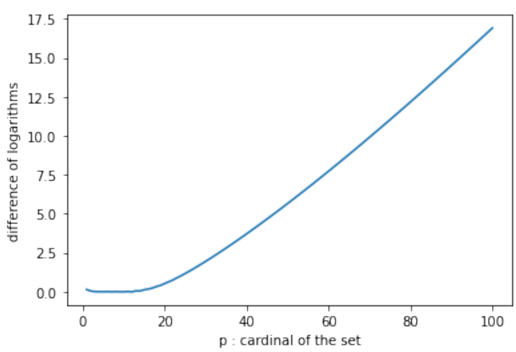

FIGURE 3. $\ln(U_{\rm bt}(p)) - \ln(U(p))$ as a function of p. When $p \geqslant 13$, $U(p)$ is a more accurate upper bound for $S_1(p,m)$ than $U_{\rm bt}(p)$ (as $\ln(U_{\rm bt}(p)) - \ln(U(p)) > 0$ for $p \geqslant 13$).

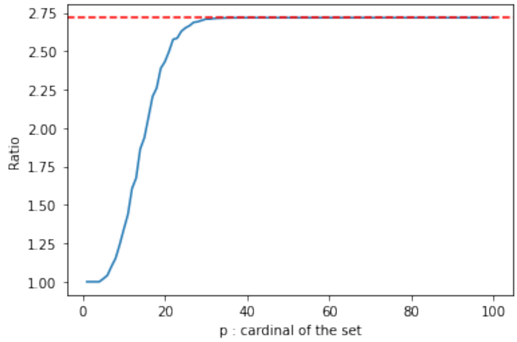

FIGURE 4. $U(p)/B(p)$ as a function of p. One may observe that $U(p)/B(p)$ is approximately equal to e when p is large.

5. PROOFS

5.1. Proof of Theorem 2.1. The identity given in Lemma 5.1 below, combined with the multinomial formula, allows us to complete the proof of Theorem 2.1.

LEMMA 5.1. *Let $m, r \in \mathbb{N}_{>0}$ and $p \in \mathbb{N}$ with $p \geqslant rm$. The r-associated Stirling numbers of the second kind satisfy the equality*

$$S_r(p,m) = \frac{p!}{m!} \sum_{i_1+\cdots+i_m=p-rm} \frac{1}{(r+i_1)!\cdots(r+i_m)!},$$

where the sum is taken over all the integers $i_1,\dots,i_m \in \{0,\dots,p-rm\}$ satisfying $i_1+\cdots+i_m = p-rm$.

Proof. Given $i_1,\dots,i_m \in \mathbb{N}$ such that $i_1+\cdots+i_m = p-rm$, let us count the number of ordered partitions of $\{1,\dots,p\}$ into m parts where the first part has $r+i_1$ elements, the second part has $r+i_2$ elements and so on.

There are $\binom{p}{r+i_1}$ possibilities for the first part, $\binom{p-r-i_1}{r+i_2}$ possibilities for the second part and so on. Therefore the relevant number is

$$\frac{p!}{(r+i_1)!\cdots(r+i_m)!}.$$

Consequently, the number of ordered partitions of $\{1,\dots,p\}$ into m parts having at least r elements is

$$\sum_{i_1+\cdots+i_m=p-rm} \frac{p!}{(r+i_1)!\cdots(r+i_m)!}.$$

Finally, when the order is not taken into account, by dividing by $m!$, one may deduce that

$$S_r(p,m) = \frac{p!}{m!} \sum_{i_1+\cdots+i_m=p-rm} \frac{1}{(r+i_1)!\cdots(r+i_m)!}. \qquad \blacksquare$$

Proof of Theorem 2.1. Let us recall the multinomial formula: for $x_1, \ldots, x_m$ in $\mathbb{R}$ and $k \in \mathbb{N}$,

$$(x_1 + \cdots + x_m)^k = \sum_{i_1+\cdots+i_m=k} \frac{k!}{i_1! \cdots i_m!} x_1^{i_1} \cdots x_m^{i_m}.$$

Let $k = p - rm$. Since $\mathbb{E}(X_1^s) = \frac{s!r!}{(s+r)!}$, the multinomial formula and Lemma 5.1 give

$$\begin{aligned}\mathbb{E}[(X_1 + \cdots + X_m)^k] &= \sum_{i_1+\cdots+i_m=k} \frac{k!}{i_1! \cdots i_m!} \mathbb{E}(X_1^{i_1}) \cdots \mathbb{E}(X_m^{i_m}) \\ &= (r!)^m k! \sum_{i_1+\cdots+i_m=k} \frac{1}{(r+i_1)! \cdots (r+i_m)!} \\ &= \frac{m!(r!)^m k!}{(k+rm)!} S_r(k+rm, m) \\ &= \frac{m!(r!)^m (p-rm)!}{p!} S_r(p, m),\end{aligned}$$

which finishes the proof. ■

5.2. Proof of Proposition 3.1. Proposition 3.1 is a consequence of the following lemma.

LEMMA 5.2. *Let $k \geqslant 2$, $a \in (0,1)$ and $f\colon [0,1] \ni x \to a^k + ka^{k-1}(x-a) + c(x-a)^2$ where $c \geqslant 0$ is such that $f(1) = 1$ (namely $c = \frac{a^{k-1}(ak-a-k)+1}{(1-a)^2}$). Then $f(x) \geqslant x^k$ for all $x \in [0,1]$.*

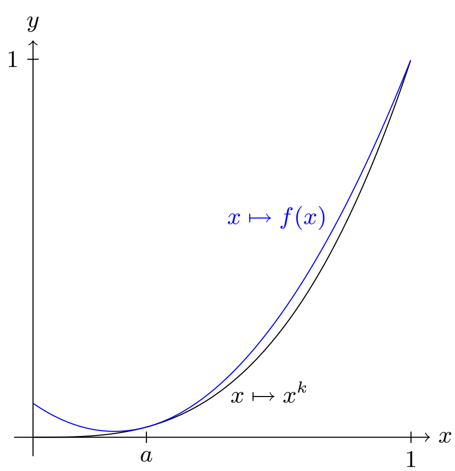

FIGURE 5. Illustration of the inequality given in Lemma 5.2.

Proof. First note that for all $x \in [0,1]$ the condition $f(x) \geqslant x^k$ is equivalent to

$$ka^{k-1}(x-a) + c(x-a)^2 \geqslant x^k - a^k = (x-a)(x^{k-1} + ax^{k-2} + \cdots + a^{k-1}).$$

Note that this inequality holds if and only if

$$\begin{aligned} ka^{k-1} + c(x-a) &\geqslant x^{k-1} + ax^{k-2} + \cdots + a^{k-1} \quad \text{for all } x \in [a,1],\\ ka^{k-1} + c(x-a) &\leqslant x^{k-1} + ax^{k-2} + \cdots + a^{k-1} \quad \text{for all } x \in [0,a]. \end{aligned}$$

Let $d(x) = ka^{k-1} + c(x-a)$ and $p(x) = x^{k-1} + ax^{k-2} + \cdots + a^{k-1}$. Then $p(a) = d(a)$ and $p(1) = d(1)$, by construction of c. Because p is convex and d is affine, one may deduce that $d(x) \leqslant p(x)$ if $x \in [0,a]$, while $d(x) \geqslant p(x)$ if $x \in [a,1]$, which completes the proof. ∎

Proof of Proposition 3.1. By Lemma 5.2, for $a = \mathbb{E}(\overline{X}_m) = \frac{1}{r+1}$, $r > 0$, for all $x \in [0,1]$ we get

$$\begin{aligned} x^k &\leqslant \mathbb{E}(\overline{X}_m)^k + k\mathbb{E}(\overline{X}_m)^{k-1}(x - \mathbb{E}(\overline{X}_m)) + c(x - \mathbb{E}(\overline{X}_m))^2\\ &\leqslant \frac{1}{(r+1)^k} + \frac{k}{(r+1)^{k-1}}\Big(x - \frac{1}{r+1}\Big) + c\Big(x - \frac{1}{r+1}\Big)^2 \end{aligned}$$

where $c = (1+\frac{1}{r})^2\big(1 - \frac{kr+1}{(r+1)^k}\big)$. This inequality implies that

$$\begin{aligned} \mathcal{M}_r(k,m) = m^k\mathbb{E}(\overline{X}_m^k) &\leqslant m^k\big(\mathbb{E}(\overline{X}_m)^k + c\operatorname{var}(\overline{X}_m)\big)\\ &\leqslant \frac{m^k}{(1+r)^k} + c\frac{rm^{k-1}}{(1+r)^2(2+r)}\\ &\leqslant \frac{m^k}{(r+1)^k} + \frac{(r+1)^k - 1 - kr}{(r+1)^k r(r+2)}m^{k-1}. \quad ∎ \end{aligned}$$

5.3. Proof of Proposition 3.2. We use a well-known beta integral (see, for example, [5]): if $a, b \in \mathbb{N}$ and $x \in \mathbb{R}_{>0}$ then

$$\int_0^x (x-t)^a t^b\,dt = \frac{a!b!}{(a+b+1)!}x^{a+b+1}. \tag{5.1}$$

To compute the density of $X_1 + \cdots + X_m$ on the tail $[m-1, m]$ explicitly, we use the following technical lemma. Let

$$h_r(x) = \begin{cases} rx^{r-1} & \text{if } x \in [0,1],\\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

Then h_r is the density of $1-X$ when the density of X is the g_r of (2.1). Convolution computations are slightly easier to handle with h_r than g_r.

LEMMA 5.3. *Let* $m \in \mathbb{N}_{>0}$. *Then*

$$h_r^{*m}(x) = \frac{(r!)^m}{(mr-1)!} x^{mr-1} \quad \textit{for all } x \in [0,1], \tag{5.3}$$

$$h_r^{*m}(x) \leqslant \frac{(r!)^m}{(mr-1)!} x^{mr-1} \quad \textit{for all } x \in \mathbb{R}_{\geqslant 0}. \tag{5.4}$$

Proof. We prove (5.3) by induction. When $m = 1$, we notice that $h_r^{*m}(x) = h_r(x)$ for all $x \in [0,1]$. Let $m \in \mathbb{N}_{>0}$ be such that

$$h_r^{*m}(x) = \frac{(r!)^m}{(mr-1)!} x^{mr-1} \quad \forall x \in [0,1].$$

Then for $x \in [0,1]$ we have

$$\begin{aligned} h_r^{*m+1}(x) &= \int_{\mathbb{R}} h_r^{*m}(x-t) h_r(t)\, dt = \int_0^x h_r^{*m}(x-t) r t^{r-1}\, dt \\ &= \frac{r(r!)^m}{(mr-1)!} \int_0^x (x-t)^{mr-1} t^{r-1}\, dt \\ &= \frac{r(r!)^m}{(mr-1)!} \frac{(mr-1)!(r-1)!}{((m+1)r-1)!} x^{(m+1)r-1} \\ &= \frac{(r!)^{m+1}}{((m+1)r-1)!} x^{(m+1)r-1}. \end{aligned}$$

The proof of (5.4) by induction is quite similar. When $m = 1$, the result is straightforward. Let $m \in \mathbb{N}_{>0}$ be such that

$$h_r^{*m}(x) \leqslant \frac{(r!)^m}{(mr-1)!} x^{mr-1} \quad \forall x \in \mathbb{R}_{\geqslant 0}.$$

Then for $x \in \mathbb{R}_{\geqslant 0}$ we have

$$\begin{aligned} h_r^{*m+1}(x) &= \int_{\mathbb{R}} h_r^{*m}(x-t) h_r(t)\, dt = \int_0^x h_r^{*m}(x-t) h_r(t)\, dt \\ &\leqslant \frac{r(r!)^m}{(mr-1)!} \int_0^x (x-t)^{mr-1} t^{r-1}\, dt \\ &\leqslant \frac{r(r!)^m}{(mr-1)!} \frac{(mr-1)!(r-1)!}{((m+1)r-1)!} x^{(m+1)r-1} \\ &\leqslant \frac{(r!)^{m+1}}{((m+1)r-1)!} x^{(m+1)r-1}. \quad \blacksquare \end{aligned}$$

Note that the only difference between the proofs of (5.3) and (5.4) is the majorization of $h_r(t)$ by rt^{r-1}. We are now ready to prove the explicit formula for the mth convolution power g_r^{*m} on $[m-1, m]$ and an upper bound on $[0, m]$.

COROLLARY 5.1. *For all $x \in [0, m]$ we have*

$$g_r^{*m}(x) \leqslant \frac{(r!)^m}{(mr-1)!}(m-x)^{mr-1}.$$

Moreover, if $x \in [m-1, m]$ then

$$g_r^{*m}(x) = \frac{(r!)^m}{(mr-1)!}(m-x)^{mr-1}.$$

Proof. By Lemma 5.3, it suffices to prove that $g_r^{*m}(x) = h_r^{*m}(m-x)$ for all $x \in [0, m]$. Let $x \in [0, m]$ and set $z = m - x$. Since the density of $(1 - X_1) + \cdots + (1 - X_m)$ is h_r^{*m} , we have

$$\begin{aligned}\frac{\partial}{\partial z}\mathbb{P}((1-X_1)+\cdots+(1-X_m) \leqslant z) &= h_r^{*m}(z)\\ \frac{\partial}{\partial z}\mathbb{P}(X_1+\cdots+X_m \geqslant m-z) &= h_r^{*m}(z)\\ g_r^{*m}(m-z) &= h_r^{*m}(z)\\ g_r^{*m}(x) &= h_r^{*m}(m-x). \quad \blacksquare\end{aligned}$$

Notice that if $r = 1$ and $x \in [m-1, m]$, then $g_r^{*m}(x) = (m-x)^{m-1}/(m-1)!$, which is the density in the tail of the Irvin–Hall distribution (see [7, 8]).

The upper and lower bounds given in Proposition 3.2 are straightforward consequences of Corollary 5.1. Indeed, the upper bound is just the beta integral

$$\int_0^m \frac{(r!)^m}{(mr-1)!}(m-x)^{mr-1}x^k\,dx.$$

Proof of Proposition 3.2. The lower bound is derived from the following computation:

$$\begin{aligned}&\frac{(r!)^m}{(mr-1)!}\int_0^{m-1} x^k(m-x)^{mr-1}\,dx\\ &\qquad= \frac{(r!)^m}{(mr-1)!}\int_0^{m-1} x^k(1+(m-1)-x)^{mr-1}\,dx\\ &\qquad= \frac{(r!)^m}{(mr-1)!}\int_0^{m-1} x^k\sum_{i=0}^{mr-1}\binom{mr-1}{i}(m-1-x)^i\,dx\\ &\qquad= \frac{(r!)^m}{(mr-1)!}\sum_{i=0}^{mr-1}\binom{mr-1}{i}\frac{k!i!(m-1)^{k+i+1}}{(k+i+1)!}\\ &\qquad= \frac{(r!)^m k!}{(k+mr)!}\sum_{i=1}^{mr}\binom{k+mr}{k+i}(m-1)^{k+i}.\end{aligned}$$

Because $\mathcal{M}_r(k,m) \geqslant \frac{(r!)^m}{(mr-1)!} \int_{m-1}^{m} x^k (m-x)^{mr-1}\, dx$ one may deduce that

$$\mathcal{M}_r(k,m) \geqslant \frac{k!(r!)^m m^{k+mr}}{(k+mr)!}\left(1 - \frac{(m-1)^k}{m^{k+mr}} \sum_{i=1}^{mr} \binom{k+mr}{k+i} (m-1)^i\right).$$

Assume that $k > mr$. Then $\binom{k+mr}{k+i} \leqslant \binom{k+mr}{k}$ for $1 \leqslant i \leqslant mr$, hence

$$\begin{aligned}
&\frac{(m-1)^k}{m^{k+mr}} \sum_{i=1}^{mr} \binom{k+mr}{k+i} (m-1)^i \\
&\qquad \leqslant \frac{1}{m^{mr}}\left(1-\frac{1}{m}\right)^k \sum_{i=1}^{mr} \binom{k+mr}{k} (m-1)^i \\
&\qquad \leqslant \left(1-\frac{1}{m}\right)^k \binom{k+mr}{k} \frac{((m-1)^{mr+1} - (m-1)}{m-2} \\
&\qquad \leqslant \left(1-\frac{1}{m}\right)^k (k+mr)^{mr} \frac{(m-1)^{2mr}}{(mr)!} \xrightarrow[k\to+\infty]{} 0. \quad \blacksquare
\end{aligned}$$

5.4. Proof of Proposition 3.3. (i) Note that the density of the random variable $Y_i = (r-1)X_i$ on $[0, r-1]$ is

$$f(x) = \frac{r}{r-1}\left(1 - \frac{x}{r-1}\right)^{r-1}.$$

Since for all $x \in [0, r-1]$,

$$\frac{r}{r-1}\left(1 - \frac{x}{r-1}\right)^{r-1} \leqslant \frac{r}{r-1} \exp(-x),$$

one may deduce that

$$\begin{aligned}
(r-1)^k \mathbb{E}\Big(\sum_{i=1}^{m} X_i\Big)^k = \mathbb{E}\Big(\sum_{i=1}^{m} Y_i\Big)^k &\leqslant \left(\frac{r}{r-1}\right)^k \mathbb{E}\Big(\sum_{i=1}^{m} \mathcal{E}_i\Big)^k \\
&\leqslant \left(\frac{r}{r-1}\right)^k \frac{(m-1+k)!}{(m-1)!}
\end{aligned}$$

because $\sum_{i=1}^{m} \mathcal{E}_i$ has an Erlang distribution (with density $h(x) = \frac{x^{m-1}\exp(-x)}{(m-1)!}$) whose moment of order k is $\frac{(m-1+k)!}{(m-1)!}$. This inequality completes the proof of (i).

(ii) The density of $(r-1)X_1$ converges pointwise to the density of $\mathcal{E}_1$, namely $\lim_{r\to+\infty} f(x) = \exp(-x)$ for all $x \in [0, +\infty)$. Therefore,

$$\lim_{r\to+\infty} x^k f^{*m}(x) = \frac{x^{m+k-1}\exp(-x)}{(m-1)!}.$$

Finally, the dominated convergence theorem gives

$$\begin{aligned}\lim_{r\to+\infty}(r-1)^k\mathbb{E}\Big(\sum_{i=1}^{m}X_i\Big)^k &= \lim_{r\to+\infty}\int_0^{+\infty}\frac{x^{m+k-1}\exp(-x)}{(m-1)!}\,dx\\ &= \mathbb{E}\Big(\sum_{i=1}^{m}\mathcal{E}_i\Big)^k = \frac{(m-1+k)!}{(m-1)!}. \quad ■\end{aligned}$$

Acknowledgments. The authors thank the anonymous referee for the suggestion concerning the on-line encyclopedia of integer sequences.

REFERENCES

[1] D. Berend and T. Tassa, *Improved bounds on Bell numbers and on moments of sums of random variables*, Probab. Math. Statist. 30 (2010), 185–205.

[2] W. E. Bleick and P. C. C. Wang, *Asymptotics of Stirling numbers of the second kind*, Proc. Amer. Math. Soc. 42 (1974), 575–580.

[3] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Springer, 1974.

[4] E. Czabarka, P. L. Erdős, V. Johnson, A. Kupczok and L. A. Székely, *Asymptotically normal distribution of some tree families relevant for phylogenetics, and of partitions without singletons*, Moscow J. Combin. Number Theory 1 (2011), 12–24.

[5] A. K. Gupta and S. Nadarajah, *Handbook of Beta Distribution and its Applications*, CRC Press, 2004.

[6] L. H. Harper, *Stirling behavior is asymptotically normal*, Ann. Math. Statist. 38 (1967), 410–414.

[7] P. Hall, *The distribution of means for samples of size n drawn from a population in which the variate takes values between 0 and 1, all such values being equally probable*, Biometrika 19 (1927), 240–244.

[8] J. O. Irwin, *On the frequency distribution of the means of samples from a population having any law of frequency with finite moments, with special reference to Pearson's type II*, Biometrika 19 (1927), 225–239.

[9] T. Mansour and M. Schork, *Commutation Relations, Normal Ordering, and Stirling Numbers*, CRC Press, 2015.

[10] B. C. Rennie and A. J. Dobson, *On Stirling numbers of the second kind*, J. Combin. Theory 7 (1969), 116–121.

[11] N. J. A. Sloane and The OEIS Foundation, *The On-Line Encyclopedia of Integer Sequences*, 2024.

Jakub Gismatullin
Instytut Matematyczny
Uniwersytet Wrocławski
50-384 Wrocław, Poland
E-mail: jakub.gismatullin@uwr.edu.pl

Patrick Tardivel
Instytut Matematyczny
Uniwersytet Wrocławski
50-384 Wrocław, Poland and
Institut de Mathématiques de Bourgogne
UMR 5584 CNRS, Université de Bourgogne
21078 Dijon, France
E-mail: Patrick.Tardivel@u-bourgogne.fr

Received 5.1.2024;
accepted 31.7.2024