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THE ESTIMATES FOR THE GREEN FUNCTION IN LIPSCHITZ DOMAINS FOR THE SYMMETRIC STABLE PROCESSES

BY

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Abstract. We give sharp global estimates for the Green function, Martin kernel and Poisson kernel in Lipschitz domains for symmetric α-stable processes. We give some applications of the estimates.

Key words and phrases: Green function, Lipschitz domain, Poisson kernel, boundary Harnack principle.

1. INTRODUCTION

Potential theory for symmetric α-stable processes has been intensively studied in recent years (see e.g. [3], [6], [14]). In particular, sharp estimates for the Green function and the Poisson kernel for bounded smooth domains with $C^{1,1}$ boundary have been obtained ([12], [19]). For example, let $G_D(x, y)$ be the Green function of a bounded $C^{1,1}$ domain $D \subset \boldsymbol{R}^d$ $(d \geqslant 2)$ for the symmetric α-stable process. Let x_0 be a fixed point in D. Define

$$\phi(x) = \min\left(G_D(x_0, x), \mathscr{C}_{d,\alpha}(r_0/4)^{\alpha-d}\right).$$

There are constants c_1, c_2 depending only on D, α such that ([12], [19])

$$c_1^{-1}[\operatorname{dist}(x, \partial D)]^{\alpha/2}(x) \leqslant \phi(x) \leqslant c_1[\operatorname{dist}(x, \partial D)]^{\alpha/2}(x), \qquad x \in D,$$

and, for $x, y \in D$, we have

$$c_2^{-1}\min\left(|x-y|^{\alpha-d}, \frac{\phi(x)\phi(y)}{|x-y|^d}\right) \leqslant G_D(x, y) \leqslant c_2\min\left(|x-y|^{\alpha-d}, \frac{\phi(x)\phi(y)}{|x-y|^d}\right),$$

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where $\mathscr{C}_{d,\alpha}$ is given by (5) below and r_0 is the localization radius of the domain (see Section 2 for definitions). From this result and the Ikeda–Watanabe formula similar estimates for the Poisson kernel of $C^{1,1}$ domains have been obtained in [12]. Later, in [5], similar estimates have been obtained for the classical Green function in Lipschitz domains. Analogous estimates have been obtained in [11] for α-stable censored processes in $C^{1,1}$ domains.

The purpose of the present paper is to give similar estimates for the Green function, the Poisson kernel and the Martin kernel for symmetric α-stable processes in bounded Lipschitz domains. The main tool in obtaining these results is the boundary Harnack principle (BHP) for α-harmonic functions ([3], cf. also [7], [21]). Our main results are the following (for the notation see Section 2).

THEOREM 1. *There is a constant $C_1 = C_1(\underline{D}, \alpha)$ such that for every $x, y \in D$ we have*

$$C_1^{-1} \frac{\phi(x)\phi(y)}{\phi^2(A)} |x-y|^{\alpha-d} \leqslant G(x, y) \leqslant C_1 \frac{\phi(x)\phi(y)}{\phi^2(A)} |x-y|^{\alpha-d}, \tag{1}$$

where $A \in \mathscr{B}(x, y)$. In fact, (1) *holds with $C_1 = C_1(d, \lambda, \alpha)$ provided $\delta(x) \vee \delta(y) \vee |x-y| \leqslant r_0/32$.*

THEOREM 2. *There is a constant $C_2 = C_2(\underline{D}, \alpha)$ such that for every $x \in D$ and $y \in \operatorname{int}(D^c)$ we have*

$$C_2^{-1} \frac{\phi(x)\phi(y')}{\phi^2(A)\delta^\alpha(y)(1+\delta(y))^\alpha} |x-y|^{\alpha-d} \leqslant P(x, y) \leqslant C_2 \frac{\phi(x)\phi(y')}{\phi^2(A)\delta^\alpha(y)(1+\delta(y))^\alpha} |x-y|^{\alpha-d}, \tag{2}$$

where $y' \in \mathscr{A}_{\delta(y)}(S)$, $A \in \mathscr{B}(x, y')$ and $S \in \partial D$ is any point such that $|y-S| = \delta(y)$.

THEOREM 3. *There is a constant $C_3 = C_3(\underline{D}, \alpha)$ such that for every $x \in D$, $Q \in \partial D$ we have*

$$C_3^{-1} \frac{\phi(x)}{\phi^2(A)} |x-Q|^{\alpha-d} \leqslant K(x, Q) \leqslant C_3 \frac{\phi(x)}{\phi^2(A)} |x-Q|^{\alpha-d}, \tag{3}$$

where $A \in \mathscr{A}_{|x-Q|}(Q)$. In fact, (3) *holds with $C_3 = C_3(d, \lambda, \alpha)$ provided $|x-Q| \leqslant r_0/32$.*

The above results show that the boundary behaviour of the Green function, the Poisson kernel and the Martin kernel can be expressed in terms of $\phi(x)$. This role of $\phi(x)$ stems from the boundary Harnack principle. We note here that unlike in $C^{1,1}$ domains, the boundary behaviour of $\phi(x)$ for bounded Lipschitz domains strongly depends on the local shape of the boundary (see Lemma 8) and estimates (1)–(3) are much more difficult than their counterparts for $C^{1,1}$ domains.

Our proofs of Theorems 1 and 3 follow closely the arguments of [5], with appropriate adjustments and simplifications. However, the estimates for the Poisson kernel for α-stable symmetric processes are new with no counterpart in [5]. We remark here that the problem of estimating the Poisson kernel is qualitatively different from that of estimating the Martin kernel (see, e.g., [16]). Our estimate for $P_D(x, y)$ is a consequence of the Ikeda–Watanabe formula and the estimate for the Green function (1).

The work is organized as follows. Section 2 sets up the notation and collects basic facts and definitions for further use. In Section 3 we prove estimates for the Green function. Section 4 deals with the Poisson kernel and the Martin kernel. In Section 5 we give applications of the main results: simple proofs of "3G Theorem" and the estimates for the Green function and Poisson kernel in $C^{1,1}$ domains ([12], [19]).

2. PRELIMINARIES

In this section we introduce basic notation and present without proofs some standard facts needed in this work. Most of the material is adopted from [1], [3] and [19].

2.1. Basic notation and terminology. For natural number $d \geqslant 1$, we denote by $\mathbf{R}^d$ the d-dimensional Euclidean space with norm $|\cdot|$. We put $\mathbf{N} = \{0, 1, 2, \ldots\}$. We write D^c, $\bar{D}$, $\operatorname{int}(D)$ and ∂D for its complement, closure, interior and boundary, respectively. For $D \subset \mathbf{R}^d$, $x \in \mathbf{R}^d$, $r > 0$, we put

$$B(x, r) = \{y \in \mathbf{R}^d \colon |x-y| < r\}, \qquad \operatorname{diam}(D) = \sup\{|x-y| \colon x, y \in D\},$$

$$\operatorname{dist}(D, x) = \inf\{|x-y| \colon y \in D\}, \qquad \delta_D(x) = \operatorname{dist}(x, \partial D).$$

We write, as usual, $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Let $m(D)$ be the d-dimensional Lebesgue measure of $D \subset \mathbf{R}^d$. Assume that $\mathscr{B}(\mathbf{R}^d)$ denotes the Borel σ-field of $\mathbf{R}^d$, and $f \in \mathscr{B}(\mathbf{R}^d)$ means that the function f is Borel measurable. The notation $c = c(\alpha, \beta, \gamma)$ means that the constant c depends *only* on α, β, γ. Constants are always strictly positive and finite.

2.2. Definitions and properties of sets. For the rest of the paper we assume that $d \geqslant 2$. A set $D \subset \mathbf{R}^d$ is called a *domain* if it is open and nonempty.

A bounded domain $D \subset \mathbf{R}^d$ is called a *Lipschitz domain* with Lipschitz character (r_0, λ), $r_0 > 0$, $\lambda > 0$, if for every $Q \in \partial D$ there exists a function $\Gamma_Q \colon \mathbf{R}^{d-1} \to \mathbf{R}$ satisfying the Lipschitz condition $|\Gamma_Q(a) - \Gamma_Q(b)| \leqslant \lambda |a-b|$ for $a, b \in \mathbf{R}^{d-1}$, and an orthonormal coordinate system CS_Q such that if $y = (y_1, y_2, \ldots, y_d)$ in CS_Q coordinates, then

$$B(Q, r_0) \cap D = B(Q, r_0) \cap \{y \colon y_d > \Gamma_Q(y_1, y_2, \ldots, y_{d-1})\}.$$

The constant r_0 is called the *localization radius* and the constant λ the *Lipschitz constant*. Note that we do not assume connectedness of D in this definition. One can choose r_0 so small that distance between connected components of disconnected Lipschitz domain with localization radius r_0 is not less than r_0.

It is not difficult to check that a ball $B(0, r)$ is a Lipschitz domain with Lipschitz character $(r, \sqrt{3})$.

For the rest of the paper, unless it is stated otherwise, the domain D is Lipschitz with Lipschitz character (r_0, λ). We denote by $\theta = \operatorname{diam}(D)$ the diameter of D, and by $\delta(x) = \delta_D(x)$ the distance between $x \in \boldsymbol{R}^d$ and the boundary of D. It can be proved that the set $\{x \in D\colon \delta(x) \geqslant r_0/2\}$ is nonempty (or with less work, one can take r_0 so small that this set is not empty). We choose one of its elements as a reference point and denote it by x_0. We also fix a point $x_1 \in D$ such that $|x_0 - x_1| = r_0/4$ (cf. [5]). The dependence of constants on D which is only through d, λ, r_0, θ will be marked in this paper by the symbol $\underline{D}$, e.g. $C(\underline{D}) = C(d, \lambda, r_0, \theta)$. Let $\kappa = 1/(2\sqrt{1+\lambda^2})$ and $Q \in \partial D$. For $t \in (0, r_0/32]$ we define

$$\mathscr{A}_t(Q) = \{A \in D\colon B(A, \kappa t) \subset D \cap B(Q, t)\}.$$

The set $\mathscr{A}_t(Q)$ is nonempty (see [15], Lemma 6.6). For $t > r_0/32$ we put $\mathscr{A}_t(Q) = \{x_1\}$.

For any $x, y \in D$ we put $r = r(x, y) = \delta(x) \vee \delta(y) \vee |x-y|$. For $r \leqslant r_0/32$ let

$$\mathscr{B}(x, y) = \{A \in D\colon B(A, \kappa r) \subset D \cap B(x, 3r) \cap B(y, 3r)\}.$$

If $r > r_0/32$ we put $\mathscr{B}(x, y) = \{x_1\}$. The set $\mathscr{B}(x, y)$ is nonempty (see [5]). Of course, by symmetry, $\mathscr{B}(x, y) = \mathscr{B}(y, x)$.

2.3. Symmetric α-stable Lévy motion. We denote by (X_t, P^x) the standard rotation invariant ("symmetric") α-stable, $\boldsymbol{R}^d$-valued, Lévy process (i.e. homogeneous with independent increments), with index of stability $\alpha \in (0, 2)$ and the characteristic function of the form

$$E^0 \exp(i\xi X_t) = \exp(-t|\xi|^\alpha), \qquad \xi \in \boldsymbol{R}^d,\ t \geqslant 0.$$

As usual, E^x denotes the expectation with respect to the distribution P^x of the process starting from $x \in \boldsymbol{R}^d$. We always assume that sample paths of X_t are right continuous and have left limits almost surely. (X_t, P^x) is a Markov process with transition probabilities given by $P_t(x, D) = P^x(X_t \in D) = \mu_t(D-x)$, where μ_t is the distribution of X_t with respect to P^0, and is strong Markov with respect to the so-called "standard filtration" $(\mathscr{F}_t, \mathscr{F})$ and quasi-left-continuous on $[0, \infty)$ (see [1]). We have $P^x(X_t \in D) = \int_D p(t, x, y)\,dy$, where $p(t, x, y)$ is the transition function of X_t.

For an open set $D \subset \boldsymbol{R}^d$, we define a Markov time $\tau_D = \inf\{t \geqslant 0\colon X_t \in D^c\}$, the *first exit time* from D. If $m(D) < \infty$, then $P^x\{\tau_D < \infty\} = 1$, $x \in \boldsymbol{R}^d$. In this case the P^x distribution of X_{τ_D} is a probability measure on D^c, called *α-harmonic measure* (in x with respect to D) and denoted by ω_D^x. If ω_D^x is absolutely continuous with respect to the Lebesgue measure on D^c, then the corresponding density function $P_D(x, y)$, $x \in D$, $y \in \mathbf{R}^d$, is called the *Poisson kernel* (we put $P_D(x, y) = 0$ for $x, y \in D$). For Lipschitz domains the α-harmonic measure ω_D^x is concentrated on $\operatorname{int}(D^c)$ and is absolutely continuous with respect to the Lebesgue measure on D^c. The Poisson kernel $P_D(x, y)$ is jointly continuous in $(x, y) \in D \times \operatorname{int}(D^c)$ (see [3], Lemma 6).

For $D = B(0, r)$, $r > 0$, and $x \in B(0, r)$, the Poisson kernel $P_{B(0,r)} = P_r$ is given explicitly by the formula

$$P_r(x, y) = C_\alpha^d \left[\frac{r^2 - |x|^2}{|y|^2 - r^2}\right]^{\alpha/2} \frac{1}{|x-y|^d} \quad \text{for } |y| > r, \tag{4}$$

with $C_\alpha^d = \Gamma(d/2)\, \pi^{-d/2-1} \sin(\pi\alpha/2)$, and equals 0 for $|y| \leqslant r$ (see [2]).

2.4. Riesz potentials and α-harmonicity. For any $x, y \in \mathbf{R}^d$, we define *potential density* or the Riesz kernel $u(\cdot, \cdot)$ by

$$u(x, y) = \int_0^\infty p(t, x, y)\, dt.$$

$u(x, y)$ is given explicitly by the formula (see [1])

$$u(x, y) = \mathscr{C}_{d,\alpha} |x-y|^{\alpha-d},$$

where

$$\mathscr{C}_{d,\gamma} = \frac{\Gamma((d-\gamma)/2)}{2^\gamma \pi^{d/2} |\Gamma(\gamma/2)|}. \tag{5}$$

For any nonnegative $f \in \mathscr{B}(\mathbf{R}^d)$ we define the *potential operator* U_α of the process X_t by

$$U_\alpha f(x) = E^x \int_0^\infty f(X_t)\, dt, \qquad x \in \mathbf{R}^d.$$

It follows that

$$U_\alpha f(x) = \int_{\mathbf{R}^d} u(x, y) f(y)\, dy.$$

For any nonnegative $f \in \mathscr{B}(\mathbf{R}^d)$ we define

$$G_D f(x) = E^x \int_0^{\tau_D} f(X_t)\, dt, \qquad x \in \mathbf{R}^d.$$

G_D is called the *Green operator* for D. We define $G_D(\cdot, \cdot)$, the *Green function* for D, by

$$G_D(x, y) = u(x, y) - E^x\{\tau_D < \infty;\, u(X(\tau_D), y)\}, \qquad x, y \in \mathbf{R}^d,\ x \neq y.$$

We put $G_D(x, x) = \infty$ if $x \in D$ and $G_D(x, x) = 0$ when $x \in D^c$. For any nonnegative $f \in \mathscr{B}(\mathbf{R}^d)$, we have

$$G_D f(x) = \int_{\mathbf{R}^d} G_D(x, y) f(y)\, dy.$$

It is well known that $G_D(x, y) > 0$ on $D \times D$, $G_D(\cdot, \cdot)$ is symmetric and $G_D(x, y) = 0$ if x or y belongs to D^c.

The following Ikeda–Watanabe formula expressing the Poisson kernel $P_D(x, y)$ in terms of Green function is known (see [17]):

$$P_D(x, y) = \mathscr{C}_{d,-\alpha} \int_D \frac{G_D(x, z)}{|z-y|^{d+\alpha}}\, dz, \qquad x \in D,\ y \in \operatorname{int}(D^c), \tag{6}$$

where $\mathscr{C}_{d,-\alpha}$ is given by (5).

Definition 4. Let $u \in \mathscr{B}(\mathbf{R}^d)$. We say that u is *α-harmonic* in an open set $D \subset \mathbf{R}^d$ if

$$u(x) = E^x u(X(\tau_U)), \qquad x \in U,$$

for every bounded open set U satisfying $\bar{U} \subset D$. We say that u is *regular α-harmonic* in D if

$$u(x) = E^x u(X(\tau_D)), \qquad x \in D.$$

By the strong Markov property of $\{X_t\}$, regular α-harmonic functions are α-harmonic.

As the consequence of the definitions presented above, for any $y \in D$ and $r > 0$ the Green function $G_D(\cdot, y)$ is α-harmonic on $D \backslash \{y\}$ and regular α-harmonic on $D \backslash B(y, r)$. Moreover, if D_1 and D_2 are domains and $D_1 \subset D_2$, then $G_{D_1}(x, y) \leqslant G_{D_2}(x, y)$ for $x, y \in D_1$ (see [19] for more details).

Now we introduce the *Martin kernel* $K_D(x, Q)$ for bounded Lipschitz domains ([4], Lemma 6; see also [20]). For every $Q \in \partial D$ and $x \in D$ we define

$$K_D(x, Q) = \lim_{D \ni \xi \to Q} \frac{G_D(x, \xi)}{G_D(x_0, \xi)}. \tag{7}$$

The mapping $(x, Q) \mapsto K_D(x, Q)$ is continuous on $D \times \partial D$. For every $Q \in \partial D$ the function $K_D(\cdot, Q)$ is α-harmonic in D with $K_D(x_0, Q) = 1$. If $Q, S \in \partial D$ and $Q \neq S$, then $K_D(x, Q) \to 0$ as $x \to S$.

We will denote by $G(x, y)$, $P(x, y)$ and $K(x, y)$ the Green function, the Poisson kernel and the Martin kernel for D, respectively.

2.5. Properties of α-harmonic functions. In this section we collect some results of [3] needed in the sequel.

Lemma 5 (Harnack inequality). *Let $x, y \in \mathbf{R}^d$, $s > 0$ and $k \in \mathbf{N}$ satisfy $|x-y| \leqslant 2^k s$. Let u be a function which is nonnegative in $\mathbf{R}^d$ and α-harmonic in*

$B(x, s) \cup B(y, s)$. *Then*

(8) $$M_1^{-1} 2^{-k(d+\alpha)} u(x) \leqslant u(y) \leqslant M_1 2^{k(d+\alpha)} u(x)$$

with $M_1 = M_1(d, \alpha)$.

The next lemma is a version of Lemma 13 in [3].

LEMMA 6 (BHP). *Let* $Z \in \partial D$ *and* $\rho \in (0, r_0]$. *Assume that functions* u, v *are nonnegative in* $\mathbf{R}^d$ *and positive, regular* α*-harmonic in* $D \cap B(Z, \rho)$. *If* u *and* v *vanish on* $D^c \cap B(Z, \rho)$, *then with a constant* $M_2 = M_2(d, \lambda, \alpha)$ *the following holds*:

(9) $$M_2^{-1} \frac{u(x)}{v(x)} \leqslant \frac{u(y)}{v(y)} \leqslant M_2 \frac{u(x)}{v(x)}$$

for $x, y \in D \cap B(Z, \rho/2)$.

The next two results are versions of Lemmas 4 and 5 from [3].

LEMMA 7 (Carleson estimate). *There exists a constant* $M_3 = M_3(d, \alpha, \lambda)$ *such that, for all* $Q \in \partial D$ *and* $s \in (0, r_0/32)$, *and functions* u *nonnegative in* $\mathbf{R}^d$, *regular* α*-harmonic in* $D \cap B(Q, 2s)$ *and satisfying* $u(x) = 0$ *on* $D^c \cap B(Q, 2s)$, *we have*

(10) $$u(x) \leqslant M_3 u(A), \quad x \in D \cap B(Q, s),$$

where $A \in \mathscr{A}_s(Q)$.

LEMMA 8. *There exist constants* $\gamma = \gamma(d, \alpha, \lambda) < \alpha$ *and* $M_4 = M_4(d, \alpha, \lambda)$ *such that for all* $Q \in \partial D$ *and* $t \in (0, r_0/32]$, *and functions* u *nonnegative in* $\mathbf{R}^d$, α*-harmonic in* $D \cap B(Q, t)$, *we have*

(11) $$u(A_1) \geqslant M_4 (|A_1 - Q|/t)^\gamma u(A_2), \quad s \in (0, t),$$

where $A_1 \in \mathscr{A}_s(Q)$, *and* $A_2 \in \mathscr{A}_t(Q)$.

For the rest of the paper we fix the constant γ in Lemma 8.

3. ESTIMATES FOR THE GREEN FUNCTION

In this section we prove Theorem 1. At first we will need an auxiliary lemma.

LEMMA 9. *Let* $N > 0$ *and* $x, y \in D$ *satisfy* $|x-y| \leqslant Ns$, *where* $s = \delta(x) \wedge \delta(y)$. *Let* u *be a function nonnegative in* $\mathbf{R}^d$ *and* α*-harmonic in* $B(x, s) \cup B(y, s)$. *Then*

(12) $$\tilde{M}_1^{-1} u(x) \leqslant u(y) \leqslant \tilde{M}_1 u(x)$$

with $\tilde{M}_1 = \tilde{M}_1(d, \alpha, N)$.

Proof. Let $k \in N$ be such that $2^{k-1} < N+1 \leqslant 2^k$. Since u is α-harmonic in $B(x, s) \cup B(y, s)$ and $|x-y| < 2^k s$, by Lemma 5 we obtain

$$(M_1 2^{k(d+\alpha)})^{-1} u(x) \leqslant u(y) \leqslant M_1 2^{k(d+\alpha)} u(x).$$

Therefore (12) holds with $\tilde{M}_1 = M_1 (2(N+1))^{d+\alpha}$. ■

We will also need the following estimates for the Green function of the ball (see [19] for $d \geqslant 3$ and [12] for $d \geqslant 2$). The estimates are consequences of an explicit formula for the function (see [2]).

PROPOSITION 10. *There exists a constant* $M_5 = M_5(d, \alpha)$ *such that*

$$M_5^{-1}\left[\frac{1}{|x-y|^{d-\alpha}} \wedge \frac{\delta_{B_r}^{\alpha/2}(x)\,\delta_{B_r}^{\alpha/2}(y)}{|x-y|^d}\right] \leqslant G_{B_r}(x, y) \leqslant M_5\left[\frac{1}{|x-y|^{d-\alpha}} \wedge \frac{\delta_{B_r}^{\alpha/2}(x)\,\delta_{B_r}^{\alpha/2}(y)}{|x-y|^d}\right],$$

where $B_r = B(a, r)$, $a \in \mathbf{R}^d$, $x, y \in B_r$.

LEMMA 11. *Let* $N > 0$ *and* $x, y \in D$ *satisfy* $|x-y| \leqslant N[\delta(x) \wedge \delta(y)]$. *Then*

$$C_4^{-1}|x-y|^{\alpha-d} \leqslant G(x, y) \leqslant C_4|x-y|^{\alpha-d} \tag{13}$$

with $C_4 = C_4(d, \alpha, N)$.

Proof. The right-hand inequality is obvious because $G(x, y) \leqslant u(x, y) = \mathscr{C}_{d,\alpha}|x-y|^{\alpha-d}$. Let $s = \delta(x) \wedge \delta(y)$. We now prove the left-hand side of (13). We first assume that $|x-y| \leqslant s/2$. We clearly have $\delta_B(y) \geqslant s/2$, where $B = B(x, s)$. By Proposition 10 we obtain

$$M_5^{-1}\left[\frac{1}{|x-y|^{d-\alpha}} \wedge \frac{\delta_B^{\alpha/2}(x)\,\delta_B^{\alpha/2}(y)}{|x-y|^d}\right] \leqslant G_B(x, y) \leqslant G(x, y),$$

where M_5 is the constant from Proposition 10. Since $\delta_B(y) \geqslant |x-y|$ and $\delta_B(x) \geqslant |x-y|$, we get

$$M_5^{-1}\frac{1}{|x-y|^{d-\alpha}} \leqslant G(x, y). \tag{14}$$

Thus (13) holds with $C_4 = C_4(d, \alpha) = M_5 \vee \mathscr{C}_{d,\alpha}$.

Now assume that $|x-y| > s/2$. Let y_0 be a point such that $|x-y_0| < s/4$. From Lemma 9 and (14) we obtain $G(x, y) \geqslant c_1 G(x, y_0) \geqslant c_2|x-y|^{\alpha-d}$, which gives the lower bound in (13). ∎

Lemma 11 yields that there exists a constant $M_6 = M_6(d, \alpha)$ such that

$$g(z) \geqslant M_6 r_0^{\alpha-d}, \quad z \in B(x_0, 2r_0/5). \tag{15}$$

To simplify the notation we will write

$$g(x) = G(x_0, x) \quad \text{and} \quad \phi(x) = G(x_0, x) \wedge [\mathscr{C}_{d,\alpha}(r_0/4)^{\alpha-d}],$$

see the Introduction. We recall that $G(x, y) \leqslant u(x, y) = \mathscr{C}_{d,\alpha}|x-y|^{\alpha-d}$. In particular, $G(x_0, x) \leqslant u(x_0, x) \leqslant \mathscr{C}_{d,\alpha}(r_0/4)^{\alpha-d}$ if $|x-x_0| \geqslant r_0/4$. Thus, for $|x-x_0| \geqslant r_0/4$, $\phi(x) = g(x)$. Note that $\delta(x_1) \geqslant r_0/4$.

First we prove the estimate for Green function assuming that x and y are not close to x_0.

LEMMA 12. *There is a constant $C_5 = C_5(\underline{D}, \alpha)$ such that if $x, y \in D \setminus B(x_0, r_0/3)$ and $A \in \mathscr{B}(x, y)$, then*

$$(16) \qquad C_5^{-1} \frac{g(x)g(y)}{g^2(A)} |x-y|^{\alpha-d} \leqslant G(x, y) \leqslant C_5 \frac{g(x)g(y)}{g^2(A)} |x-y|^{\alpha-d}.$$

In fact, (16) *holds with $C_5 = C_5(d, \lambda, \alpha)$ provided $\delta(x) \vee \delta(y) \vee |x-y| \leqslant r_0/32$.*

Proof. The proof of this lemma is the same as the proof of an analogous lemma in [5] with appropriate adjustments, so we omit most of the details. To give the reader the idea of proof we will only prove (16) under the assumption

$$5\delta(x) < 5\delta(y) < |x-y| \quad \text{and} \quad r \leqslant r_0/32,$$

where $r = r(x, y) = \delta(x) \vee \delta(y) \vee |x-y|$ (cf. [5]). To simplify the notation we will write ρ_0 for $r_0/32$. Let Q and S be points such that $|x-Q| = \delta(x)$ and $|y-S| = \delta(y)$. We have $r = |x-y|$ and

$$|Q-S| \geqslant |x-y| - \delta(x) - \delta(y) > |x-y| - |x-y|/5 - |x-y|/5 = 3r/5.$$

We choose $E \in \mathscr{A}_{r/5}(Q)$ and $F \in \mathscr{A}_{r/5}(S)$. By Lemma 6 (with $\rho = 2r/5$, $Z = S$) applied to the functions $G(x, \cdot)$, $g(\cdot)$ we obtain

$$(17) \qquad c_1^{-1} \frac{G(x, F)}{g(x)g(F)} \leqslant \frac{G(x, y)}{g(x)g(y)} \leqslant c_1 \frac{G(x, F)}{g(x)g(F)}$$

with $c_1 = c_1(d, \lambda, \alpha)$. Similarly, applying Lemma 6 to functions $G(\cdot, F)$, $g(\cdot)$ (taking $\rho = 2r/5$, $Z = Q$), we get

$$(18) \qquad c_1^{-1} \frac{G(E, F)}{g(E)g(F)} \leqslant \frac{G(x, F)}{g(x)g(F)} \leqslant c_1 \frac{G(E, F)}{g(E)g(F)}.$$

Thus we have

$$(19) \qquad c_1^{-2} \frac{G(E, F)}{g(E)g(F)} \leqslant \frac{G(x, y)}{g(x)g(y)} \leqslant c_1^2 \frac{G(E, F)}{g(E)g(F)}.$$

Since $\delta(E), \delta(F) \geqslant \kappa r/5$, $\delta(A) \geqslant \kappa r$ and $|x-y|/5 < |E-F| < 9|x-y|/5 < 5|x-y|$, we have

$$|E-F| < 9r/5 \leqslant (9/\kappa)[\delta(E) \wedge \delta(F)],$$

$$|E-A| \leqslant |E-Q| + |Q-A| < r + 4r \leqslant (25/\kappa)[\delta(E) \wedge \delta(A)],$$

$$|F-A| \leqslant |F-S| + |S-A| < r + 4r \leqslant (25/\kappa)[\delta(F) \wedge \delta(A)].$$

Hence, by Lemmas 9 and 11, we obtain

$$c_3^{-1} |E-F|^{\alpha-d} \leqslant G(E, F) \leqslant c_3 |E-F|^{\alpha-d},$$

$$c_4^{-1} g(E) \leqslant g(A) \leqslant c_4 g(E), \qquad c_4^{-1} g(F) \leqslant g(A) \leqslant c_5 g(F).$$

Therefore we obtain (16) with $C_5 = C_5(d, \lambda, \alpha)$. The proof is complete. ∎

Let us remark here that the above argument is less technical than that of [5]. This is due to the fact that the BHP for our stable processes (Lemma 6 above) has less stringent assumptions regarding the domain where the function needs to be harmonic as compared to the BHP for the classical harmonic functions.

The proof of Theorem 1 is based on Lemmas 5, 6, 11 and 12, and is analogous to the one from [5], so we omit the details.

4. THE POISSON AND THE MARTIN KERNEL

4.1. The Poisson kernel. In this section we will deal with the Poisson kernel — the density function of α-harmonic measure ω_D^x (see Section 2). Before we prove Theorem 2, we will need some estimates for the function $\phi(x)$.

Let us recall that for $x \in D \setminus B(x_0, r_0/4)$ we have $\phi(x) = g(x)$, where $g(x) = G(x_0, x)$; and $g(x)$ is an α-harmonic function in $D \setminus \{x_0\}$. Therefore, although $\phi(x)$ is not α-harmonic in $D \setminus B(x_0, r_0/4)$, it is equal to an α-harmonic function on this set. This simple observation yields useful estimates of the function $\phi(x)$. We also recall that $\gamma = \gamma(d, \lambda, \alpha) < \alpha$ is the constant from Lemma 8.

By the Harnack inequality there exists a constant $C_6 = C_6(\underline{D}, \alpha, N)$ such that

$$\phi(x) \geqslant C_6 \tag{20}$$

for all $x \in D$ satisfying $\delta(x) \geqslant N$.

LEMMA 13. *Let $x, z_1, z_2, z \in D$ and $r_i = \delta(x) \vee \delta(z_i) \vee |x - z_i|$ for $i = 1, 2$. Let N be a constant satisfying $r_1 \leqslant N r_2$ or $|x - z_1| \leqslant N |x - z_2|$. Let $A \in \mathscr{B}(x, z)$, $A_1 \in \mathscr{B}(x, z_1)$, and $A_2 \in \mathscr{B}(x, z_2)$. Then*

$$\phi(A_1) \leqslant C_7 \phi(A_2), \tag{21}$$

$$\phi(x) \leqslant C_8 \phi(A), \tag{22}$$

$$\phi(x) \geqslant C_9 \delta(x)^{\gamma}, \tag{23}$$

where $C_7 = C_7(\underline{D}, \alpha, N)$, $C_8 = C_8(\underline{D}, \alpha)$, and $C_9 = C_9(\underline{D}, \alpha)$.

Proof. We may and do assume that $N \geqslant 1$. If $N r_2 > r_0/32$, then from (20) we get (21).

Therefore we may and do assume that $r_1 \leqslant N r_2 \leqslant r_0/32$. Let $z_1' \in \mathscr{A}_{r_1}(Q)$ and $z_2' \in \mathscr{A}_{N r_2}(Q)$, where $Q \in \partial D$ is a point such that $|x - Q| = \delta(x)$. We have

$$|z_1' - A_1| < 5 r_1 \leqslant \frac{5}{\kappa} [\delta(z_1') \wedge \delta(A_1)],$$

$$|z_2' - A_2| \leqslant (N+4) r_2 \leqslant \frac{N+4}{\kappa N} [\delta(z_2') \wedge \delta(A_2)].$$

Applying Lemma 9 we have

$$\phi(A_1) \leqslant c_2 \phi(z_1'), \qquad \phi(z_2') \leqslant c_2 \phi(A_2)$$

with $c_2 = c_2(d, \lambda, \alpha, N)$. In fact, we apply Lemma 9 to the domain $D_0 = D \backslash B(x_0, r_0/4)$ and the function g. According to the remarks at the beginning of this section the results for the function ϕ follow. In the sequel we will simply pass over similar discussions. By Lemma 7 we have $\phi(z_1') \leqslant c_3 \phi(z_2')$ with $c_3 = c_3(d, \lambda, \alpha)$. Therefore we obtain (21) with $C_7 = c_2^2 c_3$.

Note that $|x-z_1| < N|x-z_2|$ implies $r_1 \leqslant 2(N+1) r_2$. Therefore the proof of (21) is completed.

To prove (22) note that if $\delta(z) \leqslant r_0/32$, then $z \in \mathscr{B}(z, z)$. Since $\delta(x) \vee \delta(z) \vee |x-z| \geqslant \delta(z)$, by (21) we get (22). The case $\delta(z) > r_0/32$ follows from (20).

Now we will prove (23). If $\delta(x) \geqslant r_0/64$, then (20) yields (23). If $\delta(x) < r_0/64$, then $x \in \mathscr{A}_{2\delta(x)}(Q)$, where $Q \in \partial D$ is a point satisfying $|x-Q| = \delta(x)$. Let $z_0 \in \mathscr{A}_{r_0/32}(Q)$. Lemma 8 applied to x and z_0 and (20) yield (23). ∎

Lemma 14. *There exists a constant $C_{11} = C_{11}(D, \alpha)$ such that for all $Q \in \partial D$ and $t > 0$ we have*

$$\phi(A_1) \geqslant C_{11} \frac{|A_1 - Q|^\gamma}{t^\gamma} \phi(A_2), \qquad s \in (0, t), \tag{24}$$

where $A_1 \in \mathscr{A}_s(Q)$, $A_2 \in \mathscr{A}_t(Q)$.

Proof. If $t \leqslant r_0/32$, then (24) holds by Lemma 8. Assume that $t > r_0/32$. Then $A_2 = x_1$.

For $s < r_0/32$ let $z' \in \mathscr{A}_{r_0/32}(Q)$. By (20) we get $\phi(z') \geqslant c_1 \phi(A_2)$, where $c_1 = c_1(D, \alpha)$. From Lemma 8 we obtain

$$\phi(A_1) \geqslant c_2 \left(\frac{|A_1 - Q|}{r_0/32} \right)^\gamma \phi(z') \geqslant c_1 c_2 \left(\frac{|A_1 - Q|}{t} \right)^\gamma \phi(A_2),$$

where $c_2 = c_2(d, \lambda, \alpha)$. If $s \geqslant r_0/32$, then (24) obviously holds. ∎

Lemma 15. *There exists a constant $C_{12} = C_{12}(D, \alpha)$ such that for all $x, z_1, z_2 \in D$ satisfying $|x-z_1| \leqslant |x-z_2|$ we have*

$$\phi(A_1) \geqslant C_{12} \frac{|x-z_1|^\gamma}{|x-z_2|^\gamma} \phi(A_2), \tag{25}$$

where $A_1 \in \mathscr{B}(x, z_1)$, $A_2 \in \mathscr{B}(x, z_2)$.

Proof. Let $r_1 = \delta(x) \vee \delta(z_1) \vee |x-z_1|$ and $r_2 = \delta(x) \vee \delta(z_2) \vee |x-z_2|$. Let $Q \in \partial D$ be a point such that $|x-Q| = \delta(x)$.

If $r_1 > r_0/32$, then by (20) we have $\phi(A_1) \geqslant c_1 \phi(A_2)$, where $c_1 = c_1(D, \alpha)$. Since $|x-z_1| \leqslant |x-z_2|$, (25) holds.

Assume that $r_1 \leqslant r_0/32$ and $r_2 > r_0/32$. Then $A_2 = x_1$. If $|x-z_2| > r_0/64$, then by (23) we have $\phi(A_1) \geqslant c_2(r_1\kappa)^\gamma \geqslant c_2\kappa^\gamma|x-z_1|^\gamma$, where $c_2 = c_2(\underline{D}, \alpha)$. Hence (25) holds because the function ϕ is bounded from above. If $|x-z_2| \leqslant r_0/64$, then $\delta(x) > r_0/64$ (because $|x-z_2|+\delta(x) \geqslant r_2 > r_0/32$). Hence, by (20) and (22), we have $\phi(A_1) \geqslant c_3\phi(x) \geqslant c_4$, where $c_3 = c_3(\underline{D}, \alpha)$ and $c_4 = c_4(\underline{D}, \alpha)$. Using the condition $|x-z_1| \leqslant |x-z_2|$, we obtain (25).

Now we assume that $r_1, r_2 \leqslant r_0/32$. Let $z_1' \in \mathscr{A}_{r_1}(Q)$ and $z_2' \in \mathscr{A}_{r_2}(Q)$. Since $|A_i - z_i'| < (5/\kappa)[\delta(A_i) \wedge \delta(z_i')]$ for $i = 1, 2$, we obtain by Lemma 9

$$c_5\phi(A_1) \geqslant \phi(z_1'), \qquad c_5\phi(z_2') \geqslant \phi(A_2),$$

where $c_5 = c_5(\underline{D}, \alpha)$. If $r_1 \geqslant r_2$, then by Lemma 7 we have $c_6\phi(z_1') \geqslant \phi(z_2')$, where $c_6 = c_6(d, \lambda, \alpha)$. Therefore (25) holds with $C_{12} = c_5^{-2}c_6^{-1}$. Let $r_1 < r_2$. By Lemma 8 we have

$$\phi(z_1') \geqslant c_7\left(\frac{|z_1'-Q|}{r_2}\right)^\gamma \phi(z_2') \geqslant c_7\kappa^\gamma\left(\frac{r_1}{r_2}\right)^\gamma \phi(z_2'),$$

where $c_7 = c_7(d, \lambda, \alpha)$. Note that $\delta(z_2) \leqslant 2(\delta(x) \vee |x-z_2|)$, and hence $r_2 \leqslant 2(r_1 \vee |x-z_2|)$. If $r_1 \geqslant |x-z_2|$, then $r_1/r_2 \geqslant 1/2 \geqslant |x-z_1|/(2|x-z_2|)$. If $r_1 < |x-z_2|$, then $r_2 \leqslant 2|x-z_2|$, and since $r_1 \geqslant |x-z_1|$, we again get $r_1/r_2 \geqslant |x-z_1|/(2|x-z_2|)$. Using this we obtain (25) with $C_{12} = c_5^{-2}c_7(\kappa/2)^\gamma$. ■

The following lemma is crucial in our considerations. Its proof depends on the fact that the constant γ in Lemma 8 is smaller than α.

LEMMA 16. *Let $y \in \operatorname{int}(D^c)$ and $S \in \partial D$ be a point such that $\delta(y) = |y-S|$. Let $t \geqslant \delta(y)$. Then for $G = B(S, t) \cap D$ and $y' \in \mathscr{A}_{\delta(y)}(S)$ we have*

$$C_{13}^{-1}\frac{\phi(y')}{\delta(y)^\alpha(1+\delta(y))^d} \leqslant \int_G \frac{\phi(z)}{|y-z|^{d+\alpha}}dz \leqslant C_{13}\frac{\phi(y')}{\delta(y)^\alpha(1+\delta(y))^d} \tag{26}$$

with $C_{13} = C_{13}(\underline{D}, \alpha)$.

Proof. For all $z \in D$ let $z' \in \mathscr{A}_{|y-z|}(S)$. From Lemma 7 it follows easily that

$$\phi(z') \geqslant c_1\phi(z), \tag{27}$$

where $c_1 = c_1(\underline{D}, \alpha)$.

Assume that $\delta(y) \leqslant r_0/32$. By Lemma 14 there exists $c_2 = c_2(\underline{D}, \alpha)$ such that for $z \in D$ we have

$$\phi(y') \geqslant c_2\left(\frac{|y'-S|}{|y-z|}\right)^\gamma \phi(z') \geqslant c_1c_2\left(\frac{\kappa\delta(y)}{|y-z|}\right)^\gamma \phi(z). \tag{28}$$

Hence we obtain

$$\int_G \frac{\phi(z)}{|y-z|^{d+\alpha}}dz \leqslant \int_G c_1^{-1} c_2^{-1} \frac{\phi(y')\,|y-z|^\gamma}{|y-z|^{d+\alpha}(\kappa\delta(y))^\gamma}dz$$

$$\leqslant c_1^{-1} c_2^{-1} \frac{(1+r_0/32)^d}{(1+\delta(y))^d} \int_{B(y,\delta(y))^c} \frac{\phi(y')}{|y-z|^{d+\alpha-\gamma}(\kappa\delta(y))^\gamma}dz \leqslant c_3 \frac{\phi(y')}{\delta(y)^\alpha(1+\delta(y))^d},$$

where $c_3 = c_3(\underline{D}, \alpha)$.

Let us put $B = B(y', \kappa\delta(y)/2)$. Note that $B \subset G$ and for every $z \in B$ we have $|y'-z| < \delta(y') \wedge \delta(z)$. Hence, by Lemma 9, we have

$$\phi(z) \geqslant c_4 \phi(y'),$$

where $c_4 = c_4(d, \alpha)$. Using this we obtain

$$\int_G \frac{\phi(z)}{|y-z|^{d+\alpha}}dz \geqslant \int_B \frac{\phi(z)}{|y-z|^{d+\alpha}}dz \geqslant \int_B \frac{c_4\phi(y')}{|y-z|^{d+\alpha}}dz$$

$$\geqslant \frac{c_4\phi(y')}{(2\delta(y))^{d+\alpha}} m(B) \geqslant c_5 \frac{\phi(y')}{\delta(y)^\alpha(1+\delta(y))^d},$$

where $c_5 = c_5(\underline{D}, \alpha)$. Taking $C_{13} = c_3 \vee c_5^{-1}$ we obtain (26).

Now assume that $\delta(y) > r_0/32$. From (22) and the fact $\int_D G_D(x, z)\,dz = E^x\{\tau_D\}$ we have

$$c_8^{-1} \leqslant c_7 \int_G \delta(z)^\gamma dz \leqslant \int_G \phi(z)\,dz \leqslant c_6 E^{x_0}\{\tau_D\} \leqslant c_8,$$

where c_6, c_7, c_8 depend only on $\underline{D}$ and α. From the last inequality we easily obtain (26). ■

LEMMA 17. *There exists a constant $C_{14} = C_{14}(\underline{D}, \alpha)$ such that*

$$C_{14}^{-1}\phi(x) \leqslant E^x\{\tau_D\} \leqslant C_{14}\phi(x), \qquad x \in D. \tag{29}$$

Proof. Let $B = B(z_0, 1)$ be such that $\delta(z_0) = \theta + 1$. Consider the function $f(x) = P^x\{X_{\tau_D} \in B\}$. Clearly, this function is α-harmonic in D. From the Ikeda–Watanabe formula (6) and the fact that $E^x\{\tau_D\} = \int_D G_D(x, y)\,dy$ there exists a constant $c_1 = c_1(d, \theta, \alpha)$ such that

$$c_1^{-1} E^x\{\tau_D\} \leqslant f(x) \leqslant c_1 E^x\{\tau_D\}.$$

The rest is the consequence of Lemma 6 applied to functions $f(\cdot)$ and $G(x_0, \cdot)$. ■

Proof of Theorem 2. In this proof we will use the convention that all constants depend only on $\underline{D}$ and α (unless it is stated otherwise). For every $z_1, z_2 \in D$ we denote by A_{z_1,z_2} any point belonging to the set $\mathscr{B}(z_1, z_2)$. For the rest of the proof we put

$$r_1 = r_1(x, z) = \delta(x) \vee \delta(z) \vee |x-z|, \qquad r_2 = r_2(x, y') = \delta(x) \vee \delta(y') \vee |x-y'|.$$

To shorten the notation we will write $\rho_0 = r_0/32$. By Theorem 1 and (6) we have

$$(30)\qquad \mathscr{C}_{d,-\alpha} C_1^{-1} \int_D \frac{\phi(x)\phi(z)}{\phi^2(A_{x,z})|x-z|^{d-\alpha}|z-y|^{d+\alpha}} dz \leqslant P(x, y)$$

$$\leqslant \mathscr{C}_{d,-\alpha} C_1 \int_D \frac{\phi(x)\phi(z)}{\phi^2(A_{x,z})|x-z|^{d-\alpha}|z-y|^{d+\alpha}} dz,$$

where $C_1 = C_1(\underline{D}, \alpha)$. Our task will be to estimate the above integral. We will consider 3 cases:

(a) $|x-y| \geqslant 5\delta(y)$ and $\delta(y) \leqslant r_0/32$;
(b) $|x-y| < 5\delta(y) \leqslant 5r_0/32$;
(c) $\delta(y) > r_0/32$.

Case (a). $|x-y| \geqslant 5\delta(y)$ and $\delta(y) \leqslant r_0/32$.

Note that $|x-y'| \leqslant |x-y|+\delta(y)+|y'-S| < 2|x-y|$ and $|x-y'| \geqslant |x-y|-\delta(y)-|y-S| > 3|x-y|/5$. Hence we get

$$3|x-y|/5 < |x-y'| < 2|x-y|.$$

Let us consider the following sets:

$$B_1 = B(y, |x-y|/2) \cap D, \qquad B_2 = B(x, |x-y|/2) \cap D,$$
$$B_3 = D \setminus (B_1 \cup B_2).$$

Let us put

$$I_i = \int_{B_i} \frac{\phi(x)\phi(z)}{\phi^2(A_{x,z})|x-z|^{d-\alpha}|z-y|^{d+\alpha}} dz \qquad \text{for } i = 1, 2, 3.$$

We first estimate I_1. For $z_1, z_2 \in B_1$ we have $|x-z_1| \geqslant |x-y|/2 \geqslant |x-z_2|/3$. By the reason of symmetry, $|x-z_2|/3 \leqslant |x-z_1| \leqslant 3|x-z_2|$. Since $y' \in B_1$, by (21) (taking $z_1 = z$ and $z_2 = y'$) there exists a constant c_1 such that

$$(31)\qquad c_1^{-1}\phi(A_{x,y'}) \leqslant \phi(A_{x,z}) \leqslant c_1 \phi(A_{x,y'}), \qquad z \in B_1.$$

By Lemma 16 there exists a constant c_2 satisfying

$$(32)\qquad c_2^{-1} \frac{\phi(y')}{\delta(y)^\alpha (1+\delta(y))^\alpha} \leqslant \int_{B_1} \frac{\phi(z)\,dz}{|y-z|^{d+\alpha}} \leqslant c_2 \frac{\phi(y')}{\delta(y)^\alpha(1+\delta(y))^\alpha}.$$

Moreover, for $z \in B_1$ we have $|x-y|/2 \leqslant |x-y| \leqslant 2|x-y|$. Using this, (31) and (32) we obtain

$$(33)\qquad m_1^{-1} \frac{\phi(x)\phi(y')}{\phi^2(A_{x,y'})|x-y|^{d-\alpha}\delta(y)^\alpha(1+\delta(y))^\alpha} \leqslant I_1$$

$$\leqslant m_1 \frac{\phi(x)\phi(y')}{\phi^2(A_{x,y'})|x-y|^{d-\alpha}\delta(y)^\alpha(1+\delta(y))^\alpha},$$

where $m_1 = c_1^2 c_2 2^{d-\alpha}$.

We now estimate I_2 from above. Let $z \in B_2$. We have $|y-z| \geqslant |x-y|/2$. Using this and (21), we have

$$I_2 \leqslant c_3 \int_{B_2} \frac{\phi(x) 2^{d+\alpha}}{\phi(A_{x,z}) |x-z|^{d-\alpha} |x-y|^{d+\alpha}} dz. \tag{34}$$

Suppose $z \in B_2$. Since $|x-z| < |x-y'| < 2|x-y|$, by Lemma 15 we have

$$\phi(A_{x,z}) \geqslant c_4 \left[\frac{|x-z|}{|x-y|}\right]^\gamma \phi(A_{x,y'}).$$

Consequently, we get

$$\int_{B_2} \frac{dz}{\phi(A_{x,z}) |x-z|^{d-\alpha}} \leqslant c_4^{-1} \int_{B_2} \frac{|x-y|^\gamma}{\phi(A_{x,y'}) |x-z|^\gamma |x-z|^{d-\alpha}} dz \tag{35}$$
$$\leqslant c_4^{-1} c_5 \frac{|x-y|^\gamma}{\phi(A_{x,y'})} |x-y|^{\alpha-\gamma} = c_4^{-1} c_5 \frac{|x-y|^\alpha}{\phi(A_{x,y'})}.$$

Note that by assumption (a) and Lemma 15 we obtain

$$\phi(A_{y',y''}) \geqslant c_6 \left[\frac{\delta(y)}{|x-y'|}\right]^\gamma \phi(A_{x,y'}) \geqslant c_6 2^{-\alpha} \left[\frac{\delta(y)}{|x-y|}\right]^\alpha \phi(A_{x,y'}),$$

where y'' is a point such that $\delta(y'') = |y'-y''| = \delta(y')/2$. Since $\phi(A_{y',y''}) \leqslant c_7 \phi(y')$ (by Lemma 9), we have

$$\phi(y') \geqslant c_6 c_7^{-1} 2^{-\alpha} \left[\frac{\delta(y)}{|x-y|}\right]^\alpha \phi(A_{x,y'}).$$

Hence

$$1 \leqslant \frac{c_7 2^\alpha \phi(y') |x-y|^\alpha}{c_6 \phi(A_{x,y'}) \delta(y)^\alpha}.$$

Applying this and (35) to (34) we obtain

$$I_2 \leqslant m_2 \frac{\phi(x) \phi(y')}{\phi^2(A_{x,y'}) |x-y|^{d-\alpha} \delta(y)^\alpha (1+\delta(y))^\alpha}, \tag{36}$$

where $m_2 = c_3 c_4^{-1} c_5 c_6^{-1} c_7 2^{d+2\alpha} (1+\rho_0)^\alpha$.

Now we estimate I_3 from above. Note that for $z \in B_3$ it follows that $|x-z| \geqslant |x-y|/2$ and $|x-z| \geqslant |x-y'|/3$. By (21) we have $c_8 \phi(A_{x,z}) \geqslant \phi(A_{x,y'})$. Using this and Lemma 16 we get

$$I_3 \leqslant c_8^2 2^{d-\alpha} \int_{B_3} \frac{\phi(x) \phi(z)}{\phi^2(A_{x,y'}) |x-y|^{d-\alpha} |z-y|^{d+\alpha}} dz$$

$$\leqslant c_8^2 2^{d-\alpha} \frac{\phi(x)}{\phi^2(A_{x,y'})|x-y|^{d-\alpha}} \int_D \frac{\phi(z)\,dz}{|z-y|^{d+\alpha}}$$

$$\leqslant c_9 c_8^2 2^{d-\alpha} \frac{\phi(x)\phi(y')}{\phi^2(A_{x,y'})|x-y|^{d-\alpha}\delta(y)^\alpha(1+\delta(y))^d}.$$

Hence we get

$$I_3 \leqslant m_3 \frac{\phi(x)\phi(y')}{\phi^2(A_{x,y'})|x-y|^{d-\alpha}|\delta(y)^\alpha(1+\delta(y))^\alpha}, \tag{37}$$

where $m_3 = c_8^2 c_9 2^{d-\alpha}$.

Using (33), (36) and (37) we obtain

$$m_1^{-1} \frac{\phi(x)\phi(y')}{\phi^2(A_{x,y'})|x-y|^{d-\alpha}\delta(y)^\alpha(1+\delta(y))^\alpha} \leqslant I_1+I_2+I_3$$

$$\leqslant (m_1+m_2+m_3) \frac{\phi(x)\phi(y')}{\phi^2(A_{x,y'})|x-y|^{d-\alpha}\delta(y)^\alpha(1+\delta(y))^\alpha}.$$

Applying this to (30) we obtain (2).

Case b. $|x-y| \leqslant 5\delta(y) \leqslant 5r_0/32$.

Note that in this case $\delta(x) < |x-y| \leqslant 5\delta(y)$. Since $\delta(y) \leqslant \rho_0$, we have $\delta(y') \geqslant (\kappa/5)\delta(x)$.

By (22) there exists a constant c_{10} such that $\phi(y') \leqslant c_{10}\phi(A_{x,y'})$. Note that $|x-y'| \leqslant |x-y|+|y-y'| \leqslant 7\delta(y) \leqslant (7/\kappa)\delta(y')$. Hence $\delta(y') \geqslant (\kappa/7)r_2$. If $r_2 > \rho_0$, we have $\delta(y') > (\kappa/7)\rho_0$, and by (20) we obtain

$$c_{11}^{-1}\phi(y') \leqslant \phi(A_{x,y'}) \leqslant c_{11}\phi(y'). \tag{38}$$

If $r_2 \leqslant \rho_0$, we have $|y'-A_{x,y'}| \leqslant 3r_2 \leqslant (21/\kappa)[\delta(A_{x,y'}) \wedge \delta(y')]$ and (38) now follows from Lemma 9.

Let us put

$$B_4 = B(y, 3|x-y|)\cap D, \quad B_5 = D\backslash B_4, \quad B_6 = B(S, \delta(y))\cap D.$$

We will consider the integrals

$$I_i = \int_{B_i} \frac{\phi(x)\phi(z)}{\phi^2(A_{x,z})|x-z|^{d-\alpha}|z-y|^{d+\alpha}}\,dz \quad \text{for } i = 4, 5, 6.$$

We first estimate I_4 from above. Suppose $z \in B_4$. Let $z_1 \in D$ be such that $4|x-y| < |x-z_1| < 5|x-y|$. We have $|x-z| < |x-z_1|$ and $|x-y'| < |x-z_1|$. Applying Lemma 15 to the points x, z, z_1 and (21) to the points x, y', z_1 we obtain

$$\phi(A_{x,z}) \geqslant c_{12}\left(\frac{|x-z|}{|x-z_1|}\right)^\gamma \phi(A_{x,z_1}) \geqslant c_{13}\left(\frac{|x-z|}{|x-y|}\right)^\gamma \phi(A_{x,y'}). \tag{39}$$

By (22) we have

$$c_{14}\,\phi(A_{x,z}) \geqslant \phi(z). \tag{40}$$

By assumption (b) we have $|y-z| \geqslant \delta(y) \geqslant |x-y|/5$. Therefore, using (38)–(40) we get

$$\begin{aligned} I_4 &\leqslant c_{14}\int_{B_4}\frac{\phi(x)}{\phi(A_{x,z})\,|x-z|^{d-\alpha}\,|y-z|^{d+\alpha}}\,dz \\ &\leqslant c_{13}^{-1}c_{14}\int_{B_4}\frac{\phi(x)\,|x-y|^{\gamma}}{\phi(A_{x,y'})\,|x-z|^{d-\alpha+\gamma}\,|y-z|^{d+\alpha}}\,dz \\ &\leqslant c_{11}c_{13}^{-1}c_{14}c_{15}\frac{\phi(x)\,\phi(y')\,|x-y|^{\alpha}}{\phi^2(A_{x,y'})\,|x-y|^{d+\alpha}} \\ &\leqslant m_4\frac{\phi(x)\,\phi(y')}{\phi^2(A_{x,y'})\,|x-y|^{d-\alpha}\,\delta(y)^{\alpha}\big(1+\delta(y)\big)^{\alpha}}, \end{aligned}$$

where $m_4 = c_{11}c_{13}^{-1}c_{14}c_{15}(1+\rho_0)^{\alpha}$.

We now estimate I_5 from above. Suppose that $z\in B_5$. We have $|x-z| > |x-y|$ and $2|x-z| > |x-y'|$. Hence, by (21) we obtain $\phi(A_{x,y'}) \leqslant c_{16}\,\phi(A_{x,z})$. Since $\alpha < d$ and $B_5 \subset D$, by Lemma 16 we get

$$\begin{aligned} I_5 &\leqslant c_{16}^2\frac{\phi(x)}{\phi^2(A_{x,y'})\,|x-y|^{d-\alpha}}\int_{B_5}\frac{\phi(z)\,dz}{|y-z|^{d+\alpha}} \\ &\leqslant m_5\frac{\phi(x)\,\phi(y')}{\phi^2(A_{x,y'})\,|x-y|^{d-\alpha}\,\delta(y)^{\alpha}\big(1+\delta(y)\big)^{\alpha}}. \end{aligned}$$

We now estimate I_6 from below. Suppose that $z\in B\big(S,\,\delta(y)\big)\cap D$. Note that $|x-z| < 3|x-y|$. Moreover, $r_1 \leqslant 4\kappa^{-1}r_2$. Indeed, $\delta(z) < \kappa^{-1}\delta(y')$ and

$$|x-z| \leqslant |x-y'|+|y'-z| < |x-y'|+2\kappa^{-1}\delta(y') \leqslant 4\kappa^{-1}\,[|x-y'| \vee \delta(y')].$$

Hence

$$\delta(x)\vee\delta(z)\vee|x-z| \leqslant 4\kappa^{-1}\big(\delta(x)\vee\delta(y')\vee|x-y'|\big).$$

Therefore, by Lemma 13 we have $c_{17}\,\phi(A_{x,z}) \leqslant \phi(A_{x,y'})$. Using this, by Lemma 16, we obtain

$$\begin{aligned} I_6 &\geqslant c_{17}^2\,3^{\alpha-d}\frac{\phi(x)}{\phi^2(A_{x,y'})\,|x-y|^{d-\alpha}}\int_{B_6}\frac{\phi(z)\,dz}{|y-z|^{d+\alpha}} \\ &\geqslant c_{17}^2\,c_{18}\,3^{\alpha-d}\frac{\phi(x)\,\phi(y')}{\phi^2(A_{x,y'})\,|x-y|^{d-\alpha}\,\delta(y)^{\alpha}\big(1+\delta(y)\big)^{d}} \\ &\geqslant m_6\frac{\phi(x)\,\phi(y')}{\phi^2(A_{x,y'})\,|x-y|^{d-\alpha}\,\delta(y)^{\alpha}\big(1+\delta(y)\big)^{\alpha}}. \end{aligned}$$

Therefore we get

$$m_6 \frac{\phi(x)\phi(y')}{\phi^2(A_{x,y'})|x-y|^{d-\alpha}\delta(y)^\alpha(1+\delta(y))^\alpha} \leqslant I_6 \leqslant I_4+I_5$$

$$\leqslant (m_4+m_5)\frac{\phi(x)\phi(y')}{\phi^2(A_{x,y'})|x-y|^{d-\alpha}\delta(y)^\alpha(1+\delta(y))^\alpha}.$$

Applying this to (30) we obtain (2).

Case (c). $\delta(y) > r_0/32$.

From the Ikeda–Watanabe formula we infer that

$$c_{19}^{-1}\frac{E^x\{\tau_D\}}{\delta(y)^{d+\alpha}} \leqslant P(x, y) \leqslant c_{19}\frac{E^x\{\tau_D\}}{\delta(y)^{d+\alpha}},$$

and (2) follows from Lemma 17. ∎

The estimates of the Martin kernel follow easily from the estimates for the Green function (Theorem 1). Since the proof is analogous to the one from [5], we will omit it.

5. APPLICATIONS

In this section we present some applications of the results obtained in this work, which simplifies proofs of some well-known results. The first one is the following "3G Theorem" (cf. [6] and [15]).

THEOREM 18 ("3G Theorem"). *There exists a constant* $C_{14} = C_{14}(\underline{D}, \alpha)$ *such that for every* $x, y, z \in D$ *we have*

$$\frac{G(x, y)G(y, z)}{G(x, z)} \leqslant C_{14}\frac{|x-y|^{\alpha-d}|y-z|^{\alpha-d}}{|x-z|^{\alpha-d}}. \tag{41}$$

Proof. The proof follows [5]. Let $x, y, z \in D$ and $R \in \mathscr{B}(x, y)$, $S \in \mathscr{B}(y, z)$ and $T \in \mathscr{B}(x, z)$. By Theorem 1 we have

$$\frac{G(x, y)G(y, z)}{G(x, z)} \leqslant C_1^3\frac{|x-y|^{\alpha-d}|y-z|^{\alpha-d}}{|x-z|^{\alpha-d}}W^2,$$

where

$$W = \frac{\phi(y)\phi(T)}{\phi(R)\phi(S)}.$$

We only need to show that W is bounded. By (22) there exists a constant $c_1 = c_1(\underline{D}, \alpha)$ such that $\phi(y) \leqslant c_1\phi(R)$ and $\phi(y) \leqslant c_1\phi(S)$. Note that $|x-z| \leqslant |x-y|+|y-z| \leqslant 2(|x-y| \vee |y-z|)$. Hence by (21) there exists a constant $c_2 = c_2(\underline{D}, \alpha)$ such that either $\phi(T) \leqslant c_2\phi(R)$ or $\phi(T) \leqslant c_2\phi(S)$. Therefore $W \leqslant c_1 c_2$. Taking $C_{14} = C_1^3 c_1^2 c_2^2$ we obtain (41). ∎

We conclude this work with short proofs of the well-known estimates for the Green function and the Poisson kernel for bounded $C^{1,1}$ domains (Theorems 21 and 22) first proved in [12], [13] and [19].

A function $F: \boldsymbol{R}^{d-1} \to \boldsymbol{R}$ is called $C^{1,1}$ if it has first derivative F' and there exists a constant η such that for all $x, y \in \boldsymbol{R}^{d-1}$ we have $|F'(x)-F'(y)| \leqslant \eta |x-y|$. A domain $D \subset \boldsymbol{R}^d$ is called a $C^{1,1}$ *domain* with constants $\eta, r_0 > 0$ if for every $Q \in \partial D$ there exists a $C^{1,1}$ function $F_Q: \boldsymbol{R}^{d-1} \to \boldsymbol{R}$ (with $C^{1,1}$ constant η), an orthonormal coordinate system CS_Q and a constant $r_0 = r_0(D)$ such that if $y = (y_1, y_2, \ldots, y_d)$ in CS_Q coordinates, then

$$B(Q, r_0) \cap D = B(Q, r_0) \cap \{y: y_d > F_Q(y_1, y_2, \ldots, y_{d-1})\}.$$

Clearly, every (bounded) $C^{1,1}$ domain is Lipschitz. $C^{1,1}$ domains have the following property ([22]):

There exists a constant s_0 such that for every $x \in D$ satisfying $\delta(x) < s_0$ there exist two balls B_x^1 and B_x^2 of radius s_0 such that $B_x^1 \subset D$, $B_x^2 \subset \operatorname{int}(D^c)$ and $\partial B_x^1 \cap \partial B_x^2 = \{x^*\}$, where $x^* \in \partial D$ is a point satisfying $\delta(x) = |x-x^*|$. The constant s_0 depends on d, η, r_0, where r_0 and η are constants defining the $C^{1,1}$ domain.

In what follows we assume that our (bounded) Lipschitz domain D with Lipschitz character (r_0, λ) is also a $C^{1,1}$ domain with constants r_0 and η. When writing $c = c(\underline{D})$, we mean $c = c(d, r_0, \lambda, \theta)$, as usual. We first need the following auxiliary results based on the explicit formula for the Green function of the complement of the ball [2].

LEMMA 19. *For any $s > 0$ there exists a constant $M_8 = M_8(d, \alpha, s)$ such that for every ball $B = B(a, s) \subset \boldsymbol{R}^d$ we have*

$$G_{B^c}(x, y) \leqslant M_8 |y-a|^{\alpha/2} \frac{\delta_B(x)^{\alpha/2}}{|x-y|^{d-\alpha/2}}, \qquad x, y \in B^c. \tag{42}$$

The proof of this lemma can be found in [12] (Lemma 2.5).

LEMMA 20. *There exists a constant $C_{15} = C_{15}(\underline{D}, s_0, \alpha)$ such that*

$$C_{15}^{-1} \delta^{\alpha/2}(x) \leqslant E^x\{\tau_D\} \leqslant C_{15} \delta^{\alpha/2}(x) \tag{43}$$

for all $x \in D$.

Proof. It is well known (see, e.g., (2.10) in [10], or [8]) that there exists a constant $M_9 = M_9(d, \alpha)$ such that for any $s > 0$ we have

$$E^x\{\tau_{B(0,s)}\} = M_9 (s^2 - |x|^2)^{\alpha/2}, \qquad x \in B(0, s). \tag{44}$$

First assume that $\delta(x) \geqslant s_0$. Note that

$$E^0\{\tau_{B(0,s_0)}\} = E^x\{\tau_{B(x,s_0)}\} \leqslant E^x\{\tau_D\} \leqslant E^x\{\tau_{B(x,\theta)}\} = E^0\{\tau_{B(0,\theta)}\}.$$

Hence (43) holds clearly because $\delta(x)$ is also bounded by two constants: $s_0 \leqslant \delta(x) \leqslant \theta$.

Now assume that $\delta(x) < s_0$. We have $E^x\{\tau_{B_x^1}\} \leqslant E^x\{\tau_D\}$. From (44) it follows that there exists a constant $c_1 = c_1(d, s_0, \alpha)$ such that $E^x\{\tau_{B_x^1}\} \geqslant c_1 \delta_{B_x^1}^{\alpha/2}(x) = c_1 \delta^{\alpha/2}(x)$. Hence we obtain the left-hand side of (43). Let x' be the center of the ball B_x^2. Note that $\delta_{(B_x^2)^c}(x) = \delta(x)$. By Lemma 19, for any $y \in D$ we get

$$G(x, y) \leqslant G_{(B_x^2)^c}(x, y) \leqslant c_2 |y-x'|^{\alpha/2} \frac{\delta^{\alpha/2}(x)}{|x-y|^{d-\alpha/2}},$$

where $c_2 = c_2(d, \alpha, s_0)$. Therefore

$$E^x\{\tau_D\} = \int_D G(x, y)\,dy \leqslant \int_D c_2 |y-x'|^{\alpha/2} \frac{\delta^{\alpha/2}(x)}{|x-y|^{d-\alpha/2}}\,dy$$

$$\leqslant \int_D c_2 |s_0+\theta|^{\alpha/2} \frac{\delta^{\alpha/2}(x)}{|x-y|^{d-\alpha/2}}\,dy \leqslant c_3 \delta^{\alpha/2}(x),$$

where $c_3 = c_3(\underline{D}, s_0, \alpha)$. This completes the proof. ∎

In connection to the fact that $E^x\{\tau_D\}$ is comparable to $G(x_0, x)$ at the boundary of D we remark here that it is known that for every $Q \in \partial D$ the limit

$$\lim_{D \ni y \to Q} \frac{G(x_0, y)}{\delta(y)^{\alpha/2}}$$

exists and is a positive number (see [9] for the proof).

THEOREM 21. *There exists a constant $C_{16} = C_{16}(\underline{D}, s_0, \alpha)$ such that*

$$(45)\quad C_{16}^{-1}\left(1 \wedge \frac{\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)}{|x-y|^{\alpha}}\right)|x-y|^{\alpha-d} \leqslant G(x, y)$$

$$\leqslant C_{16}\left(1 \wedge \frac{\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)}{|x-y|^{\alpha}}\right)|x-y|^{\alpha-d}.$$

Proof. By Lemmas 17 and 20 there exists a constant $c_1 = c_1(\underline{D}, s_0, \alpha)$ such that

$$(46)\qquad c_1^{-1} \delta^{\alpha/2}(x) \leqslant \phi(x) \leqslant c_1 \delta^{\alpha/2}(x), \qquad x \in D.$$

By Theorem 1 and (46) we get for $c_2 = c_2(\underline{D}, \alpha)$

$$c_2^{-1} W |x-y|^{\alpha-d} \leqslant G(x, y) \leqslant c_2 W |x-y|^{\alpha-d}, \qquad x, y \in D,$$

where

$$(47)\quad W = \frac{\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)}{[\delta(x) \vee \delta(y) \vee |x-y|]^{\alpha}} = \left(\frac{\delta(y)}{\delta(x)}\right)^{\alpha/2} \wedge \left(\frac{\delta(x)}{\delta(y)}\right)^{\alpha/2} \wedge \frac{\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)}{|x-y|^{\alpha}}.$$

Since $(\delta(x)/\delta(y)) \wedge (\delta(y)/\delta(x)) \leqslant 1$, we get

$$W \leqslant 1 \wedge [\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)\,|x-y|^{-\alpha}].$$

Therefore the right-hand side of (45) holds with $C_{16} = c_2$.

To estimate W from below, we assume first that $\delta(y) < \delta(x)/3$. Then $|x-y| \geqslant \delta(x) - \delta(y) > 2\delta(x)/3$. Hence

$$\delta(x)\,\delta(y)/|x-y|^2 < 9\delta(y)/(4\delta(x)) < 1.$$

Therefore we get

$$1 \wedge \frac{\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)}{|x-y|^{\alpha}} = \frac{(9\delta(y))^{\alpha/2}}{(4\delta(x))^{\alpha/2}} \wedge \frac{\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)}{|x-y|^{\alpha}}$$
$$\leqslant \left(\frac{9}{4}\right)^{\alpha/2}\left[\frac{\delta^{\alpha/2}(y)}{\delta^{\alpha/2}(x)} \wedge \frac{\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)}{|x-y|^{\alpha}}\right] < 3W.$$

The case $\delta(x) < \delta(y)/3$ is analogous to the previous one.

If $\delta(y)/3 \leqslant \delta(x) \leqslant 3\delta(y)$, then $\delta(y)/\delta(x) \geqslant 1/3$ and $\delta(x)/\delta(y) \geqslant 1/3$. Hence $1 \wedge (\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)/|x-y|^{\alpha}) < 3W$.

Therefore we obtain (45) with $C_{16} = 3c_2$. ∎

We will now give a short proof of the estimates for the Poisson kernel for bounded $C^{1,1}$ domains (see [12]).

THEOREM 22. *There exists a constant* $C_{17} = C_{17}(\underline{D}, s_0, \alpha)$ *such that*

$$(48)\qquad C_{17}^{-1}\frac{\delta^{\alpha/2}(x)}{|x-y|^{d}\,\delta^{\alpha/2}(y)\,(1+\delta(y))^{\alpha/2}} \leqslant P(x, y)$$
$$\leqslant C_{17}\frac{\delta^{\alpha/2}(x)}{|x-y|^{d}\,\delta^{\alpha/2}(y)\,(1+\delta(y))^{\alpha/2}},$$

where $x \in D$ *and* $y \in \operatorname{int}(D^c)$.

Proof. By Lemmas 17 and 20 there exists a constant $c_1 = c_1(\underline{D}, s_0, \alpha)$ such that

$$(49)\qquad c_1^{-1}\,\delta^{\alpha/2}(x) \leqslant \phi(x) \leqslant c_1\,\delta^{\alpha/2}(x), \qquad x \in D.$$

Let $x \in D$, $y \in \operatorname{int}(D^c)$, $y' \in \mathcal{A}_\delta(y)(S)$, where $S \in \partial D$ is a point such that $|y-S| = \delta(y)$. Let $s = \delta(x) \vee \delta(y') \vee |x-y'|$. By Theorem 2 and (49) we have

$$(50)\quad c_2^{-1}\frac{\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y')}{\delta^{\alpha}(A)\,|x-y|^{d-\alpha}\,\delta^{\alpha}(y)\,(1+\delta(y))^{\alpha}} \leqslant P(x, y)$$
$$\leqslant \frac{c_2\,\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y')}{\delta^{\alpha}(A)\,|x-y|^{d-\alpha}\,\delta^{\alpha}(y)\,(1+\delta(y))^{\alpha}},$$

where $A \in \mathcal{B}(x, y')$ and $c_2 = c_2(\underline{D}, s_0, \alpha)$.

First assume that $\delta(y) \leqslant r_0/32$. Note that $\kappa\delta(y) \leqslant \delta(y') \leqslant \delta(y)$. Hence there exists a constant $c_3 = c_3(\lambda)$ such that

(51) $$c_3^{-1}\delta(y) \leqslant \delta(y') \leqslant c_3\,\delta(y).$$

It suffices to prove that there exists a constant $c_4 = c_4(\lambda)$ such that

(52) $$c_4^{-1}|x-y| \leqslant \delta(A) \leqslant c_4|x-y|.$$

Indeed, $|x-y| \leqslant |x-y'|+|y-y'| \leqslant |x-y'|+2\delta(y) \leqslant 3c_3 s \leqslant (3c_3/\kappa)\delta(A)$. Moreover, $|x-y| \geqslant \delta(x) \vee \delta(y)$ and $|x-y'| \leqslant |x-y|+2\delta(y) \leqslant 3|x-y|$, and hence $|x-y| \geqslant s/3 > \delta(A)/12$. Therefore (52) holds. Applying (51) and (52) to (50) we obtain (48).

Now assume that $\delta(y) > r_0/32$. We have $y' = x_1 = A$. There exist constants $c_5 = c_5(\theta, r_0)$ and $c_6 = c_6(\theta, r_0)$ such that

$$c_5^{-1}|x-y| \leqslant \delta(y) \leqslant c_5|x-y|, \qquad c_6^{-1}\delta(y) \leqslant \delta(y)+1 \leqslant c_6\,\delta(y)$$

for all $x \in D$. These two inequalities and (50) yield (48). ∎

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