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CENTRAL LIMIT THEOREM FOR SOME DEPENDENT RANDOM ELEMENTS OF $D[0, 1]$

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Abstract. This paper gives conditions which imply that the m-dependent sequence and the martingale difference sequence of random elements of $D[0, 1]$ satisfy the central limit theorem in $D[0, 1]$. Obtained results are an extension of results of Hahn [3].

1. Introduction. Let $D \equiv D[0, 1]$ be the space, endowed with the Skorohod topology, of real-valued functions on $[0, 1]$ which are right continuous and have the left-hand limits (for details of D and the basic properties of the Skorohod topology, see [2]).

The sequence $\{X_n\} = \{X_n, n \geqslant 1\}$ of D-valued random elements satisfies the central limit theorem (CLT) in D if there exists a Gaussian random element Z in D which is the limit in distribution of the sequence of random elements

$$Z_n = n^{-1/2} \sum_{i=1}^{n} (X_i - \mathrm{E}X_i).$$

This convergence is denoted by $Z_n \xrightarrow{D} Z$, and we call Z the *limiting Gaussian element.*

In [3] Hahn gave sufficient conditions for the sequence of independent identically distributed random elements to satisfy CLT in D. This result is included in the following

THEOREM 1. *Let $\{X_n\}$ be the sequence of independent identically distributed random elements of D such that, for all $t \in [0, 1]$, $\mathrm{E}X_1(t) = 0$ and $\mathrm{E}X_1^2(t) < \infty$. Assume there exist nondecreasing continuous functions G and F on $[0, 1]$ and numbers $\alpha > 1/2$, $\beta > 1$ such that, for all $0 \leqslant s \leqslant t \leqslant u \leqslant 1$,*

$$\mathrm{E}\big(X(u) - X(t)\big)^2 \leqslant \big(G(u) - G(t)\big)^{\alpha}, \tag{1}$$

$$\mathrm{E}\big(X(u) - X(t)\big)^2 \big(X(t) - X(s)\big)^2 \leqslant \big(F(u) - F(s)\big)^{\beta}. \tag{2}$$

Then $\{X_n\}$ satisfies the CLT in D and the limiting Gaussian element is distributed on $C[0, 1]$, the space of real-valued continuous functions on $[0, 1]$.

Taking this theorem as the starting point we formulate sufficient conditions for the m-dependent sequence and martingale difference sequence of random elements of D to satisfly CLT in D.

Types of dependence of random elements are the transposition of the corresponding types of dependence of random variables and are defined in the following way:

A sequence $\{X_n\}$ of random elements is *m-dependent* if, for all $j \in N$, $i \in N$ and $k \in N$, the random vectors $(X_j, X_{j+1}, \ldots, X_k)$ and $(X_{k+n}, X_{k+n+1}, \ldots, X_i)$ are independent whenever $n > m$.

We say that $\{(X_n, \mathscr{F}_n)\} = \{(X_n, \mathscr{F}_n), n \geqslant 1\}$ is the *martingale difference sequence* if $\{X_n\}$ is the sequence of random elements adapted to σ-fields $\mathscr{F} = \{\mathscr{F}_n, n \geqslant 1\}$ such that, for every $n \in N$ and every $t \in [0, 1]$, $\mathrm{E}X_n(t) < \infty$ and $\mathrm{E}\big(X_n(t)|\mathscr{F}_{n-1}\big) = 0$.

2. CLT for the sequence of dependent random elements. In this section we give a proposition of CLT for the m-dependent sequence and martingale difference sequence of D-valued random elements.

THEOREM 2. *Let $\{X_n\}$ be a strictly stationary sequence of m-dependent random elements of D such that, for all $t \in [0, 1]$,*

$$\mathrm{E}X_1(t) = 0, \tag{3}$$

$$\mathrm{E}X_1^2(t) < \infty. \tag{4}$$

Assume there exist nondecreasing continuous functions G and F on $[0, 1]$ and numbers $\alpha > 1/2$, $\beta > 1$ such that, for all $0 \leqslant s \leqslant t \leqslant u \leqslant 1$,

$$\mathrm{E}\big(X_1(t) - X_1(s)\big)^2 \leqslant \big(G(t) - G(s)\big)^\alpha, \tag{5}$$

$$\mathrm{E}\big(X_1(t) - X_1(s)\big)^2 \big(X_k(u) - X_k(t)\big)^2 \leqslant \big(F(u) - F(s)\big)^\beta \quad \textit{for } k = 1, 2, \ldots, m. \tag{6}$$

Then $\{X_n\}$ satisfies the CLT in D and the limiting Gaussian element is distributed on C.

THEOREM 3. *Let $\{(X_n, \mathscr{F}_n)\}$ be a martingale difference sequence of random elements of D such that, for all $s, t \in [0, 1]$ and all $n \in N$,*

$$\mathrm{E}\big(\max_{i \leqslant n} |n^{-1/2} X_i(t)|\big)^2 \to 0, \tag{7}$$

$$\frac{1}{n} \sum_{i=1}^{n} X_i(t) X_i(s) \xrightarrow{\mathrm{P}} C(t, s), \tag{8}$$

where $C(t,s)$ is a function of two variables with finite values.

Assume there exist nondecreasing continuous functions G and F on $[0, 1]$ and numbers $\alpha > 1/2$, $\beta > 1$ such that, for all $n \in N$ and all $0 \leqslant s \leqslant t \leqslant u \leqslant 1$,

$$\mathrm{E}\big(X_n(t) - X_n(s)\big)^2 \big(X_n(u) - X_n(t)\big)^2 \leqslant \big(F(u) - F(s)\big)^\beta, \tag{9}$$

(10) $$\mathrm{E}\{(X_n(t)-X_n(s))^2 | \mathscr{F}_{n-1}\} \leqslant (G(t)-G(s))^{\alpha} \text{ a.s.}$$

Then $\{X_n\}$ satisfies the CLT in D and the limiting Gaussian element is distributed on C.

2. Proofs. For the proof of Theorems 2 and 3 we show the convergence of finite-dimensional distributions of the sequence $\{Z_n\}$ to the corresponding finite-dimensional distributions of the Gaussian random element, and the tightness of the sequence $\{Z_n\}$ in D.

Proof of Theorem 2. Convergence of finite-dimensional distributions of the sequence $\{Z_n\}$ is the consequence of Theorem 20.1 of [2] and Cramer-Wold technique (Theorem 7.7 of [2]).

We verify the tightness of the sequence $\{Z_n\}$. Let

$$Y_n = \sum_{i=m(n-1)+1}^{nm} X_i.$$

Since the sequence $\{X_n\}$ is m-dependent, it follows that $\{Y_{2n}, n \geqslant 1\}$ and $\{Y_{2n-1}, n \geqslant 1\}$ are sequences of independent random elements. We verify that conditions (1) and (2) of Theorem 1 hold.

From condition (5) of the theorem we get the estimation

$$\mathrm{E}(Y_n(t)-Y_n(s))^2 \leqslant (m^{2/\alpha}(G(t)-G(s)))^{\alpha}$$

and condition (1) of Theorem 1 is satisfied by the sequence $\{Y_n\}$ with the function G replaced by $m^{2/\alpha} G$.

Now we verify condition (2) of Theorem 1 for the sequence $\{Y_n\}$, using condition (6) and the Schwarz inequality:

$$\begin{aligned}
&\mathrm{E}(Y_n(t)-Y_n(s))^2(Y_n(u)-Y_n(t))^2 \\
&\quad = \mathrm{E}\{\sum_{i=m(n-1)+1}^{nm} \sum_{j=m(n-1)+1}^{nm} (X_i(t)-X_i(s))(X_j(u)-X_j(t))\}^2 \\
&\quad \leqslant m^2 \sum_{i=m(n-1)+1}^{nm} \sum_{j=m(n-1)+1}^{nm} \mathrm{E}(X_i(t)-X_i(s))^2(X_j(u)-X_j(t))^2 \\
&\quad \leqslant m^4(F(u)-F(s))^{\beta}.
\end{aligned}$$

The function, the existence of which is assumed in (2), Theorem 1, is $m^{4/\beta} F$. Thus $\{Y_{2n}\}$ and $\{Y_{2n-1}\}$ are sequences of independent random elements which satisfy conditions (1) and (2) of Theorem 1. Hence we infer that the sequences

$$Z_n' = n^{-1/2} \sum_{i=1}^{A_n} Y_{2i}, \quad Z_n'' = n^{-1/2} \sum_{i=1}^{B_n} Y_{2i-1},$$

where $A_n = [n/2m]$, $B_n = [(n+1)/2m]$, converge in distribution and the limiting elements are distributed on C. Let

$$R_n = n^{-1/2} \sum_{i=C_n}^{n} X_i, \quad \text{where } C_n = \left[\frac{n}{m}\right]+1.$$

We can notice that $R_n \xrightarrow{D} 0$. Since, moreover, $Z_n = Z_n' + Z_n'' + R_n$, the sequence $\{Z_n\}$ is tight.

Proof of Theorem 3. The convergence of finite-dimensional distribution of the sequence $\{Z_n\}$ to the corresponding finite-dimensional distribution of the Gaussian random element follows from Theorem 3.2 of [4] and Theorem 7.7 of [2]. According to Theorem 15.6 of [2] it suffices to verify that, for all $n \in N$ and all $0 \leqslant s \leqslant t \leqslant u \leqslant 1$,

$$\mathrm{E}\big(Z_n(t)-Z_n(s)\big)^2\big(Z_n(u)-Z_n(t)\big)^2 \leqslant \big(B(u)-B(s)\big)^{\gamma},$$

where $\gamma > 1$ and B is a nondecreasing continuous function on $[0, 1]$. Without loss of generality we may assume $|F(t)| \leqslant 1$, $|G(t)| \leqslant 1$ for all $t \in [0, 1]$. We have

$$\mathrm{E}\Big[n^{1/2} \sum_{i=1}^{n} \big(X_i(t)-X_i(s)\big)\Big]^2 \Big[n^{-1/2} \sum_{i=1}^{n} \big(X_i(u)-X_i(t)\big)\Big]^2$$

$$= n^{-2}\mathrm{E}\Big[\sum_{i=1}^{n} \big(X_i(t)-X_i(s)\big)\big(X_i(u)-X_i(t)\big) + \sum_{i \neq j}^{n} \big(X_i(t)-X_i(s)\big)\big(X_j(u)-X_j(t)\big)\Big]^2$$

$$= 2n^{-2}\Big[\mathrm{E}\Big\{\sum_{i=1}^{n} \big(X_i(t)-X_i(s)\big)\big(X_i(u)-X_i(t)\big)\Big\}^2 + \\ + \mathrm{E}\Big\{\sum_{i \neq j}^{n} \big(X_i(t)-X_i(s)\big)\big(X_j(u)-X_j(t)\big)\Big\}^2\Big].$$

By the Schwartz inequality and assumption (9) we get

$$n^{-2}\mathrm{E}\Big[\sum_{i=1}^{n} \big(X_i(t)-X_i(s)\big)\big(X_i(u)-X_i(t)\big)\Big]^2 \leqslant \big(F(u)-F(s)\big)^{\beta}.$$

Let us remark that sequences $\{(U_n, \mathscr{F}_n), n \geqslant 1\}$ and $\{(W_n, \mathscr{F}_n), n \geqslant 1\}$, defined as

$$U_n = \big(X_n(t)-X_n(s)\big) \sum_{i=1}^{n-1} \big(X_i(u)-X_i(t)\big),$$

$$W_n = \big(X_n(u)-X_n(t)\big) \sum_{i=1}^{n-1} \big(X_i(t)-X_i(s)\big),$$

create the sequences of martingale difference. This remark, the Schwartz inequality and assumption (10) imply

$$\mathrm{E}\big[\sum_{i\neq j}^{n}(X_i(t)-X_i(s))(X_j(u)-X_j(t))\big]^2$$

$$=\mathrm{E}\big[\sum_{i=1}^{n}(U_i+W_i)\big]^2\leqslant 2\big[\sum_{i=1}^{n}\mathrm{E}U_i^2+\sum_{i=1}^{n}\mathrm{E}W_i^2\big]$$

$$\leqslant 2\sum_{i=1}^{n}\Big[\mathrm{E}\big[\sum_{j=1}^{i-1}(X_j(u)-X_j(t))^2\,\mathrm{E}\,\{(X_i(t)-X_i(s))^2|\mathcal{F}_{i-1}\}\big]+$$

$$+\mathrm{E}\big[\sum_{j=1}^{i-1}(X_j(t)-X_j(s))^2\,\mathrm{E}\,\{(X_i(u)-X_i(t))^2\,|\mathcal{F}_{i-1}\}\big]\Big]$$

$$\leqslant 4\sum_{j=1}^{n}(i-1)\big(G(u)-G(t)\big)^{\alpha}\big(G(t)-G(s)\big)^{\alpha}$$

$$\leqslant 2n^2\big(G(u)-G(s)\big)^{2\alpha}.$$

Thus

$$\mathrm{E}\big(Z_n(t)-Z_n(s)\big)^2\big(Z_n(u)-Z_n(t)\big)^2$$

$$\leqslant 2\big(F(u)-F(s)\big)^{\beta}+4\big(G(u)-G(s)\big)^{2\alpha}\leqslant \big(B(u)-B(s)\big)^{\gamma},$$

where $B=2^{1/\gamma}F+4^{1/\gamma}G$, $\gamma=\min(2\alpha,\beta)$.

Applying Theorem 5.3. of [2] to the sequence $\{(Z_n(t)-Z_n(s))^2\}$ we see that $E\big(Z(t)-Z(s)\big)^2\leqslant\big(G(t)-G(s)\big)^{\alpha}$ and, hence, the Gaussian random element Z is sample-continuous (Theorem 1 of [3]).

4. Example. Consider the system of theory of reliability (see [1]) which consists of $n+1$ elements and has the structure function $\Phi=\Phi(x_1x_2,x_2x_3,\ldots,x_nx_{n+1})$, where $x_1,x_2,\ldots,x_{n+1}$ are binary values of elements and Φ is a "k out of n" structure function. In the case of renewed elements we assume that $x_i=X_i(t)$ are independent binary processes. Then

$$\Phi=\mathbb{1}_{\{Y_1+\ldots+Y_n\geqslant k\}},$$

where $Y_i=X_iX_{i+1}$ $(i=1,2,\ldots,n)$. In that way the asymptotic behaviour of the survival function of the system can be brought to the study of the sum of 2-dependent binary processes. The CLT for independent binary processes was studied in [5]. If X_n are independent copies of a binary process fulfilling assumptions of Theorem 3 from [5], then the sequence $\{Y_n\}$ satisfies CLT in $D[0,\infty)$, the space of all real-valued right continuous functions on $[0,\infty)$ which have left-hand limits in $(0,\infty)$, endowed with the Lindvall metric. As in [5], it is enough to verify that $\{Y_n\}$ satisfies CLT in $D[0,c]$ for all $c>0$. Corollary 1 from [5] and the estimation

$$\mathrm{E}\big(Y_n(t)-\mathrm{E}Y_n(t)-Y_n(s)+\mathrm{E}Y_n(s)\big)^2 \leqslant \mathrm{E}\big(Y_n(t)-Y_n(s)\big)^2$$
$$= P\big(X_n(t)X_{n+1}(t) \neq X_n(s)X_{n+1}(s)\big)$$
$$\leqslant P\big(X_n(t) \neq X_n(s),\ X_{n+1}(t)=1\big)+P\big(X_n(t)=1,\ X_{n+1}(t) \neq X_{n+1}(s)\big)$$
$$\leqslant 2P\big(X_1(t) \neq X_1(s)\big)$$

imply the existence of a continuous nondecreasing function G such that

$$E\big(Y_n(t)-Y_n(s)\big)^2 \leqslant \big(G(t)-G(s)\big) \quad \text{for all } t,\ s\in[0,\ c].$$

Theorem 2 and Lemma 3 from [5] and the estimation

$$\mathrm{E}\big(Y_n(t)-Y_n(s)\big)^2\big(Y_n(u)-Y_n(t)\big)^2$$
$$= \mathrm{P}\big(X_n(t)X_{n+1}(t) \neq X_n(s)X_{n+1}(s),\ X_n(u)X_{n+1}(u) \neq X_n(t)X_{n+1}(t)\big)$$
$$\leqslant \mathrm{P}\big(X_n(t) \neq X_n(s),\ X_n(u) \neq X_n(t)\big)+\mathrm{P}\big(X_n(t) \neq X_n(s),\ X_{n+1}(u) \neq X_{n+1}(t)\big)+$$
$$+\mathrm{P}\big(X_{n+1}(t) \neq X_{n+1}(s),\ X_{n+1}(u) \neq X_{n+1}(t)\big)+\mathrm{P}\big(X_n(u)$$
$$\neq X_n(t),\ X_{n+1}(t) \neq X_{n+1}(s)\big)$$
$$\leqslant 2\mathrm{P}\big(X_1(t) \neq X_1(s) = X_1(u)\big)+2\mathrm{P}\big(X_1(t) \neq X_1(s)\big)\mathrm{P}\big(X_1(t) \neq X_1(u)\big)$$

imply the existence of a continuous nondecreasing function F such that

$$\mathrm{E}\big(Y_n(t)-\mathrm{E}Y_n(t)-Y_n(s)+\mathrm{E}Y_n(s)\big)^2\big(Y_n(u)-\mathrm{E}Y_n(u)-Y_n(t)+\mathrm{E}Y_n(t)\big)^2$$
$$\leqslant \big(F(u)-F(s)\big)^{\beta}, \quad \text{where } \beta > 1.$$

By analogy,

$$\mathrm{E}\big(Y_n(t)-\mathrm{E}Y_n(t)-Y_n(s)+\mathrm{E}Y_n(s)\big)^2\big(Y_{n+1}(u)-\mathrm{E}Y_{n+1}(u)-Y_{n+1}(t)+\mathrm{E}Y_{n+1}(t)\big)^2$$
$$\leqslant \big(F(u)-F(s)\big)^{\beta}.$$

Now, by Theorem 2, we conclude that $\{Y_n\}$ satisfies CLT in $D[0,\ c]$.

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