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FACIAL SETS OF PROBABILITY MEASURES

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Abstract. This is a discussion of probability measures in a noncommutative setting as required by quantum mechanical probability theory. The concepts of a facial, orthostable and orthofacial subset of probability measures on an orthomodular poset are introduced. They provide a link between the poset and the boundary structure of convex sets of such measures.

An orthomodular poset admitting a facial subset Δ has interesting properties: e.g. it is a complete lattice and every element in Δ is a completely additive measure. We investigate the connection between orthostability and the Jordan-Hahn decomposition of measures. It is shown that the set of completely additive probability measures on the projection lattice of a von Neumann algebra is orthofacial. Finally we use the notion of orthofaciality of a subset Δ of probability measures on an orthomodular poset to give a necessary and sufficient condition for each bounded affine functional on Δ to be the expectation functional of some observable having finite spectrum.

1. PRELIMINARIES

A poset $(L, \leqslant)$, ${}^{\#}L > 1$, with smallest (o) and largest element (1) and with a map $p \in L \to p' \in L$ satisfying (i) if $p \leqslant q$, then $q' \leqslant p'$, (ii) $(p')' = p$ and (iii) $p \vee p' = 1$ is called *orthocomplemented poset*; the map $p \to p'$ is referred to as an *orthocomplementation* on $(L, \leqslant)$. A pair $p, q \in L$, $(L, \leqslant, ')$ an orthocomplemented poset, is said to be *orthogonal*, denoted by $p \perp q$, provided $p \leqslant q'$. A subset $C \subseteq L$ is said to be *orthogonal* if $p \neq q$, $p, q \in C$, implies that $p \perp q$. Clearly, every orthogonal set in $L - \{o\}$ is contained in a maximal such set. With $\mathcal{O}(L)$ denote the collection of all maximal orthogonal sets in $L - \{o\}$ and with $\mathcal{O}_\sigma(L)$ those members of $\mathcal{O}(L)$ which have countably many elements, e.g. $\{1\} \in \mathcal{O}_\sigma(L)$.

An orthocomplemented poset $(L, \leqslant, ')$ is called an *orthomodular poset* provided (i) $p \perp q$ implies that $p \vee q$ exists and (ii) if $p \perp q$ and $p \vee q = 1$, then $p = q'$. Notice that in presence of (i), condition (ii) is equivalent to (ii') if $p \leqslant q$, then $q = p \vee (p' \wedge q)$ (cf. [6]). Let $(B, \leqslant)$ be a Boolean lattice and for

$p \in B$ denote with p' the unique element in B such that $p \vee p' = 1$, $p \wedge p' = \mathrm{o}$. Then $(B, \leqslant, ')$ is an orthomodular poset.

For an orthomodular poset $(L, \leqslant, ')$ we consider $\boldsymbol{R}^L$ in the product topology τ (the topology of pointwise convergence, a locally convex Hausdorff topology). An element $\mu \in \boldsymbol{R}^L$ is called a *measure* provided

$$\mu(p \vee q) = \mu(p) + \mu(q) \text{ if } p \perp q, \quad p, q \in L.$$

With $W(L)$ we denote the subspace of measures. A measure $\mu \in W(L)$ is called *positive* if $\mu(p) \geqslant 0$ for all $p \in L$; $K(L)$ denotes the set of positive measures on L.

Clearly, $\mu(\mathrm{o}) = 0$ for all $\mu \in W(L)$. If $\mu \in K(L)$ and $p \leqslant q$, then $\mu(q) - \mu(p) = \mu(p' \wedge q) \geqslant 0$; hence positive measures are isotone maps from L to $\boldsymbol{R}$. Also $K(L)$ is a cone in $W(L)$ (i.e. (i) $K(L) + K(L) \subseteq K(L)$, (ii) $tK(L) \subseteq K(L)$ for $t \geqslant 0$ and $K(L) \cap -K(L) = \{0\}$).

A measure μ is called *normalized* if $\mu(1) = 1$. By a *probability measure* we mean a positive normalized measure; $\Omega(L) \subseteq W(L)$ denotes the convex set of probability measures. One verifies that $\Omega(L)$ is a base for the cone $K(L)$ (i.e. $\Omega(L)$ is a convex subset of the cone $K(L)$ and every element $\mu \in K(L) - \{0\}$ admits a unique representation as $\mu = t\nu$, where $\nu \in \Omega(L)$ and $t \geqslant 0$) using the fact that $\mu(1) > 0$, $\mu \in K(L) - \{0\}$. Also, $\Omega(L)$ is τ-closed and being a subset of the τ-compact Tychonoff cube $[0, 1]^L$, by isotonicity of a probability measure, $\Omega(L)$ is clearly τ-compact.

Quite often we do not consider all the probability measures on an orthomodular poset but rather a subset of $\Omega(L)$, e.g. the σ-additive measures etc. In order to have the frame to tackle problems in this context, we are now going to develop the theory in the appropriate generality. We shall assume throughout this paper that $(L, \leqslant, ')$ is such that $\Omega(L) \neq \emptyset$.

Let Δ be a non-empty and convex subset of $\Omega(L)$, $V(\Delta) := \operatorname{lin} \Delta \subseteq W(L)$ and $K(\Delta) := \{t\mu \mid \mu \in \Delta, t \geqslant 0\}$. Then clearly $K(\Delta)$ is a generating cone for $V(\Delta)$, i.e. $V(\Delta) = K(\Delta) - K(\Delta)$, and Δ is a base for $K(\Delta)$; also $\operatorname{con}(\Delta \cup -\Delta)$ is convex, circled and absorbing in $V(\Delta)$.

Let $V'(\Delta)$ be the algebraic dual of $V(\Delta)$. With every $p \in L$ we associate a map $e_\Delta(p)\colon V(\Delta) \to \boldsymbol{R}$ by setting $e_\Delta(p)(\mu) := \mu(p)$. Clearly, $P(\Delta) := \{e_\Delta(p) \mid p \in L\} \subseteq V'(\Delta)$ and $P(\Delta)$ is a total set of linear functionals on $V(\Delta)$, i.e. if $e_\Delta(p)(\mu) = 0$ for all $p \in L$, then $\mu = 0$. The topology $\sigma(V(\Delta), P(\Delta))$ on $V(\Delta)$ coincides with the topology $\tau | V(\Delta)$. Also, a subset $\Delta \subseteq \Omega(L)$ is τ-closed iff it is $\tau | V(\Omega(L))$-closed iff, of course, it is τ-compact. Also note that $K(\Omega(L)) = K(L)$ and we set $V(L) := V(\Omega(L))$.

THEOREM 1.1. *Let $\Delta \subseteq \Omega(L)$ be non-empty and convex. Then $(V(\Delta), \Delta)$ is a base normed space (i.e. $V(\Delta)$ is a real vector space, Δ is a base of a generating cone and the Minkowski functional over* $\operatorname{con}(\Delta \cup -\Delta)$ *is a norm).*

Moreover, if $\Delta_1, \Delta_2 \subseteq \Omega(L)$ are convex, non-empty and $\Delta_1 \subseteq \Delta_2$, then the corresponding norms satisfy $\|\mu\|_{\Delta_2} \leqslant \|\mu\|_{\Delta_1}$, $\mu \in V(\Delta_1)$.

Proof. It remains to prove that $\|\mu\|_\Delta = 0$, $\mu \in V(\Delta)$, implies that $\mu = 0$, where now

$$\|\mu\|_\Delta = \inf\{t > 0 \mid \mu \in t\,\mathrm{con}(\Delta \cup -\Delta)\}.$$

Suppose that $\|\mu\|_\Delta = 0$; then $0 = t\|\mu\|_\Delta = \|t\mu\|_\Delta$, $\mu \to \|\mu\|_\Delta$ being a seminorm. The set $\mathrm{con}(\Delta \cup -\Delta)$ is circled, hence $t\mu \in 1\cdot\mathrm{con}(\Delta \cup -\Delta)$. Since

$$e_\Delta(p)\big(\mathrm{con}(\Delta \cup -\Delta)\big) \subseteq [-1, 1],$$

we conclude that $|e_\Delta(p)(t\mu)| \leqslant 1$, i.e. $|e_\Delta(p)(\mu)| \leqslant 1/t$ for $p \in L$ and $t > 0$. The totality of the set $P(\Delta) \subseteq V'(\Delta)$ now implies that $\mu = 0$.

The second assertion follows from the fact that $\Delta_1 \subseteq \Delta_2$ implies

$$\mathrm{con}(\Delta_1 \cup -\Delta_1) \subseteq \mathrm{con}(\Delta_2 \cup -\Delta_2) \cap V(\Delta_1).$$

Again let $\Delta \subseteq \Omega(L)$ be non-empty and convex. Then $\mathrm{int}\,B(\Delta) \subseteq \mathrm{con}(\Delta \cup -\Delta) \subseteq B(\Delta)$, where now $B(\Delta)$ is the $n(\Delta)$-closed (norm-closed) unit ball. With $V^*(\Delta)$ we denote the subspace of $n(\Delta)$-continuous members of $V'(\Delta)$. Obviously, $f \in V'(\Delta)$ is $n(\Delta)$-continuous iff f is bounded on the base Δ; therefore $P(\Delta) \subseteq V^*(\Delta)$.

We now follow the general theory of base normed and order unit normed spaces [1, 10, 13, 15]: if we order $V^*(\Delta)$ by $f, g \in V^*(\Delta)$, $f \leqslant g :\Leftrightarrow f(\mu) \leqslant g(\mu)$ for all $\mu \in \Delta$, then $\big(V^*(\Delta), \leqslant, e_\Delta(1)\big)$ is an order unit normed space, i.e. an Archimedian ordered vector space with order unit $e_\Delta(1)$. Also

$$\begin{aligned}\|f\|_\Delta :&= \sup\{f(\mu) \mid \mu \in B(\Delta)\}\\ &= \sup\{|f(\mu)| \mid \mu \in \Delta\} = \inf\{t > 0 \mid f \in t[-e_\Delta(1), +e_\Delta(1)]\}.\end{aligned}$$

Note that $-\|f\|_\Delta \cdot e_\Delta(1) \leqslant f \leqslant \|f\|_\Delta \cdot e_\Delta(1)$, thus $[-e_\Delta(1), +e_\Delta(1)]$ is the norm-closed unit ball in $V^*(\Delta)$.

For $f \in V^*(\Delta)$ define $f' := e_\Delta(1) - f$. One verifies that $f'' = f$ and $f \leqslant g \Rightarrow g' \leqslant f'$; also, if $f \in [0, e_\Delta(1)]$, then $f' \in [0, e_\Delta(1)]$.

For convenience, we consider e_Δ as a map from L to $P(\Delta)$; it satisfies

(i) $e_\Delta(o) = 0$,
(ii) $p \leqslant q \Rightarrow e_\Delta(p) \leqslant e_\Delta(q)$,
(iii) $e_\Delta(p') = \big(e_\Delta(p)\big)'$ and
(iv) $p \perp q \Rightarrow e_\Delta(p) + e_\Delta(q) \leqslant e_\Delta(1)$.

Also, $P(\Delta) \subseteq [0, e_\Delta(1)]$ and, since $P(\Delta)$ is a total set, we conclude that

$$V^*(\Delta) = \sigma\big(V^*(\Delta), V(\Delta)\big)\text{-cl}\,\mathrm{lin}\,P(\Delta).$$

In the sequel, we are going to discuss the boundary structure of certain convex subsets of $\Omega(L)$; hence, let us introduce the appropriate notions.

Let V be a real vector space and let C be a convex subset of V with $V = \mathrm{lin}\,C$. A subset F of C is called a *face of C* provided, for $t \in (0, 1)$,

$$v, \varkappa \in C:\ tv + (1-t)\varkappa \in F \Leftrightarrow v, \varkappa \in F.$$

A point $v \in C$ is said to be an *extreme point of* C if $\{v\}$ is a face of C; ext C denotes the set of extreme points of C. Note that C and the empty subset of C are faces of C; also the intersection of a family of faces of C is again a face of C. This shows that the collection $\mathscr{F}(C)$ of faces of C, ordered by set-inclusion, is a complete lattice.

Let ϱ be a loc. c. Hausdorff topology on V. A subset F of C is called a *ϱ-exposed face of* C if there exists a ϱ-continuous linear functional f on V and $t \in R$ such that $C \subseteq f^{-1}(-\infty, t]$ and $F = f^{-1}(t) \cap C$. With $\varrho\text{-}\mathscr{E}(C)$ we denote the collection of ϱ-exposed faces of C; note that $C \in \varrho\text{-}\mathscr{E}(C)$. A point $v \in C$ is called a *ϱ-exposed point of* C provided $\{v\} \in \varrho\text{-}\mathscr{E}(C)$; $\varrho\text{-}\exp C$ denotes the set of ϱ-exposed points of C. Clearly, $\varrho\text{-}\mathscr{E}(C) \subseteq \mathscr{F}(C)$. A subset F of C is said to be a *ϱ-semi-exposed face of* C if it is the intersection of ϱ-exposed faces; $\varrho\text{-}\mathscr{S}(C)$ denotes the collection of ϱ-semi-exposed faces of C. Clearly, by the very definition, $(\varrho\text{-}\mathscr{S}(C), \subseteq)$ is a complete lattice and $\varrho\text{-}\mathscr{E}(C) \subseteq \varrho\text{-}\mathscr{S}(C) \subseteq \mathscr{F}(C)$.

One can show that $\mathscr{F}(C) = \varrho\text{-}\mathscr{E}(C) = \varrho\text{-}\mathscr{S}(C)$ provided C is a polytope [5], i.e. C is ϱ-compact and ext C is finite; then V is finite-dimensional and ϱ equals the Euclidean topology on V.

A pair of faces F, G is said to be *ϱ-parallel*, denoted by $F \|_\varrho G$, provided there exists a ϱ-continuous linear functional f and $s, t \in R$, $s < t$, such that $C \subseteq f^{-1}[s, t]$ and $G \subseteq f^{-1}(t)$, $F \subseteq f^{-1}(s)$. Clearly, $F \|_\varrho G \Rightarrow G \|_\varrho F$; $F \|_\varrho G \Rightarrow F \cap G = \emptyset$; $F \|_\varrho \emptyset$ for all $F \in \mathscr{F}(C)$; $F \|_\varrho G$, $H \subseteq G$, $H \in \mathscr{F}(C) \Rightarrow F \|_\varrho H$.

Let $\Delta \subseteq \Omega(L)$ be non-empty and convex. Then $n(\Delta)\text{-}\mathscr{E}(\Delta) = \sigma(V(\Delta), V^*(\Delta))\text{-}\mathscr{E}(\Delta)$ since $f \in V'(\Delta)$ is $\sigma(V(\Delta), V^*(\Delta))$-continuous if and only if it is $n(\Delta)$-continuous. If $f \in [0, e_\Delta(1)]$, then $f^{-1}(1) \cap \Delta$ and $f^{-1}(0) \cap \Delta = (f')^{-1}(1) \cap \Delta$ are $n(\Delta)$-exposed faces of Δ. Note that $\emptyset \subseteq \Delta$ is an $n(\Delta)$-exposed face. If $F \in n(\Delta)\text{-}\mathscr{E}(\Delta)$, then there exists an $f \in [0, e_\Delta(1)]$ such that $F = f^{-1}(1) \cap \Delta$.

Let us prove that *if $F = \Delta$, then $F = e_\Delta(1)^{-1}(1) \cap \Delta$*. If $F \neq \Delta$, then there exists a $g \in V^*(\Delta)$, $g \neq 0$, such that $F = g^{-1}(t) \cap \Delta$ and $\Delta \subseteq g^{-1}(-\infty, t]$ for some $t \in R$. If $t = 0$, then $g \leqslant 0$, thus $-g/\|g\|_\Delta \in [0, e_\Delta(1)]$ and one verifies that $(-g/\|g\|_\Delta)^{-1}(0) \cap \Delta = F$. If $t \neq 0$, then for $h := g - te_\Delta(1)$ we have $F = h^{-1}(0) \cap \Delta$ and $\Delta \subseteq h^{-1}(-\infty, 0]$. Since $F \neq \Delta$, we conclude that $h \neq 0$ and we proceed as above.

Also, *if $F \|_{n(\Delta)} G$, $F, G \in \mathscr{F}(\Delta)$, then there exists an $f \in [0, e_\Delta(1)]$ such that $G \subseteq f^{-1}(1)$ and $F \subseteq f^{-1}(0)$*. To see this, suppose that $\Delta \subseteq g^{-1}[s, t]$, $G \subseteq g^{-1}(t)$, $F \subseteq g^{-1}(s)$ for some $g \in V^*(\Delta)$ and $s < t$. Then $s \cdot e_\Delta(1) \leqslant g \leqslant t \cdot e_\Delta(1)$. Therefore $h := (g - se_\Delta(1))/(t-s) \in [0, e_\Delta(1)]$ and one immediately verifies that $G \subseteq h^{-1}(1)$, $F \subseteq h^{-1}(0)$; this proves the claim. Since we have $0 \leqslant f(\mu) \leqslant 1$ for all $\mu \in \Delta$ provided $f \in [0, e_\Delta(1)]$, we get $f^{-1}(1) \cap \Delta \|_{n(\Delta)} f^{-1}(0) \cap \Delta$.

Let $(L, \leqslant, ')$ be an orthomodular poset and $\Delta \subseteq \Omega(L)$ non-empty and convex. We associate with every element in L an $n(\Delta)$-exposed face of Δ as follows: for $p \in L$ we set $a_\Delta(p) := \{\mu \in \Delta \mid \mu(p) = 1\}$. Clearly, $a_\Delta(p) = e_\Delta(p)^{-1}(1) \cap \Delta \in n(\Delta) - \mathscr{E}(\Delta)$. The map $p \in L \to a_\Delta(p) \in n(\Delta) - \mathscr{E}(\Delta)$ has the following properties:

(i) $$a_\Delta(o) = \emptyset, \; a_\Delta(1) = \Delta,$$

(ii) $$p \leqslant q \Rightarrow a_\Delta(p) \subseteq a_\Delta(q),$$

(iii) $$p \perp q \Rightarrow a_\Delta(p) \|_{n(\Delta)} a_\Delta(q).$$

Assertions (i) and (ii) are obvious, (iii) is proved as follows. Let $p \perp q$; then $p \leqslant q'$, hence $a_\Delta(p) \subseteq e_\Delta(q)^{-1}(0)$ and $a_\Delta(q) \subseteq e_\Delta(q)^{-1}(1)$.

2. PROBABILITY MEASURES

With $\tilde{\mathscr{O}}(L)$ we denote the collection of all orthogonal sets in $L - \{o\}$ and with $\tilde{\mathscr{O}}_\sigma(L)$ those members of $\tilde{\mathscr{O}}(L)$ which have countably many elements. Clearly, $\mathscr{O}(L) \subseteq \tilde{\mathscr{O}}(L)$ and $\mathscr{O}_\sigma(L) \subseteq \tilde{\mathscr{O}}_\sigma(L)$. Now let $D \in \tilde{\mathscr{O}}(L)$ and order the collection D^f of finite subsets of D by set-inclusion. Then $(D^f, \subseteq)$ is a directed set (i.e. any two elements of D^f have an upper bound in D^f) and, for any $\mu \in W(L)$, $(\mu(\bigvee C))_{C \in D^f}$ is a net in $\boldsymbol{R}$. If $\mu \in K(L)$, then the corresponding net is isotone and bounded by $\mu(1)$; therefore it converges in $\boldsymbol{R}$. This, however, proves that for any $\mu \in V(L)$ the net $(\mu(\bigvee C))_{C \in D^f}$ converges, since $K(L)$ is a generating cone for $V(L)$.

A measure μ is said to be *completely additive*, resp. *σ-additive*, if for every $D \in \tilde{\mathscr{O}}(L)$, resp. every $D \in \tilde{\mathscr{O}}_\sigma(L)$, for which $\bigvee D$ exists, the net $(\mu(\bigvee C))_{C \in D^f}$ converges in $\boldsymbol{R}$ and converges to $\mu(\bigvee D)$.

The subspace in $W(L)$ of completely additive, resp. σ-additive, measures is denoted by $W_c(L)$, resp. $W_\sigma(L)$; clearly, $W_c(L) \subseteq W_\sigma(L) \subseteq W(L)$. We write $\Omega_c(L) := W_c(L) \cap \Omega(L)$ and $\Omega_\sigma(L) = W_\sigma(L) \cap \Omega(L)$.

LEMMA 2.1. *Let $\mu \in W(L)$. Then $\mu \in W_c(L)$, resp. $\mu \in W_\sigma(L)$, if and only if for each $D \in \mathscr{O}(L)$, resp. $D \in \mathscr{O}_\sigma(L)$,*

$$\lim(\mu(\bigvee C))_{C \in D^f}$$

exists and equals $\mu(1)$.

Proof. First note that $D \in \tilde{\mathscr{O}}(L)$ belongs to $\mathscr{O}(L)$ if and only if $\bigvee D$ exists and equals 1.

The condition holds true if $\mu \in W_c(L)$ or $\mu \in W_\sigma(L)$. Conversely, suppose that $D \in \tilde{\mathscr{O}}_\sigma(L)$ and that $\bigvee D$ exists. If $\bigvee D = 1$, we are done. If $\bigvee D \neq 1$, then $(\bigvee D)' \neq o$ and $\{D, (\bigvee D)'\} \in \mathscr{O}_\sigma(L)$. Now

$$\mu(1) = \lim(\mu(\bigvee C))_{C \in \{D, (\vee D)'\}^f} = \lim(\mu((\bigvee C) \vee (\bigvee D)'))_{C \in D^f}$$

the latter net being a subnet of the first. Therefore,

$$\lim(\mu(\bigvee C)_{C \in D^f} = \mu(1) - \mu(\bigvee D)') = \mu(\bigvee D).$$

The proof for $D \in \tilde{\mathscr{O}}(L)$ is similar.

Let $\Delta \subseteq \Omega(L)$ be non-empty and convex. By the theorem of Bourbaki-Alaoglu, the unit ball in $V^*(\Delta)$ is $\sigma(V^*(\Delta), V(\Delta))$-compact. Since $[0, e_\Delta(1)]$ is the $\sigma(V^*(\Delta), V(\Delta))$-homeomorphic image of the unit ball under the affine map $g \to (g + e_\Delta(1))/2$, we conclude that $[0, e_\Delta(1)]$ is $\sigma(V^*(\Delta), V(\Delta))$-compact too.

Let $D \in \tilde{\mathcal{O}}(L)$; then $(e_\Omega(\bigvee C))_{C \in D^f}$ is an isotone net in $[0, e_\Omega(1)]$. Therefore, there is a $\sigma(V^*(L), V(L))$-convergent subnet $(e_\Omega(\bigvee C'))_{C'}$ which converges to, say, $f \in [0, e_\Omega(1)]$. For any $\mu \in \Omega(L)$, the real net $(e_\Omega(\bigvee C)(\mu))_{C \in D^f}$ is isotone, bounded and therefore converges. Since $\lim (e_\Omega(\bigvee C')(\mu))_{C'} = f(\mu)$, we conclude that

$$\lim (e_\Omega(\bigvee C)(\mu))_{C \in D^f} = f(\mu) \quad \text{for all } \mu \in \Omega(L),$$

hence for all $\mu \in V(L) = \operatorname{lin} \Omega(L)$. Therefore, the net $(e_\Omega(\bigvee C))_{C \in D^f}$ converges to f in the $\sigma(V^*(L), V(L))$-topology.

For $D \in \mathcal{O}(L)$ define

$$d(D) := e_\Omega(1) - \lim (e_\Omega(\bigvee C))_{C \in D^f},$$

the limit being taken in the $\sigma(V^*(L), V(L))$-topology. Note that $d(D) \in [0, e_\Omega(1)]$ for all $D \in \mathcal{O}(L)$; $d(D)$ is called the *deficiency functional of* D [4].

THEOREM 2.2 (Fischer-Rüttimann [4]). *Let* $(L, \leqslant, ')$ *be an orthomodular poset with* $\Omega(L) \neq \emptyset$. *Then*

$$\Omega_c(L) = \bigcap_{D \in \mathcal{O}(L)} \ker d(D) \cap \Omega(L)$$

and

$$\Omega_\sigma(L) = \bigcap_{D \in \mathcal{O}_\sigma(L)} \ker d(D) \cap \Omega(L).$$

Moreover, $\Omega_\sigma(L)$ *and* $\Omega_c(L)$ *are* $\sigma(V(L), V^*(L))$-*semi-exposed faces of* $\Omega(L)$.

Proof. With J we denote the canonical isometry from $V(L)$ to $V^{**}(L)$, i.e. $J(\mu)(f) = f(\mu)$, $\mu \in V(L)$, $f \in V^*(L)$. Let $\mu \in \Omega_\sigma(L)$ and $D \in \mathcal{O}_\sigma(L)$. Since $J(\mu) \in V^{**}(L)$ is $\sigma(V^*(L), V(L))$-continuous and $\bigvee D = 1$, we get

$$\begin{aligned} d(D)(\mu) &= J(\mu)(d(D)) = J(\mu)(e_\Omega(1)) - \lim (J(\mu)(e_\Omega(\bigvee C)))_{C \in D^f} \\ &= \mu(1) - \lim (\mu(\bigvee C))_{C \in D^f} = 0. \end{aligned}$$

Therefore,

$$\Omega_\sigma(L) \subseteq \bigcap_{D \in \mathcal{O}_\sigma(L)} \ker d(D) \cap \Omega(L).$$

Conversely, let μ be a probability measure such that $d(D)(\mu) = 0$ for all $D \in \mathcal{O}_\sigma(L)$. Then

$$0 = J(\mu)(d(D)) = \mu(1) - \lim (\mu(\bigvee C))_{C \in D^f} \quad \text{for all } D \in \mathcal{O}_\sigma(L).$$

Thus $\mu \in \Omega_\sigma(L)$, by lemma 2.1.

So far we have shown that

$$\Omega_\sigma(L) = \bigcap_{D\in\mathcal{O}_\sigma(L)} (\ker d(D) \cap \Omega(L)).$$

Since $d(D)\in[0, e_\Omega(1)]$, we conclude that

$$\ker d(D) \cap \Omega(L) \in \sigma\big(V(L), V^*(L)\big) - \mathscr{E}\big(\Omega(L)\big),$$

thus $\Omega_\sigma(L)\in\sigma\big(V(L), V^*(L)\big) - \mathscr{S}\big(\Omega(L)\big)$.

The proof of the remaining assertion is similar.

A subset $\Delta \subseteq \Omega(L)$ is said to be *separating*, if $\mu(p) = \mu(q)$ for all $\mu\in\Delta$ implies that $p = q$; Δ is said to be *full* if $\mu(p) \leqslant \mu(q)$ for all $\mu\in\Delta$ implies that $p \leqslant q$; Δ is called *strong* if $\mu(p) = 1 \Rightarrow \mu(q) = 1$, $\mu\in\Delta$, implies that $p\leqslant q$; Δ is said to be *unital* provided for every $p\in L-\{o\}$ there exists a $\mu\in\Delta$ such that $\mu(p) = 1$. We have the following implications: Δ is strong $\Rightarrow \Delta$ is full $\Rightarrow \Delta$ is separating, Δ is strong $\Rightarrow \Delta$ is unital.

LEMMA 2.3. *Let $(L, \leqslant, ')$ be an orthomodular poset and $\Delta \subseteq \Omega(L)$ be non-empty and convex.*

(i) *If Δ is unital, then $\|e_\Delta(p)\|_\Delta = 1$ for all $p\in L$, $p\neq o$. If Δ is τ-closed and $\|e_\Delta(p)\|_\Delta = 1$ for all $p\in L$, $p\neq o$, then Δ is unital.*

(ii) *The subset Δ is separating if and only if e_Δ: $L\to P(\Delta)$ is an injection.*

(iii) *The subset Δ is full if and only if e_Δ is an order isomorphism from $(L, \leqslant)$ onto $\big(P(\Delta), \leqslant\big)$. If either is the case, then e_Δ is an ortho-order isomorphism from $(L, \leqslant, ')$ onto $\big(P(\Delta), \leqslant, '\big)$ and the latter becomes an orthomodular poset.*

(iv) *The subset Δ is strong if and only if a_Δ: $L\to n(\Delta)-\mathscr{E}(\Delta)$ is an order isomorphism from $(L, \leqslant)$ into $\big(n(\Delta)-\mathscr{E}(\Delta), \subseteq\big)$.*

Proof. For (i) simply note that

$$\|e_\Delta(p)\|_\Delta = \sup_{\mu\in\Delta} e_\Delta(p)(\mu)$$

and that τ-closedness of Δ implies $\sigma\big(V(\Delta), P(\Delta)\big)$-compactness of Δ. Statements (ii), (iii) and (iv) are straightforward.

For $\mu\in V(L)$ we define the functionals $\mu^+, \mu^-, |\mu|\in \boldsymbol{R}^L$ as follows:

$$\mu^+(p) := \sup_{q\leqslant p}\mu(q), \qquad \mu^-(p) := -\inf_{q\leqslant p}\mu(q), \qquad |\mu| = \mu^+ + \mu^-;$$

they are referred to as the *upper variation*, *lower variation* and the *total variation* of μ, respectively. One easily verifies that $\mu^\pm = (-\mu)^\mp$ and that $\mu^\pm$ are bounded and positive. Note that these functionals are not measures in general.

If $|\mu|(1) = 0$, then $\mu^\pm(1) = 0$, hence $\mu = 0$. Clearly, if $t\in\boldsymbol{R}$, then $|t\mu|(1) = |t|\,|\mu|(1)$. Also, $|\mu_1+\mu_2|(1) \leqslant |\mu_1|(1)+|\mu_2|(1)$, since $(\mu_1+\mu_2)^\pm(1)\leqslant \mu_1^\pm(1) + \mu_2^\pm(1)$. Therefore, the functional $\mu\to|\mu|(1)$ is a norm, called the *variation norm* on $V(L)$.

LEMMA 2.4. (i) $|\mu|(1) = \sup_{p\in L}(\mu(p)-\mu(p')),\ \mu\in V(L)$.

(ii) *Let* $\Delta \subseteq \Omega(L)$ *be non-empty and convex. Then* $|\mu|(1) \leqslant \|\mu\|_\Delta$ *for* $\mu\in V(\Delta)$.

Proof. (i) We have

$$|\mu|(1) = \mu^+(1)+\mu^-(1) = \sup_{p\in L} e_\Omega(p)(\mu)+\sup_{p\in L} e_\Omega(p)(-\mu).$$

Now just note that $e_\Omega(p) = (e_\Omega(p)-e_\Omega(p')+e_\Omega(1))/2$.

(ii) Since $e_\Delta(p)-e_\Delta(p')\in[-e_\Delta(1),\ e_\Delta(1)]$, we get

$$\begin{aligned}|\mu|(1) &= \sup_L(\mu(p)-\mu(p'))\\ &= \sup_L(e_\Delta(p)-e_\Delta(p'))(\mu) \leqslant \sup\{f(\mu)\,|\,f\in[-e_\Delta(1),\ e_\Delta(1)]\} = \|\mu\|_\Delta.\end{aligned}$$

A non-empty, convex subset Δ of $\Omega(L)$ is said to have the *Jordan-Hahn property* [2, 11] provided for every $\mu\in V(\Delta)$ there exists a triple $(p,\ \nu,\ \varkappa)\in L\times K(\Delta)\times K(\Delta)$ such that $\mu = \nu-\varkappa$ and $\nu(p') = \varkappa(p) = 0$.

LEMMA 2.5. *If* $\Delta \subseteq \Omega(L)$ *has the Jordan-Hahn property, then* $|\mu|(1) = \|\mu\|_\Delta$ *for* $\mu\in V(\Delta)$ *and* $B(\Delta) = \mathrm{con}(\Delta\cup-\Delta)$.

Proof. By the previous lemma, we already have $|\mu|(1)\leqslant\|\mu\|_\Delta$. Let $\mu\in V(\Delta)-\{0\}$. For $\mu/\|\mu\|_\Delta$ there exist $p\in L$, $\nu,\ \varkappa\in\Delta$, $s,\ t\geqslant 0$ such that $\mu/\|\mu\|_\Delta = s\nu-t\varkappa$ and $s\nu(p') = t\varkappa(p) = 0$. Then

$$\begin{aligned}s+t \geqslant 1 = \|\mu/\|\mu\|_\Delta\|_\Delta &\geqslant |\mu/\|\mu\|_\Delta|(1) \geqslant (\mu/\|\mu\|_\Delta)(p)-(\mu/\|\mu\|_\Delta)(p')\\ &= s\nu(p)-t\varkappa(p)-s\nu(p')+t\varkappa(p') \geqslant s+t.\end{aligned}$$

This now shows that $|\mu|(1) = \|\mu\|_\Delta$ and also that $\mu/\|\mu\|_\Delta\in\mathrm{con}(\Delta\cup-\Delta)$. It also follows that $B(\Delta) = \mathrm{con}(\Delta\cup-\Delta)$.

The next two results are concerned with the relationship between the Jordan-Hahn property of Δ, $P(\Delta)$, the extreme points and the $\sigma(V^*(\Delta),\ V(\Delta))$-exposed points of the order interval $[0,\ e_\Delta(1)]$.

Note that under the map $f\to f-f'$ the extreme points, resp. the $\sigma(V^*(\Delta),\ V(\Delta))$-exposed points, of $[0,\ e_\Delta(1)]$ are injectively sent onto the extreme points, resp. the $\sigma(V^*(\Delta),\ V(\Delta))$-exposed points, of $[-e_\Delta(1),\ e_\Delta(1)]$.

LEMMA 2.6. *Let* $(L,\ \leqslant,\ ')$ *be an orthomodular poset and* Δ *a nonempty, convex subset of* $\Omega(L)$. *If* Δ *has the Jordan-Hahn property, then* $\sigma(V^*(\Delta),\ V(\Delta))-\exp[0,\ e_\Delta(1)] \subseteq P(\Delta)$.

Proof. Let $f\in\sigma(V^*(\Delta),\ V(\Delta))-\exp[0,\ e_\Delta(1)]$. Then there exists a $\mu\in V(\Delta)$ with $\|\mu\|_\Delta = 1$ such that

$$\{f-f'\} = \{g\in[-e_\Delta(1),\ e_\Delta(1)]\,|\,g(\mu) = 1\}.$$

Now there exist $\nu,\ \varkappa\in\Delta$, $s,\ t\geqslant 0$ and $p\in L$ such that $\mu = s\nu-t\varkappa$ and $(s\nu)(p') = (t\varkappa)(p) = 0$. Using the same arguments as in the proof of lemma 2.5, we get

$$(e_\Delta(p)-e_\Delta(p'))(\mu) = s+t = 1.$$

Since $e_\Delta(p)-e_\Delta(p')\in[-e_\Delta(1), e_\Delta(\Delta)]$, we conclude that $f = e_\Delta(p)$.

THEOREM 2.7. *Let $(L, \leqslant, ')$ be an orthomodular poset and Δ a convex and strong subset of $\Omega(L)$.*

If Δ has Jordan-Hahn property and

$$\operatorname{ext}[0, e_\Delta(1)] = \sigma(V^*(\Delta), V(\Delta))-\exp[0, e_\Delta(1)],$$

then

$$P(\Delta) = \operatorname{ext}[0, e_\Delta(1)].$$

Proof. In view of lemma 2.6, it suffices to show that

$$P(\Delta) \subseteq \sigma(V^*(\Delta), V(\Delta))-\exp[0, e_\Delta(1)].$$

So, let $p\in L$; then

$$F := \{g\in[-e_\Delta(1), e_\Delta(1)] \mid a_\Delta(p)\cup -a_\Delta(p') \subseteq g^{-1}(1)\}$$

is a non-empty, $\sigma(V^*(\Delta), V(\Delta))$-closed face of the $\sigma(V^*(\Delta), V(\Delta))$-compact set $[-e_\Delta(1), e_\Delta(1)]$. By the theorem of Krein-Milman, F has an extreme point. However,

$$\operatorname{ext} F \subseteq \operatorname{ext}[-e_\Delta(1), e_\Delta(1)] \subseteq \sigma(V^*(\Delta), V(\Delta))-\exp[-e_\Delta(1), e_\Delta(1)]$$

and

$$\sigma(V^*(\Delta), V(\Delta))-\exp[0, e_\Delta(1)] \subseteq P(\Delta).$$

Therefore there exists a $q\in L$ with

$$e_\Delta(q)-e_\Delta(q')\in\sigma(V^*(\Delta), V(\Delta))-\exp[-e_\Delta(1), e_\Delta(1)]\cap F.$$

One immediately verifies that $a_\Delta(p)\subseteq a_\Delta(q)$ and $a_\Delta(p')\subseteq a_\Delta(q')$. Strongness of Δ now implies that $p = q$, thus

$$e_\Delta(p)\in\sigma(V^*(\Delta), V(\Delta))-\exp[0, e_\Delta(1)].$$

3. FACIALITY AND ORTHOSTABILITY

We now focus our attention on the relationship between an orthomodular poset $(L, \leqslant, ')$ and the boundary structure of non-empty convex subsets Δ of $\Omega(L)$. Recall from section 1 that the map a_Δ: $L\to n(\Delta)-\mathscr{E}(\Delta)$ preserves order while orthogonality goes into parallelity.

A subset $\Delta\subseteq\Omega(L)$ is said to be *facial* provided (i) Δ is convex, (ii) a_Δ is surjective and (iii) $a_\Delta(p)\subseteq a_\Delta(q)$ implies that $p\leqslant q$. Clearly, a facial subset is strong and therefore non-empty.

The subset Δ is called *orthostable*, if (i) Δ is convex and (ii) $a_\Delta(p)\|_{n(\Delta)} a_\Delta(q)$ implies that $p \perp q$.

An orthostable subset Δ is strong. To see this, suppose that $a_\Delta(p) \subseteq a_\Delta(q)$. Since $a_\Delta(q)\|_{n(\Delta)} a_\Delta(q')$, we get $a_\Delta(p)\|_{n(\Delta)} a_\Delta(q')$, hence $p \perp q'$.

An orthostable and facial subset $\Delta \subseteq \Omega(L)$ is called *orthofacial*. Clearly, $\Delta \subseteq \Omega(L)$ is orthofacial if and only if Δ is orthostable and a_Δ is surjective. In other words, with the obvious choice of the morphism, a convex subset Δ is orthofacial if and only if a_Δ is an isomorphism from $(L, \leqslant, \perp)$ onto $(n(\Delta)-\mathscr{E}(\Delta), \subseteq, \|_{n(\Delta)})$.

THEOREM 3.1. *Let $(L, \leqslant, ')$ be an orthomodular poset and $\Delta \subseteq \Omega(L)$. If Δ is facial, then $\Delta \subseteq \Omega_c(L)$.*

Proof. Since $d(D) \in [0, e_\Omega(1)]$, $D \in \mathcal{O}(L)$, we conclude that $d(D)| V(\Delta) \in [0, e_\Delta(1)]$, hence

$$\ker\big(d(D)| V(\Delta)\big) \cap \Delta \in n(\Delta)-\mathscr{E}(\Delta) \qquad \text{for all } D \in \mathcal{O}(L).$$

Therefore, for each $D \in \mathcal{O}(L)$, there exists a $p \in L$ with

$$a_\Delta(p) = \ker\big(d(D)| V(\Delta)\big) \cap \Delta.$$

For $q \in D$ and for any $\mu \in a_\Delta(q)$ we now have

$$d(D)(\mu) = 1-\lim\big(\mu(\bigvee C)\big)_{C \in D^f} = 1-\mu(q).$$

This shows that $a_\Delta(q) \subseteq a_\Delta(p)$; thus, since Δ is also strong, p is an upper bound for D. However, D is a maximal orthogonal set in $L-\{o\}$ and, therefore, $p = 1$. Then

$$\Delta = a_\Delta(p) = \ker\big((d(D)| V(\Delta)\big) \cap \Delta,$$

thus $\Delta \subseteq \ker d(D) \cap \Omega(L)$ for all $D \in \mathcal{O}(L)$. This shows that $\Delta \subseteq \Omega_c(L)$, by theorem 2.2.

THEOREM 3.2. *An orthomodular poset $(L, \leqslant, ')$ which admits a facial subset Δ of probability measures is a complete orthomodular lattice. Furthermore, if $\mu(p) = 1$ for all $p \in A$, $A \subseteq L$, $\mu \in \Delta$, then $\mu(\bigwedge A) = 1$.*

Proof. First, we prove that $(L, \leqslant)$ is a lattice. To see this, let $p, q \in L$; then

$$a_\Delta(p) \cap a_\Delta(q) = \big(\tfrac{1}{2}e_\Delta(p)+\tfrac{1}{2}e_\Delta(q)\big)^{-1}(1) \cap \Delta \in n(\Delta)-\mathscr{E}(\Delta).$$

Therefore, there exists an $r \in L$ with $a_\Delta(r) = a_\Delta(p) \cap a_\Delta(q)$. Suppose now that $u \leqslant p, q$. Then $a_\Delta(u) \subseteq a_\Delta(r)$, hence $u \leqslant r$. This shows that $p \wedge q$ exists and that $a_\Delta(p \wedge q) = a_\Delta(p) \cap a_\Delta(q)$.

For any subset A of L, $\big(e(\bigwedge B)\big)_{B \in A^f}$ is an antitone net in $[0, e_\Delta(1)]$. By arguments similar to those used in connection with deficiency functionals, one shows that this net converges to an element $g \in [0, e_\Delta(1)]$ in the $\sigma\big(V^*(\Delta), V(\Delta)\big)$-topology and, furthermore, that $g \leqslant e_\Delta(p)$ for all $p \in A$. Now,

$g^{-1}(1) \cap \Delta \in n(\Delta) - \mathscr{E}(\Delta)$ and, therefore, there exists a $q \in L$ with $a_\Delta(q) = g^{-1}(1) \cap \Delta$. We show that q is the infimum of A. Certainly, q is a lower bound for A, by virtue of Δ being strong. Suppose that $r \leqslant A$. Then $e_\Delta(r) \leqslant e_\Delta(\bigwedge B)$ for all $B \in A^f$; thus, $e_\Delta(r) \leqslant g$. Then $a_\Delta(r) \subseteq g^{-1}(1) \cap \Delta$; thus $a_\Delta(r) \subseteq a_\Delta(q)$ and finally $r \leqslant q$.

Clearly,

$$a_\Delta(q) \subseteq \bigcap_{p \in A} a_\Delta(p).$$

If

$$\mu \in \bigcap_{p \in A} a_\Delta(p),$$

then also, for $B \in A^f$,

$$\mu \in \bigcap_{p \in B} a_\Delta(p) = a_\Delta(\bigwedge B)$$

and, therefore, we have $e_\Delta(\bigwedge B)(\mu) = 1$. Then $g(\mu) = 1$, hence $\mu \in a_\Delta(q)$.

COROLLARY 3.3. *Let $(L, \leqslant, ')$ be an orthomodular poset. If $\Delta \subseteq \Omega(L)$ is facial, then $(n(\Delta) - \mathscr{E}(\Delta), \subseteq)$ is a complete lattice. The infimum of a family of $n(\Delta)$-exposed faces coincides with its set-intersection. Moreover, $n(\Delta) - \mathscr{S}(\Delta) = n(\Delta) - \mathscr{E}(\Delta)$.*

Next, we give two sufficient conditions, in terms of the Jordan-Hahn property, for a subset of $\Omega(L)$ to be orthostable.

THEOREM 3.4. *Let $(L, \leqslant, ')$ be an atomic orthomodular lattice and Δ a convex subset of $\Omega(L)$. Suppose that for every atom p in $(L, \leqslant)$ there exists a $\mu_p \in \Delta$ with the properties*: (i) $\mu_p(p) = 1$ *and* (ii) *if* $\mu_p(r) = \mu_p(q) = 1$, *then* $\mu_p(r \wedge q) = 1$.

If Δ has the Jordan-Hahn property, then Δ is orthostable.

Proof. Let p, q be atoms. Since Δ has the Jordan-Hahn property, there exist $r \in L$, $v, \varkappa \in \Delta$ and $s, t \geqslant 0$ such that $(sv)(r') = (t\varkappa)(r) = 0$ and $\mu := \frac{1}{2}(\mu_p - \mu_q) = sv - t\varkappa$. Then, $\mu(1) = 0 = s - t$.

Suppose now that $a_\Delta(p) \|_{n(\Delta)} a_\Delta(q)$. There exists an $f \in [0, e_\Delta(1)]$ with $a_\Delta(p) \subseteq f^{-1}(1)$ and $a_\Delta(q) \subseteq f^{-1}(0)$. Then $f - f' \in [-e_\Delta(1), e_\Delta(1)]$ and $(f - f')(\mu) = 1$. Since $\frac{1}{2}\mu_q - \frac{1}{2}\mu_q \in B(\Delta)$, we get $\|\mu\|_\Delta = 1$, hence $1 \leqslant 2s$, by the triangle inequality. On the other hand, $1 \geqslant (e_\Delta(r) - e_\Delta(r'))(sv - s\varkappa) = 2s$ and, therefore, $1 = (e_\Delta(r) - e_\Delta(r'))(\mu) = \mu_p(r) - \mu_q(r)$. This shows that $\mu_p(r) = \mu_q(r') = 1$. Then $\mu_p(p \wedge r) = 1 = \mu_q(q \wedge r')$. Since p, q are atoms and $p \wedge r, q \wedge r' \neq 0$, we conclude that $p \leqslant r$ and $q \leqslant r'$, thus $p \perp q$.

Clearly, if $p, q \in L - \{o\}$ and $a_\Delta(p) \|_{n(\Delta)} a_\Delta(q)$, then for all pairs of atoms u, v with $u \leqslant p$, $v \leqslant q$ we get $a_\Delta(u) \|_{n(\Delta)} a_\Delta(v)$. Therefore, by the preceding result, $u \perp v$. Using (ii') of the definition of an orthomodular poset one can show that $(L, \leqslant)$ is also atomistic. Thus

$$p = \bigvee \{u | u \leqslant p, u \text{ an atom}\} \perp \bigvee \{v | v \leqslant q, v \text{ an atom}\} = q.$$

THEOREM 3.5. *Let* $(L, \leqslant, ')$ *be an orthomodular poset with a strong convex subset* Δ *of probability measures.*

If Δ *has the Jordan-Hahn property and if*

$$\operatorname{ext}[0, e_\Delta(1)] = \sigma(V^*(\Delta), V(\Delta)) - \exp[0, e_\Delta(1)],$$

then Δ *is orthostable.*

Proof. Let $a_\Delta(p) \|_{n(\Delta)} a_\Delta(q)$, $p, q \in L$. Then

$$F := \{g \in [0, e_\Delta(1)] \mid a_\Delta(p) \subseteq g^{-1}(1) \text{ and } a_\Delta(q) \subseteq g^{-1}(0)\}$$

is a non-empty, $\sigma(V^*(\Delta), V(\Delta))$-compact face of $[0, e_\Delta(1)]$. Thus $\operatorname{ext} F \neq \emptyset$ and, therefore, by theorem 2.7, there exists an $r \in L$ with $e_\Delta(r) \in F$. Then $a_\Delta(p) \subseteq e_\Delta(r)^{-1}(1) \cap \Delta = a_\Delta(r)$ and $a_\Delta(q) \subseteq e_\Delta(r)^{-1}(0) \cap \Delta = a_\Delta(r')$. Since Δ is strong, we conclude that $p \leqslant r$ and $q \leqslant r'$, thus $p \perp q$.

THEOREM 3.6. *Let* $(L, \leqslant, ')$ *be an orthomodular poset. If* $\Delta \subseteq \Omega(L)$ *is orthofacial and* $\operatorname{con}(\Delta \cup -\Delta)$ *is* $n(\Delta)$*-closed, then* Δ *has the Jordan-Hahn property.*

Proof. Let $\mu \in V(\Delta)$ with $\|\mu\|_\Delta = 1$. We may and do assume that $\mu \notin \Delta \cup -\Delta$. Since $B(\Delta) = \operatorname{con}(\Delta \cup -\Delta)$, there exist $\nu, \varkappa \in \Delta$ and $t \in (0, 1)$ such that $\mu = t\nu - (1-t)\varkappa$. Note that $J(\mu) \in V^{**}(\Delta)$ is $\sigma(V^*(\Delta), V(\Delta))$-continuous and as such it attains its norm on the $\sigma(V^*(\Delta), V(\Delta))$-compact unit ball of $V^*(\Delta)$, say at $f \in [-e_\Delta(1), e_\Delta(1)]$. Then $1 = J(\mu)(f) = f(\mu) = tf(\nu) - (1-t)f(\varkappa)$; thus $f(\nu) = f(-\varkappa) = 1$, since $-1 \leqslant f(\nu), f(\varkappa) \leqslant 1$.

Now $(f + e_\Delta(1))/2 \in [0, e_\Delta(1)]$, hence there exist $p, q \in L$ with $a_\Delta(p) = ((f + e_\Delta(1))/2)^{-1}(1) \cap \Delta$ and $a_\Delta(q) = ((f + e_\Delta(1))/2)^{-1}(0) \cap \Delta$. Then we have $a_\Delta(p) \|_{n(\Delta)} a_\Delta(q)$, thus $p \perp q$ or $q \leqslant p'$.

Next, observe that $((f + e_\Delta(1))/2)(\nu) = 1$, $((f + e_\Delta(1))/2)(\varkappa) = 0$ and, therefore, $\nu \in a_\Delta(p)$, $\varkappa \in a_\Delta(q)$, showing that $(t\nu)(p') = 0 = ((1-t)\varkappa)(p)$.

4. τ-CLOSED SETS OF PROBABILITY MEASURES

The following two results generalize [4], thm. 3 and thm. 7. They will be used in the sequel.

THEOREM 4.1. *Let* $(L, \leqslant, ')$ *be an orthomodular poset and* Δ *a convex subset of* $\Omega(L)$.

If Δ *is* τ*-closed and has the Jordan-Hahn property, then* $(V(\Delta), \| \|_\Delta)$ *is a reflexive Banach space.*

Proof. Since Δ is τ-closed, it also is τ-compact and, therefore,

$$\operatorname{con}(\Delta \cup -\Delta) = \{t\mu - (1-t)\nu \mid \mu, \nu \in \Delta, t \in [0, 1]\}$$

is τ-compact; so, in particular, $B(\Delta) = \operatorname{con}(\Delta \cup -\Delta)$. By the theorem of Dixmier-Ng (e.g. [7], p. 211),

$$V_*(\Delta) := \{f \in V'(\Delta) \mid f \mid B(\Delta) \text{ is } \tau \mid V(\Delta)\text{-continuous}\}$$

is a norm-closed subspace of $V^*(\Delta)$, and $(V(\Delta), \|\ \|_\Delta)$ is isometric to the Banach dual of $(V_*(\Delta), \|\ \|_\Delta | V_*(\Delta))$. Note that $P(\Delta) \subseteq V_*(\Delta)$ and that $2P(\Delta) - e_\Delta(1)$ is a subset of the unit ball of $V_*(\Delta)$.

Let $\mu \in V(\Delta)$. Then there exist $\nu, \varkappa \in \Delta$, $s, t \geqslant 0$ and $p \in L$ such that $s\nu(p') = t\varkappa(p) = 0$ and $\mu = s\nu - t\varkappa$. Now

$$s+t \geqslant \|\mu\|_\Delta \geqslant \big(e_\Delta(p) - e_\Delta(p')\big)(\mu) = \big(e_\Delta(p) - e_\Delta(p')\big)(s\nu - t\varkappa) = s+t.$$

This shows that every element of $(V_*(\Delta))^* \approx V(\Delta)$ attains its supremum ($=$ norm) on the closed unit ball of $V_*(\Delta)$. By the theorem of James [8] (also see e.g. [7]), the closed unit ball of $V_*(\Delta)$ is $(V_*(\Delta), V(\Delta))$-compact, showing that $V_*(\Delta)$ and finally $V(\Delta)$ are reflexive normed linear spaces in their respective norms.

THEOREM 4.2. *Let $(L, \leqslant, ')$ be an orthomodular poset with a τ-closed, unital set Δ of probability measures.*

Then $\Delta \subseteq \Omega_\sigma(L)$ if and only if every countable maximal orthogonal subset of L is finite.

Proof. Clearly, if ${}^{\#}D < \aleph_0$ for all $D \in \mathcal{O}_\sigma(L)$, then $d(D) = 0$, hence, by theorem 2.2,

$$\Omega_\sigma(L) = \bigcap_{D \in \mathcal{O}_\sigma(L)} \ker d(D) \cap \Omega(L) = \Omega(L).$$

Suppose now that there exists a $D \in \mathcal{O}_\sigma(L)$ with ${}^{\#}D = \aleph_0$. Let $i \in N \to p_i \in D$ be a distinct enumeration of D. Then

$$\bigvee_{i=1}^{n} p_i < 1,$$

thus

$$\bigwedge_{i=1}^{n} p_i' \neq o \quad \text{for all } n \in N.$$

The subset Δ is unital and $\tau | V(L)$-closed, also, $e_\Omega(\bigwedge_{i=1}^{n} p_i')$ is $\tau | V(L)$-continuous and therefore

$$\Big(e_\Omega(\bigwedge_{i=1}^{n} p_i')^{-1}(1) \cap \Delta\Big)_{n=1}^{\infty}$$

is a decreasing sequence of non-empty, τ-closed subsets of the τ-compact set $\Omega(L)$. Consequently, the intersection of all the members of this sequence is not empty, i.e. there exists $\mu \in \Delta$ with

$$\mu(\bigwedge_{i=1}^{n} p_i') = 1 \quad \text{or} \quad \mu(\bigvee_{i=1}^{n} p_i) = 0 \quad \text{for all } n \in N.$$

Then

$$d(D)(\mu) = 1 - \lim\big(e_\Omega(\bigvee C)(\mu)\big)_{C \in D^f} = 1 - \lim\big(e_\Omega(\bigvee_{i=1}^{n} p_i)(\mu)\big)_{n \in N} = 1.$$

Hence,

$$\mu \notin \bigcap_{D \in \mathcal{O}_\sigma(L)} \ker d(D) \cap \Omega(L) = \Omega_\sigma(L),$$

i.e. $\Delta \nsubseteq \Omega_\sigma(L)$.

We now combine several of the previous results in the following

THEOREM 4.3. *If an orthomodular poset $(L, \leqslant, ')$ admits a τ-closed orthofacial subset Δ of probability measures, then*

(i) $(L, \leqslant)$ *is a lattice*;

(ii) *every orthogonal subset in L is finite*;

(iii) *if $\mu(p) = \mu(q) = 1$, then $\mu(p \wedge q) = 1$ for all $\mu \in \Delta$*;

(iv) Δ has the Jordan-Hahn property;

(v) $\|\mu\|_\Delta = \|\mu\|_{\Omega(L)}$ *for all* $\mu \in V(\Delta)$;

(vi) $\big(V(\Delta), \|\ \|_\Delta\big)$ *is a reflexive Banach space.*

Proof. (i) and (iii) follow from theorem 3.2.

(ii) Let $C \in \tilde{\mathcal{O}}(L)$ and suppose that C is not finite. Then C contains a countably infinite subset D. If $D \notin \mathcal{O}_\sigma(L)$, then, since $(L, \leqslant)$ is a complete lattice, by theorem 3.2, $\{D, (\bigvee D)'\} \in \mathcal{O}_\sigma(L)$. In this case and also when $D \in \mathcal{O}_\sigma(L)$ it follows that $\Delta \nsubseteq \Omega_\sigma(L)$, by theorem 4.2. This contradicts the assertion in theorem 3.1.

(iv) Since Δ is τ-compact, we conclude that $\operatorname{con}(\Delta \cup -\Delta)$ is $h(\Delta)$-closed. The assertion now follows by virtue of theorem 3.6.

(v) follows from (iv), lemma 2.5, lemma 2.4 (ii) and theorem 1.1.

(vi) is a consequence of (iv) and theorem 4.1.

Let us now discuss faciality in the context of Boolean lattices. First notice that if $(L, \leqslant, ')$ is an orthomodular lattice and if Δ is a convex, unital subset of $\Omega(L)$, then $a_\Delta(p)\|_{n(\Delta)} a_\Delta(q) \Rightarrow a_\Delta(p) \cap a'_\Delta(q) = \emptyset \Rightarrow p \wedge q = o$. Therefore, if $(L, \leqslant, ')$ is a Boolean lattice together with its unique orthocomplementation and Δ a convex unital subset of $\Omega(L)$, then Δ is orthostable.

THEOREM 4.4. *Let $(L, \leqslant, ')$ be an orthomodular poset. Any two of the following three conditions imply the third*:

(i) *L is finite*;

(ii) $(L, \leqslant)$ *is a Boolean lattice*;

(iii) $(L, \leqslant, ')$ *admits a τ-closed facial subset Δ of probability measures.*

Proof. (i); (iii) $\Rightarrow$ (ii). Since L is finite, we conclude that

$$V^*(\Delta) = \sigma\big(V^*(\Delta), V(\Delta)\big)\text{-cl} \operatorname{lin} P(\Delta) = \operatorname{lin} P(\Delta).$$

Hence $V^*(\Delta)$ and, finally, $V(\Delta)$ are finite dimensional; also $\tau|V(\Delta) = n(\Delta)$ and, therefore, Δ is $n(\Delta)$-compact. By [9], theorem 2.1,

$$\Delta = n(\Delta)-\mathrm{cl}\ \mathrm{con}\big(n(\Delta)-\exp\Delta\big).$$

Since Δ is also facial, we get $^{\#}n(\Delta)-\exp\Delta \leqslant {}^{\#}n(\Delta)-\mathscr{E}(\Delta) < \aleph_0$ and, therefore, $\Delta = \mathrm{con}\big(n(\Delta)-\exp\Delta\big)$, showing that Δ is a polytope [5]. Hence $n(\Delta)-\mathscr{E}(\Delta) = \mathscr{F}(\Delta)$. Since the map a_Δ: $L \to \mathscr{F}(\Delta)$ is an order-isomorphism, the map $F \in \mathscr{F}(\Delta) \to F' := a_\Delta\big(a_\Delta^{-1}(F)\big)' \in \mathscr{F}(\Delta)$ is an orthocomplementation that makes $(\mathscr{F}(\Delta), \subseteq, ')$ into an orthomodular lattice. This in turn proves that Δ is a simplex, by a theorem of the author ([12], theorem 3.5). Therefore $(L, \leqslant)$ is a Boolean lattice.

(i); (ii) $\Rightarrow$ (iii). Again, since L is finite, $V(L)$ is finite-dimensional. Moreover,

$$\Omega(L) = \bigcap_{p \in L} e_{\Omega(L)}(p)^{-1}[0, \infty) \cap e_{\Omega(L)}(1)^{-1}[1, \infty) \cap e_{\Omega(L)}(1)^{-1}(-\infty, 1],$$

hence the $n(\Omega)$-bounded set $\Omega(L)$ is the intersection of finitely many $n(\Omega)$-closed half-spaces, i.e. $\Omega(L)$ is a polytope. Again, $\mathscr{F}\big(\Omega(L)\big) = n(\Omega)-\mathscr{E}\big(\Omega(L)\big)$. Now let G be a proper face of $\Omega(L)$ and $F_1, F_2, \ldots, F_n$ the facets containing G. Since $\Omega(L)$ is a polytope, we have

$$G = \bigcap_{i=1}^{n} F_i.$$

By [12], theorem 4.2, there exist $p_1, p_2, \ldots, p_n \in L$ such that $F_i = a_{\Omega(L)}(p_i)$, $i = 1, 2, \ldots, n$. It is easily verified that a probability measure on a Boolean lattice satisfies $\mu(p) = \mu(q) = 1 \Rightarrow \mu(p \wedge q) = 1$. Therefore

$$G = \bigcap_{i=1}^{n} a_{\Omega(L)}(p_i) = a_{\Omega(L)}\big(\bigwedge_{i=1}^{n} p_i\big).$$

This shows that the map $a_{\Omega(L)}$ is surjective. It is a basic fact that for such an orthomodular poset $\Omega(L)$ is strong. Also, as remarked earlier, $\Omega(L)$ is τ-closed. Thus $\Omega(L)$ is the desired set.

(ii); (iii) $\Rightarrow$ (i). By theorem 4.3 (ii), every orthogonal set in L is finite. Therefore, using orthomodularity, every nonzero element in L covers an atom of L. Also, every non-zero element is the supremum of the atoms it covers. Since $(L, \leqslant)$ is a Boolean lattice, it follows that the collection of atoms is an orthogonal set. This proves the claim.

5. EXAMPLES

5.1. Let $\mathscr{A}$ be a von Neumann algebra [3, 14]. The real vector space $\mathscr{A}_{sa} = \{a \in \mathscr{A} | a = a^*\}$ of self-adjoint elements, ordered by the positive cone $\{a \in \mathscr{A} | a = bb^* \text{ for some } b \in \mathscr{A}\}$ and with the identity $1 \in \mathscr{A}$ as order unit is a complete order unit normed space. The (unique) pre-dual Banach space of $\mathscr{A}$ is denoted by $\mathscr{A}_*$. We consider $\mathscr{A}_*$ to be canonically embedded into $\mathscr{A}^*$; i.e. $\mathscr{A}_*$ consists of the normal linear functionals on $\mathscr{A}$. A linear functional φ on $\mathscr{A}$ is said to be *self-adjoint* provided $\varphi(a^*) = \overline{\varphi(a)}$ for all $a \in \mathscr{A}$. Let $(\mathscr{A}^*)_{sa}$, $(\mathscr{A}_*)_{sa}$ be the real vector spaces of self-adjoint elements of $\mathscr{A}^*$, $\mathscr{A}_*$, respectively. Notice that these are closed subspaces of $\mathscr{A}^*$. The convex set S of positive linear functionals on $\mathscr{A}$ which map the identity to $1 \in \boldsymbol{R}$ is a subset of $(\mathscr{A}^*)_{sa}$ and the pairs $\big((\mathscr{A}^*)_{sa}, S\big)$, $\big((\mathscr{A}_*)_{sa}, S_*\big)$, where $S_* = \mathscr{A}_* \cap S$, are base normed spaces. Then $\operatorname{con}(S \cup -S)$, $\operatorname{con}(S_* \cup -S_*)$ are the base norm unit balls, respectively. Moreover, $\|\varphi\|_s = \|\varphi\|$ for $\varphi \in (\mathscr{A}^*)_{sa}$ and $\|\varphi\|_{s_*} = \|\varphi\|_s$ for $\varphi \in (\mathscr{A}_*)_{sa}$. The order unit normed space $(\mathscr{A}_{sa}, \leqslant, 1)$ is norm and order isomorphic to the order unit normed space $\big((\mathscr{A}_*)^*_{sa}, \leqslant, e\big)$, where $e \in (\mathscr{A}_*)^*_{sa}$ is such that $e(S_*) = 1$. This isomorphism is established through the map $\gamma\colon \mathscr{A}_{sa} \to (\mathscr{A}_*)^*_{sa}$ defined by $\gamma(a)(\varphi) = \varphi(a)$, $\varphi \in (\mathscr{A}_*)_{sa}$.

In the order inherited from $\mathscr{A}_{sa}$ the set of self-adjoint idempotents (projections), $\mathscr{P}(\mathscr{A}) = \{p \in \mathscr{A} \mid p = pp^* = p^*\}$ forms a complete lattice with o as the smallest and 1 as the largest element. The map $p \to p' := 1 - p$ is an orthocomplementation which makes $\big(\mathscr{P}(\mathscr{A}), \leqslant, '\big)$ into an orthomodular poset. One can show that to each projection $p \neq$ o there exists an element $\varphi \in S_*$ with $\varphi(p) = 1$.

Let φ be a positive normal linear functional on $\mathscr{A}$. There is a largest projection p such that $\varphi(p) = 0$. The projection $s(\varphi) := 1 - p$ is called *support* of φ. Notice that for $a \in \mathscr{A}$, $\varphi(a^* a) = 0$ if and only if $a = b\big(1 - s(\varphi)\big)$ for some $b \in \mathscr{A}$. We claim that for any non-zero projection p, $q := \bigvee \{s(\varphi) | \varphi(p) = 1,\ \varphi \in S_*\} = p$ holds true. Suppose $q < p$. Then there exists a $\psi \in S_*$ such that $\psi(p - q) = 1$, thus $\psi(p) = 1$. Therefore $s(\psi) \leqslant q$, but also $s(\psi) \leqslant p - q$. This shows that $s(\psi) =$ o, hence $\psi\big(1 - s(\psi)\big) = 1$, a contradiction.

For any $a \in \mathscr{A}$ the set $\{p \in \mathscr{P}(\mathscr{A}) | pa = a\}$, resp. $\{p \in \mathscr{P}(\mathscr{A}) | ap = a\}$, has a smallest element denoted by $l(a)$, resp. $r(a)$. Notice, if $a \in \mathscr{A}_{sa}$, then $s(a) := r(a) = l(a)$; $s(a)$ is called the *support* of $a \in \mathscr{A}_{sa}$.

For $\text{o} \leqslant a$, $\varphi(a) = 0$ if and only if $\varphi\big(s(a)\big) = 0$. To see this let $\varphi(a) = \varphi(\sqrt{a}^* \sqrt{a}) = 0$; thus we get $\sqrt{a} = b\big(1 - s(\varphi)\big)$ for some $b \in \mathscr{A}$, hence $a = \big(b(1 - s(\varphi)\big)^2$. Then clearly $a\big(1 - s(\varphi)\big) = a$, thus $\text{o} \leqslant s(a) \leqslant 1 - s(\varphi)$. Since $0 \leqslant \varphi$ and $\varphi\big(1 - s(\varphi)\big) = 0$, we conclude that $\varphi\big(s(a)\big) = 0$. Conversely, suppose that $\varphi\big(s(a)\big) = 0$; then $\text{o} \leqslant s(a) \leqslant 1 - s(\varphi)$. Now $a\big(1 - s(\varphi)\big) = as(a)\big(1 - s(\varphi)\big) = as(a) = a = \big(1 - s(\varphi)\big) a$ and $0 = \varphi\big\{\big(\sqrt{a}(1 - s(\varphi))\big)^* \big(\sqrt{a}(1 - s(\varphi))\big)\big\} = \varphi\big(\big(1 - s(\varphi)\big) a\big) = \varphi(a)$.

Let $\varphi \in S_*$ and $a \in [\text{o}, 1]$. Then $\varphi(a) = 1$ implies that $\varphi\big(s(a)\big) = 1$. The proof goes as follows: if $\varphi(a) = 1$, then $\varphi(1 - a) = 0$, thus $\varphi\big(s(1 - a)\big) = 0$. From this we get $s(1 - a) \leqslant 1 - s(\varphi)$. Then $(1 - a)\big(1 - s(\varphi)\big) = 1 - a$, thus $as(\varphi) = s(\varphi)$ and, therefore, $s(a) s(\varphi) = s(a) as(\varphi) = as(\varphi) = s(\varphi)$, hence $s(\varphi) \leqslant s(a) \leqslant 1$. The assertion now follows since $\varphi\big(s(\varphi)\big) = 1$.

We define a map $\alpha\colon (\mathscr{A}^*)_{sa} \to \boldsymbol{R}^{\mathscr{P}(\mathscr{A})}$ through $\alpha(\varphi) = \varphi|\mathscr{P}(\mathscr{A})$. Notice that for $p, q \in \mathscr{P}(\mathscr{A})$ with $p \perp q$ we have $p \vee q = p+q$. For $\varphi \in (\mathscr{A}^*)_{sa}$ we then get $\alpha(\varphi)(p \vee q) = \alpha(\varphi)(p)+\alpha(\varphi)(q)$, thus $\alpha(\varphi) \in W(\mathscr{P}(\mathscr{A}))$. This map α is clearly linear and, by the spectral theorem, α is also injective. Moreover, for $\varphi \in (\mathscr{A}^*)_{sa}$, $\varphi \in S$ if and only if $\alpha(\varphi) \in \Omega(\mathscr{P}(\mathscr{A}))$. Hence $\alpha((\mathscr{A}^*)_{sa}) = \operatorname{lin}\alpha(S) \subseteq V(\Omega(\mathscr{P}(\mathscr{A})))$. Since $\alpha(S)$, resp. $\alpha(S_*)$, is convex, the pair $(\alpha(\mathscr{A}^*)_{sa}, \alpha(S))$, resp. $(\alpha(\mathscr{A}_*)_{sa}, \alpha(S_*))$, is a base normed space and for the corresponding base norm we have $\|\alpha(\varphi)\|_{\Omega(\mathscr{P}(\mathscr{A}))} \leqslant \|\alpha(\varphi)\|_{\alpha(S)}$ for all $\varphi \in (\mathscr{A}^*)_{sa}$, resp. $\|\alpha(\varphi)\|_{\alpha(S)} \leqslant \|\alpha(\varphi)\|_{\alpha(S_*)}$ for all $\varphi \in (\mathscr{A}_*)_{sa}$. Obviously, the maps α and $\alpha|(\mathscr{A}_*)_{sa}$ are norm and order isomorphisms between the corresponding base normed spaces. The adjoint map of $\alpha|(\mathscr{A}_*)_{sa}$ defined by $(\alpha|(\mathscr{A}_*)_{sa})^*(f)(\varphi) = f(\alpha(\varphi))$, $\varphi \in (\mathscr{A}_*)_{sa}, f \in (\alpha(\mathscr{A}_*)_{sa})^*$, is a norm and order isomorphism between the order unit normed spaces $((\alpha(\mathscr{A}_*)_{sa})^*, \leqslant, e_{S_*}(1))$ and $((\mathscr{A}_*)_{sa}^*, \leqslant, e)$. Notice that $\gamma(p) = (\alpha|(\mathscr{A}_*)_{sa})^*(e_{S_*}(p))$ for all projections p.

We are going to show that $\alpha(S_*) \subseteq \Omega(\mathscr{P}(\mathscr{A}))$ is orthofacial. Let $F \in n(\alpha(S_*)) - \mathscr{E}(\alpha(S_*))$. Then there exists an $a \in [0, 1]$ such that

$$\alpha^{-1}(F) = \gamma(a)^{-1}(0) \cap S_* = \{\varphi \in S_* | \varphi(a) = 0\}.$$

Thus

$$\alpha^{-1}(F) = \{\varphi \in S_* | \varphi(s(a)') = 1\} = \{\varphi \in S_* | \gamma(s(a)')(\varphi) = 1\}$$

as previously remarked. Therefore $F = a_{\alpha(S_*)}(s(a)')$, showing that the map $a_{\alpha(S_*)}$ is surjective. If $a_{\alpha(S_*)}(p) \subseteq a_{\alpha(S_*)}(q)$, then

$$\{\varphi \in S_* | \varphi(p) = 1\} \subseteq \{\varphi \in S_* | \varphi(q) = 1\},$$

hence $p \leqslant q$, by a remark made above, and faciality of $\alpha(S_*)$ follows.

Next suppose that for $F, G \in n(\alpha(S_*)) - \mathscr{E}(\alpha(S_*))$, $F \|_{n(\alpha(S_*))} G$ holds true. Then there exists an $a \in [o, 1]$ such that $\alpha^{-1}(F) \subseteq \gamma(a)^{-1}(1) \cap S_*$ and $\alpha^{-1}(G) \subseteq \gamma(0)^{-1}(0) \cap S_*$. However,

$$\gamma(a)^{-1}(1) \cap S_* = \{\varphi \in S_* | \varphi(a) = 1\} \subseteq \{\varphi \in S_* | \varphi(s(a)) = 1\}$$

and

$$\gamma(a)^{-1}(0) \cap S_* = \{\varphi \in S_* | \varphi(s(a)) = 0\}.$$

Thus

$$F \subseteq e_{\alpha(S_*)}(s(a))^{-1}(1) \cap \alpha(S_*)$$

and

$$G \subseteq e_{\alpha(S_*)}(s(a)')^{-1}(1) \cap \alpha(S_*).$$

Let $p, q \in \mathscr{P}(\mathscr{A})$ be such that $a_{\alpha(S_*)}(p) = F$ and $a_{\alpha(S_*)}(q) = G$. Then $p \leqslant s(a)$, $q \leqslant s(a)'$, hence $p \perp q$. This proves orthofaciality of $\alpha(S_*)$.

One final remark. Since $\operatorname{con}(\alpha(S_*) \cup -\alpha(S_*))$ equals the unit ball in

$V(\alpha(S_*))$, it follows from theorem 3.6 that $\alpha(S_*)$ has the Jordan-Hahn property. Hence, by lemma 2.4(ii) and lemma 2.5, it follows that $\|\varphi\| = \|\alpha(\varphi)\|_{\alpha(S_*)} = \|\alpha(\varphi)\|_{\alpha(S)} = \|\alpha(\varphi)\|_{\Omega(\mathscr{P}(\mathscr{A}))}$ for all $\varphi \in (\mathscr{A}_*)_{sa}$.

5.2. We give an example of an orthomodular poset with an orthofacial subset which does not have the Jordan-Hahn property. Let $(L, \leqslant, ')$ be given through the Hasse-diagram in Fig. 1.

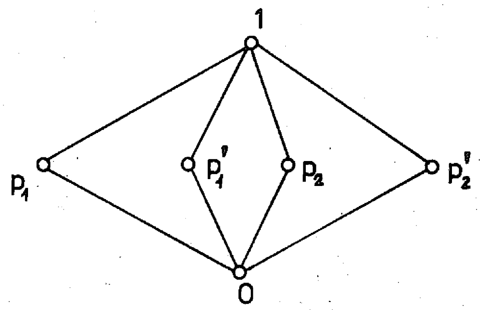

Fig. 1

Let μ_i, resp. ν_i, be the unique probability measures such that $\mu_i(p_i) = 1$ and $\mu_i(p_j) = 1/2$ $(i \neq j)$, resp. $\nu_i(p_i) = 0$ and $\nu_i(p_j) = 1/2$ $(i \neq j)$, for $i = 1, 2$. Then

$$\Delta := \{\mu \in \Omega(L) | 0 < \mu(p_i) < 1,\ i = 1, 2\} \cup \{\mu_1, \mu_2, \nu_1, \nu_2\}$$

is a convex subset of $\Omega(L)$,

$$n(\Delta) - \mathscr{E}(\Delta) = \{\emptyset, \Delta, \{\mu_1\}, \{\mu_2\}, \{\nu_1\}, \{\nu_2\}\},$$

$$a_\Delta(p_i) = \{\mu_i\},\ a_\Delta(p_i') = \{\nu_i\} \quad \text{for } i = 1, 2$$

and clearly $a_\Delta(1) = \Delta$, $a_\Delta(\mathrm{o}) = \emptyset$. It follows that the map a_Δ is surjective.

As for parallelity, we only have $\{\mu_i\} \|_{n(\Delta)} \{\nu_i\}$, $i = 1, 2$, and $\emptyset$ is $n(\Delta)$-parallel to all the elements of $n(\Delta) - \mathscr{E}(\Delta)$. Therefore, Δ is orthostable.

Let now $\varrho := \frac{1}{2}(\mu_1 + \mu_2) - \frac{1}{2}(\nu_1 + \nu_2) \in V(\Delta)$ and suppose that there exist $r \in L$, $\varkappa, \omega \in \Delta$, $s, t \geqslant 0$ such that $\varrho = s\varkappa - t\omega$ and $(s\varkappa)(r') = (t\omega)(r) = 0$. Then, clearly, $s, t > 0$, hence $\varkappa(r) = 1$ and $\omega(r) = 0$, also $r \neq \mathrm{o}, 1$. If $r = p_1$, then $\varkappa = \mu_1$ and $\omega = \nu_1$. Furthermore, $s = t$ since $\mu_i(1) = \nu_i(1) = \varkappa(1) = \omega(1) = 1$, $i = 1, 2$. Hence, $\varrho = s(\mu_1 - \nu_1)$. Since $\frac{1}{2}\mu_1 - \frac{1}{2}\nu_1 \in \mathrm{con}(\Delta \cup -\Delta)$, we have $\|\mu_1 - \nu_1\|_\Delta \leqslant 2$; on the other hand

$$2 = (e_\Delta(p_1) - e_\Delta(p_1'))(\mu_1 - \nu_1)$$
$$\leqslant \sup\{f(\mu_1 - \nu_1) | f \in [-e_\Delta(1), +e_\Delta(1)]\} = \|\mu_1 - \nu_1\|_\Delta.$$

Consequently, $\|\mu_1 - \nu_1\|_\Delta = 2$. Next, we are going to show that $\|\varrho\|_\Delta = 1$. One verifies that for $0 \leqslant \bar{t} < \frac{1}{2}$, $\frac{1}{2}(\mu_1 + \mu_2) + \bar{t}\varrho$ and $\frac{1}{2}(\nu_1 + \nu_2) - \bar{t}\varrho$ belong to Δ.

Therefore

$$\tfrac{1}{2}\{\tfrac{1}{2}(\mu_1+\mu_2)+\bar{t}\varrho\}-\tfrac{1}{2}\{\tfrac{1}{2}(\nu_1+\nu_2)-\bar{t}\varrho\} = \varrho(\tfrac{1}{2}+\bar{t})\in \operatorname{con}(\Delta \cup -\Delta).$$

This proves that $\|\varrho\|_\Delta \leqslant 1$. However, $(e_\Delta(p_1)-e_\Delta(p_1'))(\varrho) = 1$.

Now, $1 = \|\varrho\|_\Delta = s\|\mu_1-\nu_1\|_\Delta = 2s$. Substituting $\frac{1}{2}$ for s above, yields $\mu_2-\nu_2 = 0$, a contradiction. The remaining possibilities for r are treated in a similar manner.

5.3. Let $(L, \leqslant, ')$ be as above. Denote with μ_1, resp. μ_2, ν_1, ν_2, the unique probability measure with $\mu_1(p_1) = 1$ and $\mu_1(p_2) = \frac{1}{4}$, resp. $\mu_2(p_1) = \frac{1}{2}$ and $\mu_2(p_2) = 1$, $\nu_1(p_1) = \frac{3}{4}$ and $\nu_1(p_2) = 0$, $\nu_2(p_1) = 0$ and $\nu_2(p_2) = \frac{1}{2}$. Define $\Delta := \{\mu\in\Omega(L)|0 < \mu(p_1),\ \mu(p_2) < 1$ and $-\frac{2}{3}\mu(p_1)+\frac{1}{2} < \mu(p_2) < -\frac{3}{2}\mu(p_1)+\frac{7}{4}\}$ $\cup\{\mu_1, \mu_2, \nu_1, \nu_2\}$; clearly, Δ is a convex subset of $\Omega(L)$. Also

$$n(\Delta)-\mathscr{E}(\Delta) = \{\emptyset, \Delta, \{\mu_1\}, \{\mu_2\}, \{\nu_1\}, \{\nu_2\}\},$$

$$a_\Delta(p_i) = \{\mu_i\}, \quad a_\Delta(p_i') = \{\nu_i\}, \quad i = 1, 2,$$

and $a_\Delta(1) = \Delta$, $a_\Delta(\mathrm{o}) = \emptyset$. Therefore Δ is facial. Now, Δ is 2-dimensional, thus $V(\Delta)$ is a 3-dimensional vector space. One verifies that $\{\nu_1, \nu_2, \mu_1\}$ is a linear basis for $V(\Delta)$ and that $\operatorname{aff}\Delta = \operatorname{aff}\{\nu_1, \nu_2, \mu_1\}$. We define a linear functional f on $V(\Delta)$ as follows: $f(\nu_1) = \frac{1}{4}$, $f(\nu_2) = 0$, $f(\mu_1) = \frac{3}{4}$. A simple but lengthy computation shows that $0\leqslant f(\mu)\leqslant 1$ for all $\mu\in\Delta$, hence $f\in[\mathrm{o}, e_\Delta(1)]$. Since $\mu_2 = -2\nu_1+\nu_2+2\mu_1$, we get $f(\mu_2) = 1$. Then $a_\Delta(p_2)\subseteq f^{-1}(1)$ and $a_\Delta(p_1')\subseteq f^{-1}(0)$, hence $a_\Delta(p_2)\|_{n(\Delta)}a_\Delta(p_1')$ but p_2 non $\perp p_1'$.

Therefore Δ is facial but not orthostable.

5.4. Let $(L, \leqslant, ')$ be as above. By similar methods one shows that $\Omega(L)$ is orthostable but not facial (see thm. 4.4).

6. A SPECTRAL THEOREM

Let $(L, \leqslant, ')$ be an orthomodular poset. With $\mathscr{B}(\boldsymbol{R})$ we denote the class of Borel sets of $\boldsymbol{R}$. By a *Varadarajan observable* [16] we mean a map x: $\mathscr{B}(\boldsymbol{R})\to L$ satisfying:

(i) $x(\emptyset) = \mathrm{o}$, $x(\boldsymbol{R}) = 1$;

(ii) if $u_1\cap u_2 = \emptyset$, $u_1, u_2\in\mathscr{B}(\boldsymbol{R})$, then $x(u_1)\perp x(u_2)$;

(iii) for every sequence $(u_i)_{i=1}^\infty$ of pairwise disjoint elements in $\mathscr{B}(\boldsymbol{R})$, $x(\bigcup_{i=1}^\infty u_i)$ is the supremum of $\{x(u_i)|i = 1, 2, \ldots\}$ in $(L, \leqslant)$.

With $S(L)$ we denote the collection of Varadarajan observables. Notice, if $(u_i)_{i=1}^\infty$ is a sequence in $\mathscr{B}(\boldsymbol{R})$, then for all $x\in S(L)$, $x(\bigcup_{i=1}^\infty u_i)$ is the supremum, resp. $x(\bigcap_{i=1}^{\infty} u_i)$ is the infimum, of $\{x(u_i)|i=1, 2, \ldots\}$ in $(L, \leqslant)$.

For a Borel function χ: $\boldsymbol{R} \to \boldsymbol{R}$ and a Varadarajan observable $x \in S(L)$, the map $u \in \mathscr{B}(\boldsymbol{R}) \to x(\chi^{-1}(u)) \in L$, denoted by $\chi(x)$, is an element of $S(L)$. By the *spectrum* of a Varadarajan observable we mean the set

$$s(x):\ = \bigcap \{\Delta \subseteq \boldsymbol{R} | \Delta \text{ closed, } x(\Delta)=1\}.$$

One verfies that $x(s(x))=1$, $s(x) \neq \emptyset$ and $s(\chi(x)) \subseteq \mathrm{cl}\,\chi(s(x))$. A Varadarajan observable x is said to be *bounded* provided the spectrum of x is bounded; $S^b(L)$ denotes the collection of bounded Varadarajan observables.

Let Δ be a non-empty convex subset of $\Omega(L)$, $x \in S^b(L)$ and $\mu \in \Delta$. Then $t \in \boldsymbol{R} \to \mu(x(-\infty, t]) \in \boldsymbol{R}$ is a bounded and isotonic function. The map

$$\mu \in \Delta \to \int_{\min s(x)-\varepsilon}^{\max s(x)} id_{\boldsymbol{R}}\, d\mu(x(-\infty, t]) \in \boldsymbol{R},$$

where the integral is taken in the sense of Stieltjes and $\varepsilon > 0$, is affine and bounded by the interval $[\min s(x), \max s(x)]$. Therefore, this map admits a unique extension to an ($n(\Delta)$-continuous) linear functional on $V(\Delta)$, called the *expectation functional of* x *on* Δ and denoted by $E_\Delta(x)$.

Here, as it turns out, we are concerned with Varadarajan observables having finite spectrum. If, for $x \in S(L)$, $s(x) = \{t_1, t_2, \ldots, t_n\}$ then, clearly, $s \in S^b(L)$ and, as is easily shown,

$$E_\Delta(x) = \sum_{i=1}^{n} t_i\, e_\Delta(x(\{t_i\})).$$

Also $x(\{t_i\}) \neq o$, $i = 1, 2, \ldots, n$. Thus, whenever Δ is unital, then there exist $\mu, \nu \in \Delta$ such that $E_\Delta(x)(\mu) = \max\{t_i\}$ and $E_\Delta(x)(\nu) = \min\{t_i\}$. Therefore,

$$\|E_\Delta(x)\|_\Delta = \sup_{\mu \in \Delta} |E_\Delta(x)(\mu)| = \max\{\max\{t_i\}, -\min\{t_i\}\}.$$

In particular, $E_\Delta(x)$ attains its norm on $B(\Delta)$. Also, $x(\{t\}) \neq o$ implies $t \in s(x)$.

From [6], theorem 3.19, it follows that $E_\Delta(x) = E_\Delta(y)$ if and only if $x = y$ provided that Δ is strong and x, y have finite spectra. Using standard techniques one shows that an orthomodular poset, in which orthogonal subsets are finite, admits only Varadarajan observables with finite spectrum.

Examples show that in general an $n(\Delta)$-continuous linear functional is not the expectation functional of some Varadarajan observable. Orthofaciality is now being used to give a necessary and sufficient condition for this to be true for a certain class of orthomodular posets.

We shall make use of the following technical lemmata.

LEMMA 6.1. *Let* $(L, \leqslant, ')$ *be an orthomodular poset with a strong convex* *set Δ of probability measures. Furthermore, suppose that*

$$P(\Delta) \supseteq \operatorname{ext}[0, e_\Delta(1)].$$

If $a_\Delta(p) \subseteq f^{-1}(1)$, $p \in L$ and $f \in [0, e_\Delta(1)]$, then $e_\Delta(p) \leqslant f$.

Proof. Consider

$$F := \{g \in [0, e_\Delta(1)] \mid a_\Delta(p) \subseteq g^{-1}(1)\},$$

which is a non-empty $\sigma(V^*(\Delta), V(\Delta))$-compact face of $[0, e_\Delta(1)]$. Then $\emptyset \neq \operatorname{ext} F \subseteq P(\Delta)$. Thus for any $g \in \operatorname{con} \operatorname{ext} F$ there exist $q_1, q_2, \ldots, q_m \in L$ and $t_1, t_2, \ldots, t_m \in (0, 1]$ with $\sum_{i=1}^{m} t_i = 1$, such that

$$g = \sum_{i=1}^{m} t_i e_\Delta(q_i).$$

Then

$$g^{-1}(1) \cap \Delta = \bigcap_{i=1}^{m} e_\Delta(q_i)^{-1}(1) \cap \Delta = \bigcap_{1=i}^{m} a_\Delta(q_i),$$

hence $a_\Delta(p) \subseteq a_\Delta(q_i)$ or $p \leqslant q_i$ for $i = 1, 2, \ldots, m$. This shows that $e_\Delta(p) \leqslant e_\Delta(q_i)$, hence $t_i e_\Delta(p) \leqslant t_i e_\Delta(q_i)$ and, finally,

$$e_\Delta(p) = \sum_{i=1}^{m} t_i e_\Delta(p) \leqslant \sum_{1=i}^{m} t_i e_\Delta(q_i) = g.$$

By the theorem of Krein-Milman, $F = \sigma(V^*(\Delta), V(\Delta))-\mathrm{cl}\ \operatorname{con} \operatorname{ext} F$. If $f \in F$, then there exists a net $(g_\delta)_\delta$ in $\operatorname{con} \operatorname{ext} F$ converging to f in the $\sigma(V^*(\Delta), V(\Delta))$-topology. Since $e_\Delta(p)(\mu) \leqslant g_\delta(\mu)$ for all $\mu \in \Delta$, we conclude that $e_\Delta(p) \leqslant f$.

In the sequel we assume that $(L, \leqslant, ')$ is an orthomodular poset with a τ-closed orthofacial set Δ of probability measures and that

$$\operatorname{ext}[0, e_\Delta(1)] \subseteq \sigma(V^*(\Delta), V(\Delta))-\exp[0, e_\Delta(1)].$$

Then, by theorem 4.3, Δ has the Jordan-Hahn property, also $\operatorname{con}(\Delta \cup -\Delta) = B(\Delta)$. Together with theorem 2.7 we then conclude that $\operatorname{ext}[0, e_\Delta(1)] = P(\Delta)$.

We define mappings α, β, γ from $[0, e_\Delta(1)]$ into $[0, e_\Delta(1)]$ as follows: for $f \in [0, e_\Delta(1)]$ let

$$\alpha(f) = \{0, \text{ if } f = 0; f/\|f\|_\Delta \text{ otherwise};$$

$$\beta(f) = e_\Delta(a_\Delta^{-1}(f^{-1}(1) \cap \Delta));$$

$$\gamma(f) = f - \beta(f) \text{ (see lemma 6.1).}$$

We set $(\gamma\alpha)^0 := \mathrm{id}_{[0, e_\Delta(1)]}$. Notice that $\beta(f)^{-1}(1) \cap \Delta = f^{-1}(1) \cap \Delta$ and that $\beta(f) \in P(\Delta)$.

LEMMA 6.2. *Let* $g\in[0, e_\Delta(1)]$ *and* $\mu\in\Delta$. *Then*

(i) $g(\mu)=0\Rightarrow\alpha(g)(\mu)=0$;

(ii) $g(\mu)=0\Rightarrow\gamma(g)(\mu)=0$;

(iii) $g(\mu)=1\Rightarrow\gamma(g)(\mu)=0$.

Proof. (i) is obvious.

(ii) If $g(\mu)=0$, then $\mu\in g^{-1}(0)\cap\Delta$. Since

$$g^{-1}(0)\cap\Delta\|_{n(\Delta)}g^{-1}(1)\cap\Delta,$$

we conclude, by orthostability, that

$$a_\Delta^{-1}(g^{-1}(0)\cap\Delta)\perp a_\Delta^{-1}(g^{-1}(1)\cap\Delta).$$

Now $\mu(a_\Delta^{-1}(g^{-1}(0)\cap\Delta))=1$, thus $\mu(a_\Delta^{-1}(g^{-1}(1)\cap\Delta))=0$. Showing that $\beta(g)(\mu)=0$. Now $\gamma(g)(\mu)=g(\mu)-\beta(g)(\mu)=0$.

(iii) If $g(\mu)=1$, then $\mu\in g^{-1}(1)\cap\Delta$, hence $\mu(a_\Delta^{-1}(g^{-1}(1)\cap\Delta))=1$ and thus $e_\Delta(a_\Delta^{-1}(g^{-1}(1)\cap\Delta))(\mu)=1$. Showing that $\beta(g)(\mu)=1$. Now $\gamma(g)(\mu)=g(\mu)-\beta(g)(\mu)=0$.

LEMMA 6.3. *For* $i, k\in N_0$, $i\neq k$, *and* $g\in[0, e_\Delta(1)]$ *we have*

$$\beta\alpha(\gamma\alpha)^i(g)+\beta\alpha(\gamma\alpha)^k(g)\leqslant e_\Delta(1).$$

Proof. It suffices to show that $\beta\alpha(\gamma\alpha)^i(g)+\beta\alpha(\gamma a)^0(g)\leqslant e_\Delta(1)$ for $i\geqslant 1$ and $g\in[0, e_\Delta(1)]$. Let $\mu\in\Delta$ and $\beta\alpha(\gamma\alpha)^0(g)(\mu)=1$; then

$$\mu\in[\beta\alpha(g)]^{-1}(1)\cap\Delta=[\alpha(g)]^{-1}(1)\cap\Delta,$$

i.e. $\alpha(g)(\mu)=1$. Then $\gamma\alpha(g)(\mu)=0$, by lemma 6.2(iii), thus $\alpha\gamma\alpha(g)(\mu)=0$, by (i), hence $\gamma\alpha\gamma\alpha(g)(\mu)=0$, by (ii). Repeated use of (i) and (ii) yields $(\gamma\alpha)^{i+1}(g)(\mu)=\alpha(\gamma\alpha)^i(g)(\mu)=0$, hence

$$\beta\alpha(\gamma\alpha)^i(g)(\mu)=\alpha(\gamma a)^i(g)(\mu)-(\gamma\alpha)^{i+1}(g)(\mu)=0.$$

To this end we have shown that $\beta\alpha(\gamma\alpha)^0(g)(\mu)=1$ implies $(\beta\alpha(\gamma\alpha)^i(g))'(\mu)=1$ for $\mu\in\Delta$, but Δ being strong, we conclude that $\beta\alpha(\gamma a)^0(g)\leqslant(\beta\alpha(\gamma\alpha)^i(g))'$.

LEMMA 6.4. *Let* $g\in[0, e_\Delta(1)]$. *Then*

(i) $\beta\alpha(g)=0\Leftrightarrow g=0$;

(ii) *there exists an* $i\in N_0$ *such that* $\beta\alpha(\gamma\alpha)^i(g)=0$.

Proof. (i) If $g=0$, then $\alpha(g)=0$ and, therefore, $\alpha(g)^{-1}(1)\cap\Delta=\emptyset$. Conversely, suppose that $g\neq 0$. Then $\|\alpha(g)\|_\Delta=1$. Since $\alpha(g)$ is $\sigma(V(\Delta), V^*(\Delta))$-continuous and $B(\Delta)$ is $\sigma(V(\Delta), V^*(\Delta))$-compact by reflexivity of $V(\Delta)$ (theorem 4.3), $\alpha(g)$ attains its norm at an extreme point of $B(\Delta)$. But $\operatorname{ext}B(\Delta)\subseteq\Delta\cup-\Delta$, since $B(\Delta)=\operatorname{con}(\Delta\cup-\Delta)$, and we conclude that $\alpha(g)^{-1}(1)\cap\Delta\neq\emptyset$, thus $\beta\alpha(g)\neq 0$.

(ii) By lemma 6.3, the set

$$\{e_\Delta^{-1}(\beta\alpha(\gamma\alpha)^i)(g) \mid i \in N_0\} \subseteq L$$

is orthogonal and, therefore, by theorem 4.3 (ii), has finitely many elements. Hence there exist $i, k \in N_0$, $i \neq k$, with

$$e_\Delta^{-1}(\beta\alpha(\gamma\alpha)^i)(g) = e_\Delta^{-1}(\beta\alpha(\gamma\alpha)^k)(g).$$

Having an orthogonal pair, we now get $\beta\alpha(\gamma\alpha)^k(g) = 0$.
Let $g \in [0, e_\Delta(1)]$ and define

$$m(g) := \min\{i \in N_0 \mid \beta\alpha(\gamma\alpha)^i(g) = 0\}.$$

Note that $m(g) \geqslant 1$ if and only if $g \in (0, e_\Delta(1)]$, by lemma 6.4 (i).

THEOREM 6.5. *Let* $(L, \leqslant, ')$ *be an orthomodular poset with a* τ*-closed orthofacial set* Δ *of probability measures and suppose that*

$$\operatorname{ext}[0, e_\Delta(1)] \subseteq \sigma(V^*(\Delta), V(\Delta)) - \exp[0, e_\Delta(1)].$$

Then, for every $g \in (0, e_\Delta(1)]$,

$$g = \sum_{k=0}^{m(g)-1} \prod_{i=0}^{k} \|(\gamma\alpha)^i(g)\|_\Delta \beta\alpha(\gamma\alpha)^k(g)$$

holds true.

Moreover,

$$0 < \|(\gamma\alpha)^i(g)\|_\Delta \leqslant 1 \quad \textit{for } i = 0, 1, 2, \ldots, m(g)-1.$$

Proof. For $g \in [0, e_\Delta(1)]$, we show by induction that

$$(*) \qquad g = \sum_{k=0}^{m-1} \prod_{i=0}^{k} \|(\gamma\alpha)^i(g)\|_\Delta \beta\alpha(\gamma\alpha)^k(g) + \prod_{i=0}^{m-1} \|(\gamma\alpha)^i(g)\|_\Delta (\gamma\alpha)^m(g)$$

holds true for all $m \in N$.

Certainly, $\gamma\alpha(g) = \alpha(g) - \beta\alpha(g)$, thus $\alpha(g) = \beta\alpha(g) + \gamma\alpha(g)$. Then

$$\|g\|_\Delta \alpha(g) = g = \|(\gamma\alpha)^0(g)\|_\Delta \beta\alpha(\gamma\alpha)^0(g) + \|(\gamma\alpha)^0(g)\|_\Delta (\gamma\alpha)^1(g).$$

Suppose that $(*)$ holds true for $m \in N$. Also

$$(\gamma\alpha)^{m+1}(g) = \gamma\alpha(\gamma\alpha)^m(g) = \alpha(\gamma\alpha)^m(g) - \beta\alpha(\gamma\alpha)^m(g).$$

Then

$$\alpha(\gamma\alpha)^m(g) = \beta\alpha(\gamma\alpha)^m(g) + (\gamma\alpha)^{m+1}(g)$$

and thus

$$(\gamma\alpha)^m(g) = \|(\gamma\alpha)^m(g)\|_\Delta \beta\alpha(\gamma\alpha)^m(g) + \|(\gamma\alpha)^m(g)\|_\Delta (\gamma\alpha)^{m+1}(g).$$

Then

$$g = \sum_{k=0}^{m-1} \prod_{i=0}^{k} \|(\gamma\alpha)^i(g)\|_\Delta \beta\alpha(\gamma\alpha)^k(g) +$$

$$+ \prod_{i=0}^{m-1} \|(\gamma\alpha)^i(g)\|_\Delta \{\|(\gamma\alpha)^m(g)\|_\Delta \beta\alpha(\gamma\alpha)^m(g) + \|(\gamma\alpha)^m(g)\|_\Delta (\gamma\alpha)^{m+1}(g)\}$$

$$= \sum_{k=0}^{m} \prod_{i=0}^{k} \|(\gamma\alpha)^i(g)\|_\Delta \beta\alpha(\gamma\alpha)^k(g) + \prod_{i=0}^{m} \|(\gamma\alpha)^i(g)\|_\Delta (\gamma\alpha)^{m+1}(g).$$

Now $\beta\alpha(\gamma\alpha)^{m(g)}(g) = 0$; we conclude, by lemma 6.4(i), that $(\gamma\alpha)^{m(g)}(g) = 0$. Choosing $g \neq 0$, we may set $m = m(g)$ and get the desired representation for g.

Clearly, if $g \in [0, e_\Delta(1)]$, then $\|(\gamma\alpha)^i(g)\|_\Delta \leqslant 1$ for all $i \in N_0$. If $\|(\gamma\alpha)^i(g)\|_\Delta = 0$, then $\beta\alpha(\gamma\alpha)^i(g) = 0$, by lemma 6.4(i). Hence$m(g) \leqslant i$.

THEOREM 6.6. *Let $(L, \leqslant, ')$ be an orthomodular poset with a τ-closed orthofacial set Δ of probability measures and suppose that*

$$\mathrm{ext}\,[0, e_\Delta(1)] \subseteq \sigma(V^*(\Delta), V(\Delta)) - \exp\,[0, e_\Delta(1)].$$

Then every Varadarajan observable has finite spectrum and the mapping

$$x \in S(L) \to E_\Delta(x) \in V^*(\Delta)$$

is bijective.

Proof. Since every orthogonal subset of L is finite, by theorem 4.3 (ii), we conclude that every observable has finite spectrum. This implies that $S^b(L) = S(L)$.

Suppose that $f \in V^*(\Delta)$ and that $f \neq te_\Delta(1)$, $t \in R$. One is easily convinced that $g := \frac{1}{2}(f/\|f\|_\Delta + e_\Delta(1))$ belongs to $(0, e_\Delta(1)]$. By virtue of theorem 6.5 and lemma 6.3, there exist pairwise orthogonal elements $p_1, p_2, \ldots, p_n$ in $L - \{0\}$ and $t_1, t_2, \ldots, t_n > 0$ such that

$$g = \sum_{i=1}^{n} t_i e_\Delta(p_i).$$

We may assume that $t_i < t_j$ for $i < j$ and set

$$t_0 := 0, \qquad p_0 := (\bigvee_{i=1}^{n} p_i)'.$$

Denote with x the unique Varadarajan observable such that $x(\{t_i\}) = p_i$ for $i = 0, 1, 2, \ldots, n$. Then clearly $g = E_\Delta(x)$. Let $\chi(t) = 2\|f\|_\Delta t - \|f\|_\Delta$, $t \in R$; then

$$E_\Delta(\chi(x)) = \sum_{i=0}^{n} \chi(t_i) e_\Delta(p_i)$$

$$= 2\|f\|_\Delta \sum_{i=0}^{n} t_i e_\Delta(p_i) - \|f\|_\Delta \sum_{i=0}^{n} e_\Delta(p_i) = 2\|f\|_\Delta g - \|f\|_\Delta e_\Delta(1) = f.$$

The case where f is a multiple of $e_\Delta(1)$ is easily dealt with.

This proves surjectivity of the map; injectivity follows by [6], theorem 3.19.

The following result gives us a converse to the aforementioned theorem.

THEOREM 6.7. *Let* $(L, \leqslant, ')$ *be an orthomodular poset with a strong convex set* Δ *of probability measures.*

Suppose that each Varadarajan observable has finite spectrum and that the mapping

$$x \in S(L) \to E_\Delta(x) \in V^*(\Delta)$$

is surjective.

Then Δ *and* $\bar{\Delta}: = \tau\text{-cl}\,\Delta \subseteq \boldsymbol{R}^L$ *both are orthofacial,* $(V(\bar{\Delta}), \|\ \|_{\bar{\Delta}})$ *is the Banach space completion of* $(V(\Delta), \|\ \|_\Delta)$ *and*

$$\operatorname{ext}[0, e_\Delta(1)] \subseteq \sigma\big(V^*(\Delta), V(\Delta)\big) - \exp[0, e_\Delta(1)],$$

$$\operatorname{ext}[0, e_{\bar{\Delta}}(1)] \subseteq \sigma\big(V^*(\bar{\Delta}), V(\bar{\Delta})\big) - \exp[0, e_{\bar{\Delta}}(1)].$$

Proof. (i) We define a map $R\colon V^{**}(\Delta) \to \boldsymbol{R}^L$ as follows:

$$R(\tilde{v})(p): = \tilde{v}\big(e_\Delta(p)\big) \quad \text{for all } p \in L,\ \tilde{v} \in V^{**}(\Delta).$$

Since $V^*(\Delta) = \operatorname{lin} P(\Delta)$, this map is injective, clearly it is linear. Recall from the general theory of base normed and order unit normed spaces, that $\big(V^{**}(\Delta), \tilde{\Delta}\big)$, where

$$\tilde{\Delta}: = \{\tilde{v} \in V^{**}(\Delta) \,|\, \tilde{v}(f) \geqslant 0 \text{ for all } f \geqslant 0,\ \tilde{v}\big(e_\Delta(1)\big) = 1\},$$

is a base normed space with unit ball equals to $\operatorname{con}(\tilde{\Delta} \cup -\tilde{\Delta})$. Then $R(\tilde{\Delta}) \subseteq \Omega(L)$, thus $R\big(V^{**}(\Delta)\big) \subseteq V\big(\Omega(L)\big)$. Now $\big(R\big(V^{**}(\Delta)\big), R(\tilde{\Delta})\big)$ is a base normed space and $\big(R\big(V^{**}(\Delta)\big), \|\ \|_{R(\tilde{\Delta})}\big)$ becomes a Banach space isometric to $\big(V^{**}(\Delta), \|\ \|_{\tilde{\Delta}}\big)$ under the map R.

Denote with J the canonical embedding map from $V(\Delta)$ into $V^{**}(\Delta)$, an isometry. Then for $v \in V(\Delta)$ we get

$$R\big(J(v)\big)(p) = J(v)\big(e_\Delta(p)\big) = e_\Delta(p)(v) = v(p) \quad \text{for all } p \in L,$$

showing that $R \circ J = \mathrm{id}_{V(\Delta)}$, hence $V(\Delta) \subseteq R\big(V^{**}(\Delta)\big)$. Also $\big(V(\Delta), \|\ \|_\Delta\big)$ is a subspace of $\big(R(V^{**}(\Delta)), \|\ \|_{R(\tilde{\Delta})}\big)$ since, for $v \in V\Delta)$.

$$\|v\|_\Delta = \|J(v)\|_{\tilde{\Delta}} = \big\|R\big(J(v)\big)\big\|_{R(\tilde{\Delta})} = \|v\|_{R(\tilde{\Delta})}.$$

We set $\bar{V} := n\big(R(\Delta)\big) - \operatorname{cl} V(\Delta)$. Then $(\bar{V}, \|\ \|_{\bar{V}})$ is the Banach space completion of $\big(V(\Delta), \|\ \|_\Delta\big)$. Clearly, $T\colon \bar{V}^* \to V^*(\Delta)$ defined by $T(f) = f\,|\,V(\Delta)$ is an isometry from $\bar{V}^*$ onto $V^*(\Delta)$. We set

$$\bar{P} = \{e_{R(\tilde{\Delta})}(p)\,|\,\bar{V}|\, p \in L\} \subseteq \bar{V}^*.$$

Then $\sigma(\bar{V}, \bar{P}) = \tau\,|\,\bar{V}$; also note that $T(\bar{P}) = P(\Delta)$.

Let $f \in \bar{V}^*$; then

$$T(f) = \sum_{i=1}^{n} t_i e_\Delta(p_i)$$

for some maximal orthogonal set $p_1, p_2, \ldots, p_n$ in $L-\{o\}$ and scalars $t_1, t_2, \ldots, t_n$, $n > 0$. As is easily seen, $T(f)$ attains its norm on $B(\Delta)$ and, since the unit ball $\bar{B}$ of $\bar{V}$ contains $B(\Delta)$ and $\|T(f)\|_\Delta = \|f\|_{\bar{V}}$, we conclude that f attains its norm on $\bar{B}$. This implies that $\bar{B}$ is $\sigma(\bar{V}, \bar{V}^*)$-compact, by the theorem of James.

Notice that $\sigma(\bar{V}, \bar{V}^*)|V(\Delta) = \sigma(V(\Delta), V^*(\Delta))$. Since $V^*(\Delta) = \operatorname{lin} P(\Delta)$, we get $\bar{V}^* = \operatorname{lin} \bar{P}$, hence $\sigma(\bar{V}, \bar{V}^*) = \tau|\bar{V}$. This has the following consequence: $\Delta \subseteq \bar{B}$, hence $\sigma(\bar{V}, \bar{V}^*)-\mathrm{cl}\,\Delta$ is $\tau|\bar{V}$-compact, thus also τ-compact. Furthermore, $\bar{\Delta} \subseteq \sigma(\bar{V}, \bar{V}^*)-\mathrm{cl}\,\Delta = \tau|\bar{V}-\mathrm{cl}\,\Delta \subseteq \tau-\mathrm{cl}\,\Delta$, hence $\bar{\Delta} = \sigma(\bar{V}, \bar{V}^*)$ $-\mathrm{cl}\,\Delta$.

Now

$$\begin{aligned} B(\Delta) &= \sigma(V(\Delta), V^*(\Delta))-\mathrm{cl}\,\mathrm{con}(\Delta \cup -\Delta) = \sigma(\bar{V}, \bar{V}^*)|V(\Delta)-\mathrm{cl}\,\mathrm{con}(\Delta \cup -\Delta) \\ &\subseteq \sigma(\bar{V}, \bar{V}^*)-\mathrm{cl}\,\mathrm{con}(\Delta \cup -\Delta) \subseteq \mathrm{con}(\bar{\Delta} \cup -\bar{\Delta}) \subseteq \bar{B}, \end{aligned}$$

$\bar{\Delta}$ being convex and $\sigma(\bar{V}, \bar{V}^*)$-compact. Since $\bar{B} = \|\ \|_{\bar{V}}-\mathrm{cl}\,B(\Delta)$, we get $\bar{B} = \mathrm{con}(\bar{\Delta} \cup -\bar{\Delta})$.

Clearly $\bar{\Delta}$ is a convex subset of $\Omega(L)$. It now follows that $\bar{V} = V(\bar{\Delta})$ and $\|\ \|_{\bar{V}} = \|\ \|_{\bar{\Delta}}$. Also $T\colon V^*(\bar{\Delta}) \to V^*(\Delta)$ is an order-isomorphism, since

$$\bar{\Delta} = \sigma(V(\bar{\Delta}), V^*(\bar{\Delta}))-\mathrm{cl}\,\Delta.$$

(ii) We show that $\bar{\Delta}$ and Δ are orthofacial. First note that now $Te_{\bar{\Delta}}(p)$ $= e_\Delta(p)$. Again, for $f \in V^*(\bar{\Delta})$, there exists an $x \in S(L)$ such that

$$T(f) = E_\Delta(x) = \sum_{i=1}^{n} t_i e_\Delta(x(\{t_i\})), \qquad \text{where } s(x) = \{t_1, t_2, \ldots, t_n\}.$$

Then

$$f = \sum_{i=1}^{n} t_i T^{-1} e_\Delta(x(\{t_i\})) = \sum_{i=1}^{n} t_i e_{\bar{\Delta}}(x(\{t_i\})) = E_{\bar{\Delta}}(x).$$

Let $x \in S(L)$, $s(x) = \{t_1, t_2, \ldots, t_n\}$, $t_i < t_j$ for $i < j$, and suppose that $E_{\bar{\Delta}}(x) \in [0, e_{\bar{\Delta}}(1)]$. Then

$$0 \leqslant t_1 \leqslant E_{\bar{\Delta}}(x)(\mu) \leqslant t_n \leqslant 1 \qquad \text{for all } \mu \in \bar{\Delta}.$$

We claim that

$$E_{\bar{\Delta}}(x)^{-1}(1) \cap \bar{\Delta} = a_{\bar{\Delta}}(x(\{1\})) \quad \text{and} \quad E_{\bar{\Delta}}(x)^{-1}(0) \cap \bar{\Delta} = a_{\bar{\Delta}}(x(\{0\})).$$

Let $\mu \in \bar{\Delta}$ with $\mu(x\{1\}) = 1$; then $x\{1\} \neq 0$, hence $1 \in s(x)$ and $t_n = 1$.

Now

$$E_{\bar{\Delta}}(x)(\mu) = \sum_{i=1}^{n} t_i \mu(x\{t_i\}) = 1.$$

Conversely, suppose that, for $\mu \in \bar{\Delta}$, $E_{\bar{\Delta}}(x)(\mu) = 1$. Then $t_n = 1$ and clearly $\mu(x(\{1\})) = 1$ since $s(x) \subseteq [0, 1]$ and

$$\sum_{i=1}^{n} \mu(x(\{t_i\})) = 1.$$

The second claim is proved in a similar manner.

Let $F \in n(\bar{\Delta}) - \mathscr{E}(\bar{\Delta})$; then there exists an $x \in S(L)$ with $E_{\bar{\Delta}}(x) \in [0, e_{\bar{\Delta}}(1)]$ such that $F = E_{\bar{\Delta}}(x)^{-1}(1) \cap \bar{\Delta}$. Therefore $F = a_{\bar{\Delta}}(x(\{1\}))$, showing that the map $a_{\bar{\Delta}}$ is surjective.

Next, let $E \|_{n(\bar{\Delta})} F$, $E, F \in n(\bar{\Delta}) - \mathscr{E}(\bar{\Delta})$. Then there exists an $x \in S(L)$ with $E_{\bar{\Delta}}(x) \in [0, e_{\bar{\Delta}}(1)]$ such that $E \subseteq E_{\bar{\Delta}}(x)^{-1}(1) \cap \bar{\Delta} = a_{\bar{\Delta}}(x(\{1\}))$ and $F \subseteq E_{\bar{\Delta}}(x)^{-1}(0) \cap \bar{\Delta} = a_{\bar{\Delta}}(x(\{0\}))$. Now if $E = a_{\bar{\Delta}}(p)$ and $F = a_{\bar{\Delta}}(q)$, $p, q \in L$, then $p \leqslant x(\{1\})$ and $q \leqslant x(\{0\})$, hence $p \perp q$ since $\{1\} \cap \{0\} = \emptyset$. This proves orthostability of $\bar{\Delta}$. The proof that Δ is orthofacial is quite analogous.

(iii) We show that $\mathrm{ext}[0, e_{\Delta}(1)] \subseteq P(\Delta)$ holds true. Let $f \in [0, e_{\Delta}(1)]$, $f \notin P(\Delta)$. If $\|f\|_{\Delta} \neq 1$, then also $f/\|f\|_{\Delta} \in (0, e_{\Delta}(1)]$ and $f = \|f\|_{\Delta}(f/\|f\|_{\Delta}) + (1-\|f\|_{\Delta})0$, thus $f \notin \mathrm{ext}[0, e_{\Delta}(1)]$. Now suppose that $\|f\|_{\Delta} = 1$. There exist pairwise orthogonal elements $p_1, p_2, \ldots, p_n$ in $L - \{o\}$ and scalars $0 < t_1 < t_2 \ldots < t_{n-1} < t_n = 1$ such that

$$f = \sum_{i=1}^{n} t_i e_{\Delta}(p_i).$$

Notice that $n > 1$. We set

$$t_0 := 0,\ p_0 := \Big(\bigvee_{j=1}^{n} p_j\Big)' \quad \text{and} \quad q_i := \bigvee_{j=0}^{i} p_j \text{ for } i = 0, 1, 2, \ldots, n.$$

Then

$$\begin{aligned} f &= \sum_{i=0}^{n} t_i e_{\Delta}(p_i) = t_0 e_{\Delta}(q_0) + \sum_{j=1}^{n} t_j \big(e_{\Delta}(q_j) - e_{\Delta}(q_{j-1})\big) \\ &= \sum_{j=0}^{n-1} (t_j - t_{j+1}) e_{\Delta}(q_j) + t_n e_{\Delta}(q_n) = e_{\Delta}(1) - \sum_{j=0}^{n-1} (t_{j+1} - t_j) e_{\Delta}(q_j) \\ &= \sum_{j=0}^{n-1} (t_{j+1} - t_j) e_{\Delta}(q_j'). \end{aligned}$$

Since

$$\sum_{j=0}^{n-1} (t_{j+1} - t_j) = 1, \quad 0 < t_{j+1} - t_j < 1,\ j = 1, 2, \ldots, n > 1,$$

we conclude that $f \notin \mathrm{ext}[0, e_{\Delta}(1)]$.

(iv) We claim that

$$P(\Delta) \subseteq \sigma(V^*(\Delta), V(\Delta)) - \exp[0, e_\Delta(1)].$$

First notice that every orthogonal subset in L is finite since $\bar{\Delta}$ is τ-closed and orthofacial. Therefore the poset $(L, \leqslant)$ is atomistic. Suppose now that $p \in L - \{0, 1\}$ and let $q_1, q_2, \ldots, q_n$, resp. $r_1, r_2, \ldots, r_m$, be a maximal set of pairwise orthogonal atoms majorized by p, resp. p'. Then

$$p = \bigvee_{i=1}^{n} q_i \quad \text{and} \quad p' = \bigvee_{j=1}^{m} r_j.$$

Select $\mu_i \in a_\Delta(q_i)$, $i = 1, 2, \ldots, n$, resp. $\nu_j \in a_\Delta(r_j)$, $j = 1, 2, \ldots, m$, and define

$$\varkappa = \frac{1}{n} \sum_{i=1}^{n} \mu_i - \frac{1}{m} \sum_{j=1}^{m} \nu_j.$$

One verifies that $[0, e_\Delta(1)] \subseteq J(\varkappa)^{-1}(-\infty, 1]$ and that $J(\varkappa)(e_\Delta(p)) = 1$. Suppose that, for $g \in [0, e_\Delta(1)]$, $J(\varkappa)(g) = 1$ holds true. Then $g(\mu_i) = 1$, $i = 1, 2, \ldots, n$, and $g(\nu_j) = 0$, $j = 1, 2, \ldots, m$. Thus $\mu_i \in g^{-1}(1) \cap \Delta$ and $\nu_j \in g^{-1}(0) \cap \Delta$. By faciality of Δ and theorem 4.3(iii), we conclude that $a_\Delta(q_i) \subseteq g^{-1}(1) \cap \Delta \in n(\Delta) - \mathscr{E}(\Delta)$ and $a_\Delta(r_j) \subseteq g^{-1}(0) \cap \Delta \in n(\Delta) - \mathscr{E}(\Delta)$. Hence $a_\Delta(p) \subseteq g^{-1}(1) \cap \Delta$ and $a_\Delta(p') \subseteq g^{-1}(0) \cap \Delta$. But $g = E(x)$ for some $x \in S(L)$. By a previous remark in (ii), we then get $a_\Delta(p) \subseteq a_\Delta(x(\{1\}))$ and $a_\Delta(p') \subseteq a_\Delta(x(\{0\}))$, thus $p \leqslant x(\{1\})$ and $p' \leqslant x(\{0\})$. This shows that $p = x(\{1\})$ and $p' = x(\{0\})$, hence, since $o < p, p' < 1$, $0, 1 \in s(x)$ and finally $\{0, 1\} = s(x)$. Then

$$g = 0 \cdot e_\Delta(x(\{0\})) + 1 \cdot e_\Delta(x(\{1\})) = e_\Delta(p).$$

If $p = 1$, select a maximal orthogonal subset of atoms $p_1, p_2, \ldots, p_n$ in L and $\mu_i \in a_\Delta(p_i)$, $i = 1, 2, \ldots, n$. Set

$$\varkappa := \frac{1}{n} \sum_{i=1}^{n} \mu_i \in \Delta.$$

Then $J(\varkappa)(e_\Delta(1)) = 1$ and $[0, e_\Delta(1)] \subseteq J(\varkappa)^{-1}(-\infty, 1]$. Now proceed as above to show that $e_\Delta(1)$ is an $n(\Delta)$-exposed point of $[0, e_\Delta(1)]$. If $p = 0$, then $J(-\varkappa)(0) = 0$ and $[0, e_\Delta(1)] \subseteq J(-\varkappa)^{-1}(-\infty, 0]$. If $J(-\varkappa)(g) = 0$, $g \in [0, e_\Delta(1)]$, then $J(\varkappa)(e_\Delta(1) - g) = 1$, hence $e_\Delta(1) - g = e_\Delta(1)$, thus $g = 0$.

(v) Notice that T^{-1} is a norm- and order-isomorphism from $V^*(\Delta)$ to $V^*(\bar{\Delta})$. Then, with (iii) and the previous remarks, it follows that $\operatorname{ext}[0, e_{\bar{\Delta}}(1)] \subseteq P(\bar{\Delta})$. As a straightforward excercise one shows that if $f \in \sigma(V^*(\Delta), V(\Delta)) - \exp[0, e_\Delta(1)]$, then $T^{-1} f$ is a $\sigma(V^*(\bar{\Delta}), V(\bar{\Delta}))$-exposed point of $[0, e_{\bar{\Delta}}(1)]$. Therefore, using (iv), we conclude that

$$\operatorname{ext}[0, e_{\bar{\Delta}}(1)] \subseteq P(\bar{\Delta}) \subseteq \sigma(V^*(\bar{\Delta}), V(\bar{\Delta})) - \exp[0, e_{\bar{\Delta}}(1)].$$

It can be shown that, in theorem 6.6, the condition concerning the exposed points is not redundant for the assertion to hold true.

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