PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 42, Fasc. 2 (2022), pp. 319–356
Published online 6.12.2022
doi:10.37190/0208-4147.00069

QUENCHED ASYMPTOTICS FOR SYMMETRIC LÉVY PROCESSES INTERACTING WITH POISSONIAN FIELDS

BY

ZHI-HE CHEN (FUZHOU) AND JIAN WANG (FUZHOU)

Abstract. We establish the quenched large time asymptotics for the Feynman–Kac functional

$$\mathbb{E}_x\Big[\exp\Big(-\int_0^t V^\omega(Z_s)\,ds\Big)\Big]$$

associated with a pure-jump symmetric Lévy process $(Z_t)_{t\geqslant 0}$ in general Poissonian random potentials V^ω on $\mathbb{R}^d$, which is closely related to the large time asymptotic behavior of solutions to the nonlocal parabolic Anderson problem with Poissonian interaction. In particular, when the density function with respect to the Lebesgue measure of the associated Lévy measure is given by

$$\rho(z) = \frac{1}{|z|^{d+\alpha}}\mathbb{1}_{\{|z|\leqslant 1\}} + e^{-c|z|^\theta}\mathbb{1}_{\{|z|>1\}}$$

for some $\alpha \in (0,2)$, $\theta \in (0,\infty]$ and $c > 0$, an explicit quenched asymptotics is derived for potentials with the shape function given by $\varphi(x) = 1 \wedge |x|^{-d-\beta}$ for $\beta \in (0,\infty]$ with $\beta \neq 2$, and it is completely different for $\beta > 2$ and $\beta < 2$. We also discuss the quenched asymptotics in the critical case (e.g., $\beta = 2$ in the example above). The work fills the gaps of the related work for pure-jump symmetric Lévy processes in Poissonian potentials, where only the case that the shape function is compactly supported (e.g., $\beta = \infty$ in the example above) has been handled in the literature.

2020 Mathematics Subject Classification: Primary 60G52; Secondary 60J25, 60J55, 60J35, 60J75.

Key words and phrases: symmetric Lévy process, Poissonian potential, quenched asymptotic, nonlocal parabolic Anderson problem, spectral theory.

1. BACKGROUND AND MAIN RESULTS

This paper is devoted to the analysis of large time asymptotic behavior of solutions to the nonlocal parabolic Anderson problem with Poissonian interaction:

$$\frac{\partial u^\omega}{\partial t} = Lu^\omega - V^\omega u^\omega \tag{1.1}$$

on $[0,\infty)\times\mathbb{R}^d$ with the initial condition $u^\omega(0,x)=1$. Here, L is the infinitesimal generator of a pure-jump symmetric Lévy process $Z := (Z_t,\mathbb{P}_x)_{t\geqslant 0, x\in\mathbb{R}^d}$ with characteristic exponent

$$\psi(\xi) = \int\limits_{\mathbb{R}^d\setminus\{0\}} (1-\cos\langle\xi,z\rangle)\,\nu(dz) \tag{1.2}$$

for some symmetric Lévy measure ν (i.e., ν is a Radon measure on $\mathbb{R}^d\setminus\{0\}$ with $\int_{\mathbb{R}^d\setminus\{0\}}(|z|^2\wedge 1)\,\nu(dz)<\infty$ and $\nu(A)=\nu(-A)$ for any $A\in\mathscr{B}(\mathbb{R}^d\setminus\{0\})$); the potential is

$$V^\omega(x) = \int\limits_{\mathbb{R}^d} \varphi(x-y)\,\mu^\omega(dy), \tag{1.3}$$

where μ^ω is a Poissonian random measure on $\mathbb{R}^d$ with intensity measure $\rho\,dx$, for a constant $\rho>0$, on a given probability space $(\Omega,\mathbb{Q})$, and φ is a non-negative shape function on $\mathbb{R}^d$. We refer to the monographs [14, 21] for background on this topic. Throughout this paper, $\mathbb{Q}$ and $\mathbb{E}_\mathbb{Q}$ denote the probability and expectation corresponding to the Poissonian field, while $\mathbb{P}_x$ and $\mathbb{E}_x$ denote the probability and expectation corresponding to the Lévy process Z with starting point $x\in\mathbb{R}^d$.

Under mild assumptions (so that the potential V^ω belongs $\mathbb{Q}$-almost surely to the local Kato class of the process Z; see Subsection 2.2 for more details), the solution to the problem (1.1) enjoys the Feynman–Kac representation

$$u^\omega(t,x) = \mathbb{E}_x\Big[\exp\Big(-\int\limits_0^t V^\omega(Z_s)\,ds\Big)\Big]. \tag{1.4}$$

Thus, the analysis of properties of the solution to (1.1) can be done via (1.4) by estimating $u^\omega(t,x)$. There are a number of works on the large time behavior of $u^\omega(t,x)$ in both the *annealed* sense (averaged with respect to $\mathbb{Q}$) and the *quenched* sense (almost sure with respect to $\mathbb{Q}$). In this paper we will mainly analyse the quenched behavior of $u^\omega(t,x)$ for pure-jump symmetric Lévy processes in Poissonian potentials with a non-negative shape function φ.

Let us begin by recalling the history of related topics. The annealed asymptotics of $u^\omega(t,x)$ was first established by Donsker and Varadhan [7] for symmetric (but not necessarily isotropic) non-degenerate α-stable processes (including Brownian motion). They proved in [7, Theorem 3] that, when the shape function $\varphi(x)$ is of order $o(1/|x|^{d+\alpha})$ as $|x|\to\infty$, which is later referred to as the *light tailed case*,

$$\lim_{t\to\infty}\frac{\log\mathbb{E}_\mathbb{Q}[u^\omega(t,x)]}{t^{d/(d+\alpha)}} = -\rho^{\alpha/(d+\alpha)}\left(\frac{d+\alpha}{\alpha}\right)\left(\frac{\alpha\lambda_{(\alpha)}(B(0,1))}{d}\right)^{d/(d+\alpha)}, \tag{1.5}$$

where

$$\lambda_{(\alpha)}(B(0,1)) = \inf_{\text{open } U,\,|U|=w_d}\lambda_1^{(\alpha)}(U),$$

w_d is the volume of the unit ball $B(0,1)$, and $\lambda_1^{(\alpha)}(U)$ is the principal Dirichlet eigenvalue for the generator of the symmetric α-stable process on U. In particular, when the symmetric α-stable process is isotropic, it follows from the Faber–Krahn isoperimetric inequality that the infimum in the definition of $\lambda_{(\alpha)}(B(0,1))$ above is attained on the ball of radius $r_d = w_d^{-1/d}$ and so we have $\lambda_{(\alpha)}(B(0,1)) = w_d^{\alpha/d}\lambda_1^{(\alpha)}(B(0,1))$, where $\lambda_1^{(\alpha)}(U)$ is the principal Dirichlet eigenvalue for the fractional Laplacian on U. Thus, in this case (1.5) is reduced to

(1.6)
$$\lim_{t\to\infty}\frac{\log \mathbb{E}_{\mathbb{Q}}[u^\omega(t,x)]}{t^{d/(d+\alpha)}} = -(\rho w_d)^{\alpha/(d+\alpha)}\left(\frac{d+\alpha}{\alpha}\right)\left(\frac{\alpha\lambda_1^{(\alpha)}(B(0,1))}{d}\right)^{d/(d+\alpha)}.$$

Later Ôkura [17] extended [7, Theorem 3] to a large class of symmetric Lévy processes whose exponent ψ satisfies $\exp(-t\psi(\cdot)^{1/2}) \in L^1(\mathbb{R}^d; dx)$ for all $t > 0$ and can be written as

$$\psi(\xi) = \psi^{(\alpha)}(\xi) + o(|\xi|^\alpha), \quad |\xi| \to 0, \tag{1.7}$$

for some $\alpha \in (0,2]$. Here, $\psi^{(\alpha)}(\xi)$ is the characteristic exponent of a symmetric non-degenerate α-stable process $Z^{(\alpha)}$ (see (2.2) below) satisfying some kind of summability condition for $\psi_*^{(\alpha)}(\xi) := \inf_{t\geqslant 1} t^\alpha \psi^{(\alpha)}(t^{-1}\xi)$; see Subsection 2.1 for more details. More explicitly, it was shown in [17, Theorem 4.1] that (1.5) still holds for symmetric Lévy processes above with $\lambda_{(\alpha)}(B(0,1))$ defined via the principal Dirichlet eigenvalue for the generator of the symmetric α-stable process $Z^{(\alpha)}$ with exponent $\psi^{(\alpha)}$ given in (1.7). When the characteristic exponent of the Lévy process Z further satisfies

$$\psi(\xi) = O(|\xi|^\alpha), \quad |\xi| \to 0,$$

and the shape function φ fulfills $K := \lim_{|x|\to\infty}\varphi(x)|x|^{d+\beta} \in (0,\infty)$ for some $0 < \beta < \alpha$ (which is referred as the *heavy tailed case*), Ôkura proved in [16, Theorem 6.3′] that

$$\lim_{t\to\infty}\frac{\log \mathbb{E}_{\mathbb{Q}}[u^\omega(t,x)]}{t^{d/(d+\beta)}} = -\rho w_d \Gamma\left(\frac{\beta}{d+\beta}\right)K^{d/(d+\beta)} \tag{1.8}$$

for symmetric Lévy processes satisfying (1.7). See Pastur [19] for the first result in this direction when Z is Brownian motion. The reader can also be referred to [16, Theorem 6.4′] and [18, Theorem 1 and Remarks] for the study in the critical case, e.g., $K := \lim_{|x|\to\infty}\varphi(x)|x|^{d+\alpha} \in (0,\infty)$; see the Appendix for related discussion. In particular, according to all the conclusions above, the annealed asymptotics of $u^\omega(t,x)$ is of order $t^{-d/(d+\beta\wedge\alpha)}$ when $\varphi(x) = K(1\wedge |x|^{-d-\beta})$. However, in the light tailed case, the right hand side of (1.5) for the annealed asymptotics of $u^\omega(t,x)$ is independent of K and β, while in the heavy tailed case the right hand side of (1.8) only depends on the constants K and β.

Compared with the annealed asymptotics, the study of the quenched asymptotics of $u^\omega(t,x)$ is less developed. The first result for the quenched asymptotics of $u^\omega(t,x)$ for Brownian motions moving in a Poissonian potential was established by Sznitman [20, Theorem], who showed that when φ is compactly supported (which in particular corresponds to the shape function $\varphi(x) = K(1 \wedge |x|^{-d-\beta})$ with $\beta = \infty$, and so belongs to the special light tailed case), $\mathbb{Q}$-almost surely for all $x \in \mathbb{R}^d$ we have

$$\lim_{t\to\infty} \frac{\log u^\omega(t,x)}{t/(\log t)^{2/d}} = -\left(\frac{\rho w_d}{d}\right)^{2/d} \lambda_{\mathrm{BM}}(B(0,1)),$$

where $\lambda_{\mathrm{BM}}(B(0,1))$ is the principal Dirichlet eigenvalue for the Laplacian on $B(0,1)$. More recently, the quenched asymptotics of $u^\omega(t,x)$ for symmetric Lévy processes satisfying (1.7) has been extensively studied in [11]; see [11, Table 1, p. 165] for results concerning explicit Lévy processes.

Concerning Brownian motions in a heavy tailed Poissonian potential, for example, $\varphi(x) = 1 \wedge |x|^{-d-\beta}$ with $\beta \in (0,2)$, it was shown in [9, Theorem 2] that $\mathbb{Q}$-almost surely for any $x \in \mathbb{R}^d$,

$$\lim_{t\to\infty} \frac{\log u^\omega(t,x)}{t/(\log t)^{\beta/d}} = -\frac{d}{d+\beta}\left(\frac{\beta}{d(d+\beta)}\right)^{\beta/d} \left(\rho w_d \Gamma\left(\frac{\beta}{d+\beta}\right)\right)^{(d+\beta)/d},$$

where $\Gamma(x)$ is the Gamma function. (In fact, the second order asymptotics of $u^\omega(t,x)$ was also established in [9, Theorem 1.2].)

However, the quenched asymptotics of $u^\omega(t,x)$ for symmetric Lévy processes in heavy tailed cases, as well as in light tailed cases when φ does not have compact support, are still unknown. The goal of this paper is to fill these gaps. To state our main contribution, in the following two results we restrict ourselves to the special shape function $\varphi(x) = 1 \wedge |x|^{-d-\beta}$ with $\beta \in (0,\infty]$.

THEOREM 1.1. *Let Z be a rotationally symmetric α-stable process on $\mathbb{R}^d$ with $\alpha \in (0,2)$. Then the following statements hold:*

(i) *If $\beta \in (\alpha,\infty]$, then for all $x \in \mathbb{R}^d$, $\mathbb{Q}$-almost surely,*

$$\begin{aligned} &-(d+\alpha)^{\alpha/(d+\alpha)}\left[\left(\frac{\alpha}{d}\right)^{d/(d+\alpha)} + \left(\frac{d}{\alpha}\right)^{\alpha/(d+\alpha)}\right] A_1 \\ &\quad \leqslant \liminf_{t\to\infty} \frac{\log u^\omega(t,x)}{t^{d/(d+\alpha)}} \leqslant \limsup_{t\to\infty} \frac{\log u^\omega(t,x)}{t^{d/(d+\alpha)}} \leqslant -\alpha(\alpha+d/2)^{-d/(\alpha+d)} A_1, \end{aligned}$$

where

$$A_1 = \left(\frac{\rho w_d}{d}\right)^{\alpha/(d+\alpha)} [\lambda_1^{(\alpha)}(B(0,1))]^{d/(d+\alpha)}.$$

(ii) *If $\beta \in (0, \alpha)$, then for all $x \in \mathbb{R}^d$, $\mathbb{Q}$-almost surely,*

$$-(d+\alpha)^{\beta/(d+\beta)}\left[\left(\frac{\beta}{d}\right)^{d/(d+\beta)}+\left(\frac{d}{\beta}\right)^{\beta/(d+\beta)}\right]A_2 \leqslant \liminf_{t\to\infty}\frac{\log u^\omega(t,x)}{t^{d/(d+\beta)}} \leqslant \limsup_{t\to\infty}\frac{\log u^\omega(t,x)}{t^{d/(d+\beta)}} \leqslant -\alpha^{\beta/(d+\beta)}A_2,$$

where

$$A_2=\left(\frac{d}{d+\beta}\right)^{d/(\beta+d)}\left(\frac{\beta}{d(d+\beta)}\right)^{\beta/(\beta+d)}\Gamma\left(\frac{\beta}{d+\beta}\right)\rho w_d.$$

THEOREM 1.2. *Suppose that the Lévy measure $\nu(dz)=\rho(|z|)\,dz$ satisfies*

$$\rho(|z|)\asymp\frac{1}{|z|^{d+\alpha}}\mathbb{1}_{\{|z|\leqslant 1\}}+e^{-c|z|^\theta}\mathbb{1}_{\{|z|>1\}}$$

for some $\alpha \in (0,2)$, $\theta \in (0,\infty]$ and $c>0$, where $f \asymp g$ means that there is a constant $c_0 \geqslant 1$ such that $c_0^{-1}g \leqslant f \leqslant c_0 g$. Then the following statements hold:

(i) *If $\beta \in (2,\infty]$, then for all $x \in \mathbb{R}^d$, $\mathbb{Q}$-almost surely,*

$$\lim_{t\to\infty}\frac{\log u^\omega(t,x)}{t/(\log t)^{2/d}}=-\left(\frac{\rho w_d(1\wedge\theta)}{d}\right)^{2/d}\lambda_1^{(2)}(B(0,1)),$$

where $\lambda_1^{(2)}(B(0,1))$ is the first Dirichlet eigenvalue for the generator of the killed Brownian motion when exiting the ball $B(0,1)$ and with the covariance matrix $(a_{ij})_{1\leqslant i,j\leqslant d}$ given by

$$a_{ii}=\int_{\mathbb{R}^d\setminus\{0\}} z_i^2\,\nu(dz)=\int_{\mathbb{R}^d\setminus\{0\}} z_i^2\rho(|z|)\,dz,\quad a_{ij}=0,\quad 1\leqslant i\neq j\leqslant d.$$

(ii) *If $\beta \in (0,2)$, then for all $x \in \mathbb{R}^d$, $\mathbb{Q}$-almost surely,*

$$\lim_{t\to\infty}\frac{\log u^\omega(t,x)}{t/(\log t)^{\beta/d}}=-\frac{d}{d+\beta}\left(\frac{\beta(1\wedge\theta)}{d(d+\beta)}\right)^{\beta/d}\left[\rho w_d\Gamma\left(\frac{\beta}{d+\beta}\right)\right]^{(d+\beta)/d}.$$

Theorems 1.1 and 1.2 show that the quenched asymptotics of $u^\omega(t,x)$ for pure-jump symmetric Lévy processes in Poissonian potentials depends not only on the shape function φ in the potential, but also on the properties of the Lévy measure (for large jumps) of the Lévy process Z. This dependence appears for the annealed asymptotics of $u^\omega(t,x)$ as well, and it is much more evident in the quenched asymptotics. For instance, considering the example with $\theta \in (0,1)$ in Theorem

1.2 which satisfies (1.7) with $\alpha = 2$, in the light tailed case the precise value of the annealed asymptotics of $u^\omega(t,x)$ is independent of θ by (1.6), but that of the quenched asymptotics of $u^\omega(t,x)$ does depend on θ by Theorem 1.2(i). The same occurs for the heavy tailed case. On the other hand, in both light tailed and heavy tailed cases, for rotationally symmetric α-stable processes, by Theorem 1.1 the correct order of the quenched asymptotics of $u^\omega(t,x)$ is the same as that of the annealed asymptotics; however, according to Theorem 1.2, this is not true for symmetric Lévy processes with exponential decay for large jumps. Actually, in the case of compactly supported shape functions one has different quenched and annealed rates as long as the decay of the Lévy measure at infinity is faster than polynomial; see [11, Section 1; in partciular, Table 1, p. 165] for more discussion of this point.

Next, we briefly comment on our proofs for the quenched asymptotics of $u^\omega(t,x)$ for pure-jump symmetric Lévy processes in general Poissonian potentials.

(i) As indicated by [11], because of the light tail of the potential, $V^\omega(x)$ is comparable to $\tilde{V}^\omega(x)$ whose associated shape function is compactly supported. This enables us to use the classical approach of [20, 11]; that is, when the shape function has compact support, $\mathbb{Q}$-almost surely there exists a large area where the potential is zero and so the principal Dirichlet eigenvalue for the generator of the process Z is naturally involved in the quenched asymptotics of $u^\omega(t,x)$. When $\beta = \infty$ (this is just the case that the shape function φ has compact support), Theorems 1.1(i) and 1.2(i) have been proven in [11]; see [11, Table 1, p. 165] for more details. Based on this and the strategy of the approach mentioned above, we believe that assertions of [11] should hold true for all light tailed cases.

(ii) In heavy tailed cases, the potential $V^\omega(x)$ will play a dominating role in the quenched asymptotics of $u^\omega(t,x)$. Similar to the Brownian motion case studied in [9], it is natural to expect that the main contribution of $u^\omega(t,x)$ defined by (1.4) comes from the process Z which spends most of the time in the area where $V^\omega(x)$ takes small values. Motivated by this fact, we partly adopt the argument in [9] to treat upper bounds of the principal Dirichlet eigenvalue for the random Schrödinger operator associated with equation (1.1), which in turn yield explicit quenched asymptotics of $u^\omega(t,x)$ in a general heavy tailed setting.

(iii) To consider quenched asymptotics for pure-jump symmetric Lévy processes in both light tailed and heavy tailed potentials at the same time, we give a unified approach which is inspired by [3] (which studied quenched asymptotics for Brownian motions in renormalized Poissonian potentials) and based on recent development on (Dirichlet) heat kernel estimates for symmetric jump processes. We emphasize that the argument for lower bounds for the quenched asymptotics of $u^\omega(t,x)$ here is different from that in [11]. In particular, the lower bound for the quenched asymptotics of $u^\omega(t,x)$ in Theorem 1.1(i) for symmetric rotationally α-stable process slightly improves that in [11]; see [11, Remark 5.1(4)]. We also note that for rotationally symmetric α-stable processes the associated liminf and limsup constants do not coincide for the quenched asymptotics. The reason is partly due to the polynomial decay of the distribution for exit times of rotationally symmetric α-stable processes. The corresponding result in [11, Table 1, p. 165] has this kind of gap between the upper and lower bounds too.

We further mention that our main results for quenched estimates of $u^\omega(t,x)$ hold (see Theorems 3.1 and 3.2) for pure-jump symmetric Lévy processes in general Poissonian potentials, so the results should apply to various examples discussed in [11, Section 5]. It is also possible to extend them to symmetric Lévy processes with non-degenerate Gaussian part as in [11], and the details are left to interested readers. Instead, to highlight the power of our approach, we will obtain fairly general estimates of $u^\omega(t,x)$ for critical potentials (for example, $\varphi(x) = 1 \wedge |x|^{-d-\alpha}$ with α being in (1.7)); see Theorems 5.1 and 5.2 in the Appendix. Specifically, we can prove the following quenched estimates of $u^\omega(t,x)$ in the critical cases.

PROPOSITION 1.1.

(i) *Let Z be a rotationally symmetric α-stable process on $\mathbb{R}^d$ with $\alpha \in (0,2)$, and $\varphi(x) = 1 \wedge |x|^{-d-\alpha}$. Then, $\mathbb{Q}$-almost surely for all $x \in \mathbb{R}^d$,*

$$-\infty < \liminf_{t\to\infty} \frac{\log u^\omega(t,x)}{t^{d/(d+\alpha)}} \leqslant \limsup_{t\to\infty} \frac{\log u^\omega(t,x)}{t^{d/(d+\alpha)}} < 0.$$

(ii) *Let Z be a pure-jump rotationally symmetric Lévy process given in Theorem 1.2, and $\varphi(x) = 1 \wedge |x|^{-d-2}$. Then, $\mathbb{Q}$-almost surely for all $x \in \mathbb{R}^d$,*

$$-\infty < \liminf_{t\to\infty} \frac{\log u^\omega(t,x)}{t/(\log t)^{2/d}} \leqslant \limsup_{t\to\infty} \frac{\log u^\omega(t,x)}{t/(\log t)^{2/d}} < 0.$$

The rest of the paper is arranged as follows. In Section 2, we give some preliminaries and main assumptions. Section 3 is the main part, and it is split into three subsections. In particular, after establishing quenched bounds for $u^\omega(t,x)$ and estimates for the principal Dirichlet eigenvalue, we derive general quenched estimates of $u^\omega(t,x)$. Section 4 is devoted to the proofs of Theorems 1.1 and 1.2. Finally, in the Appendix we present quenched upper bounds for the principal Dirichlet eigenvalue in the heavy tailed case, and quenched estimates of $u^\omega(t,x)$ with critical potentials.

2. PRELIMINARIES AND ASSUMPTIONS

2.1. Lévy processes. Let $Z := (Z_t, \mathbb{P}_x)_{t\geqslant 0, x\in\mathbb{R}^d}$ be a pure-jump symmetric Lévy process on $\mathbb{R}^d$ with characteristic exponent ψ given by (1.2). Throughout the paper, we will assume the following two conditions hold for the exponent ψ:

(i)

$$\lim_{|\xi|\to\infty} \frac{\psi(\xi)}{\log^2|\xi|} = \infty;$$

(ii)

$$\psi(\xi) = \psi^{(\alpha)}(\xi) + o(|\xi|^\alpha), \qquad |\xi| \to 0, \tag{2.1}$$

for some $\alpha \in (0, 2]$, where

$$\psi^{(\alpha)}(\xi) = \begin{cases} \int_0^\infty \int_{\mathbb{S}^{d-1}} \frac{1-\cos(r\langle \xi, z\rangle)}{r^{1+\alpha}} \mu(dz)\,dr, & \alpha \in (0,2), \\ \sum_{1\leqslant i,j\leqslant d} a_{ij}\xi_i\xi_j, & \alpha = 2, \end{cases} \tag{2.2}$$

with μ being a symmetric finite measure on the unit sphere $\mathbb{S}^{d-1}$ and $(a_{ij})_{1\leqslant i,j\leqslant d}$ a symmetric non-negative definite matrix. Moreover, $\inf_{|\xi|=1} \psi^{(\alpha)}(\xi) > 0$, and, for all $\delta, r > 0$,

$$\sum_{\xi \in r\mathbb{Z}^d} \exp(-\delta \psi_*^{(\alpha)}(\xi)) < \infty,$$

where $\psi_*^{(\alpha)}(\xi) = \inf_{t\geqslant 1} t^\alpha \psi^{(\alpha)}(t^{-1}\xi)$.

It is clear that, under (i), $e^{-t\psi^{1/2}(\cdot)} \in L^1(\mathbb{R}^d; dx)$ for all $t > 0$. In particular, $e^{-t\psi(\cdot)} \in L^1(\mathbb{R}^d; dx)$ for all $t > 0$, which implies that the process Z has the transition density function $p(t, x - y) = p(t, x, y)$ with respect to the Lebesgue measure such that $p(t, 0) = \sup_{x\in\mathbb{R}^d} p(t, x) < \infty$ for all $t > 0$; see [15, Theorem 1]. We further suppose that $p(t, x) > 0$ for all $t > 0$ and $x \in \mathbb{R}^d$. Note that the asymptotic condition (2.1) on $\psi(\xi)$ is essentially based on the property of the Lévy measure ν on $\{z \in \mathbb{R}^d : |z| > 1\}$. For example, according to [11, Proposition 5.2(i)], if ν has finite second moment, i.e., $\int_{\{|z|>1\}} |z|^2 \nu(dz) < \infty$, then (2.1) holds with $\alpha = 2$ and $a_{ij} = \frac{1}{2}\int_{\mathbb{R}^d\setminus\{0\}} z_i z_j \nu(dz)$.

In this paper, we always let D be a bounded domain (i.e., a connected open subset) of $\mathbb{R}^d$. Let $Z^D := (Z_t^D, \mathbb{P}^x)_{t\geqslant 0, x\in D}$ be the subprocess of Z killed upon exiting D. Then Z^D has the transition density function

$$p^D(t, x, y) = p(t, x, y) - \mathbb{E}_x\big(p(t - \tau_D, Z_{\tau_D}, y)\mathbb{1}_{\{\tau_D\leqslant t\}}\big), \qquad t > 0,\ x, y \in D,$$

where $\tau_D = \inf\{t > 0 : Z_t \notin D\}$. Denote by $(P_t^D)_{t\geqslant 0}$ the Dirichlet semigroup associated with the process Z^D. Since D is bounded and $p^D(t, x, y) \leqslant p(t, x, y) = p(t, x-y) \leqslant p(t, 0) < \infty$ for all $t > 0$ and $x, y \in D$, the operators P_t^D are compact and the spectrum of $(-L)|_D$, the generator of the semigroup $(P_t^D)_{t\geqslant 0}$, is discrete:

$$0 < \lambda_1(D) < \lambda_2(D) \leqslant \lambda_3(D) \leqslant \cdots \to \infty.$$

When Z is a symmetric α-stable process with $\alpha \in (0, 2]$, the eigenvalues will be denoted by $\lambda_i^{(\alpha)}(D)$ for $i \geqslant 1$.

2.2. Random potential. Consider the random potential V^ω given by (1.3), which can be written as

$$V^\omega(x) = \sum_i \varphi(x - \omega_i), \qquad x \in \mathbb{R}^d,$$

and the points $\{\omega_i\}$ are from a realization of a homogeneous Poisson point process in $\mathbb{R}^d$ with parameter $\rho > 0$. In this paper, we assume that the non-negative shape function φ is continuous, and satisfies

$$\int_{\mathbb{R}^d} (e^{\bar{\varphi}(x)} - 1)\, dx < \infty, \tag{2.3}$$

where $\bar{\varphi}(x) = \sup_{z \in B(x,1)} \varphi(z)$. Then, following the proof of [9, Lemma 5], we know that $\mathbb{Q}$-almost surely there is $r(\omega) > 0$ such that for all $r \geqslant r(\omega)$,

$$\sup_{x \in B(0,r)} V^\omega(x) \leqslant 3d \log r. \tag{2.4}$$

A typical example that satisfies (2.3) is the function $\varphi(x) = K(1 \wedge |x|^{-d-\theta})$ for some positive constants K and θ. Indeed, if there are constants $c_0, \theta > 0$ such that

$$\varphi(x) \leqslant \frac{c_0}{(1 + |x|)^{d+\theta}}, \qquad x \in \mathbb{R}^d,$$

then, according to [1, Lemma 2.1], we even find that $\mathbb{Q}$-almost surely there is $r(\omega) > 0$ such that for all $r \geqslant r(\omega)$,

$$\sup_{x \in B(0,r)} V^\omega(x) \leqslant c\left(1 + \frac{\log r}{\log \log r}\right),$$

where $c > 0$ is independent of $r(\omega)$ and r. In particular, (2.4) shows that $\mathbb{Q}$-almost surely, V^ω belongs to the local Kato class relative to the process Z, i.e., $\mathbb{Q}$-almost surely,

$$\lim_{t \to 0} \sup_{x \in \mathbb{R}^d} \int_0^t \mathbb{E}_x(V^\omega(Z_s) \mathbb{1}_{\{Z_s \in B(0,R)\}})\, ds = 0$$

for all $R > 0$.

2.3. Feynman–Kac semigroup. Since $\mathbb{Q}$-almost surely V^ω belongs to the local Kato class relative to the process Z, we can well define the random Feynman–Kac semigroups $(T_t^{V^\omega})_{t \geqslant 0}$ and $(T_t^{V^\omega, D})_{t \geqslant 0}$ as follows:

$$T_t^{V^\omega} f(x) = \mathbb{E}_x\left[f(Z_t) \exp\left(-\int_0^t V^\omega(Z_s)\, ds\right)\right], \qquad f \in L^2(\mathbb{R}^d; dx),\ t > 0,$$

$$T_t^{V^\omega, D} f(x) = \mathbb{E}_x\left[f(Z_t) \exp\left(-\int_0^t V^\omega(Z_s)\, ds\right) \mathbb{1}_{\{\tau_D > t\}}\right], \qquad f \in L^2(D; dx),\ t > 0.$$

Below, we will discuss some properties of $(T_t^{V^\omega})_{t\geqslant 0}$ and $(T_t^{V^\omega,D})_{t\geqslant 0}$, which are understood to hold $\mathbb{Q}$-almost surely. First, in our setting both $(T_t^{V^\omega})_{t\geqslant 0}$ and $(T_t^{V^\omega,D})_{t\geqslant 0}$ admit strictly positive and bounded symmetric kernels $p^{V^\omega}(t,x,y)$ and $p^{V^\omega,D}(t,x,y)$ with respect to the Lebesgue measure such that $\mathbb{Q}$-almost surely,

$$p^{V^\omega}(t,x,y)\leqslant p(t,x,y)=p(t,x-y),\qquad x,y\in\mathbb{R}^d,\ t>0,$$

and

$$p^{V^\omega,D}(t,x,y)\leqslant p^D(t,x,y)\leqslant p(t,x-y),\qquad x,y\in D,\ t>0.$$

On the other hand, it is known that $(T_t^{V^\omega})_{t\geqslant 0}$ can be generated by the random non-local Schrödinger operator $H^\omega := -L+V^\omega$, where L is the infinitesimal generator of the Lévy process Z. Hence, the semigroup $(T_t^{V^\omega,D})_{t\geqslant 0}$ corresponds to the Schrödinger operator H^ω with the Dirichlet conditions on D^c. In particular, the operators $T_t^{V^\omega,D}$ are compact, so that $\mathbb{Q}$-almost surely the spectrum of the operator H^ω with the Dirichlet conditions on D^c is discrete:

$$0<\lambda_1^{V^\omega,D}<\lambda_2^{V^\omega,D}\leqslant\lambda_3^{V^\omega,D}\leqslant\cdots\to\infty.$$

For simplicity, below we write $\lambda_1^{V^\omega,D}$ as $\lambda_{V^\omega,D}$; it will play an important role in our paper. Denote by $\|\cdot\|_{L^2(D;dx)\to L^2(D;dx)}$ the operator norm from $L^2(D;dx)$ to $L^2(D;dx)$. It then follows that $\mathbb{Q}$-almost surely

$$\|T_t^{V^\omega,D}\|_{L^2(D;dx)\to L^2(D;dx)}=e^{-t\lambda_1^{V^\omega,D}}=e^{-t\lambda_{V^\omega,D}},\qquad t>0, \tag{2.5}$$

which implies that $\mathbb{Q}$-almost surely,

$$\int_D \mathbb{E}_x\Big[\exp\Big(-\int_0^t V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D>t\}}\Big]\,dx\leqslant |D|e^{-t\lambda_{V^\omega,D}},\qquad t>0. \tag{2.6}$$

Furthermore, following [16, Section 2] we can construct the conditional process of the Lévy process Z starting from $x\in\mathbb{R}^d$ and terminating in $y\in\mathbb{R}^d$ at time $t>0$, for all t,x,y. This conditional process is denoted by $((Z_s)_{s\in[0,t]},\mathbb{P}_{0,x}^{t,y})$ and referred to as the $(0,x;t,y)$-*pinned process* of Z. In the literature, $\mathbb{P}_{0,x}^{t,y}$ refers to the bridge law of the pinned process; see [2] for more details. The fundamental relation between the original Lévy process Z and the pinned process of $((Z_s)_{s\in[0,t]},\mathbb{P}_{0,x}^{t,y})$ is the following (see [16, Theorem 2.2]):

$$\mathbb{P}_{0,x}^{t,y}(A)=p(t,x,y)^{-1}\mathbb{E}_x(p(t-s,Z_s,y)\mathbb{1}_A) \tag{2.7}$$

for each $A\in\sigma\{Z_u: 0\leqslant u\leqslant s\}$ with $0\leqslant s<t$. According to [16, Propositions 4.2 and 4.3], $\mathbb{Q}$-almost surely for any $x\in D$ and $t>0$,

$$\begin{aligned} p(t,x,x)\mathbb{E}_{0,x}^{t,x}\Big[\exp\Big(-\int_0^t V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D>t\}}\Big]&\\ =p^{V^\omega,D}(t,x,x)=\sum_{k=1}^\infty e^{-t\lambda_k^{V^\omega,D}}e_k^{\omega,D}(x)^2&, \end{aligned} \tag{2.8}$$

where $\{e_k^{\omega,D}(x)\}_{k\geqslant 1}$ are the normalized eigenfunctions corresponding to $\{\lambda_k^{V^\omega,D}\}_{k\geqslant 1}$ with $\|e_k^{\omega,D}\|_{L^2(D;dx)} = 1$ for all $k \geqslant 1$. Indeed, $p^{V^\omega,D}(t,x,y)$ is the fundamental solution to the Dirichlet boundary value problem for (1.1), i.e., for any fixed $y \in D$, $p^{V^\omega,D}(t,\cdot,y)$ is the solution to the equation

$$\partial_t u^\omega(t,x) = Lu^\omega(t,x) - V^\omega(x)u(t,x), \qquad (t,x) \in (0,\infty) \times D,$$

with $u^\omega(0,\cdot) = \delta_y(\cdot)$ and $u^\omega(t,x) = 0$ for all $(t,x) \in (0,\infty) \times D^c$. Then the second equality in (2.8) is a direct consequence of the Fourier expansion for $p^{V^\omega,D}(t,x,x)$; see [10, (2.31) in Section 2.3] or [3, last line of p. 1461] for related discussion when $L = \Delta$. Thanks to (2.8), $\mathbb{Q}$-almost surely we have

$$e^{-t\lambda_{V^\omega,D}} \leqslant \int_D p(t,x,x)\mathbb{E}_{0,x}^{t,x}\Big[\exp\Big(-\int_0^t V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D>t\}}\Big]\,dx, \qquad t>0. \tag{2.9}$$

3. GENERAL BOUNDS FOR QUENCHED ASYMPTOTICS OF $u^\omega(t,x)$

In this section, we establish general bounds for the quenched asymptotics of $u^\omega(t,x)$. Let Z be a pure-jump symmetric Lévy process on $\mathbb{R}^d$ and V^ω be the random potential given by (1.3), both of which satisfy all the assumptions in the previous section. For the index $\alpha \in (0,2]$ given in (2.1), we will consider the following two cases:

- *Light tailed case* (L): The shape function φ in the random potential $V^\omega(x)$ satisfies
$$\lim_{|x|\to\infty} \varphi(x)|x|^{d+\alpha} = 0.$$
- *Heavy tailed case* (H): The characteristic exponent $\psi(\xi)$ of the process Z fulfills $\psi(\xi) = O(|\xi|^\alpha)$ as $|\xi| \to 0$, and there are constants $\beta \in (0,\alpha)$ and $K > 0$ such that, for the shape function φ in the random potential $V^\omega(x)$,

$$\lim_{|x|\to\infty} \varphi(x)|x|^{d+\beta} = K. \tag{3.1}$$

The section is split into three parts. We first show quenched bounds for $u^\omega(t,x)$, and then present estimates for the principal Dirichlet eigenvalue $\lambda_{V^\omega,D}$. General explicit results for quenched estimates of $u^\omega(t,x)$ are given in Subsection 3.3. For simplicity, in the following we take $x = 0$ in the proof, and the arguments work for all $x \in \mathbb{R}^d$ with small modifications.

3.1. Quenched bounds for $u^\omega(t,0)$. Here, we derive some pointwise quenched bounds for $u^\omega(t,0)$. Some of the arguments below are motivated by those in [3, Section 4].

3.1.1. *Upper bounds*

PROPOSITION 3.1. *For any bounded domain D with $0 \in D$, $0 < \delta < t$, $R > 0$ and $a > 1$, and for $\mathbb{Q}$-almost every $\omega \in \Omega$,*

$$\begin{aligned}u^\omega(t,0) \leqslant \mathbb{P}_0(\tau_D \leqslant t) + \min\big\{&p(\delta,0)^{1/2}|D|^{1/2}\exp(-(t-\delta/2)\lambda_{V^\omega,D}),\\ &p(\delta,0)^{1/a}|D|^{1/a}\exp(-a^{-1}(t-\delta)\lambda_{aV^\omega,D})\big\}.\end{aligned}$$

Proof. We mainly follow the idea of [8, Lemma 2.1]. For any bounded domain D with $0 \in D$, $t > 0$, and $\mathbb{Q}$-almost every $\omega \in \Omega$,

$$\begin{aligned}u^\omega(t,0) &= \mathbb{E}_0\Big[\exp\Big(-\int_0^t V^\omega(Z_s)\,ds\Big)\Big]\\ &\leqslant \mathbb{E}_0\Big[\exp\Big(-\int_0^t V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D > t\}}\Big] + \mathbb{P}_0(\tau_D \leqslant t) =: I_1 + I_2.\end{aligned}$$

Next, we will estimate I_1 in two different ways.

First, we repeat the proof of [11, Lemma 3.1]: for any $0 < \delta < t$,

$$\begin{aligned}I_1 &= T_t^{V^\omega,D}\mathbb{1}_D(0) = T_{\delta/2}^{V^\omega,D}T_{t-\delta/2}^{V^\omega,D}\mathbb{1}_D(0)\\ &= \langle p^{V^\omega,D}(\delta/2,0,\cdot), T_{t-\delta/2}^{V^\omega,D}\mathbb{1}_D\rangle_{L^2(D;dx)}\\ &\leqslant \|p^{V^\omega,D}(\delta/2,0,\cdot)\|_{L^2(D;dx)}\|T_{t-\delta/2}^{V^\omega,D}\mathbb{1}_D\|_{L^2(D;dx)}\\ &\leqslant \|p(\delta/2,0,\cdot)\|_{L^2(\mathbb{R}^d;dx)}e^{-(t-\delta/2)\lambda_{V^\omega,D}}\|\mathbb{1}_D\|_{L^2(D;dx)}\\ &= p(\delta,0)^{1/2}|D|^{1/2}\exp(-(t-\delta/2)\lambda_{V^\omega,D}),\end{aligned}$$

where in the first inequality we have used the Cauchy–Schwarz inequality and the second inequality follows from (2.5).

Second, for any $0 < \delta < t$, by the Hölder inequality with $a, b > 1$ satisfying $1/a + 1/b = 1$,

$$\begin{aligned}I_1 &\leqslant \Big(\mathbb{E}_0\Big[\exp\Big(-b\int_0^\delta V^\omega(Z_s)\,ds\Big)\Big]\Big)^{1/b}\Big(\mathbb{E}_0\Big[\exp\Big(-a\int_\delta^t V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D > t\}}\Big]\Big)^{1/a}\\ &\leqslant \Big(\int_D p^D(\delta,0,x)\mathbb{E}_x\Big[\exp\Big(-a\int_0^{t-\delta} V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D > t-\delta\}}\Big]\,dx\Big)^{1/a}\\ &\leqslant p(\delta,0)^{1/a}\Big(\int_D \mathbb{E}_x\Big[\exp\Big(-a\int_0^{t-\delta} V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D > t-\delta\}}\Big]\,dx\Big)^{1/a}\\ &\leqslant p(\delta,0)^{1/a}|D|^{1/a}\exp(-a^{-1}(t-\delta)\lambda_{aV^\omega,D}),\end{aligned}$$

where in the last inequality we have used (2.6).

Therefore, the assertion follows from all the estimates above. ∎

REMARK 3.1. The proof above essentially shows that for any bounded domain D with $0 \in D$, for any $0 < \delta < t$, $a > 1$, and for $\mathbb{Q}$-almost every $\omega \in \Omega$,

$$\mathbb{E}_0\Big[\exp\Big(-\int_0^t V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D > t\}}\Big] \leqslant I(D,t,V^\omega,\delta,a),$$

where

$$I(D,t,V^\omega,\delta,a) := \min\big\{p(\delta,0)^{1/2}|D|^{1/2}\exp(-(t-\delta/2)\lambda_{V^\omega,D}),\\ p(\delta,0)^{1/a}|D|^{1/a}\exp(-a^{-1}(t-\delta)\lambda_{aV^\omega,D})\big\}.$$

By this estimate, we can slightly improve Proposition 3.1 by local refinement. Indeed, let $\{D_k\}_{k\geqslant 1}$ be a sequence of increasing bounded domains such that $0 \in D_k$ for all $k \geqslant 1$ and $\bigcup_{k\geqslant 1} D_k = \mathbb{R}^d$. Then, for any $t > 0$ and $\mathbb{Q}$-almost every $\omega \in \Omega$,

$$\begin{aligned} u^\omega(t,0) &\leqslant \mathbb{E}_0\Big[\exp\Big(-\int_0^t V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_{D_1} > t\}}\Big] \\ &\quad + \sum_{k=1}^\infty \mathbb{E}_0\Big[\exp\Big(-\int_0^t V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_{D_k}\leqslant t < \tau_{D_{k+1}}\}}\Big] \\ &=: J_0 + \sum_{k=1}^\infty J_k. \end{aligned}$$

It is clear that $J_0 \leqslant I(D_1,t,V^\omega,\delta,a)$. On the other hand, by the Hölder inequality, for any $k \geqslant 1$ and $\xi,\eta > 1$ with $1/\xi + 1/\eta = 1$,

$$\begin{aligned} J_k &\leqslant [\mathbb{P}_0(\tau_{D_k} \leqslant t < \tau_{D_{k+1}})]^{1/\xi}\Big[\mathbb{E}_0\Big(\exp\Big(-\eta\int_0^t V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_{D_{k+1}} > t\}}\Big)\Big]^{1/\eta} \\ &\leqslant [\mathbb{P}_0(\tau_{D_k} \leqslant t < \tau_{D_{k+1}})]^{1/\xi}[I(D_{k+1},t,\eta V^\omega,\delta,a)]^{1/\eta}. \end{aligned}$$

Therefore, $\mathbb{Q}$-almost surely,

$$\begin{aligned} u^\omega(t,0) &\leqslant I(D_1,t,V^\omega,\delta,a) \\ &\quad + \sum_{k=1}^\infty [\mathbb{P}_0(\tau_{D_k} \leqslant t < \tau_{D_{k+1}})]^{1/\xi}[I(D_{k+1},t,\eta V^\omega,\delta,a)]^{1/\eta}. \end{aligned}$$

In particular, letting $\eta \to \infty$ (i.e., $\xi \to 1$), the estimate above is reduced to Proposition 3.1.

3.1.2. *Lower bounds*

LEMMA 3.1. *For any bounded domain $D \subset \mathbb{R}^d$, for any $0 < \delta < t$ and $a, b > 1$ with $1/a + 1/b = 1$, and for $\mathbb{Q}$-almost every $\omega \in \Omega$,*

$$\int_D \mathbb{E}_x\Big[\exp\Big(-\int_0^t V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D > t\}}\Big]\,dx \geqslant p(\delta,0)^{-1}p(t,0)^{-ab^{-1}}|D|^{-2ab^{-1}} \\ \times \exp(-a(t+\delta)\lambda_{a^{-1}V^\omega,D}).$$

Proof. We start from (2.9), i.e.,

$$e^{-t\lambda_{V^\omega,D}} \leqslant \int_D p(t,x,x)\mathbb{E}_{0,x}^{t,x}\Big[\exp\Big(-\int_0^t V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D>t\}}\Big]\,dx, \qquad t>0.$$

Replacing t and V^ω by $t+\delta$ and $a^{-1}V^\omega$ respectively in the inequality above, we see by the Hölder inequality that for all $a,b>1$ with $1/a+1/b=1$,

$$\begin{aligned}
&e^{-(t+\delta)\lambda_{a^{-1}V^\omega,D}}\\
&\quad\leqslant \int_D p(t+\delta,x,x)\mathbb{E}_{0,x}^{t+\delta,x}\Big[\exp\Big(-a^{-1}\int_0^{t+\delta} V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D>t+\delta\}}\Big]\,dx\\
&\quad\leqslant \Big(\int_D p(t+\delta,x,x)\mathbb{E}_{0,x}^{t+\delta,x}\Big[\exp\Big(-\int_0^{t} V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D>t+\delta\}}\Big]\,dx\Big)^{1/a}\\
&\qquad\times\Big(\int_D p(t+\delta,x,x)\mathbb{E}_{0,x}^{t+\delta,y}\Big[\exp\Big(-\frac{b}{a}\int_t^{t+\delta} V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D>t+\delta\}}\Big]\,dx\Big)^{1/b}\\
&\quad=: I_1\times I_2.
\end{aligned}$$

On the one hand, by (2.7),

$$\begin{aligned}
I_1 &\leqslant \Big(\int_D p(t+\delta,x,x)\mathbb{E}_{0,x}^{t+\delta,x}\Big[\exp\Big(-\int_0^{t} V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D>t\}}\Big]\,dx\Big)^{1/a}\\
&= \Big(\int_D \mathbb{E}_x\Big[\exp\Big(-\int_0^{t} V^\omega(Z_s)\,ds\Big)p(\delta,x-Z_t)\mathbb{1}_{\{\tau_D>t\}}\Big]\,dx\Big)^{1/a}\\
&\leqslant p(\delta,0)^{1/a}\Big(\int_D \mathbb{E}_x\Big[\exp\Big(-\int_0^{t} V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D>t\}}\Big]\,dx\Big)^{1/a}.
\end{aligned}$$

On the other hand, also due to (2.7) (see also [16, Theorem 2.2(iii)]),

$$\begin{aligned}
I_2 &\leqslant \Big(\int_D \mathbb{E}_x\Big[\exp\Big(-\frac{b}{a}\int_t^{t+\delta} V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D>t+\delta\}}\Big]\,dx\Big)^{1/b}\\
&= \Big(\int_D\int_D p^D(t,x,y)\mathbb{E}_y\Big[\exp\Big(-\frac{b}{a}\int_0^{\delta} V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D>\delta\}}\Big]\,dy\,dx\Big)^{1/b}\\
&\leqslant p(t,0)^{1/b}|D|^{1/b}\Big(\int_D \mathbb{E}_y\Big[\exp\Big(-\frac{b}{a}\int_0^{\delta} V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D>\delta\}}\Big]\,dy\Big)^{1/b}\\
&\leqslant p(t,0)^{1/b}|D|^{2/b}.
\end{aligned}$$

Combining both estimates above, we find that $\mathbb{Q}$-almost surely

$$\begin{aligned}
&\int_D \mathbb{E}_x\Big[\exp\Big(-\int_0^t V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D>t\}}\Big]\,dt\\
&\qquad\qquad \geqslant p(\delta,0)^{-1}p(t,0)^{-a/b}|D|^{-2a/b}\exp(-a(t+\delta)\lambda_{a^{-1}V^\omega,D}).
\end{aligned}$$

The proof is complete. ∎

PROPOSITION 3.2. *For any bounded domain $D \subset \mathbb{R}^d$ with $0 \in D$, for every subdomain $D_1 \subset D$, for all $0 < \delta < t$, $a, b > 1$ with $1/a + 1/b = 1$ and for $\mathbb{Q}$-almost every $\omega \in \Omega$,*

$$\begin{aligned} u^\omega(t,0) &\geqslant \mathbb{E}_0\Big[\exp\Big(-\int_0^t V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D>t\}}\Big] \\ &\geqslant p(\delta,0)^{-a}p(t-\delta,0)^{-a^2/b}|D_1|^{-2a^2/b} \\ &\quad\times\bigg(\mathbb{E}_0\bigg[\exp\bigg(\frac{b}{a}\int_0^\delta V^\omega(Z_s)\,ds\bigg)\mathbb{1}_{\{\tau_D>\delta\}}\bigg]\bigg)^{-a/b} \\ &\quad\times\Big(\inf_{x\in D_1} p^D(\delta,0,x)\Big)^a \exp(-a^2t\lambda_{a^{-2}V^\omega,D_1}). \end{aligned}$$

Proof. For $0 < \delta < t$, by the Hölder inequality, for any $a, b > 1$ with $1/a+1/b = 1$ we have

$$\begin{aligned} &\mathbb{E}_0\Big[\exp\Big(-a^{-1}\int_\delta^t V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D>t\}}\Big] \\ &\qquad\leqslant\Big(\mathbb{E}_0\Big[\exp\Big(-\int_0^t V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D>t\}}\Big]\Big)^{1/a} \\ &\qquad\quad\times\bigg(\mathbb{E}_0\bigg[\exp\bigg(\frac{b}{a}\int_0^\delta V^\omega(Z_s)\,ds\bigg)\mathbb{1}_{\{\tau_D>\delta\}}\bigg]\bigg)^{1/b}. \end{aligned}$$

Note that, according to the Markov property,

$$\begin{aligned} &\mathbb{E}_0\Big[\exp\Big(-a^{-1}\int_\delta^t V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D>t\}}\Big] \\ &\quad= \int_D p^D(\delta,0,x)\mathbb{E}_x\Big[\exp\Big(-a^{-1}\int_0^{t-\delta} V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D>t-\delta\}}\Big]\,dx \\ &\quad\geqslant\Big(\inf_{x\in D_1} p^D(\delta,0,x)\Big)\int_{D_1}\mathbb{E}_x\Big[\exp\Big(-a^{-1}\int_0^{t-\delta} V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_{D_1}>t-\delta\}}\Big]\,dx. \end{aligned}$$

Hence,

$$\begin{aligned} &\mathbb{E}_0\Big[\exp\Big(-\int_0^t V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_D>t\}}\Big] \\ &\quad\geqslant\Big(\inf_{x\in D_1} p^D(\delta,0,x)\Big)^a\Big(\int_{D_1}\mathbb{E}_x\Big[\exp\Big(-a^{-1}\int_0^{t-\delta} V^\omega(Z_s)\,ds\Big)\mathbb{1}_{\{\tau_{D_1}>t-\delta\}}\Big]\,dx\Big)^a \\ &\quad\quad\times\bigg(\mathbb{E}_0\bigg[\exp\bigg(\frac{b}{a}\int_0^\delta V^\omega(Z_s)\,ds\bigg)\mathbb{1}_{\{\tau_D>\delta\}}\bigg]\bigg)^{-a/b} \\ &\quad\geqslant\Big(\inf_{x\in D_1} p^D(\delta,0,x)\Big)^a p(\delta,0)^{-a}p(t-\delta,0)^{-a^2b^{-1}}|D_1|^{-2a^2b^{-1}} \\ &\quad\quad\times\exp(-a^2t\lambda_{a^{-2}V^\omega,D_1})\bigg(\mathbb{E}_0\bigg[\exp\bigg(\frac{b}{a}\int_0^\delta V^\omega(Z_s)\,ds\bigg)\mathbb{1}_{\{\tau_D>\delta\}}\bigg]\bigg)^{-a/b}, \end{aligned}$$

where in the last inequality we have used Lemma 3.1. The proof is finished. ∎

3.2. Estimates for the principal Dirichlet eigenvalue. In order to apply Propositions 3.1 and 3.2 to obtain explicit quenched asymptotics for $u^\omega(t,0)$, we need to estimate the principal Dirichlet eigenvalue $\lambda_{V^\omega,D}$.

It is known that the large time asymptotic behavior of solutions to (1.1) is closely connected to the integrated density of states of the random Schrödinger operator $H^\omega = -L + V^\omega$, which is defined by

$$N(\lambda) = \lim_{R\to\infty} \frac{1}{(2R)^d} \mathbb{E}_{\mathbb{Q}}[\sharp\{k \in \mathbb{N} : \lambda_k^{V^\omega,B(0,R)} \leqslant \lambda\}] \tag{3.2}$$

with $\lambda_k^{V^\omega,B(0,R)}$ being the kth smallest eigenvalue of H^ω with the Dirichlet conditions on $B(0,R)^c$. See [16, Section 5] for the existence of the limit above. Indeed, the existence of the limit in (3.2) was proved by using the spatial superadditivity property of $\mathbb{E}_{\mathbb{Q}}[\sharp\{k \in \mathbb{N} : \lambda_k^{V^\omega,B(0,R)} \leqslant \lambda\}]$, and so it is in fact the supremum over $R > 0$. Furthermore, it was observed that the Laplace transform of $N(\lambda)$ shares the large time behavior with the expectation of $u^\omega(t,x)$ given by (1.4) on $(\Omega,\mathbb{Q})$; see [8] and references therein. Then, in this sense an appropriate Tauberian theorem can be used to derive the information on the tail of $N(\lambda)$ as $\lambda \to 0$ from the large time behavior of $\mathbb{E}_{\mathbb{Q}}[u^\omega(t,x)]$. Due to the corresponding Abelian theorem, the converse is also true. We note that the study of $N(\lambda)$ requires the use of the associated pinned process rather than the symmetric Lévy process Z itself; see [16, Sections 5 and 6].

3.2.1. *Lower bounds of $\lambda_{V^\omega,B(0,R)}$ for R large enough.* To estimate lower bounds of $\lambda_{V^\omega,B(0,R)}$, we now recall some known results about the integral density $N(\lambda)$ of states of the random Schrödinger operator $H^\omega = -L + V^\omega$ defined by (3.2). It has been proved in [16, Theorems 6.2 and 6.3] that

$$\lim_{\lambda\to 0} \lambda^{d/(\beta\wedge\alpha)} \log N(\lambda) = -k_0,$$

where

$$k_0 := \begin{cases} \rho\lambda_{(\alpha)}(B(0,1))^{d/\alpha}, & \text{case (L)}, \\ \frac{\beta}{d+\beta}\left(\frac{d}{d+\beta}\right)^{d/\beta}\left(\Gamma\left(\frac{\beta}{d+\beta}\right)\rho w_d\right)^{(d+\beta)/\beta} K^{d/\beta}, & \text{case (H)}, \end{cases} \tag{3.3}$$

where

$$\lambda_{(\alpha)}(B(0,1)) = \inf_{\text{open } U, |U|=w_d} \lambda_1^{(\alpha)}(U),$$

w_d is the volume of the unit ball $B(0,1)$, and $\lambda_1^{(\alpha)}(U)$ is the principal Dirichlet eigenvalue for the generator of the symmetric α-stable process on U with exponent $\psi^{(\alpha)}(\xi)$ given in (2.1). In particular, when this symmetric α-stable process is isotropic, $\lambda_{(\alpha)}(B(0,1)) = w_d^{\alpha/d}\lambda_1^{(\alpha)}(B(0,1))$. With this at hand, we can see from the arguments in [8, (2.3)–(2.6)] that for any $\varepsilon \in (0,1)$, $\mathbb{Q}$-almost surely there is $R_\varepsilon(\omega) > 0$ such that for every $R \geqslant R_\varepsilon(\omega)$,

$$\lambda_{V^\omega, B(0,R)} \geqslant (1-\varepsilon)\left(\frac{k_0}{d\log R}\right)^{(\alpha\wedge\beta)/d}. \tag{3.4}$$

3.2.2. *Upper bounds of $\lambda_{V^\omega,B(z,r)}$ for r large enough with some z.* The following proposition is crucial for lower bounds of the quenched asymptotic of $u^\omega(t,0)$.

PROPOSITION 3.3. *The following two statements hold:*

(i) *In the light tailed case* (L)*, for any $\kappa > 1$ and $\eta, \varsigma \in (0,1)$, $\mathbb{Q}$-almost surely there exists $r_{\kappa,\eta,\varsigma}(\omega) > 0$ such that for all $r \geqslant r_{\kappa,\eta,\varsigma}(\omega)$, there is $z := z(r,\omega) \in \mathbb{R}^d$ with $|z| \leqslant M_{\kappa,\eta}(r)$,*

$$\lambda_{V^\omega, B(z,r)} \leqslant (1+\varsigma)\lambda_1^{(\alpha)}(B(0,1))r^{-\alpha}, \tag{3.5}$$

where

$$M_{\kappa,\eta}(r) = r^{-\kappa}\exp\left(\frac{w_d\rho}{d}((1+2\eta)r)^d\right),$$

and $\lambda_1^{(\alpha)}(B(0,1))$ is the principal Dirichlet eigenvalue for the generator of the symmetric α-stable process on $B(0,1)$ and with the exponent $\psi^{(\alpha)}(\xi)$ given in (2.1)*.*

(ii) *In the heavy tailed case* (H)*, for any $l > 1$ large enough, $\kappa > 1$ and $\varsigma \in (0,1)$, $\mathbb{Q}$-almost surely there exists $r_{l,\kappa,\varsigma}(\omega) > 0$ such that for all $r \geqslant r_{l,\kappa,\varsigma}(\omega)$, there is $z := z(r,\omega) \in \mathbb{R}^d$ with $|z| \leqslant M_\kappa(r)$,*

$$\lambda_{V^\omega, B(z, lr^{\beta/\alpha})} \leqslant (1+\varsigma)q_1 r^{-\beta}, \tag{3.6}$$

where $M_\kappa(r) = r^{-\kappa}e^{r^d}$ and

$$q_1 = \frac{d}{d+\beta}\left(\frac{\beta}{d(d+\beta)}\right)^{\beta/d}\left[\rho w_d \Gamma\left(\frac{\beta}{d+\beta}\right)\right]^{(d+\beta)/d} K. \tag{3.7}$$

Proof. The proof of assertion (ii) is a little more delicate, and we postpone it to the Appendix. Here we only give the proof of (i). Note that the argument for (i) with some modifications works for the critical case; see Proposition 5.2. Fix $\kappa > 1$ and $\eta \in (0,1)$, and set $I_r := ((2(1+\eta)r)\mathbb{Z}^d) \cap \{z \in \mathbb{R}^d : |z| \leqslant M_{\kappa,\eta}(r)\}$ for any $r > 0$. Define $\varphi_0(r) = \sup_{|x|\geqslant r}\varphi(x)$ for all $r \geqslant 0$ and $\varphi_0(x) = \varphi_0(|x|)$ for $x \in \mathbb{R}^d$. It is clear that $\varphi(x) \leqslant \varphi_0(x)$ for all $x \in \mathbb{R}^d$, and $\varphi_0(r)$ is a decreasing function on $[0,\infty)$ such that

$$\lim_{r\to\infty}\varphi_0(r)r^{d+\alpha} = 0. \tag{3.8}$$

For any $z \in I_r$ and $\varepsilon \in (0,1)$, define

$$F_r(z) = \{\text{the ball } B(z,(1+\eta)r) \text{ contains at least one Poisson point}\},$$
$$G_r(z) = \Big\{ \sup_{y \in B(z,r)} \sum_{\omega_i \notin B(z,(1+\eta)r)} \varphi_0(y-\omega_i) \geqslant \varepsilon r^{-\alpha} \Big\}.$$

We will estimate $\mathbb{Q}(\bigcap_{z\in I_r}(F_r(z) \cup G_r(z)))$.

Note that $\{F_r(z)\}_{z\in I_r}$ are i.i.d., and $\mathbb{Q}(F_r(0)) = 1 - e^{-w_d\rho((1+\eta)r)^d}$. Hence, there is $r_0(\kappa,\eta) > 0$ such that for all $r \geqslant r_0(\kappa,\eta)$,

$$\begin{aligned}
\mathbb{Q}\Big(\bigcap_{z\in I_r} F_r(z)\Big) &\leqslant (1-e^{-w_d\rho((1+\eta)r)^d})^{(\frac{M_{\kappa,\eta}(r)}{2(1+\eta)r})^d} \\
&\leqslant \exp\bigg(-e^{-w_d\rho((1+\eta)r)^d}\Big(\frac{M_{\kappa,\eta}(r)}{2(1+\eta)r}\Big)^d\bigg) \\
&\leqslant \exp\big(-2^{-d}(1+\eta)^{-d} r^{-d(1+\kappa)} e^{-w_d\rho((1+\eta)r)^d} e^{w_d\rho((1+2\eta)r)^d}\big) \\
&\leqslant \exp(-r^d),
\end{aligned}$$

where the first inequality follows from $\sharp I_r \geqslant \big[\frac{M_{\kappa,\eta}(r)}{(1+\eta)r}\big]^d \geqslant \big(\frac{M_{\kappa,\eta}(r)}{2(1+\eta)r}\big)^d$ for $r \geqslant 1$ large enough, and in the second inequality we have used the fact that $1-x \leqslant e^{-x}$ for all $x>0$.

On the other hand, for $y \in B(0,r)$ and $\omega_i \notin B(0,(1+\eta)r)$, we find that $|y-\omega_i| \geqslant \eta|\omega_i|/(1+\eta)$. By the fact that $\varphi_0(x) = \varphi_0(|x|)$ and the decreasing property of $\varphi_0(r)$,

$$\sup_{y\in B(0,r)} \sum_{\omega_i \notin B(0,(1+\eta)r)} \varphi_0(y-\omega_i) \leqslant \sum_{\omega_i \notin B(0,(1+\eta)r)} \varphi_0(\eta|\omega_i|/(1+\eta)).$$

Hence,

$$\begin{aligned}
&\mathbb{Q}\bigg[\exp\bigg(\frac{1}{\varphi_0(\eta r)} \sup_{y\in B(0,r)} \sum_{\omega_i\notin B(0,(1+\eta)r)} \varphi_0(y-\omega_i)\bigg)\bigg] \\
&\qquad \leqslant \mathbb{Q}\bigg[\exp\bigg(\frac{1}{\varphi_0(\eta r)} \sum_{\omega_i\notin B(0,(1+\eta)r)} \varphi_0(\eta|\omega_i|/(1+\eta))\bigg)\bigg] \\
&\qquad = \exp\bigg(\rho \int_{\mathbb{R}^d\setminus B(0,(1+\eta)r)} (e^{\varphi_0(\eta r)^{-1}\varphi_0(\eta|z|/(1+\eta))} - 1)\,dz\bigg) \\
&\qquad \leqslant \exp\bigg(e\rho(1+\eta)^d \int_{\mathbb{R}^d\setminus B(0,r)} \frac{\varphi_0(\eta z)}{\varphi_0(\eta r)}\,dz\bigg),
\end{aligned}$$

where in the last inequality we have used the fact that $e^x - 1 \leqslant ex$ for all $x \in (0,1]$. By (3.8), for any $\varepsilon \in (0,1)$ there is a constant $r_1(\eta,\varepsilon) \geqslant r_0(\kappa,\eta)$ such that for all $r \geqslant r_1(\eta,\varepsilon)$,

$$\varphi_0(\eta r) \leqslant \varepsilon^2(\eta r)^{-d-\alpha} \tag{3.9}$$

and so

$$\mathbb{Q}\left[\exp\left(\frac{1}{\varphi_0(\eta r)}\sup_{y\in B(0,r)}\sum_{\omega_i\notin B(0,(1+\eta)r)}\varphi_0(y-\omega_i)\right)\right]\leqslant\exp\left[\frac{c_1\varepsilon^2 r^{-\alpha}}{\varphi_0(\eta r)}\right],$$

where $c_1 := c_1(\eta) > 0$ depends on η but is independent of ε and r.

Below we let $\varepsilon \in (0, 1\wedge(1/(2c_1))$. Hence, according to the Markov inequality and (3.9), for r large enough,

$$\begin{aligned}
&\mathbb{Q}(G_r(0))\\
&\quad\leqslant\mathbb{Q}\left[\exp\left(\frac{1}{\varphi_0(\eta r)}\sup_{y\in B(0,r)}\sum_{\omega_i\notin B(0,(1+\eta)r)}\varphi_0(y-\omega_i)\right)\geqslant\exp\left(\frac{\varepsilon r^{-\alpha}}{\varphi_0(\eta r)}\right)\right]\\
&\quad\leqslant\exp\left[\frac{c_1\varepsilon^2 r^{-\alpha}}{\varphi_0(\eta r)}-\frac{\varepsilon r^{-\alpha}}{\varphi_0(\eta r)}\right]\leqslant\exp\left[-\frac{\varepsilon r^{-\alpha}}{2\varphi_0(\eta r)}\right]\leqslant\exp(-\eta^{d+\alpha}r^d/(2\varepsilon)).
\end{aligned}$$

Since $\{G_r(z)\}_{z\in I_r}$ have the same distribution (but are not independent of each other), we find that for any $0<\varepsilon\leqslant\varepsilon_0:=\min\{1,1/(2c_1),\eta^{d+\alpha}/(4w_d\rho(1+2\eta)^d)\}$ and r large enough,

$$\begin{aligned}
\mathbb{Q}\Big(\bigcup_{z\in I_r}G_r(z)\Big)&\leqslant 2\left(\frac{M_{\kappa,\eta}(r)}{(1+\eta)r}\right)^d\exp(-\eta^{d+\alpha}r^d/(2\varepsilon))\\
&=2(1+\eta)^{-d}r^{-(1+\kappa)d}\exp\big(w_d\rho(1+2\eta)^d r^d-\eta^{d+\alpha}r^d/(2\varepsilon)\big)\\
&\leqslant 2(1+\eta)^{-d}r^{-(1+\kappa)d}\exp\big(-\eta^{d+\alpha}r^d/(4\varepsilon)\big).
\end{aligned}$$

Combining with both estimates above, we find that for any $\varepsilon\in(0,\varepsilon_0]$ and any r large enough,

$$\begin{aligned}
\mathbb{Q}\Big(\bigcap_{z\in I_r}(F_r(z)\cup G_r(z))\Big)&\leqslant\mathbb{Q}\Big(\bigcap_{z\in I_r}F_r(z)\Big)+\mathbb{Q}\Big(\bigcup_{z\in I_r}G_r(z)\Big)\\
&\leqslant\exp(-r^d)+2(1+\eta)^{-d}r^{-(1+\kappa)d}\exp(-\eta^{d+\alpha}r^d/(4\varepsilon))\\
&\leqslant c_2\exp(-c_3 r^d),
\end{aligned}$$

where $c_2, c_3 > 0$ (which depend on $\eta, \kappa, \varepsilon$). The Borel–Cantelli lemma implies that $\mathbb{Q}$-almost surely there exists $r_{\kappa,\eta,\varepsilon}(\omega) > 0$ such that for all $r \geqslant r_{\kappa,\eta,\varepsilon}(\omega)$, there is $z := z(r,\omega)\in\mathbb{R}^d$ with $|z|\leqslant M_{\kappa,\eta}(r)$ such that both $F_r(z)$ and $G_r(z)$ fail to hold.

Below, we fix this z for all $r\geqslant r_{\kappa,\eta,\varepsilon}(\omega)$. Since $G_r(z)$ fails to occur,

$$\sup_{y\in B(z,r)}\sum_{\omega_i\notin B(z,(1+\eta)r)}\varphi_0(y-\omega_i)\leqslant\varepsilon r^{-\alpha},$$

and so, also thanks to $\varphi(x)\leqslant\varphi_0(x)$, $\lambda_{V^\omega,B(z,r)}\leqslant\lambda_{\tilde V^\omega,B(z,r)}+\varepsilon r^{-\alpha}$, where

$$\tilde V^\omega(x)=\sum_{\omega_i\in B(z,(1+\eta)r)}\varphi_0(x-\omega_i).$$

On the other hand, because $F_r(z)$ does not happen, $\tilde V^\omega(x)=0$ for all $x\in\mathbb{R}^d$, and so $\lambda_{\tilde V^\omega,B(z,r)}=\lambda_1(B(z,r))$. Therefore, for any $\varsigma\in(0,1)$ and $r\geqslant r_{\kappa,\eta,\varepsilon}(\omega)$ large enough,

$$\begin{aligned}\lambda_{V^\omega,B(z,r)}&\leqslant \lambda_1(B(z,r))+\varepsilon r^{-\alpha}=\lambda_1(B(0,r))+\varepsilon r^{-\alpha}\\&\leqslant (1+\varsigma/2)r^{-\alpha}\lambda_1^{(\alpha)}(B(0,1))+\varepsilon r^{-\alpha},\end{aligned}$$

where in the last inequality we have used Lemma 3.2 below. The proof is completed by taking $\varepsilon\leqslant\min\{\varepsilon_0,\varsigma\lambda_1^{(\alpha)}(B(0,1))/2\}$. ∎

The following was proved in [11, Proposition 5.1].

LEMMA 3.2. *Let Z be a symmetric Lévy process satisfying* (1.7)*, and $\lambda_1(D)$ be the principal Dirichlet eigenvalue for the generator of the process Z on D. Then, for any fixed $\varsigma>0$, there is $r_0:=r_0(\varsigma)>0$ such that for all $r\geqslant r_0$,*

$$\lambda_1(B(0,r))\leqslant(1+\varsigma)r^{-\alpha}\lambda_1^{(\alpha)}(B(0,1)),$$

where $\lambda_1^{(\alpha)}(B(0,1))$ is the principal Dirichlet eigenvalue for the generator of the symmetric α-stable process $Z^{(\alpha)}$ on $B(0,1)$ with characteristic exponent $\psi^{(\alpha)}(\xi)$ given in (2.1).

REMARK 3.2. Below we will consider $a^{-2}V^\omega$ with $a>1$ instead of V^ω. Here, we record the following conclusions for the potential $a^{-2}V^\omega$, which immediately follow from the proof of Proposition 3.3.

(i) In the light tailed case (L), for any $a>1$, (3.5) holds for $\lambda_{a^{-2}V^\omega,B(z,r)}$ in place of $\lambda_{V^\omega,B(z,r)}$ with some $\kappa>1$ and $\eta,\varsigma\in(0,1)$ (independent of a) and for all $r\geqslant r_{\kappa,\eta,\varsigma,a}(\omega)$ (which depends on a).

(ii) In the heavy tailed case (H), for any $a>1$, (3.6) holds for $\lambda_{a^{-2}V^\omega,B(z,r)}$ in place of $\lambda_{V^\omega,B(z,r)}$ with $\kappa,l>1$ and $\varsigma\in(0,1)$ (all of which are independent of a),

$$q_1^*=a^{-2}q_1=\frac{d}{d+\beta}\left(\frac{\beta}{d(d+\beta)}\right)^{\beta/d}\left[\rho w_d\Gamma\left(\frac{\beta}{d+\beta}\right)\right]^{(d+\beta)/d}a^{-2}K$$

(in place of q_1), and for all $r\geqslant r_{l,\kappa,\varsigma,a}(\omega)$ (which depends on a).

3.3. Quenched estimates of $u^\omega(t,0)$

THEOREM 3.1. *Let ϕ be an increasing function on $[1,\infty)$ with $\phi(1)\geqslant1$. For any $t,R\geqslant1$ with $R\geqslant\phi(t)$, set*

$$\Phi(t,R):=\mathbb{P}_0(\tau_{B(0,R)}\leqslant t). \tag{3.10}$$

Let k_0 be the constant defined in (3.3)*. Then the following statements hold:*

(i) *In the light tailed case* (L), *for any* $\varepsilon > 0$ *there is a constant* $C(\varepsilon) > 0$ *such that* $\mathbb{Q}$*-almost surely there exists a random variable* $R_\varepsilon(\omega) \geqslant 1$ *with the property that for any* $R \geqslant \max\{R_\varepsilon(\omega), \phi(t)\}$ *and* $t \geqslant 1$,

$$u^\omega(t,0) \leqslant \Phi(t,R) + C(\varepsilon) R^{d/2} \exp\left(-t(1-2\varepsilon)\left(\frac{k_0}{d\log R}\right)^{\alpha/d}\right).$$

(ii) *In the heavy tailed case* (H), *for any* $\varepsilon > 0$ *and* $a > 1$ *there is a constant* $C(\varepsilon, a) > 0$ *such that* $\mathbb{Q}$*-almost surely there exists a random variable* $R_{\varepsilon,a}(\omega) \geqslant 1$ *with the property that for any* $R \geqslant \max\{R_{\varepsilon,a}(\omega), \phi(t)\}$ *and* $t \geqslant 1$,

$$u^\omega(t,0) \leqslant \Phi(t,R) + C(\varepsilon,a) R^{d/a} \exp\left(-t(1-2\varepsilon)\left(\frac{k_0}{d\log R}\right)^{\beta/d}\right).$$

Proof. It is clear that $\Phi(v_1, v_2)$ is a non-negative function defined on $[1,\infty)^2$ such that $v_1 \mapsto \Phi(v_1, v_2)$ is increasing for fixed v_2 and $v_2 \mapsto \Phi(v_1, v_2)$ is decreasing for fixed v_1.

We first consider the light tailed case. According to Proposition 3.1 with δ small enough and (3.4), for any $\varepsilon > 0$, $\mathbb{Q}$-almost surely there is $R_\varepsilon(\omega) \geqslant 1$ such that for any $t \geqslant 1$ and $R \geqslant \max\{R_\varepsilon(\omega), \phi(t)\}$,

$$u^\omega(t,0) \leqslant \Phi(t,R) + C_1(\varepsilon) R^{d/2} \exp\left(-t(1-2\varepsilon)\left(\frac{k_0}{d\log R}\right)^{\alpha/d}\right).$$

In the heavy tailed case, we note that, from the argument for (3.4), for all $\varepsilon \in (0,1)$ and $a > 1$, $\mathbb{Q}$-almost surely there is $R_{\varepsilon,a}(\omega) \geqslant 1$ such that for every $R \geqslant R_{\varepsilon,a}(\omega)$,

$$a^{-1}\lambda_{aV^\omega, B(0,R)} \geqslant (1-\varepsilon)\left(\frac{k_0}{d\log R}\right)^{\beta/d},$$

where the right hand side is independent of a. With this, we can obtain the desired assertion by following the arguments in the light tailed case. ∎

THEOREM 3.2. *Assume that for any* $\delta \in (0,1/2)$ *and* $r \geqslant 1$,

$$\inf_{z\in B(0,r)} p^{B(0,2r)}(\delta, 0, z) \geqslant \Psi_\delta(r), \tag{3.11}$$

where Ψ_δ *is a non-negative decreasing function on* $[1,\infty)$. *Then the following statements hold:*

(i) *In the light tailed case* (L), *for any* $\delta \in (0,1/2)$, $\kappa, a > 1$, $\eta, \varsigma \in (0,1)$, $\mathbb{Q}$*-almost surely there is* $R_{\kappa,a,\eta,\varsigma}(\omega) \geqslant 1$ *such that for any* $R \geqslant R_{\kappa,a,\eta,\varsigma}(\omega)$ *and* $t \geqslant 1$,

$$\begin{aligned} u^\omega(t,0) \geqslant{}& C(\kappa,\delta,\eta,a) M_{\kappa,\eta}(R)^{-4\delta d}[\Psi_\delta(2M_{\kappa,\eta}(R))]^a \\ &\times \exp\big(-a^2(1+\varsigma)\lambda_1^{(\alpha)}(B(0,1)) t R^{-\alpha}\big), \end{aligned}$$

where

$$M_{\kappa,\eta}(R) = R^{-\kappa} \exp\left(\frac{w_d \rho}{d} ((1+2\eta)R)^d \right),$$

and $\lambda_1^{(\alpha)}(B(0,1))$ *is the principal Dirichlet eigenvalue for the generator of the symmetric* α*-stable process* $Z^{(\alpha)}$ *on* $B(0,1)$ *with characteristic exponent* $\psi^{(\alpha)}(\xi)$ *given in* (2.1).

(ii) *In the heavy tailed case* (H)*, for any* $\delta \in (0,1/2)$*,* $\kappa > 1$ *large enough,* $a > 1$ *and* $\varsigma \in (0,1)$*,* $\mathbb{Q}$*-almost surely there is* $R_{\kappa,a,\varsigma}(\omega) \geqslant 1$ *such that for any* $R \geqslant R_{\kappa,a,\varsigma}(\omega)$ *and* $t \geqslant 1$*,*

$$u^\omega(t,0) \geqslant C(\kappa,\delta,\varsigma,a) M_\kappa(R)^{-4\delta d} [\Psi_\delta(2M_\kappa(R))]^a \exp(-(1+\varsigma) q_1 t R^{-\beta}),$$

where $M_\kappa(R) = R^{-\kappa}\exp(R^d)$ *and* q_1 *is given by* (3.7).

Proof. We only prove assertion (i), since (ii) can be verified similarly by applying Proposition 3.3(ii) and Remark 3.2(ii) instead of Proposition 3.3(i) and Remark 3.2(i), respectively.

For any $a, \kappa > 1$ and $\eta, \varsigma \in (0,1)$, let $D = B(0, 2M_{\kappa,\eta}(r))$ and $D_1 = B(z,(1+\eta)r)$ for $r \geqslant r_{\kappa,\eta,\varsigma,a}(\omega)$, where $r_{\kappa,\eta,\varsigma,a}(\omega)$, $M_{\kappa,\eta}(r)$ and $z := z(r,\omega)$ are given in Proposition 3.3(i) and Remark 3.2(i). Since $|z| \leqslant M_{\kappa,\eta}(r)$, we have $D_1 \subset D$ for r large enough. Then, according to Propositions 3.2 and 3.3(i) as well as Remark 3.2(i), for any $\delta \in (0,1/2)$, $t \geqslant 1$ and r large enough,

$$\begin{aligned}
u^\omega(t,0) &\geqslant p(\delta,0)^a p(t-\delta,0)^{-a^2/b} (w_d((1+\eta)r)^d)^{-2a^2/b} \\
&\quad \times \exp(-3d\delta \log(2M_{\kappa,\eta}(r)))[\Psi_\delta(2M_{\kappa,\eta}(r))]^a \\
&\quad \times \exp\big(-a^2(1+\varsigma)tr^{-\alpha}\lambda_1^{(\alpha)}(B(0,1))\big) \\
&\geqslant C_1(\delta,\eta,a) r^{-2a^2 d/b} M_{\kappa,\eta}(r)^{-3\delta d} [\Psi_\delta(2M_{\kappa,\eta}(r))]^a \\
&\quad \times \exp\big(-a^2(1+\varsigma)tr^{-\alpha}\lambda_1^{(\alpha)}(B(0,1))\big) \\
&\geqslant C_2(\kappa,\delta,\eta,a) M_{\kappa,\eta}(r)^{-4\delta d} [\Psi_\delta(2M_{\kappa,\eta}(r))]^a \\
&\quad \times \exp\big(-a^2(1+\varsigma)tr^{-\alpha}\lambda_1^{(\alpha)}(B(0,1))\big),
\end{aligned}$$

where in the first inequality $b > 1$ is such that $1/a + 1/b = 1$ and we have used (2.4) and (3.11), and the second inequality follows from the fact that for all $t \geqslant 1$ and $\delta \in (0,1/2)$, $p(t-\delta,0) \leqslant p(\delta,0)$, since $t \mapsto p(t,0)$ is decreasing. The proof is finished. ■

4. PROOFS OF THEOREMS 1.1 AND 1.2

In this section, we will present the proofs of Theorems 1.1 and 1.2. Note that, both symmetric Lévy processes in Theorems 1.1 and 1.2 are rotationally invariant and satisfy the assumptions of Subsection 2.1. Moreover, the shape function $\varphi(x) = 1 \wedge |x|^{-d-\beta}$ with $\beta \in (0,\infty]$ fulfills the assumptions of Subsection 2.2 as well.

4.1. Rotationally symmetric α-stable processes

Proof of Theorem 1.1. For a rotationally symmetric α-stable process Z with $\alpha \in (0,2)$, we have $\psi(\xi) = c_0|\xi|^\alpha$ for some $c_0 > 0$, and so (1.7) holds with $\psi^{(\alpha)}(\xi) = \psi(\xi)$. Thus, for the shape function $\varphi(x) = 1 \wedge |x|^{-d-\beta}$ with $\beta \in (0,\infty]$, the light tailed case (resp. the heavy tailed case) corresponds to $\beta > \alpha$ (resp. $\beta \in (0,\alpha)$). Furthermore, it is well known that, for rotationally symmetric α-stable process Z, (3.10) holds with $\Phi(t,r) \leqslant C^* t r^{-\alpha}$ and $\phi(t) = t^{1/\alpha}$, and (3.11) holds with

$$\Psi_\delta(r) \geqslant \frac{C_*\delta}{r^{d+\alpha}};$$

see [4, 6].

(i) We first consider $\beta > \alpha$, which is referred to as the light tailed case. According to Theorem 3.1(i), for any $\varepsilon > 0$, $\mathbb{Q}$-almost surely there is $R_\varepsilon(\omega) \geqslant 1$ such that for any $t \geqslant 1$ and $R \geqslant \max\{R_\varepsilon(\omega), t^{1/\alpha}\}$,

$$u^\omega(t,0) \leqslant \frac{C_1 t}{R^\alpha} + C_2(\varepsilon) R^{d/2} \exp\left(-t(1-2\varepsilon)\left(\frac{k_0}{d\log R}\right)^{\alpha/d}\right),$$

where $C_2(\varepsilon) > 0$ is a constant independent of R and t, and where $k_0 = \rho w_d [\lambda_1^{(\alpha)}(B(0,1))]^{d/\alpha}$ with $\lambda_1^{(\alpha)}(B(0,1))$ being the principal Dirichlet eigenvalue for the fractional Laplacian on $B(0,1)$. Letting

$$R = \exp\left((1-2\varepsilon)^{d/(\alpha+d)}(\alpha + d/2)^{-d/(\alpha+d)}\left(\frac{k_0}{d}\right)^{\alpha/(d+\alpha)} t^{d/(d+\alpha)}\right)$$

for t large enough, we arrive at the desired upper bound by letting $\varepsilon \to 0$.

On the other hand, by Theorem 3.2(i), for any $\kappa, a > 1$, $\delta \in (0,1/2)$ and $\eta, \varsigma \in (0,1)$, $\mathbb{Q}$-almost surely there is $R_{\kappa,a,\eta,\varsigma}(\omega) \geqslant 1$ such that for any $R \geqslant R_{\kappa,a,\eta,\varsigma}(\omega)$ and $t \geqslant 1$,

$$u^\omega(t,0) \geqslant C R^{(4\delta d + (d+\alpha)a)\kappa} \exp(-AR^d - BtR^{-\alpha}),$$

where

$$A = \frac{w_d \rho}{d}(1+2\eta)^d [a(d+\alpha) + 4\delta d], \qquad B = a^2(1+\varsigma)\lambda_1^{(\alpha)}(B(0,1))$$

and $C > 0$ is a constant independent of t and R. Letting

$$R = \left(\frac{\alpha B}{dA}\right)^{1/(d+\alpha)} t^{1/(d+\alpha)}$$

for t large enough, we prove the lower bound by taking $\delta, \eta, \varsigma \to 0$ and $a \to 1$.

(ii) For the heavy tailed case (i.e., $\beta \in (0,\alpha)$), it follows from Theorem 3.1(ii) that for all $a > 1$ and $\varepsilon \in (0,1)$, $\mathbb{Q}$-almost surely there is $R_{a,\varepsilon}(\omega) \geqslant 1$ such that for any $t \geqslant 1$ and $R \geqslant \max\{R_{a,\varepsilon}(\omega), t^{1/\alpha}\}$,

$$u^\omega(t,0) \leqslant \frac{C_1 t}{R^\alpha} + C_2(a,\varepsilon) R^{d/a} \exp\biggl(-t(1-2\varepsilon)\biggl(\frac{k_0}{d\log R}\biggr)^{\beta/d}\biggr),$$

where k_0 is given by (3.3) in case (H). Then, choosing

$$R = \exp\biggl((1-2\varepsilon)^{d/(\beta+d)}(\alpha + d/a)^{-d/(\beta+d)}\biggl(\frac{k_0}{d}\biggr)^{\beta/(d+\beta)} t^{d/(d+\beta)}\biggr)$$

for t large enough, we arrive at the upper bound by letting $\varepsilon \to 0$ and $a \to \infty$.

Due to Theorem 3.2(ii), for any $\kappa, a > 1$, $\delta \in (0,1/2)$ and $\varsigma \in (0,1)$, $\mathbb{Q}$-almost surely there is $R_{\kappa,a,\varsigma}(\omega) \geqslant 1$ such that for any $R \geqslant R_{\kappa,a,\varsigma}(\omega)$ and $t \geqslant 1$,

$$u^\omega(t,0) \geqslant C R^{(4\delta d + (d+\alpha)a)\kappa} \exp(-AR^d - BtR^{-\beta}),$$

where

$$A = a(d+\alpha) + 4\delta d, \qquad B = (1+\varsigma) q_1.$$

Letting

$$R = \biggl(\frac{\beta B}{dA}\biggr)^{1/(d+\beta)} t^{1/(d+\beta)}$$

for t large enough, we prove the lower bound by taking $\varsigma, \delta \to 0$ and $a \to 1$. ∎

REMARK 4.1. (i) We used two different ways to estimate I_1 in the proof of Proposition 3.1, which yield two different quenched upper bounds for $u^\omega(t,0)$ in Theorem 3.1. For the proof of Theorem 1.1, if we follow the argument for the light tailed case (i.e., $\beta \in (\alpha,\infty]$) when dealing with the heavy tailed case (i.e., $\beta \in (0,\alpha)$), we can only deduce that when $\beta \in (0,\alpha)$, for all $x \in \mathbb{R}^d$, $\mathbb{Q}$-almost surely,

$$\limsup_{t\to\infty} \frac{\log u^\omega(t,x)}{t^{d/(d+\beta)}} \leqslant -\frac{\alpha}{(\alpha + d/2)^{d/(d+\beta)}} A_2,$$

which is weaker than the desired assertion for the upper bound in Theorem 1.1(ii).

(ii) As mentioned in Section 1, the rate functions for the quenched and annealed asymptotics of $u^\omega(t,x)$ for rotationally symmetric α-stable processes are the same. However, we cannot infer that the associated liminf and limsup constants agree for the quenched asymptotics. The reason why our argument cannot yield precise results is that both estimates (3.10) and (3.11) are of the polynomial form for symmetric α-stable processes. The corresponding result in [11, Table 1, p. 165] has also this kind of gap between the upper and lower bounds.

4.2. Rotationally symmetric processes with large jumps of exponential decay

Proof of Theorem 1.2. For a rotationally symmetric pure jump Lévy process Z with Lévy measure ν given in Theorem 1.2, by [11, Proposition 5.2(i)], (2.2) holds with $\alpha = 2$ and $(a_{ij})_{1\leqslant i,j\leqslant d}$ defined by

$$a_{ii} = \int_{\mathbb{R}^d\setminus\{0\}} z_i^2\,\nu(dz) = \int_{\mathbb{R}^d\setminus\{0\}} z_i^2\rho(|z|)\,dz, \qquad a_{ij}=0, \qquad 1\leqslant i\neq j\leqslant d.$$

Thus, for the shape function $\varphi(x) = 1\wedge |x|^{-d-\beta}$ with $\beta\in(0,\infty]$, the light tailed case (resp. the heavy tailed case) corresponds to $\beta>2$ (resp. $\beta\in(0,2)$).

Furthermore, according to [5, Theorems 1.2 and 1.4], for any $t\geqslant 1$ and $x\in\mathbb{R}^d$ with $|x|\geqslant 2t^{1/((2-\theta)\vee 1)}$, the transition density function $p(t,x)$ of the process X satisfies

$$p(t,x)\leqslant c_1\exp\left(-c_2|x|^{\theta\wedge 1}\left(\log\frac{|x|}{t}\right)^{(\theta-1)^+/\theta}\right).$$

This along with Lemma 4.1 below implies (3.10) holds with $\Phi(t,r)\leqslant c_3\exp(-c_4 r^{\theta\wedge 1})$ and $\phi(t)=2t$. On the other hand, by [13, Theorem 1.1], we know that (3.11) holds with

$$\Psi_\delta(r)\geqslant c_5\exp(-c_6 r^{\theta\wedge 1}(\log r)^{(\theta-1)^+/\theta}).$$

For brevity, we only deal with the light tailed case (i.e., $\beta\in(2,\infty)$), since the heavy tailed case can be treated similarly. First, by Theorem 3.1(i), for any $\varepsilon>0$, $\mathbb{Q}$-almost surely there is $R_\varepsilon(\omega)\geqslant 1$ so that for any $t\geqslant 1$ and $R\geqslant \max\{R_\varepsilon(\omega),2t\}$,

$$u^\omega(t,0)\leqslant c_3\exp(-c_4R^{\theta\wedge 1}) + C_1(\varepsilon)R^{d/2}\exp\left(-t(1-2\varepsilon)\left(\frac{k_0}{d\log R}\right)^{2/d}\right),$$

where $C_1(\varepsilon)>0$ is a constant independent of R and t, and where $k_0 = \rho w_d[\lambda_1^{(2)}(B(0,1))]^{d/2}$ with $\lambda_1^{(2)}(B(0,1))$ being the principal Dirichlet eigenvalue for the generator of a Brownian motion killed upon exiting $B(0,1)$ and with the covariance matrix $(a_{ij})_{1\leqslant i,j\leqslant d}$ above. Letting $R=Ct^{1/(1\wedge\theta)}$ for large C and t, we prove the desired upper bound by taking $\varepsilon\to 0$.

On the other hand, according to Theorem 3.2(i), for any $\kappa,a>1$ and $\eta,\varsigma\in(0,1)$, $\mathbb{Q}$-almost surely there is $R_{\kappa,a,\eta,\varsigma}(\omega)\geqslant 1$ such that for any $R\geqslant R_{\kappa,a,\eta,\varsigma}(\omega)$ and $t\geqslant 1$,

$$\begin{aligned}
u^\omega(t,0) &\geqslant C_2\exp\big(-C_3(M_{\kappa,\eta}(R))^{a(\theta\wedge 1)}(\log M_{\kappa,\eta}(R))^{(\theta-1)^+/\theta}\big)\\
&\qquad\times\exp\big(-a^2(1+\varsigma)\lambda_1^{(2)}(B(0,1))tR^{-2}\big)\\
&\geqslant C_4\exp\left[-C_5R^{-\kappa a(\theta\wedge 1)+d(\theta-1)^+/\theta}\exp\left(a(\theta\wedge 1)\frac{w_d\rho}{d}((1+2\eta)R)^d\right)\right]\\
&\qquad\times\exp\big(-a^2(1+\varsigma)\lambda_1^{(2)}(B(0,1))tR^{-2}\big).
\end{aligned}$$

Choosing κ large enough and

$$R = \frac{1}{1+2\eta}\left(\frac{d}{a(1\wedge\theta)w_d\rho}\right)^{1/d}(\log t)^{1/d}$$

for t large enough, we prove the lower bound by taking $\eta, \varsigma \to 0$ and $a \to 1$. ■

LEMMA 4.1. *For any Lévy process Z and all $t, R > 0$,*

$$\mathbb{P}_0(\tau_{B(0,R)} \leqslant t) \leqslant 2 \sup_{s\in[t,2t]} \mathbb{P}_0(|Z_s| \geqslant R/2).$$

Proof. For any $t, R > 0$,

$$\begin{aligned}
\mathbb{P}_0(\tau_{B(0,R)} \leqslant t) &= \mathbb{P}_0\Big(\sup_{s\in(0,t]} |Z_s| \geqslant R\Big) \\
&= \mathbb{P}_0\Big(\sup_{s\in(0,t]} |Z_s| \geqslant R, |Z_{2t}| \geqslant R/2\Big) + \mathbb{P}_0\Big(\sup_{s\in(0,t]} |Z_s| \geqslant R, |Z_{2t}| \leqslant R/2\Big) \\
&\leqslant \mathbb{P}_0(|Z_{2t}| \geqslant R/2) + \mathbb{E}_0\big(\mathbb{1}_{\{\tau_{B(0,R)}\leqslant t\}}\mathbb{P}_{Z_{\tau_{B(0,R)}}}(|Z_{2t} - Z_{\tau_{B(0,R)}}| \geqslant R/2)\big) \\
&\leqslant 2 \sup_{s\in[t,2t]} \mathbb{P}_0(|Z_s| \geqslant R/2).
\end{aligned}$$

The proof is complete. ■

5. APPENDIX

5.1. Proof of Proposition 3.3(ii). The statement mainly follows from the arguments in [9, Section 4.1]. Note that since [9] studied second order asymptotics for Brownian motions in a heavy tailed Poissonian potential, the proof there is much more demanding. In particular, the argument in [9, Section 4.1] only works for part of heavy tailed potentials (i.e., for the shape function $\varphi(x) = 1 \wedge |x|^{-(d+\beta)}$ with $\beta \in (0,2)$ and $d + \beta \geqslant 2$). In our setting, we can prove Proposition 3.3(ii) for all heavy tailed potentials, because only the first order asymptotics for the first Dirichlet eigenvalue is considered.

To highlight the differences from the argument in [9, Section 4.1], we rewrite Proposition 3.3(ii) as follows, where the notations are those of [9].

PROPOSITION 5.1. *In the heavy tailed case* (H)*, for $M > 1$ large enough, any $\kappa > 1$ and $\varepsilon > 0$, $\mathbb{Q}$-almost surely there exists $t_{M,\kappa,\varepsilon}(\omega) > 0$ such that, for all $t \geqslant t_{M,\kappa,\varepsilon}(\omega)$ there is $z := z(t,\omega) \in \mathbb{R}^d$ such that $|z| \leqslant t(\log t)^{-\kappa}$ and*

$$\lambda_{B(z,M(\log t)^{\beta/(d\alpha)})} \leqslant (1+\varepsilon)\lambda(t),$$

where $\lambda(t) = q_1(\log t)^{-\beta/d}$, and q_1 is given by (3.7)*.*

In the heavy tailed case, by the continuity of φ and (3.1), for any $\theta > 0$, there exists a constant $C(\theta) > 0$ such that for all $x \in \mathbb{R}^d$,

$$\varphi(x) \leqslant \varphi_0(x) := (K+\theta)(C(\theta) \wedge |x|^{-d-\beta}).$$

Thus, to consider upper bounds for the first Dirichlet eigenvalue corresponding to the shape function φ, it suffices to study the eigenvalue associated with φ_0. For simplicity, in the proof below we just take

$$\varphi_0(x) := 1 \wedge |x|^{-d-\beta}, \quad x \in \mathbb{R}^d,$$

since the argument goes through for $\varphi_0(x) = (K+\theta)(C(\theta) \wedge |x|^{-d-\beta})$, and then the desired assertion follows by taking θ small enough.

Let $N > 1/d$, and $M > 1$ large enough. Define $\Lambda_N(t) = [-(\log t)^N, (\log t)^N]$ and $B_M(t) = B(0, M(\log t)^{\beta/(d\alpha)})$. First, we have

LEMMA 5.1. *For any $\varepsilon > 0$, there is a constant $c(\varepsilon) > 0$ such that for all t large enough,*

$$\mathbb{Q}\Big(\sup_{y \in B_M(t)} \sup_{\omega_i \notin \Lambda_N(t)} |y-\omega_i|^{-d-\beta} > \varepsilon(\log t)^{-\beta/d}\Big) \leqslant \exp(-c(\varepsilon)(\log t)^{dN+\beta(N-1/d)}).$$

Proof. For t large enough, and for $\omega_i \notin \Lambda_N(t)$ and $y \in B_M(t)$, as $N > 1/d$ and $\beta \in (0, \alpha)$, we have $|\omega_i - y| \geqslant |\omega_i|/2$, and so

$$\sup_{y \in B_M(t)} \sum_{\omega_i \notin \Lambda_N(t)} |y-\omega_i|^{-d-\beta} \leqslant 2^{d+\beta} \sum_{\omega_i \notin \Lambda_N(t)} |\omega_i|^{-d-\beta}.$$

Note that, since $\{\omega_i\}$ are from a realization of a homogeneous Poisson point process on $\mathbb{R}^d$ with parameter ρ, for t large enough we have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}} \exp\Big\{(\log t)^{(d+\beta)N} \sum_{\omega_i \notin \Lambda_N(t)} |\omega_i|^{-d-\beta}\Big\} \\
&\quad = \exp\Big(\rho \int_{\mathbb{R}^d \setminus \Lambda_N(t)} (e^{(\log t)^{(d+\beta)N}|z|^{-(d+\beta)}} - 1)\, dz\Big) \\
&\quad \leqslant \exp\Big(\rho e \int_{\mathbb{R}^d \setminus \Lambda_N(t)} (\log t)^{(d+\beta)N}|z|^{-(d+\beta)}\, dz\Big) \leqslant \exp(c_1(\log t)^{dN}),
\end{aligned}$$

where in the first inequality we have used the fact that $e^x - 1 \leqslant ex$ for all $x \in (0, 1]$. Therefore, by the Markov inequality, for t large enough,

$$\begin{aligned}
&\mathbb{Q}\Big(\sup_{y \in B_M(t)} \sup_{\omega_i \notin \Lambda_N(t)} |y-\omega_i|^{-d-\beta} > \varepsilon(\log t)^{-\beta/d}\Big) \\
&\qquad \leqslant \mathbb{Q}\Big(\sum_{\omega_i \notin \Lambda_N(t)} |\omega_i|^{-d-\beta} \geqslant 2^{-d-\beta}\varepsilon(\log t)^{-\beta/d}\Big) \\
&\qquad \leqslant \exp\Big(c_1(\log t)^{dN} - \varepsilon 2^{-d-\beta}(\log t)^{(d+\beta)N-\beta/d}\Big) \\
&\qquad \leqslant \exp(-c_2(\log t)^{dN+\beta(N-1/d)}),
\end{aligned}$$

where in the last inequality we have used the fact that $N > 1/d$. The proof is complete. ∎

For any $t > 0$, define

$$H(t) = \log \mathbb{E}_{\mathbb{Q}}[\exp(-tV^{\omega}(0))], \quad \rho_0(t) = \left(\frac{(d+\beta)t}{da_1}\right)^{-(d+\beta)/\beta},$$

where

$$a_1 = \rho w_d \Gamma\left(\frac{\beta}{d+\beta}\right).$$

In particular,

$$\rho_0(\lambda(t)) = \left(\frac{a_1\beta}{d(d+\beta)}\right)^{-(d+\beta)/d} (\log t)^{(d+\beta)/d}$$

and

$$H(\rho_0(\lambda(t))) + \lambda(t)\rho_0(\lambda(t)) = -d\log t + o(1), \quad t \to \infty, \tag{5.1}$$

where in the latter equality we have used the fact that

$$H(t) = -a_1 t^{d/(d+\beta)} + O(e^{-t}), \quad t \to \infty;$$

see [9, Lemma 1]. Next, we introduce a transformed measure defined by

$$\tilde{\mathbb{Q}}_t(d\omega) = [e^{-H(\rho_0(\lambda(t)))-\rho_0(\lambda(t))V^{\omega}(0)}]\mathbb{Q}(d\omega), \quad t > 0.$$

Then it follows from [9, Lemma 7(1)] that $(\omega, \tilde{\mathbb{Q}}_t)$ is a Poisson point process on $\mathbb{R}^d$ with intensity $\rho e^{-\rho_0(\lambda(t))\varphi_0(z)}\,dz$. Furthermore, we have

LEMMA 5.2. *As $t \to \infty$, uniformly for all $x \in B_M(t)$,*

$$\mathbb{E}_{\tilde{\mathbb{Q}}_t}[V^{\omega}(x)] = \lambda(t) + o((\log t)^{-\beta/d}).$$

Proof. For any $x \in B_M(t)$,

$$\begin{aligned}\mathbb{E}_{\tilde{\mathbb{Q}}_t}[V^{\omega}(x)] &= \rho \int_{\mathbb{R}^d} \varphi_0(x-z)e^{-\rho_0(\lambda(t))\varphi_0(z)}\,dz \\ &= \rho \int_{B_{2M}(t)} \varphi_0(x-z)e^{-\rho_0(\lambda(t))\varphi_0(z)}\,dz \\ &\quad + \rho \int_{\mathbb{R}^d\setminus B_{2M}(t)} \varphi_0(x-z)e^{-\rho_0(\lambda(t))\varphi_0(z)}\,dz.\end{aligned}$$

It is easy to see that for t large enough,

$$\begin{aligned}&\rho \sup_{x\in B_M(t)} \int_{B_{2M}(t)} \varphi_0(x-z)e^{-\rho_0(\lambda(t))\varphi_0(z)}\,dz \\ &\quad \leqslant \rho \int_{B_{2M}(t)} e^{-\rho_0(\lambda(t))\varphi_0(z)}\,dz \leqslant c_1 \exp(-c_2(\log t)^{(d+\beta)(\alpha-\beta)/(d\alpha)}),\end{aligned} \tag{5.2}$$

where $c_1, c_2 > 0$ are independent of t (but depend on M). Thus, for $x \in B_M(t)$ and for t large enough,

$$\begin{aligned}\mathbb{E}_{\tilde{\mathbb{Q}}_t}[V^\omega(x)] \leqslant \rho \int_{\mathbb{R}^d \setminus B_{2M}(t)} & |x-z|^{-d-\beta} e^{-\rho_0(\lambda(t))|z|^{-d-\beta}}\, dz \\ & + c_1 \exp(-c_2 (\log t)^{(d+\beta)(\alpha-\beta)/(d\alpha)}).\end{aligned}$$

On the other hand, we can check that

$$\rho \int_{B_{2M}(t)} |z|^{-d-\beta} e^{-\rho_0(\lambda(t))|z|^{-d-\beta}}\, dz \leqslant c_3 \exp(-c_4 (\log t)^{(d+\beta)(\alpha-\beta)/(d\alpha)}).$$

Then, for $t > 0$ large enough,

$$\begin{aligned}&\mathbb{E}_{\tilde{\mathbb{Q}}_t}[V^\omega(0)] \\ &\quad = \rho \int_{\mathbb{R}^d} |z|^{-d-\beta} e^{-\rho_0(\lambda(t))|z|^{-d-\beta}}\, dz + c_5 \exp(-c_6 (\log t)^{(d+\beta)(\alpha-\beta)/(d\alpha)}) \\ &\quad = \lambda(t) + O\big(\exp(-c(\log t)^{(d+\beta)(\alpha-\beta)/(d\alpha)})\big)\end{aligned}$$

for some constant $c > 0$.

Next, for any $x \in B_M(t)$ and t large enough, by the fact that $\varphi_0(z) = 1 \wedge |z|^{-(d+\beta)}$ and the mean value theorem,

$$\begin{aligned}&|\mathbb{E}_{\tilde{\mathbb{Q}}_t}(V^\omega(x) - V^\omega(0))| \\ &\qquad \leqslant \rho \int_{\mathbb{R}^d \setminus B_{2M}(t)} |\varphi_0(x-z) - \varphi_0(z)| e^{-\rho_0(\lambda(t))\varphi_0(z)}\, dz \\ &\qquad\quad + O\big(\exp(-c(\log t)^{(d+\beta)(\alpha-\beta)/(d\alpha)})\big) \\ &\qquad \leqslant c_7 (\log t)^{\beta/(d\alpha)} \int_{\mathbb{R}^d \setminus B_{2M}(t)} |z|^{-d-\beta-1} e^{-c_8 (\log t)^{(d+\beta)/d} |z|^{-d-\beta}}\, dz \\ &\qquad\quad + O\big(\exp(-c(\log t)^{(d+\beta)(\alpha-\beta)/(d\alpha)})\big) \\ &\qquad \leqslant c_9 (\log t)^{-\beta/d - (1-\beta/\alpha)/d},\end{aligned}$$

thanks to $\beta \in (0, \alpha)$ again. This proves the desired assertion. ∎

Now, we are back to the probability estimate for $V^\omega(x)$ under the probability measure $\mathbb{Q}$.

LEMMA 5.3. *There is a constant $\delta \in (0, 1/2)$ such that for any $\varepsilon, M > 0$ there exists $t_{\delta,\varepsilon,M} > 0$ such that for all $t \geqslant t_{\delta,\varepsilon,M}$,*

$$\mathbb{Q}\Big(\sup_{x \in B_M(t)} |V^\omega(x) - \lambda(t)| \leqslant \varepsilon (\log t)^{-\beta/d}\Big) \geqslant c(\delta, \varepsilon, M) t^{-d} \exp((\log t)^\delta).$$

Proof. For any given $\varepsilon > 0$, by Lemma 5.2, for t large enough,

$$\sup_{x \in B_M(t)} |\mathbb{E}_{\tilde{\mathbb{Q}}_t}(V^\omega(x) - V^\omega(0))| \leqslant \frac{\varepsilon}{4} (\log t)^{-\beta/d}.$$

For any $\gamma \in (1/2, 1)$, we further define

$$E_1 = \big\{V^\omega(0) - \lambda(t) \in \big[(\log t)^{-\beta/d-\gamma}, \frac{\varepsilon}{4}(\log t)^{-\beta/d}\big]\big\},$$
$$E_2 = \left\{\sup_{x \in B_M(t)} |V^\omega(x) - V^\omega(0) - \mathbb{E}_{\tilde{\mathbb{Q}}_t}(V^\omega(x) - V^\omega(0))| \geqslant \frac{\varepsilon}{2}(\log t)^{-\beta/d}\right\}.$$

Then, for t large enough,

$$E_1 \cap E_2^c \subset \left\{\sup_{x \in B_M(t)} |V^\omega(x) - \lambda(t)| \leqslant \varepsilon(\log t)^{-\beta/d}\right\}.$$

Hence,

$$\begin{aligned}
(5.3) \qquad & \mathbb{Q}\Big(\sup_{x \in B_M(t)} |V^\omega(x) - \lambda(t)| \leqslant \varepsilon(\log t)^{-\beta/d}\Big) \\
& \geqslant e^{H(\rho_0(\lambda(t)))}\mathbb{E}_{\tilde{\mathbb{Q}}_t}(e^{\rho_0(\lambda(t))V^\omega(0)}\mathbb{1}_{E_1 \setminus E_2}) \\
& \geqslant \exp\big(H(\rho_0(\lambda(t))) + \rho_0(\lambda(t))(\lambda(t) + (\log t)^{-\beta/d-\gamma})\big)\tilde{\mathbb{Q}}_t(E_1 \setminus E_2) \\
& \geqslant \exp\big(-d \log t + \rho_0(\lambda(t))(\log t)^{-\beta/d-\gamma} + o(1)\big)(\tilde{\mathbb{Q}}_t(E_1) - \tilde{\mathbb{Q}}_t(E_2)) \\
& \geqslant c_1 t^{-d} \exp(c_2(\log t)^{1-\gamma})(\tilde{\mathbb{Q}}_t(E_1) - \tilde{\mathbb{Q}}_t(E_2)),
\end{aligned}$$

where in the third inequality we have used (5.1).

As shown in [9, Lemma 7(iii)],

$$(\log t)^{(d+2\beta)/(2d)}(V^\omega(0) - \lambda(t))$$

under $\tilde{\mathbb{Q}}_t$ converges in law to a non-degenerate Gaussian random variable as $t \to \infty$. Then

$$\tilde{\mathbb{Q}}_t(E_1) = \tilde{\mathbb{Q}}_t\left((\log t)^{(d+2\beta)/(2d)}(V^\omega(0) - \lambda(t)) \in \left[(\log t)^{1/2-\gamma}, \frac{\varepsilon}{4}(\log t)^{1/2}\right]\right)$$

is bounded from below by a positive constant for t large enough, thanks to $\gamma \in (1/2, 1)$.

On the other hand, defining

$$\bar{\mu}_t^\omega(dz) := \mu^\omega(dz) - \rho e^{-\rho_0(\lambda(t))\varphi_0(z)}\, dz,$$

we write

$$\begin{aligned}
V^\omega(x) - V^\omega(0) - \mathbb{E}_{\tilde{\mathbb{Q}}_t}(V^\omega(x) - V^\omega(0)) & \\
&= \int_{\mathbb{R}^d} (\varphi_0(x - z) - \varphi_0(-z))\bar{\mu}_t^\omega(dz) \\
&= \int_{B_{2M(t)}} (\varphi_0(x - z) - \varphi_0(-z))\bar{\mu}_t^\omega(dz) \\
&\quad + \int_{\mathbb{R}^d \setminus B_{2M(t)}} (\varphi_0(x - z) - \varphi_0(-z))\bar{\mu}_t^\omega(dz).
\end{aligned}$$

Note that, by the fact that $\varphi_0(x) = 1 \wedge |x|^{-d-\beta}$,

$$\begin{aligned}\sup_{x\in B_M(t)}\Big|\int_{B_{2M(t)}}(\varphi_0(x-z)-\varphi_0(-z))\,\bar\mu_t^\omega(dz)\Big| &\leqslant \sup_{x\in B_M(t)}\int_{B_{2M(t)}}|\varphi_0(x-z)-\varphi_0(-z)|\,\mu^\omega(dz)\\ &\quad+\rho\sup_{x\in B_M(t)}\int_{B_{2M(t)}}|\varphi_0(x-z)-\varphi_0(-z)|e^{-\rho_0(\lambda(t))\varphi_0(z)}\,dz\\ &\leqslant \int_{B_{2M(t)}}\bar\mu_t^\omega(dz)+2\rho\int_{B_{2M(t)}}e^{-\rho_0(\lambda(t))\varphi_0(z)}\,dz.\end{aligned}$$

Hence, according to the second inequality in (5.2), for t large enough,

$$\begin{aligned}\tilde{\mathbb{Q}}_t\Big(\sup_{x\in B_M(t)}\Big|\int_{B_{2M(t)}}(\varphi_0(x-z)-\varphi_0(-z))\bar\mu_t^\omega(dz)\Big|\geqslant\frac{\varepsilon}{4}(\log t)^{-\beta/d}\Big)\\ \leqslant \tilde{\mathbb{Q}}_t\Big(\int_{B_{2M(t)}}\bar\mu_t^\omega(dz)\geqslant\frac{\varepsilon}{8}(\log t)^{-\beta/d}\Big)\\ \leqslant\Big[\frac{\varepsilon}{8}(\log t)^{-\beta/d}\Big]^{-2}\mathbb{E}_{\tilde{\mathbb{Q}}_t}\Big[\int_{B_{2M(t)}}\bar\mu_t^\omega(dz)\Big]^2\\ =\Big[\frac{\varepsilon}{8}(\log t)^{-\beta/d}\Big]^{-2}\rho\int_{B_{2M(t)}}e^{-\rho_0(\lambda(t))\varphi_0(z)}\,dz\\ \leqslant c_3\exp(-c_4(\log t)^{(d+\beta)(\alpha-\beta)/(d\alpha)}),\end{aligned}$$

where in the second inequality we have used the Markov inequality, and the equality above follows from the fact that the $\tilde{\mathbb{Q}}_t$-mean of $\int_{B_{2M(t)}}\bar\mu_t^\omega(dz)$ is zero.

Furthermore, according to the mean value theorem, for t large enough,

$$\begin{aligned}\sup_{x\in B_M(t)}\Big|\int_{\mathbb{R}^d\setminus B_{2M(t)}}(\varphi_0(x-z)-\varphi_0(-z))\bar\mu_t^\omega(dz)\Big|\\ =\sup_{x\in B_M(t)}\Big|\int_{\mathbb{R}^d\setminus B_{2M(t)}}\int_0^1\frac{d}{d\theta}\varphi_0(\theta x-z)\,d\theta\,\bar\mu_t^\omega(dz)\Big|\\ \leqslant\int_{\mathbb{R}^d\setminus B_{2M(t)}}\sup_{x\in B_M(t),\,\theta\in(0,1)}\Big|\frac{d}{d\theta}\varphi_0(\theta x-z)\Big|\,\mu_t^\omega(dz)\\ +\rho\int_{\mathbb{R}^d\setminus B_{2M(t)}}\sup_{x\in B_M(t),\,\theta\in(0,1)}\Big|\frac{d}{d\theta}\varphi_0(\theta x-z)\Big|e^{-\rho_0(\lambda(t))\varphi_0(z)}\,dz\\ =\int_{\mathbb{R}^d\setminus B_{2M(t)}}\sup_{x\in B_M(t),\,\theta\in(0,1)}\Big|\frac{d}{d\theta}\varphi_0(\theta x-z)\Big|\,\bar\mu_t^\omega(dz)\\ +2\rho\int_{\mathbb{R}^d\setminus B_{2M(t)}}\sup_{x\in B_M(t),\,\theta\in(0,1)}\Big|\frac{d}{d\theta}\varphi_0(\theta x-z)\Big|e^{-\rho_0(\lambda(t))\varphi_0(z)}\,dz\end{aligned}$$

$$\leqslant c_5(\log t)^{\beta/(d\alpha)} \int\limits_{\mathbb{R}^d\setminus B_{2M(t)}} |z|^{-d-\beta-1}\,\bar\mu_t^\omega(dz)$$
$$+\, c_5(\log t)^{\beta/(d\alpha)} \int\limits_{\mathbb{R}^d\setminus B_{2M(t)}} |z|^{-d-\beta-1} e^{-\rho_0(\lambda(t))\varphi_0(z)}\,dz.$$

Note that

$$(\log t)^{\beta/(d\alpha)} \int\limits_{\mathbb{R}^d\setminus B_{2M(t)}} |z|^{-d-\beta-1} e^{-\rho_0(\lambda(t))\varphi_0(z)}\,dz \leqslant c_6(\log t)^{-\beta/d-(1-\beta/\alpha)/d}$$

for t large enough, and that the $\tilde{\mathbb{Q}}_t$-mean of $\int_{\mathbb{R}^d\setminus B_{2M(t)}} |z|^{-d-\beta-1}\,\bar\mu_t^\omega(dz)$ is zero and its variance is bounded above by $c_7(\log t)^{-(d+2\beta+2)/d}$. Hence, for t large enough, by the Markov inequality,

$$\tilde{\mathbb{Q}}_t\Bigg(\sup_{x\in B_M(t)}\Bigg|\int\limits_{\mathbb{R}^d\setminus B_{2M(t)}} (\varphi_0(x-z)-\varphi_0(-z))\bar\mu_t^\omega(dz)\Bigg| \geqslant \frac{\varepsilon}{4}(\log t)^{-\beta/d}\Bigg)$$
$$\leqslant \tilde{\mathbb{Q}}_t\Bigg(c_5(\log t)^{\beta/(d\alpha)} \int\limits_{\mathbb{R}^d\setminus B_{2M(t)}} |z|^{-d-\beta-1}\,\bar\mu_t^\omega(dz) \geqslant \frac{\varepsilon}{8}(\log t)^{-\beta/d}\Bigg)$$
$$\leqslant c_8(\log t)^{2(\beta+\beta/\alpha)/d}\,\mathbb{E}_{\tilde{\mathbb{Q}}_t}\Bigg[\int\limits_{\mathbb{R}^d\setminus B_{2M(t)}} |z|^{-d-\beta-1}\,\bar\mu_t^\omega(dz)\Bigg]^2$$
$$\leqslant c_9(\log t)^{-1-2(1-\beta/\alpha)/d}.$$

Combining all the estimates above, we find that $\tilde{\mathbb{Q}}_t(E_2)$ tends to zero as $t\to\infty$, and so $\tilde{\mathbb{Q}}_t(E_1)-\tilde{\mathbb{Q}}_t(E_2)$ is bounded below by a positive constant for t large enough. This along with (5.3) yields the desired assertion. ■

Proof of Proposition 5.1. Fix $\kappa>1$, and set $I_t := ((2(\log t)^N)\mathbb{Z}^d)\cap\{z\in\mathbb{R}^d : |z|\leqslant t(\log t)^{-\kappa}\}$ for any $t>0$. For any $z\in I_t$ and $\varepsilon>0$, define

$$F_r(z) = \Bigg\{\sup_{x\in z+B_M(t)} |\tilde V^\omega(x)-\lambda(t)| \geqslant \frac{\varepsilon}{2}(\log t)^{-\beta/d}\Bigg\},$$
$$G_r(z) = \Bigg\{\sup_{x\in z+B_M(t)} \sum_{\omega_i\notin z+\Lambda_N(t)} |z-\omega_i|^{-d-\beta} \geqslant \frac{\varepsilon}{4}(\log t)^{-\beta/d}\Bigg\},$$

where

$$\tilde V^\omega(x) = \sum_{\omega_i\in z+\Lambda_N(t)} |x-\omega_i|^{-d-\beta}.$$

We will estimate $\mathbb{Q}(\bigcap_{z\in I_t}(F_t(z)\cup G_t(z)))$.

Note that $\{G_t(z)\}_{z\in I_t}$ have the same distribution such that for any $z\in I_t$ and t large enough,

$$\mathbb{Q}(G_t(z)) = \mathbb{Q}(G_t(0)) \leqslant \exp(-c_1(\varepsilon)(\log t)^{dN+\beta(N-1/d)}),$$

thanks to Lemma 5.1. On the other hand, $\{F_t(z)\}_{z\in I_t}$ are i.i.d., and, according to Lemmas 5.3 and 5.1, for any $z \in I_t$ and t large enough,

$$\begin{aligned}\mathbb{Q}(F_t(z)) &= \mathbb{Q}(F_t(0)) \leqslant \mathbb{Q}(F_t(0)\setminus G_t(0)) + \mathbb{Q}(G_t(0))\\ &= \mathbb{Q}\Big(\sup_{x\in B_M(t)} |V^\omega(x)-\lambda(t)| \geqslant \frac{\varepsilon}{4}(\log t)^{-\beta/d}\Big) + \mathbb{Q}(G_t(0))\\ &\leqslant 1 - c_2(\varepsilon)t^{-d}\exp((\log t)^\delta) + \exp(-c_1(\varepsilon)(\log t)^{dN+\beta(N-1/d)})\\ &\leqslant 1 - c_3(\varepsilon)t^{-d}\exp((\log t)^\delta).\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{Q}\Big(\bigcap_{z\in I_t}(F_t(z)\cup G_t(z))\Big) &\leqslant \mathbb{Q}\Big(\bigcap_{z\in I_t} F_t(z)\Big) + \mathbb{Q}\Big(\bigcup_{z\in I_t} G_t(z)\Big)\\ &\leqslant [1-c_3(\varepsilon)t^{-d}\exp((\log t)^\delta)]^{c_4 t^d(\log t)^{-(\kappa+N)d}}\\ &\quad + \exp(-c_5(\varepsilon)(\log t)^{dN+\beta(N-1/d)})\\ &\leqslant \exp\big(-c_6(\varepsilon)\exp((\log t)^\delta)(\log t)^{-(\kappa+N)d}\big)\\ &\quad + \exp(-c_5(\varepsilon)(\log t)^{dN+\beta(N-1/d)})\\ &\leqslant \exp(-c_7(\varepsilon)(\log t)^{dN+\beta(N-1/d)}),\end{aligned}$$

where in the third inequality we have used the fact that $1-x \leqslant e^{-x}$ for all $x>0$. The Borel–Cantelli lemma implies that $\mathbb{Q}$-almost surely for all t large enough there exists $z := z(t,\omega) \in I_t$ for which both $F_t(z)$ and $G_t(z)$ fail to happen.

Below, we will fix such a $z \in I_t$ for all t large enough. Then

$$\begin{aligned}\lambda_{V^\omega, B(z,B_M(t))} &\leqslant \lambda_1(B(z,B_M(t))) + \sup_{x\in B(z,B_M(t))} V^\omega(x)\\ &\leqslant \lambda_1(B(0,B_M(t))) + \sup_{x\in B(z,B_M(t))} \tilde V^\omega(x)\\ &\quad + \sup_{x\in B(z,B_M(t))} \sum_{\omega_i\notin z+\Lambda_N(t)} |z-\omega_i|^{-d-\beta}\\ &\leqslant 2M^{-\alpha}(\log t)^{-\beta/d}\lambda_1^{(\alpha)}(B(0,1)) + \lambda(t) + \frac{3\varepsilon}{4}(\log t)^{-\beta/d},\end{aligned}$$

where in the last inequality we have used Lemma 3.2. Letting ε small enough and M large enough in the inequality above, we obtain the desired assertion. ■

5.2. Quenched estimates of $u^\omega(t,0)$: critical case. In this part, we will briefly show that the arguments of Theorems 3.1 and 3.2 with some modifications still work for the following

- *Critical case* (C): The characteristic exponent $\psi(\xi)$ of the pure-jump symmetric Lévy process Z fulfills $\psi(\xi) = O(|\xi|^\alpha)$ as $|\xi| \to 0$, and the shape function φ in the random potential $V^\omega(x)$ satisfies

$$0 < \liminf_{|x|\to\infty} \varphi(x)|x|^{d+\alpha} \leqslant \limsup_{|x|\to\infty} \varphi(x)|x|^{d+\alpha} < \infty. \tag{5.4}$$

In the critical case, it was shown in [16, Theorem 6.4] that the integrated density $N(\lambda)$ of states of the random Schrödinger operator H defined by (3.2) satisfies

$$-\infty < \liminf_{\lambda\to 0} \lambda^{d/\alpha} \log N(\lambda) \leqslant \limsup_{\lambda\to 0} \lambda^{d/\alpha} \log N(\lambda) < 0.$$

Then, according to the arguments in Subsection 3.2.1 and the proof of Theorem 3.1, we have

THEOREM 5.1. *In the critical case* (C)*, there is a constant* $k_0 > 0$ *such that for any* $\varepsilon > 0$ *there is* $C(\varepsilon) > 0$ *such that* $\mathbb{Q}$*-almost surely there exists a random variable* $R_\varepsilon(\omega) \geqslant 1$ *with the property that for any* $R \geqslant \max\{R_\varepsilon(\omega), \phi(t)\}$ *and* $t \geqslant 1$,

$$u^\omega(t,0) \leqslant \Phi(t,R) + C(\varepsilon)R^{d/2} \exp\left(-t(1-2\varepsilon)\left(\frac{k_0}{d\log R}\right)^{\alpha/d}\right),$$

where $\Phi(t,R)$ *and* ϕ *are given in Theorem* 3.1.

When Z is a symmetric α-stable process with exponent $\psi^{(\alpha)}(\xi)$ given in (2.2) for some $\alpha \in (0,2]$, and $K := \lim_{|x|\to\infty} \varphi(x)|x|^{d+\alpha} \in (0,\infty)$, Ôkura proved the precise annealed asymptotics of $u^\omega(t,x)$ in [18, Theorem and Remark ii]: for all $x \in \mathbb{R}^d$,

$$\lim_{t\to\infty} \frac{\log \mathbb{E}_{\mathbb{Q}}[u^\omega(t,x)]}{t^{d/(d+\alpha)}} = -C(\rho,K),$$

where

$$C(\rho,K) = \inf_{f\in L^2(\mathbb{R}^d;dx)\cap B_c(\mathbb{R}^d):\, \|f\|_{L^2(\mathbb{R}^d;dx)}=1} \{D(f,f) + W_\rho(f^2)\}$$

with $B_c(\mathbb{R}^d)$ being the set of measurable functions with compact support, $D(f,f)$ being the Dirichlet form associated with the symmetric α-stable process Z, and

$$W_\rho(f^2) = \rho \int_{\mathbb{R}^d} \left[1 - \exp\left(-K \int_{\mathbb{R}^d} \frac{f(z)^2}{|x-z|^{d+\alpha}}\, dz\right)\right] dx.$$

Then, by the Tauberian theorem of exponential type (see [12, Theorem 3]), we have

$$\lim_{\lambda\to 0} \lambda^{d/\alpha} \log N(\lambda) = -k_0 := -\frac{\alpha}{d+\alpha}\left(\frac{d}{d+\alpha}\right)^{d/\alpha} C(\rho,K).$$

So, in this case we have a precise expression for the constant k_0 in Theorem 5.1.

For quenched lower bounds of $u^\omega(t,x)$, we have the following statement.

THEOREM 5.2. *In the critical case* (C), *assume that* (3.11) *holds. Then there is a constant $C_0 > 0$ such that for any $\delta \in (0, 1/2)$, $\kappa, a > 1$ and $\eta, \varsigma \in (0,1)$, $\mathbb{Q}$-almost surely there is $R_{\kappa,a,\eta,\varsigma}(\omega) \geqslant 1$ such that for any $R \geqslant R_{\kappa,a,\eta,\varsigma}(\omega)$ and $t \geqslant 1$,*

$$u^{\omega}(t,0) \geqslant C(\kappa,\delta,\eta,a) M_{\kappa,\eta}(R)^{-4\delta d} [\Psi_{\delta}(2M_{\kappa,\eta}(R))]^{a} \exp(-a^2(1+\varsigma)C_0 t R^{-\alpha}),$$

where

$$M_{\kappa,\eta}(R) = R^{-\kappa} \exp\bigg(\frac{w_d \rho}{d}((1+2\eta)R)^d\bigg). \tag{5.5}$$

To prove Theorem 5.2, we need the following proposition, which is analogous to Proposition 3.3.

PROPOSITION 5.2. *In the critical case* (C), *for any $\kappa > 1$ and $\eta, \varsigma \in (0,1)$, $\mathbb{Q}$-almost surely there exists $r_{\kappa,\eta,\varsigma}(\omega) > 0$ such that for all $r \geqslant r_{\kappa,\eta,\varsigma}(\omega)$, there is $z := z(r,\omega) \in \mathbb{R}^d$ with $|z| \leqslant M_{\kappa,\eta}(r)$,*

$$\lambda_{V^{\omega},B(z,r)} \leqslant \big((1+\varsigma)\lambda_1^{(\alpha)}(B(0,1)) + C_1(\eta)\big) r^{-\alpha},$$

where $M_{\kappa,\eta}(r)$ is defined by (5.5), *$C_1(\eta)$ is a positive constant depending on η only, and $\lambda_1^{(\alpha)}(B(0,1))$ is the principal Dirichlet eigenvalue for the generator of the symmetric α-stable process $Z^{(\alpha)}$ on $B(0,1)$ with characteristic exponent $\psi^{(\alpha)}(\xi)$ given in* (2.1).

Proof. We use some notations from the proof of Proposition 3.3(i). For any $\kappa > 1$ and $\eta \in (0,1)$, let $I_r := ((2(1+\eta)r)\mathbb{Z}^d) \cap \{z \in \mathbb{R}^d : |z| \leqslant M_{\kappa,\eta}(r)\}$ for any $r > 0$. Define $\varphi_0(x) = \varphi_0(|x|)$ for any $x \in \mathbb{R}^d$, where $\varphi_0(r) = \sup_{|x| \geqslant r} \varphi(x)$ for $r \in [0,\infty)$. It is clear that $\varphi(x) \leqslant \varphi_0(x)$, and $\varphi_0(r)$ is a decreasing function on $[0,\infty)$ that there are constants $c_1, c_2 > 0$ such that for r large enough,

$$c_1 r^{-d-\alpha} \leqslant \varphi_0(r) \leqslant c_2 r^{-d-\alpha}, \tag{5.6}$$

thanks to (5.4).

Now, for any $z \in I_r$, define $F_r(z)$ as in the proof of Proposition 3.3(i), and

$$G_r(z) = \Big\{ \sup_{y \in B(z,r)} \sum_{\omega_i \notin B(z,(1+\eta)r)} \varphi_0(y - \omega_i) \geqslant C_* r^{-\alpha} \Big\}$$

for some constant $C_* > 0$ to be chosen later. As shown in the proof of Proposition 3.3(i), for $r > 1$ large enough, $\mathbb{Q}(\bigcap_{z \in I_r} F_r(z)) \leqslant \exp(-r^d/2)$.

On the other hand, by the decreasing property of $\varphi_0(r)$ and (5.6), for all r large enough,

$$\begin{aligned}\mathbb{Q}\Big[\exp\Big(\frac{1}{\varphi_0(\eta r)}\sup_{y\in B(0,r)}\sum_{\omega_i\notin B(0,(1+\eta)r)}\varphi_0(y-\omega_i)\Big)\Big]\\ &\leqslant \mathbb{Q}\Big[\exp\Big(\frac{1}{\varphi_0(\eta r)}\sum_{\omega_i\notin B(0,(1+\eta)r)}\varphi_0(\eta|\omega_i|/(1+\eta))\Big)\Big]\\ &= \exp\Big(\rho\int_{\mathbb{R}^d\setminus B(0,(1+\eta)r)}(e^{\varphi_0(\eta r)^{-1}\varphi_0(\eta|z|/(1+\eta))}-1)\,dz\Big)\\ &\leqslant \exp\Big(e\rho(1+\eta)^d\int_{\mathbb{R}^d\setminus B(0,r)}\frac{\varphi_0(\eta|z|)}{\varphi_0(\eta r)}\,dz\Big)\leqslant \exp(c_3r^d),\end{aligned}$$

where $c_3 := c_3(\eta)$ is independent of r. This along with the Markov inequality and (5.6) implies that for r large enough,

$$\begin{aligned}&\mathbb{Q}(G_r(0))\\ &\quad\leqslant \mathbb{Q}\Big[\exp\Big(\frac{1}{\varphi_0(\eta r)}\sup_{y\in B(0,r)}\sum_{\omega_i\notin B(0,(1+\eta)r)}\varphi_0(y-\omega_i)\Big)\geqslant \exp\Big(\frac{C_*r^{-\alpha}}{\varphi_0(\eta r)}\Big)\Big]\\ &\quad\leqslant \exp(c_3r^d-C_*c_2^{-1}\eta^dr^d).\end{aligned}$$

Since $\{G_r(z)\}_{z\in I_r}$ have the same distribution (but are not independent of each other), we find that

$$\begin{aligned}\mathbb{Q}\Big(\bigcup_{z\in I_r}G_r(z)\Big)&\leqslant 2\Big(\frac{M_{\kappa,\eta}(r)}{(1+\eta)r}\Big)^d\exp(c_3r^d-C_*c_2^{-1}\eta^dr^d)\\ &= 2(1+\eta)^{-d}r^{-(\kappa+1)d}\exp[(\rho w_d(1+2\eta)^d+c_3-C_*c_2^{-1}\eta^d)r^d].\end{aligned}$$

Now, we take $C_* = 2c_2\eta^{-d}(\rho w_d(1+2\eta)^d+c_3)$, and so

$$\mathbb{Q}\Big(\bigcup_{z\in I_r}G_r(z)\Big)\leqslant 2(1+\eta)^{-d}r^{-(\kappa+1)d}\exp\Big(-\frac{1}{2c_2}C_*\eta^dr^d\Big).$$

Therefore, for all r large enough,

$$\begin{aligned}&\mathbb{Q}\Big(\bigcap_{z\in I_r}(F_r(z)\cup G_r(z))\Big)\\ &\qquad\leqslant \mathbb{Q}\Big(\bigcap_{z\in I_r}F_r(z)\Big)+\mathbb{Q}\Big(\bigcup_{z\in I_r}G_r(z)\Big)\\ &\qquad\leqslant \exp(-r^d/2)+2(1+\eta)^{-d}r^{-(\kappa+1)d}\exp\Big(-\frac{1}{2c_2}C_*\eta^dr^d\Big).\end{aligned}$$

Hence, in view of the Borel–Cantelli lemma, $\mathbb{Q}$-almost surely there exists $z := z(r,\omega)\in\mathbb{R}^d$ such that $|z|\leqslant M_{\kappa,\eta}(r)$, and both $F_r(z)$ and $G_r(z)$ fail to hold.

With this at hand, one can follow the proof of Proposition 3.3(i) to get the desired assertion. ■

According to Proposition 5.2, one can repeat the argument for Theorem 3.2 to prove Theorem 5.2. Furthermore, as an application of Theorems 5.1 and 5.2, we also can obtain Proposition 1.1. The details are omitted.

Acknowledgments. We thank the anonymous referee for valuable comments and corrections. The research is supported by the National Natural Science Foundation of China (Nos. 11831014, 12071076 and 12225104), and the Education and Research Support Program for Fujian Provincial Agencies.

REFERENCES

[1] R. A. Carmona and S. A. Molchanov, *Stationary parabolic Anderson models and intermittency*, Probab. Theory Related Fields 102 (1995), 433–453.

[2] L. Chaumont and G. Uribe Bravo, *Markovian bridges: Weak continuity and pathwise constructions*, Ann. Probab. 39 (2011), 609–647.

[3] X. Chen, *Quenched asymptotics for Brownian motion of renormalized Poisson potential and for the related parabolic Anderson models*, Ann. Probab. 40 (2012), 1436–1482.

[4] Z.-Q. Chen and T. Kumagai, *Heat kernel estimates for stable-like processes on d-sets*, Stoch. Process. Appl. 108 (2003), 27–62.

[5] Z.-Q. Chen, P. Kim and T. Kumagai, *Global heat kernel estimates for symmetric jump processes*, Trans. Amer. Math. Soc. 363 (2011), 5021–5055.

[6] Z.-Q. Chen, P. Kim and R. Song, *Heat kernel estimates for Dirichlet fractional Laplacian*, J. Eur. Math. Soc. 124 (2010), 1307–1329.

[7] M. D. Donsker and S. R. S. Varadhan, *Asymptotics for the Wiener sausage*, Comm. Pure Appl. Math. 28 (1975), 525–565.

[8] R. Fukushima, *From the Lifshitz tail to the quenched survival asymptotics in the trapping problem*, Electron. Comm. Probab. 14 (2009), 435–446.

[9] R. Fukushima, *Second order asymptotics for Brownian motion in a heavy tailed Poissonian potential*, Markov Process. Related Fields 17 (2011), 447–482.

[10] J. P. Gärtner, W. König and S. A. Molchanov, *Almost sure asymptotics for the continuous parabolic Anderson model*, Probab. Theory Related Fields 118 (2000), 547–573.

[11] K. Kaleta and K. Pietruska-Pałuba, *The quenched asymptotics of nonlocal Schrödinger operators with Poissonian potentials*, Potential Anal. 52 (2020), 161–202.

[12] Y. Kasahara, *Tauberian theorems of exponential type*, J. Math. Kyoto Univ. 18 (1978), 209–219.

[13] K.-Y. Kim and P. Kim, *Two-sided estimates for the transition densities of symmetric Markov processes dominated by stable-like processes in $C^{1,\eta}$ open sets*, Stoch. Process. Appl. 124 (2014), 3055–3083.

[14] W. König, *The Parabolic Anderson Model: Random Walk in Random Potential*, Birkhäuser, Cham, 2016.

[15] V. Knopova and R. L. Schilling, *A note on the existence of transition probability densities for Lévy processes*, Forum Math. 25 (2013), 125–149.

[16] H. Ôkura, *On the spectral distributions of certain integro-differential operators with random potential*, Osaka J. Math. 16 (1979), 633–666.

[17] H. Ôkura, *Some limit theorems of Donsker–Varadhan type for Markov processes expectations*, Z. Wahrsch. Verw. Gebiete 57 (1981), 419–440.

[18] H. Ôkura, *An asymptotic property of a certain Brownian motion expectataion for large time*, Proc. Japan Acad. Ser. A. Math. Sci. 57 (1981), 155–159.

[19] L. A. Pastur, *The behavior of certain Wiener integrals as $t \to \infty$ and the density of states of Schrödinger equations with random potential*, Teoret. Mat. Fiz. 32 (1977), 88–95 (in Russian).
[20] A.-S. Sznitman, *Brownian asymptotics in a Poisson environment*, Probab. Theory Related Fields 95 (1993), 155–174.
[21] A.-S. Sznitman, *Brownian Motion, Obstacles and Random Media*, Springer, Berlin, 1998.

Zhi-He Chen
School of Mathematics and Statistics
Fujian Normal University
350007 Fuzhou, P.R. China
E-mail: chenzhihe1995@126.com

Jian Wang
School of Mathematics and Statistics
& Fujian Key Laboratory of Mathematical
Analysis and Applications (FJKLMAA)
& Center for Applied Mathematics
of Fujian Province (FJNU)
Fujian Normal University
350007 Fuzhou, P.R. China
E-mail: jianwang@fjnu.edu.cn

Received 26.2.2022;
accepted 10.11.2022