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POTENTIAL THEORY OF SCHRÖDINGER OPERATOR BASED ON FRACTIONAL LAPLACIAN

BY

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Abstract. We develop potential theory of Schrödinger operators based on fractional Laplacian on Euclidean spaces of arbitrary dimension. We focus on questions related to gaugeability and existence of q-harmonic functions. Results are obtained by analyzing properties of a symmetric α-stable Lévy process on $\boldsymbol{R}^d$, including the recurrent case. We provide some relevant techniques and apply them to give explicit examples of gauge functions for a general class of domains.

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1. INTRODUCTION

The paper deals with Schrödinger type operators corresponding to symmetric α-stable Lévy processes X_t on $\boldsymbol{R}^d$ equipped with a multiplicative functional $e_q(t) = \exp\left(\int_0^t q(X_s)\, ds\right)$, where q is a given function (in a Kato class). We study the existence and properties of q-harmonic functions. In particular, we address ourselves to problems related to gaugeability.

Many potential-theoretic properties of X_t for $\alpha \in (0, 2)$ are dramatically different from those of Brownian motion yet they may be regarded as typical for a general class of Lévy processes on $\boldsymbol{R}^d$. This motivates a thorough study of the Feynman–Kac semigroups related to the symmetric stable Lévy processes, especially that the explicit calculations are very often feasible in this particular case, which stimulates and enriches the general theory.

Results of this paper complement the earlier ones contained in [6]. Results of [6] were basically restricted to bounded Lipschitz domains and were based

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on the Conditional Gauge Theorem (**CGT**). This rather sophisticated result with a difficult and technical proof allows to derive the potential theory for the considered Schrödinger operators on Lipschitz domains directly from the existing one for fractional Laplacian. More general domains were dealt with those in [6] by approximating them by Lipschitz domains. Note that the case $\alpha \geqslant d = 1$, when X_t is recurrent, was not considered in [6].

In the present paper we cover also the recurrent case. Furthermore, we aim to give some general but explicit examples of the gauge function and in order to accomplish it we develop techniques based on a study of Green potentials. In comparison with [6] we now rely on a different methodology consisting in focusing on local properties of q-harmonic functions, which are then extended by some general procedures. Thus, we only need to use the local version of **CGT** for the small ball, which is a simple consequence of Khasminski's lemma and 3G Theorem for the ball. We also depart in various situations from the gaugeability assumption and study potential theory on unbounded open sets without Lipschitz character.

We now briefly describe the contents of the paper. Section 2 is preliminary; we collect here basic facts concerning potential theory of symmetric α-stable Lévy processes with special emphasis on the recurrent case. In Section 3 we provide relevant estimates for the Green function for the ball, most notably the so-called 3G Theorem. Although results pertaining to the transient case ($\alpha < d$) are known, they are included in unified proofs, original at least in the recurrent case. As a consequence of 3G Theorem we formulate a "small" **CGT** for balls.

In Section 4 we discuss problems related to gaugeability. By means of the "small" **CGT** we prove a Harnack inequality for nonnegative q-harmonic functions. Then we give an extension of the Gauge Theorem. We also summarize the connections between the existence of q-harmonic functions and gaugeability. Our results are related to but more general than the ones presented in Section 4 in [10].

In Section 5 we include auxiliary results on the Green potentials and weak fractional Laplacian needed in the subsequent sections.

Section 6 contains characterizations of q-harmonic functions u as solutions of the equation $\mathscr{S}^{\alpha} u = 0$ under appropriate gaugeability or nonnegativity conditions, which supplements and augments earlier results in [6]. We also give examples of the gauge function based on Green potentials.

In Section 7 we apply results of Section 4 to investigate the gauge function of half-lines $(-\infty, y) \subseteq \boldsymbol{R}^1$. Although results obtained in this section are motivated by those in [10], Section 9, the approach of [10] is not applicable here, and we use alternative methods of proof. The difference between the Brownian motion case and the case of the general symmetric α-stable Lévy process with $\alpha \in (0, 2)$ resulting from the discontinuity of the paths of the latter is very plain to see in this section.

In Section 8 we describe the action of Kelvin transform on q-harmonic functions, which allows us to construct easily the examples of the gauge function for "large" domains based upon gauge functions for bounded domains.

2. PRELIMINARIES

For convenience of the reader we collect here information and references necessary to understand the paper.

2.1. Notation and terminology. Most of the notation and terminology is adopted here from [3] and [6]. However, we often consider simultaneously both cases: the transient ($\alpha < d$) and the recurrent one ($\alpha \geqslant d = 1$). In the recurrent case many previously considered objects either have different properties or take on a different meaning. We remark that all functions considered in this paper are defined on the whole of $\boldsymbol{R}^d$ due to non-locality of the theory of α-harmonic functions for $\alpha < 2$. We always require Borel measurability on $\boldsymbol{R}^d$. Thus, for an open set $D \subseteq \boldsymbol{R}^d$, by $L^\infty(D)$ we denote the class of all Borel measurable functions on $\boldsymbol{R}^d$ that are bounded on D. A similar convention applies to the definition of $L^p(D)$ for $1 \leqslant p < \infty$. As usual, $f \in L^1_{\text{loc}}(D)$ means that $f\mathbf{1}_K \in L^1(\boldsymbol{R}^d)$ for every compact $K \subset D$. Analogously, $C(D)$ ($C_b^k(D)$, respectively) denotes the class of Borel functions on $\boldsymbol{R}^d$ that are continuous (have bounded continuous derivatives up to order k, respectively) on D, and $C_0(D)$ is a subclass of $C(D)$ consisting of functions that are continuous everywhere and vanish on D^c. $C_c(D)$ ($C_c^\infty(D)$, respectively) is the class of continuous functions with compact support contained in D (and infinitely differentiable, respectively). We write $A \in \mathscr{B}(\boldsymbol{R}^d)$ if A is a Borel subset of $\boldsymbol{R}^d$ and $f \in \mathscr{B}(\boldsymbol{R}^d)$ ($f \in \mathscr{B}_+(\boldsymbol{R}^d)$, respectively) if the function f is Borel measurable (and nonnegative, respectively) on $\boldsymbol{R}^d$.

The notation $C(a, b, \ldots, z)$ means that C is a constant depending *only* on $a, b, \ldots, z$. We adopt the convention that constants may change their value but their dependence does not change from one use to another. Constants are always positive and finite. As usual we write $\operatorname{diam}(A) = \sup\{|v-w|: v, w \in A\}$, $\operatorname{dist}(x, A) = \sup\{|x-v|: v \in A\}$, and $\operatorname{dist}(A, B) = \sup\{|v-w|: v \in A, w \in B\}$, where $x \in \boldsymbol{R}^d$ and $A, B \subseteq \boldsymbol{R}^d$.

2.2. Symmetric α-stable processes and α-harmonic functions. Throughout the paper we assume, unless stated otherwise, that $\alpha \in (0, 2)$. Occasionally, as in Section 3, we consider the case of $\alpha = 2$. We denote by (X_t, P^x) the standard rotation invariant ("symmetric") α-stable Lévy process in $\boldsymbol{R}^d$, $d \in \{1, 2, \ldots\}$ (i.e. homogeneous, with independent increments), with the index of stability α, and the characteristic function of the form $E^0 \exp(iuX_t) = \exp(-t|u|^\alpha)$, $u \in \boldsymbol{R}^d$, $t \geqslant 0$. The index of stability $\alpha = 2$ corresponds to the process of Brownian motion. As usual, E^x denotes the expectation with respect to the distribution P^x of the process starting from $x \in \boldsymbol{R}^d$. We always assume that sample paths of X_t are right-continuous and have left-hand limits a.s. The process (X_t) is Markov with transition probabilities given by $P_t(x, A) = P^x(X_t \in A) = \mu_t(A-x)$, where μ_t is the one-dimensional distribution of X_t with respect to P^0. We have $P_t(x, A) = \int_A p(t; x, y)\,dy$, where $p(t; x, y) = p_t(y-x)$ are the *transition densities* of X_t. The function $p_t(x) = p_t(-x)$ is continuous in (t, x) for $t > 0$, and has the following useful *scaling property*: $p_t(x) = t^{-d/\alpha} p_1(x/t^{1/\alpha})$. The process (X_t, P^x) is strong Markov with respect to the so-called "standard filtration" $\{\mathscr{F}_t;\, t \geqslant 0\}$ and quasi-left-continuous on $[0, \infty]$. The *shift operator* is denoted by θ_t. The operator θ_t is also extended to Markov times τ and is denoted then by θ_τ. For $\alpha < 2$ the process X_t has the *infinitesimal generator* $\Delta^{\alpha/2}$ given as

$$\Delta^{\alpha/2} u(x) = \mathscr{A}(d, -\alpha)\, PV \int_{\boldsymbol{R}^d} \frac{u(x+y)-u(x)}{|y|^{d+\alpha}}\,dy, \qquad u \in C_b^2(\boldsymbol{R}^d),$$

where $\mathscr{A}(d, \gamma) = \Gamma((d-\gamma)/2)/(2^\gamma \pi^{d/2} |\Gamma(\gamma/2)|)$. For $\alpha < d$ the process X_t is transient and the potential kernel of X_t is given by

$$K_\alpha(y-x) = \int_0^\infty p(t; x, y)\,dt = \frac{\mathscr{A}(d, \alpha)}{|y-x|^{d-\alpha}}, \qquad x, y \in \boldsymbol{R}^d;$$

see [1] and [15]. Whenever $\alpha \geqslant d$ the process X_t is recurrent (pointwise recurrent if $\alpha > d = 1$) and it is appropriate to consider the so-called *compensated kernels* [2]. Namely, for $\alpha \geqslant d$ we put

$$K_\alpha(y-x) = \int_0^\infty \big(p(t; x, y) - p(t; 0, x_0)\big)\,dt,$$

where $x_0 = 0$ for $\alpha > d = 1$, $x_0 = 1$ for $\alpha = d = 1$ and $x_0 = (0, 1)$ for $\alpha = d = 2$. It turns out that for $\alpha = d = 1$ or 2

$$K_\alpha(x) = \frac{1}{\pi} \ln \frac{1}{|x|};$$

and for $\alpha > d = 1$

$$K_\alpha(x) = \frac{\mathscr{A}(1, \alpha)}{|x|^{1-\alpha}} = \frac{|x|^{\alpha-1}}{2\Gamma(\alpha)\cos(\pi\alpha/2)}, \qquad x \in \boldsymbol{R}^d.$$

For $A \in \mathscr{B}(\boldsymbol{R}^d)$, we put $T_A = \inf\{t \geqslant 0;\, X_t \in A\}$, the first *entrance* time of A^c, $\tau_A = \inf\{t \geqslant 0;\, X_t \notin A\}$, the first *entrance* time of A^c, and $\tau'_A = \inf\{t > 0;\, X_t \notin A\}$, the first *hitting* time of A^c. A point $x \in \boldsymbol{R}^d$ is called *regular* for a (Borel) set A if $P^x\{\tau'_A = 0\} = 1$; A itself is called *regular* if all points $x \in A^c$ are regular for A. We say that $u \in \mathscr{B}(\boldsymbol{R}^d)$ is *α-harmonic* in an open set $D \subseteq \boldsymbol{R}^d$ if

$$u(x) = E^x u(X_{\tau_U}), \qquad x \in U, \tag{2.1}$$

for every open bounded set U with the closure $\bar{U}$ contained in D. It is called *regular α-harmonic* in D if (2.1) holds for $U = D$. If D is unbounded, then by a usual convention $E^x u(X_{\tau_D}) = E^x[\tau_D < \infty; u(X_{\tau_D})]$. Under (2.1) it is always assumed that the expectation in (2.1) is absolutely convergent; in particular, finite.

By the strong Markov property of X_t, a regular α-harmonic function u is necessarily α-harmonic. The converse is not generally true [3].

When $r > 0$, $B = B(0, r) \subset \boldsymbol{R}^d$ and $|x| < r$, the P^x-distribution of X_{τ_B} has the density function $P_r(x, \cdot)$ (the *Poisson kernel*), explicitly given by the formula

$$P_r(x, y) = C_\alpha^d \left[\frac{r^2 - |x|^2}{|y|^2 - r^2}\right]^{\alpha/2} |x-y|^{-d}, \quad |y| > r, \tag{2.2}$$

with $C_\alpha^d = \Gamma(d/2)\pi^{-d/2-1}\sin(\pi\alpha/2)$, and equal to 0 otherwise [2].

2.3. Killed symmetric α-stable Lévy motion. Let D be a bounded domain. We often assume that D is regular. By (P_t^D) we denote the semigroup generated by the process (X_t) killed on exiting D. The semigroup (P_t^D) is determined by transition densities $p_t^D(x, y)$ which are symmetric, that is $p_t^D(x, y) = p_t^D(y, x)$, and continuous in (t, x, y) for $t > 0$ and $x, y \in D$. Thus, for any $f \in \mathscr{B}_+(\boldsymbol{R}^d)$ we have

$$P_t^D f(x) = E^x[t < \tau_D; f(X_t)] = \int_D f(y) p_t^D(x, y)\, dy.$$

We call $L^p(D)$ $(1 \leqslant p < \infty)$ or, for regular D, $C_0(D)$, an *appropriate space* for the semigroup $(P_t^D)_{t>0}$. The semigroup acts on each of the appropriate spaces as a strongly continuous semigroup of contractions.

The *Green operator* for D is denoted by G_D. We set

$$G_D(x, y) = \int_0^\infty p_t^D(x, y)\, dt$$

and call $G_D(x, y)$ the *Green function* for D. We have

$$G_D f(x) = E^x\Big[\int_0^{\tau_D} f(X_t)\, dt\Big] = \int_D G_D(x, y) f(y)\, dy$$

for, e.g., nonnegative Borel functions f on $\boldsymbol{R}^d$. When D is fixed, we often write $G(x, y)$ instead of $G_D(x, y)$. If D is regular, then $G_D(x, y)$ has the following properties: $G_D(x, y) = G_D(y, x)$; $G_D(x, y)$ is positive for $x, y \in D$ and continuous at $x, y \in \boldsymbol{R}^d$ for $x \neq y$; $G_D(x, y) = 0$ if x or y belongs to D^c. For $x, y \in \boldsymbol{R}^d$ we have (unless $x = y \in D^c$)

$$G_D(x, y) = K_\alpha(x, y) - E^x K_\alpha(X_{\tau_D}, y),$$

where $K_\alpha(x, y) = K_\alpha(x-y)$.

Let $B = B(0, 1) = \{x \in \mathbf{R}^d: |x| < 1\}$, $\alpha \in (0, 2]$. It is well known that B is a regular domain and its Green function is given by the formula

(2.3) $$G(x, y) = \mathcal{B}^d_\alpha |x-y|^{\alpha-d} \int_0^{w(x,y)} \frac{r^{\alpha/2-1}}{(r+1)^{d/2}} dr, \qquad x, y \in B,$$

where

$$w(x, y) = (1-|x|^2)(1-|y|^2)/|x-y|^2,$$

and $\mathcal{B}^d_\alpha = \Gamma(d/2)/(2^\alpha \pi^{d/2} [\Gamma(\alpha/2)]^2)$; see [2] and [10]. If $\alpha > d = 1$, then $G(x, y)$ is bounded and continuous on $B \times B$ and, for $x = y$, the right-hand side of (2.3) is equal, in the limiting sense, to $(1-x^2)^{\alpha-1}/[2^{\alpha-1}\Gamma^2(\alpha/2)(\alpha-1)]$. A domain $D \subset \mathbf{R}^d$ is called *Green-bounded* if $\sup_{x \in \mathbf{R}^d} G_D \mathbf{1}(x) < \infty$. We have

(2.4) $$\|G_D \mathbf{1}\|_\infty = \sup_{x \in \mathbf{R}^d} E^x \tau_D \leqslant Cm(D)^{\alpha/d}$$

by a direct modification of the proof of Theorem 1.17 from [10]. Thus, sets of finite Lebesgue measure (in particular, bounded sets) are Green-bounded.

2.4. Kato class $\mathcal{J}^\alpha$. We say that a Borel function q belongs to the *Kato class* $\mathcal{J}^\alpha$ if q satisfies either of the two equivalent conditions (see [17]):

(2.5) $$\lim_{r \to 0} \sup_{x \in \mathbf{R}^d} \int_{|x-y| \leqslant r} |q(y) K_\alpha(x-y)| \, dy = 0,$$

(2.6) $$\lim_{t \to 0} \sup_{x \in \mathbf{R}^d} \int_0^t P_s |q|(x) \, ds = 0.$$

For open $D \subset \mathbf{R}^d$ we write $q \in \mathcal{J}^\alpha_{\mathrm{loc}}(D)$ if for every compact $K \subset D$ we have $\mathbf{1}_K q \in \mathcal{J}^\alpha$ and we put $\mathcal{J}^\alpha_{\mathrm{loc}} = \mathcal{J}^\alpha_{\mathrm{loc}}(\mathbf{R}^d)$. For $d = 1 < \alpha < 2$ it follows that $q \in \mathcal{J}^\alpha$ if and only if

(2.7) $$\sup_{x \in \mathbf{R}^d} \int_{|x-y| \leqslant 1} |q(y)| \, dy < \infty.$$

For general α and $d \in \mathbf{N}$, (2.7) is necessary (but not sufficient) for q to belong to $\mathcal{J}^\alpha$. In particular, $\mathcal{J}^\alpha_{\mathrm{loc}} \subseteq L^1_{\mathrm{loc}}$ for all $\alpha \in (0, 2)$ and $d \geqslant 1$. If $f \in L^\infty(\mathbf{R}^d)$ and $q \in \mathcal{J}^\alpha$, then $f, fq \in \mathcal{J}^\alpha$.

Let D be a Green-bounded domain in $\mathbf{R}^d$ and $q \in \mathcal{J}^\alpha$. For any $b > 0$ there exists $a > 0$ depending only on α, q and b such that

$$G_D |q| \leqslant a G_D \mathbf{1} + b.$$

Consequently, for a fixed $q \in \mathcal{J}^\alpha$, but a variable domain D, we have

(2.8) $$\sup_{x \in \mathbf{R}^d} \left| E^x \left[\int_0^{\tau_D} q(X_s) \, ds \right] \right| = \|G_D q\|_\infty \to 0 \quad \text{if } \|G_D \mathbf{1}\|_\infty \to 0.$$

Also, $G_D q \in L^\infty(\mathbf{R}^d) \cap C(D)$, and we have $\lim_{x\to z} G_D q(x) = 0$ if z is regular for D. In particular, $G_D q \in C_0(D)$ if D is regular.

2.5. Feynman–Kac semigroups. For $q \in \mathscr{J}^\alpha$ we define the additive functional

$$A(t) = \int_0^t q(X_s)\,ds, \qquad t \geqslant 0.$$

The corresponding multiplicative functional $e_q(t)$ is defined as

$$e_q(t) = \exp\big(A(t)\big), \quad t \geqslant 0.$$

For all $s, t \geqslant 0$ we have $e_q(s+t) = e_q(s)\{e_q(t) \circ \theta_s\}$. If now τ is a Markov time such that for every $t \geqslant 0$ we have $\tau \leqslant t + \tau \circ \theta_t$ on $\{t < \tau\}$, then for $q \geqslant 0$ we obtain the following important fact, referred to in the sequel as *Khasminski's lemma*:

(2.9) If $\sup_{x\in\mathbf{R}^d} E^x A(\tau) = \varepsilon < 1$, then $\sup_{x\in\mathbf{R}^d} E^x e_q(\tau) < (1-\varepsilon)^{-1}$.

Note that (2.9) applies to constant times $\tau = t_0$ and to exit times $\tau = \tau_B$. By (2.8) and (2.4) applied to B instead of D, we infer that for a given $\varepsilon > 0$ there exists $\delta = \delta(\alpha, q, \varepsilon)$ such that if $m(B) < \delta$, then

$$\sup_{x\in\mathbf{R}^d} |E^x A(\tau_B)| = \|G_B q\|_\infty < \varepsilon.$$

As a standard application of (2.9) and (2.6) we get

(2.10) $\lim_{s\to 0} \sup_{x\in\mathbf{R}^d} E^x e_{|q|}(s) = 1$ and $\sup_{x\in\mathbf{R}^d} E^x e_{|q|}(t) \leqslant \exp(C_0 + C_1 t)$

for some $C_0,\ C_1 > 0$ and all $t > 0$. In particular, it follows that

$$0 < e_{-|q|}(\tau) \leqslant e_q(\tau) \leqslant e_{|q|}(\tau) < \infty \text{ a.s. on } \{\tau < \infty\}$$

for every Markov time τ. (Note that the same conclusion holds more generally for $q \in \mathscr{J}^\alpha_{\mathrm{loc}}$ because trajectories of X_t are a.s. bounded on finite time intervals.)

By $(T_t) = (T_t^D)$ we denote the Feynman–Kac semigroup killed on exiting D. Thus, for nonnegative Borel f we have

$$T_t f(x) = E^x[t < \tau_D;\ e_q(t) f(X_t)].$$

(T_t) is a strongly continuous semigroup of bounded operators on each of the spaces appropriate for the semigroup (P_t^D). Furthermore, for every $1 \leqslant p \leqslant \infty$, we have $\|T_t\|_p \leqslant \|T_t\|_\infty \leqslant \exp(C_0 + C_1 t)$. For every $t > 0$, T_t is a bounded operator from L^p into L^∞ determined by a symmetric kernel function u_t, which is in $C_0(D \times D)$ for regular D. For each $f \in L^p$ $(1 \leqslant p \leqslant \infty)$, we thus have

$$T_t f(x) = \int_D u_t(x, y) f(y)\,dy, \qquad t > 0.$$

Moreover, if D is regular, T_t maps $L^1(D)$ into $C_0(D)$ for $t > 0$. The potential operator V for (T_t) is introduced as follows:

$$Vf(x) = \int_0^\infty T_t f(x)\,dt = E^x\Big[\int_0^{\tau_D} e_q(t) f(X_t)\,dt\Big],$$

where f is nonnegative and Borel measurable on D. We call V the *q-Green operator*. If $\int_0^\infty \|T_t\|_\infty\,dt < \infty$ with the operator norm taken in $L^\infty(D)$, then V is bounded on L^p, $1 \leqslant p \leqslant \infty$. In particular, $V1 \in L^\infty(D)$ and the operator V has a symmetric kernel $V(x, y)$ called the *q-Green function* which is given by the formula

$$V(x, y) = \int_0^\infty u_t(x, y)\,dt.$$

Thus, we have $Vf(x) = \int_D V(x, y) f(y)\,dy$.

2.6. Stopped Feynman–Kac functional. The Gauge Theorem. Let D be a domain in $\boldsymbol{R}^d$ and let $q \in \mathcal{J}^\alpha$. We will usually assume that D is bounded or of finite Lebesgue measure. Then by **2.3** we obtain $\tau_D < \infty$ a.s. Since we also have $\int_0^t |q(X_s)|\,ds < \infty$ a.s. for each $t > 0$, the random variable $e_q(\tau_D)$ is well-defined a.s. The function

$$u(x) = E^x e_q(\tau_D)$$

is called the *gauge* (function) for (D, q). When it is bounded in D, hence in $\boldsymbol{R}^d$, we say that (D, q) is *gaugeable*. For a fixed $q \in \mathcal{J}^\alpha$ but a variable domain D we use the alternative notation u_D for the gauge for (D, q). If $G_D q$ is bounded from below, then by Jensen's inequality we obtain

$$\inf_{x \in \boldsymbol{R}^d} u_D(x) > 0. \tag{2.11}$$

In particular, (2.11) holds when D is Green-bounded and $q \in \mathcal{J}^\alpha$.

If (2.11) holds and (D, q) is gaugeable, then (E, q) is gaugeable for any domain $E \subseteq D$. In fact, $\|u_E\|_\infty \leqslant \|u_D\|_\infty \|u_D^{-1}\|_\infty$.

For domains $D \subset \boldsymbol{R}^d$ of finite Lebesgue measure it follows that if $u(x) < \infty$ for some $x \in D$, then u is bounded in $\boldsymbol{R}^d$ (see [10]). This important fact will be referred to as the *Gauge Theorem*.

2.7. Conditional α-stable Lévy motion. As in [6], the conditional α-stable Lévy motion remains here to be an indispensable important technical tool. For definition and properties of this process in $\boldsymbol{R}^d$ for $\alpha < d$ we refer to [6]. We only recall here that for a bounded domain D and $y \in D$ by the *α-stable y-Lévy motion* we understand the process conditioned by the Green function $G_D(\cdot, y)$ of D, while for $\xi \in \partial D$ the *α-stable ξ-Lévy motion* is, in turn, the process conditioned by the Martin kernel $K(\cdot, \xi)$ of D.

We examine properties of these processes briefly when $\alpha \geqslant d = 1$. For this purpose we now assume that D is an open bounded interval in $\boldsymbol{R}^1$.

We observe that if $\alpha = 1 = d$, then single points are polar [1] and this case can be dealt with as in [6]. In particular, the behavior of the y-Lévy motion and ξ-Lévy motion at their lifetimes is similar as for $\alpha < d$.

When $\alpha > 1 = d$, the situation is different: single points are not polar in this case and we have to modify our arguments.

First of all, observe that when $\alpha > 1 = d$, the Green function $G_D(x, y)$ of the interval D is bounded on $D \times D$. Thus, the whole of D remains to be the state space of the conditional process in this case. As another consequence, $G_D(\cdot, y)$ is regular α-harmonic in $D\backslash\{y\}$, so $E^x G_D(X_{\tau_{D\backslash\{y\}}}, y) = G_D(x, y)$ for $x \neq y$. Thus, for $x \neq y$ we have

$$P_y^x\{\tau_{D\backslash\{y\}} < \tau_D\} = G_D(x, y)^{-1} E^x[\tau_{D\backslash\{y\}} < \tau_D;\ G_D(X_{\tau_{D\backslash\{y\}}}, y)]$$
$$= G_D(x, y)^{-1} E^x G_D(X_{\tau_{D\backslash\{y\}}}, y) = G_D(x, y)^{-1} G_D(x, y) = 1.$$

We have obtained

$$P_y^x\{T_{\{y\}} < \tau_D\} = 1. \tag{2.12}$$

On the other hand, in the same way as in the proof of Lemma 4.3 in [6] we obtain for $U \subseteq D$ and $x, y \in D$:

$$P_y^x\{\tau_U = \tau_D\} = \frac{G_U(x, y)}{G_D(x, y)}.$$

Observe that the above formula remains valid also when $x = y$, where $G_D(y, y)$ is defined in the limiting sense by (2.3). We can also show that $P_y^x\{T_U < \tau_D;\ X_{T_U} = y\} = 0$ for $y \in U$, $x \in D$, $x \neq y$. The result may give the reader some insight into the evolution of the y-process trajectories near y.

Next, using Corollary 1 from [2] we can show for $\alpha > 1 = d$ that the ξ-conditioned α-stable process exits D only through the point ξ, exactly as in the case $\alpha \leqslant d$. We define the *lifetime* of the α-stable y-Lévy motion to be ζ with $\zeta = \tau_{D\backslash\{y\}}$ for $\alpha \leqslant d$ and $\zeta = \tau_D$ when $\alpha > d = 1$. For the ξ-Lévy motion we always have $\zeta = \tau_D$.

Hence, for all $\alpha \in (0, 2)$ and $d \geqslant 1$ we have $P_y^x\{\lim_{t\uparrow\uparrow\zeta} X_t = y\} = 1$ for $x \neq y$, $x, y \in D$. Analogously, for every $\xi \in \partial D$ and $x \in D$ we have $P_\xi^x\{\lim_{t\uparrow\uparrow\tau_D} X_t = \xi\} = 1$.

The following result is very important in the sequel (see [12] and [3], Lemma 6 and Lemma 17 for justification). Let D be a bounded domain with the exterior cone property. Then the distribution of the pair $(X_{\tau_D -}, X_{\tau_D})$ with respect to P^x ($x \in D$) is concentrated on $D \times D^c$ with the density function $g^x(v, y)$ given by the following explicit formula:

$$g^x(v, y) = \frac{\mathscr{A}(d, -\alpha)}{|v-y|^{d+\alpha}} G_D(x, v), \qquad (v, y) \in D \times D^c. \tag{2.13}$$

Integrating (2.13) over D we obtain the density function

$$g^x(y) = \int_D \frac{\mathcal{A}(d, -\alpha) G_D(x, v)}{|v-y|^{d+\alpha}} dv, \quad y \in D^c, \tag{2.14}$$

of the *α-harmonic measure* $\omega_D^x(dy) = P^x\{X_{\tau_D} \in dy\}$ of the set D.

By (2.13) and routine arguments we obtain for $\Phi \geqslant 0$, measurable with respect to $\mathcal{F}_{\tau_D -}$, and any Borel $f \geqslant 0$, the following important formula:

$$E^x[f(X_{\tau_D})\Phi] = E^x[f(X_{\tau_D}) E^x_{X_{\tau_D -}}[\Phi]], \quad x \in D. \tag{2.15}$$

3. GREEN FUNCTION FOR THE BALL

The purpose of the present section is to provide some relevant estimates for the Green function for the ball given by (2.3) above. We put

$$I_d^\alpha(t) = \int_0^t \frac{r^{\alpha/2-1}}{(r+1)^{d/2}} dr, \quad t \geqslant 0.$$

The integrand is decreasing in r and

$$I_d^\alpha(kt) \leqslant k I_d^\alpha(t), \quad t \geqslant 0,\ k \geqslant 1. \tag{3.1}$$

LEMMA 3.1. *There is a constant $C_1 = C_1(d, \alpha)$ such that for all $t > 0$*

$$C_1^{-1} \leqslant I_d^\alpha(t)/[t^{\alpha/2} \wedge 1] \leqslant C_1 \qquad \text{if } \alpha < d,$$
$$C_1^{-1} \leqslant I_d^\alpha(t)/[t^{\alpha/2} \wedge t^{(\alpha-1)/2}] \leqslant C_1 \qquad \text{if } \alpha > d = 1$$

and for $\alpha = d = 1$ or 2

$$C_1^{-1} \leqslant I_d^\alpha(t)/\log(t^{\alpha/2}+1) \leqslant C_1,$$
$$C_1^{-1} \leqslant I_d^\alpha(t)/t^{\alpha/2} \leqslant C_1 \qquad \text{if } t \leqslant 2.$$

The proof of Lemma 3.1 is elementary and will be omitted. A calculation allows us for the choice of $C_1 = (2d/[\alpha(d-\alpha)]) \vee (2^{d/2-1}\alpha)$ if $\alpha < d$, $C_1 = 4/(\alpha-1)$ if $\alpha > d = 1$ and $C_1 = 8$ or 3 if $\alpha = d = 1$ or $\alpha = d = 2$, respectively.

We put $\delta(x) = \operatorname{dist}(x, B^c) = (1-|x|) \vee 0$, $x \in \mathbf{R}^d$. We clearly have

$$\frac{\delta(x)\delta(y)}{|x-y|^2} \leqslant w(x, y) \leqslant 4\frac{\delta(x)\delta(y)}{|x-y|^2}, \quad x, y \in B. \tag{3.2}$$

We also write

$$\varrho(x, y) = \frac{\delta(x)\delta(y)}{|x-y|^2}, \quad x, y \in B.$$

COROLLARY 3.2. *For all $x, y \in B$ $(x \neq y)$ we have*

$$C_1^{-1} \leqslant G(x, y)/(\mathcal{B}_\alpha^d |x-y|^{\alpha-d}[\varrho(x, y)^{\alpha/2} \wedge 1]) \leqslant 4C_1 \qquad \text{if } \alpha < d,$$

$$C_1^{-1} \leqslant G(x, y)/\left(\mathscr{B}_d^d \log\left(\varrho(x, y)^{\alpha/2}+1\right)\right) \leqslant 4C_1 \quad \text{if } \alpha = d = 1 \text{ or } 2,$$

$$C_1^{-1} \leqslant G(x, y)/\left(\mathscr{B}_\alpha^d |x-y|^{\alpha-d}\left[\varrho(x, y)^{\alpha/2} \wedge \varrho(x, y)^{(\alpha-1)/2}\right]\right) \leqslant 4C_1 \quad \text{if } \alpha > d = 1.$$

The above estimates for $\alpha < d$ and $\alpha = d = 2$ are well known (see [13], [9], and Lemma 6.19 of [10]) and we state them only for completeness.

The following result is an extension, for B, of the so-called 3G Theorem (see, e.g., [11], [5], [6], [7]). Although (3.4) with $\alpha = d = 2$ and (3.3) below are special cases of more general results formulated in those papers, we obtain the full proof of (3.4) and (3.3) as a by-product of estimates needed for (3.5) and (3.4) with $\alpha = d = 1$. Since the case of the ball is of primary interest (see the proof of Theorem 4.1 below), it deserves an independent elementary proof analogous to the proof of Proposition 5.15 in [10] and we give the arguments in detail.

Proposition 3.3. *There is $C_2 = C_2(d, \alpha)$ such that for all $x, y, z \in B$ we have*

(3.3)
$$\frac{G(x, y)G(y, z)}{G(x, z)} \leqslant C_2 \frac{|x-y|^{\alpha-d}|y-z|^{\alpha-d}}{|x-z|^{\alpha-d}} \qquad \text{if } \alpha < d,$$

(3.4)
$$\frac{G(x, y)G(y, z)}{G(x, z)} \leqslant C_2\left(\log\frac{4}{|x-y|}+\log\frac{4}{|y-z|}\right) \qquad \text{if } \alpha = d = 1 \text{ or } 2,$$

(3.5)
$$\frac{G(x, y)G(y, z)}{G(x, z)} \leqslant C_2\{\delta(y)[\delta(x) \vee \delta(y) \vee \delta(z)]\}^{(\alpha-1)/2} \qquad \text{if } \alpha > d = 1,$$

unless $x = y = z$ in (3.3) or (3.4).

Proof. Let $\alpha \in (0, 2]$, $d \in \mathbf{N}$, and $x, y, z \in B$. We may and do assume that $x \neq y$, $y \neq z$, $z \neq x$. By (2.3) we have

$$3G := \frac{G(x, y)G(y, z)}{G(x, z)} = \frac{|x-y|^{\alpha-d}|y-z|^{\alpha-d}}{|x-z|^{\alpha-d}} W,$$

where

$$W = \mathscr{B}_\alpha^d \frac{I_d^\alpha(w(x, y))\, I_d^\alpha(w(y, z))}{I_d^\alpha(w(x, z))}.$$

We reduce the number of variables by the following application of the Kelvin transform. For $v \in \mathbf{R}^d\setminus\{y\}$ we write $v^* = y-(1-|y|^2)|v-y|^{-2}(v-y)$. Note that for $v, v_1, v_2 \in \mathbf{R}^d\setminus\{y\}$ we have

(3.6) $$|v^*-y|\,|v-y| = 1-|y|^2,$$

(3.7) $$|v^*|^2-1 = (1-|v|^2)(1-|y|^2)/|v-y|^2,$$

(3.8) $$|v_1^*-v_2^*|\,|v_1-y| = |v_1-v_2|\,|v_2^*-y|;$$

see, e.g., Appendix in [15]. It follows that $w(x, y) = |x^*|^2-1$, $w(y, z) = |z^*|^2-1$ and $|x^*| > 1$, $|z^*| > 1$. By (3.7), (3.8), and (3.6) we obtain

$$w(x, z) = \frac{(1-|x|^2)(1-|z|^2)}{|x-z|^2} = \frac{(|x^*|^2-1)|x-y|^2(|z^*|^2-1)|z-y|^2}{(1-|y|^2)^2|x-z|^2} = \frac{(|x^*|^2-1)(|z^*|^2-1)}{|x^*-z^*|^2}.$$

Assume that $|x^*| \geqslant |z^*|$ and $|x^*| \geqslant \sqrt{2}$. Then $|x^*|^2-1 \geqslant |x^*|^2-|x^*|^2/2 = |x^*|^2/2$ and

$$w(x, z) \geqslant \frac{|x^*|^2(|z^*|^2-1)}{2|x^*-z^*|^2} \geqslant \frac{|x^*|^2(|z^*|^2-1)}{8|x^*|^2} = (|z^*|^2-1)/8.$$

This implies, by (3.1), that $I_d^\alpha(w(x, z)) \geqslant I_d^\alpha(w(y, z))/8$; hence $W \leqslant 8\mathscr{B}_\alpha^d I_d^\alpha(w(x, y))$ and

$$W \leqslant 8\mathscr{B}_\alpha^d [I_d^\alpha(w(x, y)) \vee I_d^\alpha(w(y, z))]. \tag{3.9}$$

By symmetry, (3.9) holds provided $|z^*| \geqslant |x^*|$ and $|z^*| \geqslant \sqrt{2}$.

We now assume that $|x^*| \leqslant \sqrt{2}$ and $|z^*| \leqslant \sqrt{2}$. Then

$$w(x, z) = \frac{(|x^*|^2-1)(|z^*|^2-1)}{|x^*-z^*|^2} \geqslant (|x^*|^2-1)(|z^*|^2-1)/8. \tag{3.10}$$

We obtain, by Lemma 3.1,

$$I_d^\alpha(w(x, z)) \geqslant C_1^{-1}[(|x^*|^2-1)(|z^*|^2-1)/8]^{\alpha/2} \geqslant C_1^{-3}8^{-\alpha/2} I_d^\alpha(w(x, y)) I_d^\alpha(w(y, z));$$

thus

$$W \leqslant 8^{\alpha/2}\mathscr{B}_\alpha^d C_1^3 \tag{3.11}$$

provided $|x^*| \leqslant \sqrt{2}$ and $|z^*| \leqslant \sqrt{2}$.

In particular, by (3.9), Lemma 3.1 and (3.11), the inequality (3.3) holds with $C_2 = 8\mathscr{B}_\alpha^d C_1^3$.

For $\alpha = d = 1$ or 2 we note that if $|x^*| \geqslant \sqrt{2}$ or $|z^*| \geqslant \sqrt{2}$, then $w(x, y) \geqslant 1$ or $w(y, z) \geqslant 1$ and, by (3.9) and Lemma 3.1, we have

$$3G = W \leqslant 8\mathscr{B}_d^d C_1[\log(2w(x, y)^{\alpha/2}) \vee \log(2w(y, z)^{\alpha/2})] \leqslant 8\mathscr{B}_d^d C_1\alpha\left[\left(\log\frac{2}{|x-y|}+\log\sqrt{(1-|x|^2)(1-|y|^2)}\right) \vee \left(\log\frac{2}{|y-z|}+\log\sqrt{(1-|z|^2)(1-|y|^2)}\right)\right]$$

$$\leqslant 8\mathscr{B}_d^d C_1 \alpha\left[\log\frac{2}{|x-y|}\vee\log\frac{2}{|y-z|}\right]$$
$$=\frac{4C_1}{\pi}\left[\log\frac{2}{|x-y|}\vee\log\frac{2}{|y-z|}\right].$$

By this and (3.11) we obtain

$$3G\leqslant\left\{\frac{4C_1}{\pi}\vee 8^{\alpha/2}\mathscr{B}_d^d C_1^3\right\}\left[\log\frac{2}{|x-y|}\vee\log\frac{2}{|y-z|}\vee 1\right]$$
$$\leqslant\frac{4C_1^3}{\pi}\left[\log\frac{4}{|x-y|}+\log\frac{4}{|y-z|}\right].$$

The proof of (3.4) is complete.

We now consider the case $\alpha > d = 1$. Since x, y, z are now real numbers, a simpler notation is possible, but we keep up the one used above.

Let $|x^*|\leqslant 2$ and $|z^*|\leqslant 2$. By (3.1) and Lemma 3.1 we get

$$W\leqslant\mathscr{B}_\alpha^1 4C_1^3\frac{(|x^*|^2-1)^{\alpha/2}(|z^*|^2-1)^{\alpha/2}}{\left[\frac{(|x^*|^2-1)(|z^*|^2-1)}{|x^*-z^*|^2}\right]^{\alpha/2}\wedge\left[\frac{(|x^*|^2-1)(|z^*|^2-1)}{|x^*-z^*|^2}\right]^{(\alpha-1)/2}}$$
$$=4\mathscr{B}_\alpha^1 C_1^3\left(|x^*-z^*|^\alpha\vee\{[(|x^*|^2-1)(|z^*|^2-1)]^{1/2}|x^*-z^*|^{\alpha-1}\}\right)$$
$$\leqslant 12\mathscr{B}_\alpha^1 C_1^3(|x^*-z^*|^\alpha\vee|x^*-z^*|^{\alpha-1})\leqslant 48\mathscr{B}_\alpha^1 C_1^3|x^*-z^*|^{\alpha-1}.$$

By (3.8) and (3.6) we obtain

$$|x^*-z^*|=\frac{|z^*-y|\,|x-z|}{|x-y|}=\frac{(1-|y|^2)\,|x-z|}{|x-y|\,|z-y|}; \tag{3.12}$$

hence

$$3G\leqslant 48\mathscr{B}_\alpha^1 C_1^3\left\{\frac{|x-y|\,|y-z|}{|x-z|}\frac{(1-|y|^2)\,|x-z|}{|x-y|\,|y-z|}\right\}^{\alpha-1} \tag{3.13}$$
$$=48\mathscr{B}_\alpha^1 C_1^3(1-|y|^2)^{\alpha-1}.$$

We now assume that $|x^*|\geqslant|z^*|\geqslant\sqrt{2}$. Recall that in this case $w(x, z)\geqslant(|z^*|^2-1)/8\geqslant 1/8$. By Lemma 3.1 we have

$$I_1^\alpha(w(x, z))\geqslant C_1^{-1}[w(x, z)^{\alpha/2}\wedge w(x, z)^{(\alpha-1)/2}]\geqslant C_1^{-1}8^{-1/2}w(x, z)^{(\alpha-1)/2}.$$

Using this, Lemma 3.1 and (3.12) we obtain

$$3G\leqslant 2\sqrt{2}\mathscr{B}_\alpha^1 C_1^3\left[\frac{|x-y|\,|y-z|}{|x-z|}\right]^{\alpha-1}\left[\frac{(|x^*|^2-1)(|z^*|^2-1)}{\frac{(|x^*|^2-1)(|z^*|^2-1)}{|x^*-z^*|^2}}\right]^{(\alpha-1)/2} \tag{3.14}$$

$$= 2\sqrt{2}\,\mathscr{B}_\alpha^1 C_1^3 \left[\frac{|x-y|\,|y-z|}{|x-z|}\right]^{\alpha-1} \left[\frac{(1-|y|^2)\,|x-z|}{|x-y|\,|y-z|}\right]^{\alpha-1}$$

$$= 2\sqrt{2}\,\mathscr{B}_\alpha^1 C_1^3 (1-|y|^2)^{\alpha-1}.$$

If $|z^*| \geqslant |x^*| \geqslant \sqrt{2}$, then we obtain the same conclusion.

We now assume that $|x^*| \geqslant 2$ and $|z^*| \leqslant \sqrt{2}$. Note that $|x^*-z^*| \leqslant 2|x^*|$ and $|x^*-z^*| \geqslant |x^*|-|z^*| \geqslant |x^*|(1-1/\sqrt{2})$; hence

$$w(x, z) = \frac{(|x^*|^2-1)(|z^*|^2-1)}{|x^*-z^*|^2} \leqslant \frac{|x^*|^2(|z^*|^2-1)}{(1-1/\sqrt{2})^2\,|x^*|^2}$$

$$= \frac{2}{(\sqrt{2}-1)^2}(|z^*|^2-1) \leqslant \frac{2}{(\sqrt{2}-1)^2},$$

$$w(x, z) \geqslant \frac{(|z^*|^2-1)\,3\,|x^*|^2/4}{4|x^*|^2} = \frac{3}{16}(|z^*|^2-1) \geqslant (|z^*|^2-1)/8.$$

These estimates and Lemma 3.1 yield

$$I_1^\alpha(w(x, z)) \geqslant I_1^\alpha(w(x, z)/[2/(\sqrt{2}-1)^2])$$

$$\geqslant C_1^{-1}(w(x, z)/[2/(\sqrt{2}-1)^2])^{\alpha/2}$$

$$\geqslant C_1^{-1}[(\sqrt{2}-1)^2/16]^{\alpha/2}(|z^*|^2-1)^{\alpha/2}.$$

By (3.12) and (3.6) we have

$$\frac{|y-z|}{|x-z|} = \frac{1-|y|^2}{|x-y|\,|x^*-z^*|} \leqslant \frac{1-|y|^2}{|x-y|\,|x^*|(1-1/\sqrt{2})}$$

$$\leqslant \frac{2}{1-1/\sqrt{2}}\,\frac{1-|y|^2}{|x-y|\,|x^*-y|} = 4+2\sqrt{2}.$$

We finally obtain

$$3G \leqslant \mathscr{B}_\alpha^1 \left[\frac{|x-y|\,|y-z|}{|x-z|}\right]^{\alpha-1} C_1^3 \frac{(|x^*|^2-1)^{(\alpha-1)/2}(|z^*|^2-1)^{\alpha/2}}{((\sqrt{2}-1)/4)^\alpha(|z^*|^2-1)^{\alpha/2}}$$

$$= (4\sqrt{2}+4)^\alpha\,\mathscr{B}_\alpha^1 C_1^3 [(1-|x|^2)(1-|y|^2)]^{(\alpha-1)/2} \left[\frac{|y-z|}{|x-z|}\right]^{\alpha-1}$$

$$\leqslant (4\sqrt{2}+4)^\alpha(4+2\sqrt{2})^{\alpha-1}\,\mathscr{B}_\alpha^1 C_1^3 [(1-|x|^2)(1-|y|^2)]^{(\alpha-1)/2}.$$

By analogy, if $|z^*| \geqslant 2$ and $|x^*| \leqslant \sqrt{2}$, then

$$3G \leqslant (4\sqrt{2}+4)^\alpha(4+2\sqrt{2})^{\alpha-1}\,\mathscr{B}_\alpha^1 C_1^3 [(1-|z|^2)(1-|y|^2)]^{(\alpha-1)/2}.$$

Combined with (3.13) and (3.14) the estimates yield (3.5) with

$$C_2 = \mathscr{B}_\alpha^1 C_1^3 [48 \vee 2\sqrt{2} \vee (4\sqrt{2}+4)^\alpha (4+2\sqrt{2})^{\alpha-1}].$$

The proof is complete. ∎

LEMMA 3.4. *Let $q \in \mathscr{J}^\alpha$ and $\varepsilon > 0$. There is $r_0 = r_0(q, \varepsilon, \alpha) > 0$ such that*

$$\int_B \frac{G_B(x, y) G_B(y, v)}{G_B(x, v)} |q(y)| \, dy \leqslant \varepsilon, \qquad x, v \in B, \tag{3.15}$$

for every ball $B \subset \mathbf{R}^d$ of radius $r \leqslant r_0$.

Proof. Let G be the Green function for the unit ball $B(0, 1) \subset \mathbf{R}^d$ (see (2.3)). Let $x_0 \in \mathbf{R}^d$, $r \in (0, r_0]$ and $B = B(x_0, r)$. By scaling we have

$$G_B(z, w) = r^{\alpha-d} G\big((z-x_0)/r, (w-x_0)/r\big), \qquad z, w \in B. \tag{3.16}$$

Let $v, x, y \in B$. To prove the lemma we consider three cases.

If $\alpha < d$, then by (3.16) and (3.3) we have

$$\begin{aligned} \frac{G_B(x, y) G_B(y, v)}{G_B(x, v)} &\leqslant C_2 r^{\alpha-d} \frac{|(x-y)/r|^{\alpha-d} |(y-v)/r|^{\alpha-d}}{|(x-v)/r|^{\alpha-d}} \\ &= C_2 \left[\frac{|x-v|}{|x-y| \, |y-v|}\right]^{d-\alpha} \leqslant C_2 \left[\frac{|x-y|+|y-v|}{|x-y| \, |y-v|}\right]^{d-\alpha} \\ &\leqslant 2^{d-\alpha} C_2 [|x-y|^{\alpha-d} + |y-v|^{\alpha-d}]. \end{aligned} \tag{3.17}$$

Note that the constant $2^{d-\alpha} C_2$ in (3.17) does not depend on r, v, x, y. By (2.5) there is $r_0 = r_0(q, \varepsilon, \alpha) > 0$ such that (3.15) holds if $0 < r \leqslant r_0$.

If $\alpha = d = 1$ or 2, then, by (3.16) and (3.4) we have

$$\begin{aligned} \frac{G_B(x, y) G_B(y, v)}{G_B(x, v)} &\leqslant C_2 [\log(4r/|x-y|) + \log(4r/|y-v|)] \\ &\leqslant C_2 [\log(1/|x-y|) + \log(1/|y-v|)] \end{aligned}$$

for every $r \leqslant 1/4$, and the result follows as above.

If $\alpha > d = 1$, then by (3.16) and (3.5) we simply have

$$\frac{G_B(x, y) G_B(y, v)}{G_B(x, v)} \leqslant C_2 r^{\alpha-1}.$$

Our result follows by (2.7). ∎

In what follows E_v^x denotes the expectation for the α-stable v-Lévy process conditioned by $G_B(\cdot, v)$, where B is a given ball in $\mathbf{R}^d$ and $x, v \in B$ (cf. **2.7** in Preliminaries). We also recall that $\zeta = \tau_B$ if $\alpha > d = 1$ and $\zeta = \tau_{B \setminus \{v\}}$ if $\alpha \leqslant d$.

The following technical result on the *conditional gauge function* $E_v^x e_q(\zeta)$, $x, v \in B$, which may be regarded as an analogue of (2.10), is a tool for studying local properties of q-harmonic functions (see, e.g., Theorem 4.1 below).

LEMMA 3.5. *Assume that $q \in \mathcal{J}^\alpha$ and $\varepsilon > 0$. Let $r_0 = r_0(q, \varepsilon, \alpha) > 0$ be the constant of Lemma 3.4. Then for every ball $B \subset \boldsymbol{R}^d$ of radius $r \leqslant r_0$ we have*

$$\exp(-\varepsilon) \leqslant E_v^x e_q(\zeta) \leqslant (1-\varepsilon)^{-1}, \quad x, v \in B. \tag{3.18}$$

Proof. We put $G = G_B$. We have

$$E_v^x\Big[\int_0^\zeta q(X_t)\,dt\Big] = G(x, v)^{-1} E^x\Big[\int_0^{\tau_B} q(X_t)\,G(X_t, v)\,dt\Big]$$
$$= \int_B \frac{G(x, y)\,G(y, v)}{G(x, v)} q(y)\,dy, \quad x, v \in B.$$

The result follows by Lemma 3.4 and (2.9). ■

Remark 3.6. If $B \subset \boldsymbol{R}^d$ is a *fixed* ball and $Q \in \mathcal{J}^\alpha_{\mathrm{loc}}$, then for $q = \delta Q$ with $\delta > 0$ the expression

$$\sup_{x,v \in B} \int_B \frac{G_B(x, y)\,G_B(y, v)}{G_B(x, v)} |q(y)|\,dy$$

can be made arbitrarily small provided δ is chosen small enough. If, say, the supremum is bounded by 1/2, then, by (2.9), it follows as above that the conditional gauge function $E_v^x e_q(\zeta)$ is bounded by 2 for $x, v \in B$.

Remark 3.7. We consider the general finite interval $D = (a, b) \subset \boldsymbol{R}^1$ and $q \in \mathcal{J}^\alpha(D)$. It can be proved that if (D, q) is gaugeable, then the conditional gauge function $u(x, v) = E_v^x e_q(\zeta)$, $x, v \in D$, is bounded away from zero and infinity. Furthermore, u has a jointly continuous symmetric extension $\bar{u}$ to $\bar{D} \times \bar{D}$ such that $\bar{u}(x, x) = 1$ for $x \in \bar{D}$. This Conditional Gauge Theorem (**CGT**) complements Theorem 4.10 in [6]. Its proof carries over from [6] with minor changes due to the different nature of the conditional processes for $\alpha \geqslant d = 1$ and is left for the interested reader. We note that we make no use of this result in our development. In fact, we focus in this paper on local results such as "small" **CGT** given in Lemma 3.5, which turn out to be sufficient to develop substantial potential theory.

4. GAUGEABILITY AND q-HARMONIC FUNCTIONS

Throughout this section we assume, unless stated otherwise, that $q \in \mathcal{J}^\alpha_{\mathrm{loc}}$. As usual, $\alpha \in (0, 2)$, $d \in \boldsymbol{N}$, X_t is the rotation invariant α-stable Lévy process in $\boldsymbol{R}^d$ and $D \subseteq \boldsymbol{R}^d$ is open.

Let $u \in \mathcal{B}(\boldsymbol{R}^d)$. We say that u is *q-harmonic* in an open set $D \subseteq \boldsymbol{R}^d$ if

$$u(x) = E^x[\tau_D < \infty;\ e_q(\tau_U)\,u(X_{\tau_U})], \quad x \in U,$$

for every bounded open set U with the closure $\bar{U}$ contained in D. It is called *regular q-harmonic* in D if the above equality holds for $U = D$ and *singular q-harmonic* in D if it is q-harmonic in D and $u(x) = 0$ for $x \in D^c$.

We always understand that the expectation in the above condition is absolutely convergent. For $q \equiv 0$ we obtain the previous definition of α-harmonicity. By the strong Markov property of X_t a regular q-harmonic function u is necessarily q-harmonic.

For $f \in \mathscr{B}(D^c)$ and $x \in \boldsymbol{R}^d$ we put

$$u_f(x) = E^x[\tau_D < \infty; e_q(\tau_D) f(X_{\tau_D})]$$

whenever the expectation is well defined, e.g., if $f \geqslant 0$ or $u_{|f|}(x) < \infty$.

THEOREM 4.1. *Let $f \in \mathscr{B}_+(D^c)$ and $K \subset D$ be compact. The following Harnack inequality holds*:

$$C^{-1} u_f(x) \leqslant u_f(y) \leqslant C u_f(x), \quad x, y \in K, \tag{4.1}$$

with $C = C(K, D, q, \alpha)$. If $u_f(x) = 0$ for some $x \in D$, then $u_f = 0$ on D and $u_f = 0$ a.e. on D^c.

Let $g \in \mathscr{B}(D^c)$. If $u_{|g|}(x) < \infty$ for some $x \in D$, then $u_{|g|}(x) < \infty$ for every $x \in D$ and u_g is continuous and regular q-harmonic in D.

Proof. Let $f \in \mathscr{B}_+(D^c)$, $u = u_f$. Note that $u = f$ on D^c and for $x \in \boldsymbol{R}^d$ we have $u(x) = E^x[\tau_D < \infty; e_q(\tau_D) f(X_{\tau_D})]$. Let $V \subseteq D$ be open. By the strong Markov property we obtain

$$\begin{aligned} E^x&[\tau_V < \infty; e_q(\tau_V) u(X_{\tau_V})] \\ &= E^x\big[\tau_V < \infty; e_q(\tau_V) E^{X_{\tau_V}}[\tau_D < \infty; e_q(\tau_D) f(X_{\tau_D})]\big] \\ &= E^x\big[\tau_V < \infty; e_q(\tau_V) E^x[(\mathbf{1}_{\{\tau_D < \infty\}} e_q(\tau_D) f(X_{\tau_D}))\circ\theta_{\tau_V} \mid \mathscr{F}_{\tau_V}]\big] \\ &= E^x\big[\tau_V < \infty; \tau_D\circ\theta_{\tau_V} < \infty; e_q(\tau_V)[e_q(\tau_D)\circ\theta_{\tau_V}] f(X_{\tau_D})\big] \\ &= E^x\big[\tau_D < \infty; e_q(\tau_D) f(X_{\tau_D})\big] = u(x), \quad x \in \boldsymbol{R}^d. \end{aligned}$$

Let $K \subset\subset D$ and $\delta_K = \operatorname{dist}(K, D^c)$. Put $F = \{x \in D: \operatorname{dist}(x, K) \leqslant \delta_K/2\}$. Clearly, F is a compact subset of D. In particular, $q\mathbf{1}_F \in \mathscr{J}^\alpha$. Let $\varrho_0 = r_0 \wedge (\delta_K/2)$, where $r_0 = r_0(q\mathbf{1}_F, \alpha)$ is the constant from Lemma 3.4, where, say, $\varepsilon = 1/2$. Let $x \in K$, $0 < r \leqslant \varrho_0$, and $B = B(x, r)$. Note that $B \subseteq F$. By (2.15), we have

$$u(y) = E^y[e_q(\tau_B) u(X_{\tau_B})] = E^y[u(X_{\tau_B}) E^y_{X_{\tau_B-}} e_q(\zeta)], \quad y \in B, \tag{4.2}$$

where $\zeta = \tau_B$ if $\alpha > d = 1$ and $\zeta = \tau_{B\setminus\{v\}}$ for $\alpha \leqslant d$ and $v = X_{\tau_B-}$ (see **2.7** in Preliminaries).

Lemma 3.5 and (4.2) yield

$$\tfrac{1}{2} E^y u(X_{\tau_B}) \leqslant u(y) \leqslant 2E^y u(X_{\tau_B}), \quad y \in B. \tag{4.3}$$

If $|y - x| < r/2$, then by (2.2) we have

$$\begin{aligned} E^x u(X_{\tau_B}) &= \int_{B^c} P_r(0, z-x) u(z)\, dz \\ &\leqslant \sup_{|z-x|>r} \frac{P_r(0, z-x)}{P_r(y-x, z-x)} \int_{B^c} P_r(y-x, z-x) u(z)\, dz \leqslant 3^{d+1} E^y u(X_{\tau_B}). \end{aligned}$$

Similarly,

$$E^x u(X_{\tau_B}) \geqslant 3^{-d-1} E^y u(X_{\tau_B}), \quad y \in B(x, r/2).$$

By (4.3) and the above we obtain

$$3^{-d-3} u(x) \leqslant u(y) \leqslant 3^{d+3} u(x), \quad y \in B(x, r/2). \tag{4.4}$$

In particular, we may take $r = \varrho_0$ in (4.3). We now consider $z \in K$ such that $|z-x| \geqslant \varrho_0/2$. Let $B_1 = B(z, \varrho_0/4)$. Note that $B_1 \subset F$ and $B_1 \cap B(x, \varrho_0/4) = \emptyset$. By (4.3) and by (4.4) with $r = \varrho_0$ we obtain

$$\begin{aligned} u(z) &\geqslant \tfrac{1}{2} E^z u(X_{\tau_{B_1}}) \geqslant \tfrac{1}{2} \int_{B(x,\varrho_0/4)} P_{\varrho_0/4}(0, y-z) u(y)\, dy \\ &\geqslant \tfrac{1}{2} C_\alpha^d [\varrho_0/4]^\alpha 3^{-d-3} u(x) \int_{B(x,\varrho_0/4)} |y-z|^{-d-\alpha} dy \\ &\geqslant 3^{-d-3} 2^{-1-2\alpha} C_\alpha^d \varrho_0^\alpha |B(0, \varrho_0)| (3|z-x|/2)^{-d-\alpha} u(x) \geqslant cu(x), \end{aligned}$$

with $c = 3^{-2d-3-\alpha} 2^{-d-1-\alpha} C_\alpha^d (\omega_d/d)(\varrho_0/\mathrm{diam}\, K)^{d+\alpha}$. Similarly, $u(x) \geqslant cu(z)$. By this and (4.4) with $r = \varrho_0$, the inequality (4.1) holds true.

We now assume that $x \in D$ and $u(x) = 0$. By the first part of the proof, for every $B = B(x, r)$ with $r > 0$ small enough we have

$$0 = u(x) \geqslant \tfrac{1}{2} E^x u(X_{\tau_B}) \geqslant \tfrac{1}{2} \int_{D^c} P_r(0, y-x) u(y)\, dy;$$

see (4.3). It follows that $u = f = 0$ a.e. on D^c. The pointwise equality $u = 0$ on D is a consequence of (4.1).

To prove the last assertion of the lemma let $g \in \mathscr{B}(D^c)$ and $u_{|g|}(x) < \infty$ for some $x \in D$. By considering u_{g_+} and u_{g_-}, where $g_+ = g \vee 0$ and $g_- = -(g \wedge 0)$, we may and do assume that $g \geqslant 0$ in what follows. Let $u = u_g$. By the Harnack inequality, u is locally bounded, hence finite in D, therefore it is regular q-harmonic in D by definition. To verify continuity of u on D, let $S = B(\xi, r)$ be such that $\bar{S} \subset D$. We have

$$u(x) = E^x u(X_{\tau_S}) + G_S(qu), \quad x \in S. \tag{4.5}$$

The proof of (4.5) is standard (see, e.g., Theorem 4.7 in [10] or Theorem 5.3 in [6]); we provide it only for reader's convenience. We let

$$\Phi(t) = \mathbf{1}_{\{t<\tau_S\}} q(X_t) u(X_{\tau_S}) \exp \int_t^{\tau_S} q(X_s)\, ds,$$

$$\Psi(t) = \mathbf{1}_{\{t<\tau_S\}} |q(X_t)| u(X_{\tau_S}) \exp \int_t^{\tau_S} q(X_s)\, ds, \quad t > 0.$$

We have

$$\int_0^\infty E^x[\Phi(t)]\,dt = E^x\Big[\int_0^{\tau_S} q(X_t)\,u(X_{\tau_S})\exp\int_t^{\tau_S} q(X_s)\,ds\,dt\Big]$$
$$= E^x\Big[\int_0^{\tau_S} q(X_t)\,E^{X_t}[e_q(\tau_S)\,u(X_{\tau_S})]\,dt\Big]$$
$$= E^x\Big[\int_0^{\tau_S} q(X_t)\,u(X_t)\,dt\Big] = G_S(qu)(x).$$

To justify the above application of Fubini's theorem we observe that $qu\mathbf{1}_S \in \mathscr{J}^\alpha$ because u is bounded on S. Therefore, by similar calculations as above,

$$\int_0^\infty E^x[\Psi(t)]\,dt = G_S(|q|\,u)(x) < \infty.$$

On the other hand, we observe that $\int_0^{\tau_S} |q(X_s)|\,ds < \infty$, and so the function

$$[0, \tau_S] \ni t \mapsto \exp\int_t^{\tau_S} q(X_s)\,ds \tag{4.6}$$

is absolutely continuous (a.s.). The derivative of the functions a.s. equals $-q(X_t)\exp\int_t^{\tau_S} q(X_s)\,ds$ a.e. Therefore

$$\int_0^\infty E^x[\Phi(t)]\,dt = E^x\Big[u(X_{\tau_S})\int_0^{\tau_S} q(X_t)\exp\int_t^{\tau_S} q(X_s)\,ds\,dt\Big]$$
$$= E^x[u(X_{\tau_S})\{e_q(\tau_S)-1\}] = u(x) - E^x u(X_{\tau_S}),$$

each term being finite. This proves (4.5). Recall that $qu\mathbf{1}_S \in \mathscr{J}^\alpha$ yields $G_S(qu) \in C_0(S)$, and that $E^x u(X_{\tau_S})$ is smooth in $x \in S$ because it is α-harmonic. By (4.5), u is continuous in S. The proof is complete. ∎

We note that if u is q-harmonic in open $D \neq \emptyset$, then

$$\int_{\mathbf{R}^d} \frac{|u(x)|\,dx}{(1+|x|)^{d+\alpha}} < \infty; \tag{4.7}$$

see, e.g., (4.3) in the proof above.

The next result is a very useful complement of the Gauge Theorem.

THEOREM 4.2. *Let $m(D) < \infty$, $q \in \mathscr{J}^\alpha$ and $f \in \mathscr{B}_+(D^c)$. If there is $x \in D$ such that $0 < u_f(x) < \infty$, then (D, q) is gaugeable.*

Proof. We fix $x_0 \in D$. Assume that $0 < u_f(x_0) < \infty$. Let $u_1(x) = E^x e_q(\tau_D)$, $x \in \mathbf{R}^d$. Our aim is to verify that $u_1(x_0) < \infty$.

By replacing f with $f \wedge 1$ we may and do restrict our considerations to the case $f \in L_+^\infty(D^c)$, $\|f\|_\infty \leqslant 1$. Then, clearly, $u_f \leqslant u_1$. However, f may equal zero on a large part of D^c and a reverse inequality $u_1(x_0) \leqslant c\,u_f(x_0)$, which we prove below, is by no means obvious.

There exist open bounded sets A, B, C such that $x_0 \in A$, $\bar{A} \subset B$, $\bar{B} \subset C$, $\bar{C} \subset D$, and the Lebesgue measure of $\Delta = D \backslash A$ is so small that

$$1/2 \leqslant E^x e_q(\tau_\Delta) \leqslant 2, \quad x \in \mathbf{R}^d; \tag{4.8}$$

see (2.9) and comments below it. We may also assume that B has the exterior cone property, e.g. $B = \{x \in D: \operatorname{dist}(x, D^c) > \delta\}$ for a suitable $\delta > 0$.

We define an auxiliary sequence of stopping times. Let $S_0 = 0$ and for $n = 0, 1, \ldots$

$$S_{2n+1} = \tau_B \circ \theta_{S_{2n}} + S_{2n},$$
$$S_{2n+2} = \tau_\Delta \circ \theta_{S_{2n+1}} + S_{2n+1} = \tau_\Delta \circ \theta_{\tau_B \circ \theta_{S_{2n}} + S_{2n}} + \tau_B \circ \theta_{S_{2n}} + S_{2n}.$$

We note that, e.g., $S_{2n} \leqslant \tau_D$ and $S_{2n} < \tau_D$ if and only if $X_{S_{2n}} \in \bar{A}$, $n = 0, 1, \ldots$ We put

$$u(x) = E^x[X_{\tau_\Delta} \in D^c;\ e_q(\tau_\Delta) f(X_{\tau_\Delta})], \quad x \in \mathbf{R}^d.$$

By the strong Markov property, (4.8) and the assumption $\|f\|_\infty \leqslant 1$ we obtain

$$\begin{aligned} u_f(x_0) &= \sum_{n=0}^{\infty} E^{x_0}[S_{2n} < \tau_D,\ S_{2n+2} = \tau_D;\ e_q(\tau_D) f(X_{\tau_D})] \\ &= \sum_{n=0}^{\infty} \big[E^{x_0}[S_{2n} < \tau_D,\ S_{2n+1} = \tau_D;\ e_q(\tau_D) f(X_{\tau_D})] \\ &\quad + E^{x_0}[S_{2n} < \tau_D,\ X_{S_{2n+1}} \in D \backslash C,\ S_{2n+2} = \tau_D;\ e_q(\tau_D) f(X_{\tau_D})] \\ &\quad + E^{x_0}[S_{2n} < \tau_D,\ X_{S_{2n+1}} \in C,\ S_{2n+2} = \tau_D;\ e_q(\tau_D) f(X_{\tau_D})]\big] \\ &= \sum_{n=0}^{\infty} \big[E^{x_0}[S_{2n} < \tau_D;\ e_q(S_{2n}) E^{X_{S_{2n}}}[X_{\tau_B} \in D^c;\ e_q(\tau_B) f(X_{\tau_B})]] \\ &\quad + E^{x_0}[S_{2n} < \tau_D;\ e_q(S_{2n}) E^{X_{S_{2n}}}[X_{\tau_B} \in D \backslash C;\ e_q(\tau_B) u(X_{\tau_B})]] \\ &\quad + E^{x_0}[S_{2n} < \tau_D;\ e_q(S_{2n}) E^{X_{S_{2n}}}[X_{\tau_B} \in C;\ e_q(\tau_B) u(X_{\tau_B})]]\big] \end{aligned} \tag{4.9}$$

$$\begin{aligned} &\leqslant \sum_{n=0}^{\infty} \big[E^{x_0}[S_{2n} < \tau_D;\ e_q(S_{2n}) E^{X_{S_{2n}}}[X_{\tau_B} \in C^c;\ 2e_q(\tau_B)]] \\ &\quad + E^{x_0}[S_{2n} < \tau_D;\ e_q(S_{2n}) E^{X_{S_{2n}}}[X_{\tau_B} \in C;\ e_q(\tau_B) u(X_{\tau_B})]]\big]. \end{aligned} \tag{4.10}$$

We write

$$v(x) = E^x[X_{\tau_\Delta} \in D^c; f(X_{\tau_\Delta})], \quad x \in \mathbf{R}^d,$$

and we claim that $v > 0$ on Δ. Indeed, if $f > 0$ on a set of positive Lebesgue measure in D^c and $x \in \Delta$, then the mean value property of v on a ball $B \subset \Delta$ centered at x, and the explicit form of the Poisson kernel for B yield $v(x) > 0$. If $f = 0$ a.e. on D^c, we proceed as follows. Let $w(y) = E^y f(X_{\tau_D})$, $y \in \mathbf{R}^d$. Observe that $w = f$ on D^c and since w is finite ($f \leqslant 1$), it is α-harmonic in D. Also $w > 0$

on D because $u_f > 0$ on D. If w attained a global maximum on D, say at y_0, we would obtain

$$0 = \Delta^{\alpha/2} w(y_0) = \mathscr{A}(d, -\alpha) \int_{\mathbf{R}^d} \frac{w(y)-w(y_0)}{|y-y_0|^{d+\alpha}} dy$$

$$= \mathscr{A}(d, -\alpha) \int_D \frac{w(y)-w(y_0)}{|y-y_0|^{d+\alpha}} dy - \mathscr{A}(d, -\alpha) \int_{D^c} \frac{w(y_0)}{|y-y_0|^{d+\alpha}} dy < 0,$$

which is a contradiction. It follows that

$$v(x) = E^x[X_{\tau_\Delta} \in D^c; w(X_{\tau_\Delta})] = E^x w(X_{\tau_\Delta}) - E^x[X_{\tau_\Delta} \in \bar{A}; w(X_{\tau_\Delta})]$$

$$= w(x) - E^x[X_{\tau_\Delta} \in \bar{A}; w(X_{\tau_\Delta})] \geqslant w(x) - \sup_{y \in \bar{A}} w(y) > 0,$$

provided $x \in \Delta$ is such that $w(x)$ is sufficiently close to $\sup_{y \in D} w(y)$. Since $e_q(\tau_D) > 0$ a.s., it follows that $u > 0$ on Δ. In particular, by Theorem 4.1, there is a constant $\kappa > 0$ such that $u(x) > \kappa$ for $x \in C \backslash B$.

Let $g \in \mathscr{B}_+(B^c)$ be bounded, $y \in B$, and $Z = E^y[e_q(\tau_B) g(X_{\tau_B})]$. We have

$$Z = \int_B \int_{B^c} E_v^y [e_q(\zeta)] G_B(y, v) \frac{\mathscr{A}(d, -\alpha)}{|v-z|^{d+\alpha}} g(z)\, dz\, dv,$$

where $\mathscr{A}(d, -\alpha)|z|^{-d-\alpha} dz$ is the Lévy measure of X_t, G_B is the Green function for B and E_v^y denotes the expectation with respect to the process conditioned by $G_B(\cdot, v)$; see (2.13), (2.14). As in **2.7** in Preliminaries, $\zeta = \tau_{B \backslash \{v\}}$ if $\alpha \leqslant d$ and $\zeta = \tau_B$ if $\alpha > d = 1$.

Assume that $g = 0$ on $C \backslash B$. Then there is a constant $c_1 > 0$ such that $|v-z| \geqslant c_1(1+|z|)$ a.e. on $B^c \times B$ with respect to $g(z)\, dz\, dv$; hence

$$Z \leqslant c_1^{-d-\alpha} \|g\|_\infty \int_B \int_{C^c} E_v^y [e_q(\zeta)] G_B(y, v) \frac{\mathscr{A}(d, -\alpha)}{(1+|z|)^{d+\alpha}} dz\, dv$$

$$\leqslant c_2 \|g\|_\infty \int_B \mathscr{A}(d, -\alpha) E_v^y [e_q(\zeta)] G_B(y, v)\, dv,$$

where $c_2 = c_1^{-d-\alpha} \int_{\mathbf{R}^d} (1+|z|)^{-d-\alpha} dz < \infty$. With $c_3 = (\operatorname{diam}(C))^{d+\alpha}/m(C \backslash B)$ we then have

$$Z \leqslant c_2 c_3 \|g\|_\infty \int_B \int_{C \backslash B} E_v^y [e_q(\zeta)] G_B(y, v) \frac{\mathscr{A}(d, -\alpha)}{|v-z|^{d+\alpha}} dz\, dv$$

$$\leqslant c_2 c_3 \|g\|_\infty E^y [X_{\tau_B} \in C; e_q(\tau_B)].$$

By this, (4.10), (4.8) and the definition of κ we obtain

$$u_f(x_0) \leqslant \sum_{n=0}^{\infty} E^{x_0} \left[S_{2n} < \tau_D; e_q(S_{2n}) E^{X_{S_{2n}}} [X_{\tau_B} \in C; e_q(\tau_B) u(X_{\tau_B})] \right] (1 + 2c_2 c_3 \kappa^{-1}).$$

Note that we have not yet used the fact that $u_f(x_0) < \infty$ to obtain the estimate and the same arguments apply to $u_1(x_0)$. Namely, there is $\eta > 0$ such that

$$\begin{aligned} u_1(x_0) &\leqslant \sum_{n=0}^{\infty} E^{x_0}\big[S_{2n} < \tau_D;\ e_q(S_{2n})\, E^{X_{S_{2n}}}\big[X_{\tau_B} \in C;\ e_q(\tau_B) \\ &\qquad \times E^{X_{\tau_B}}[X_{\tau_\Delta} \in D^c;\ e_q(\tau_\Delta)]\big]\big]\,(1+2c_2c_3\eta^{-1}) \\ &\leqslant \sum_{n=0}^{\infty} E^{x_0}\big[S_{2n} < \tau_D;\ e_q(S_{2n})\, E^{X_{S_{2n}}}[X_{\tau_B} \in C;\ e_q(\tau_B)\, u(X_{\tau_B})]\big] \\ &\qquad \times 2\kappa^{-1}(1+2c_2c_3\eta^{-1}) \\ &\leqslant 2\kappa^{-1}(1+2c_2c_3\eta^{-1})\, u_f(x_0) < \infty, \end{aligned}$$

where we used (4.8), the definition of κ, (4.9) and the finiteness of $u_f(x_0)$. The proof is complete. ∎

The multidimensional Brownian analogue of Theorem 4.2 also holds true for connected D but is not needed here. The proof is similar, only the set Δ above should be chosen connected, which is possible in dimension $d \geqslant 2$. For the one-dimensional Brownian result with a simpler proof, see [10], Theorem 9.9.

We recall that for (bounded open) gaugeable set $B \subset \mathbf{R}^d$, the gauge function $u(x) = E^x e_q(\tau_B)$ is q-harmonic in B. The following simple "converse" result clarifies the role of gaugeability for the existence of q-harmonic functions.

LEMMA 4.3. *Let $q \in \mathcal{J}^{\alpha}_{\mathrm{loc}}(D)$ and assume that $u \in \mathcal{B}(\mathbf{R}^d)$ is q-harmonic in D. Then, unless $u = 0$ on D and $u = 0$ a.e. on D^c, it follows that (B, q) is gaugeable for every open bounded set B such that $\overline{B} \subset D$.*

Proof. Let $x \in B$. Our assumptions and the definition of q-harmonicity yield

$$|u(x)| = |E^x[e_q(\tau_B)\, u(X_{\tau_B})]| \leqslant E^x[e_q(\tau_B)\, |u(X_{\tau_B})|] < \infty.$$

If $u(x) \neq 0$ or $u \neq 0$ on a subset of D^c of positive Lebesgue measure, then the second expectation above is positive. By Theorem 4.2, (B, q) is gaugeable in either case. ∎

For completeness we remark that if $u = 0$ on D, then, under the assumptions of Lemma 4.3, it follows that $u = 0$ a.e. on D^c. However, our exposition does not depend on this uniqueness property and it will be more convenient to prove it later on; see Remark 8.5 below.

LEMMA 4.4. *Let $q \in \mathcal{J}^{\alpha}_{\mathrm{loc}}(D_1 \cup D_2)$, where D_1, D_2 are open sets in $\mathbf{R}^d$. If $u \in \mathcal{B}_+(\mathbf{R}^d)$ is q-harmonic on D_1 and on D_2, then u is q-harmonic on $D_1 \cup D_2$.*

Proof. Let $D = D_1 \cup D_2$. We first assume that $u(x) = 0$ for some $x \in D_1$. By Theorem 4.1, $u = 0$ on D_1 and $u = 0$ a.e. on D_1^c. In particular, $u = 0$ a.e. on D_2. Using Theorem 4.1 once more we see that $u = 0$ on D_2, so that, finally,

$u = 0$ on $D_1 \cup D_2$ and $u = 0$ a.e. on $\mathbf{R}^d$. We obtain the same conclusion if $u(x) = 0$ for some $x \in D_2$. Such a function u is clearly q-harmonic in $D_1 \cup D_2$.

In what follows we may and do assume that $u(x) > 0$ for all $x \in D_1 \cup D_2$. Recall that by the definition of q-harmonic functions $u(x) < \infty$, $x \in D_1 \cup D_2$. To prove Lemma 4.4 we need to verify that for every open bounded U such that $\bar{U} \subset D$ and every $x \in U$ we have

$$u(x) = E^x[e_q(\tau_U) u(X_{\tau_U})]. \tag{4.11}$$

Let U_1, U_2 be open bounded sets such that $\bar{U}_1 \subset D_1$, $\bar{U}_2 \subset D_2$ and $U = U_1 \cup U_2$. By q-harmonicity we have

$$u(x) = E^x[e_q(\tau_{U_1}) u(X_{\tau_{U_1}})], \quad x \in U_1 \ (x \in \mathbf{R}^d) \tag{4.12}$$

and

$$u(x) = E^x[e_q(\tau_{U_2}) u(X_{\tau_{U_2}})], \quad x \in U_2 \ (x \in \mathbf{R}^d). \tag{4.13}$$

Let $x \in U$ be fixed. We define

$$\begin{aligned} T_0 &= 0, \\ T_{2n-1} &= T_{2n-2} + \tau_{U_1} \circ \theta_{T_{2n-2}}, \\ T_{2n} &= T_{2n-1} + \tau_{U_2} \circ \theta_{T_{2n-1}}, \quad n = 1, 2, \ldots \end{aligned}$$

We first show that

$$u(x) = E^x[e_q(T_m) u(X_{T_m})], \quad m \geqslant 0. \tag{4.14}$$

Clearly, (4.14) holds for $m = 0$ and 1. Suppose that for some $n \in N$ it follows that $u(x) = E^x[e_q(T_{2n-1}) u(X_{T_{2n-1}})]$. On the set $\{T_{2n-1} < \tau_U\}$ we have $X_{T_{2n-1}} \in U_2$. By (4.13) and the strong Markov property we have

$$\begin{aligned} u(x) &= E^x[T_{2n-1} = \tau_U;\ e_q(T_{2n-1}) u(X_{T_{2n-1}})] \\ &\quad + E^x[T_{2n-1} < \tau_U;\ e_q(T_{2n-1})\, E^{X_{T_{2n-1}}}[e_q(\tau_{U_2}) u(X_{\tau_{U_2}})]] \\ &= E^x[T_{2n-1} = \tau_U;\ e_q(T_{2n}) u(X_{T_{2n}})] + E^x[T_{2n-1} < \tau_U;\ e_q(T_{2n}) u(X_{T_{2n}})] \\ &= E^x[e_q(T_{2n}) u(X_{T_{2n}})]. \end{aligned}$$

Similarly, if for some $n \in N$

$$u(x) = E^x[e_q(T_{2n}) u(X_{T_{2n}})],$$

then

$$u(x) = E^x[e_q(T_{2n+1}) u(X_{T_{2n+1}})].$$

By induction, (4.14) holds true. By quasi-left-continuity, it follows P^x-a.s. that $T_m \to \tau_U$ and $X_{T_m} \to X_{\tau_U}$ as $m \to \infty$. By continuity of u on $\bar{U}$ we obtain P^x-a.s.

$$e_q(T_m) u(X_{T_m}) \to e_q(\tau_U) u(X_{\tau_U}) \quad \text{as } m \to \infty. \tag{4.15}$$

By (4.14) and Fatou's lemma,

$$u(x) \geqslant E^x[e_q(\tau_U)u(X_{\tau_U})]. \tag{4.16}$$

Since $u > 0$ on $D\backslash U$ and $P^x\{X_{\tau_U} \in D\backslash U\} > 0$, the right-hand side of (4.16) is positive. Since $u(x) < \infty$, by Theorem 4.2, (U, q) is gaugeable. To complete the proof we note that, for $m \in N$, P^x-a.s. we have

$$E^x[e_q(\tau_U) \mid \mathscr{F}_{T_m}] = e_q(T_m) E^{X_{T_m}} e_q(\tau_U);$$

hence $e_q(T_m) \leqslant cE^x[e_q(\tau_U) \mid \mathscr{F}_{T_m}]$, where $c = [\inf_{y \in \mathbf{R}^d} E^y e_q(\tau_U)]^{-1}$ is finite because U is Green-bounded. We put $\mathscr{P} = \{T_m < \tau_U,\ m = 1, 2, \ldots\}$ and $\mathscr{O} = \{T_m = \tau_U$ for some $m \in N\}$. Since u is bounded on U, by uniform integrability and (4.15) we obtain

$$\lim_{m\to\infty} E^x[T_m < \tau_U;\ e_q(T_m)u(X_{T_m})] = E^x[\mathscr{P};\ e_q(\tau_U)u(X_{\tau_U})]. \tag{4.17}$$

By the monotone convergence theorem we have

$$\lim_{m\to\infty} E^x[T_m = \tau_U;\ e_q(T_m)u(X_{T_m})] = E^x[\mathscr{O};\ e_q(\tau_U)u(X_{\tau_U})]. \tag{4.18}$$

By (4.14), (4.17) and (4.18), the equality (4.11) holds true. ∎

LEMMA 4.5. *Let D_1, D_2 be open sets in $\mathbf{R}^d$. Let $q \in \mathscr{J}^\alpha_{\mathrm{loc}}(D_1 \cup D_2)$ and let u be q-harmonic on D_1 and on D_2. If bounded open subsets of $D_1 \cup D_2$ are gaugeable, then u is q-harmonic on $D_1 \cup D_2$.*

Proof. Let $D = D_1 \cup D_2 \neq \emptyset$. Since u is continuous on D, it is locally bounded in D. Let U, V be open bounded and such that $\bar{U} \subset V$, $\bar{V} \subset D$. We also assume that U has the outer cone property. Let

$$c_1 = \sup_{x\in V} |u(x)| < \infty, \qquad c_2 = \int_{\mathbf{R}^d} |u(x)|/(1+|x|)^{d+\alpha}\,dx < \infty$$

(see (4.7)) and note that there is $c_3 < \infty$ such that

$$|y-v| \geqslant c_3(1+|y|) \quad \text{for } y \in V^c,\ v \in U.$$

For $v \in U$ we denote by ζ the lifetime of the v-Lévy motion on U (see Preliminaries). The following estimates are similar to those in the proof of Theorem 4.2. We have for $x \in U$

$$E^x[e_q(\tau_U)|u(X_{\tau_U})|]$$

$$= E^x[X_{\tau_U} \in V;\ e_q(\tau_U)|u(X_{\tau_U})|] + \int_{V^c}\int_U G_U(x, v)\frac{\mathscr{A}(d, -\alpha)}{|y-v|^{d+\alpha}} E^x_v[e_q(\zeta)]\,|u(y)|\,dv\,dy$$

$$\leqslant c_1 E^x e_q(\tau_U) + c_3^{-d-\alpha}\int_{V^c}\int_U G_U(x, v)\frac{\mathscr{A}(d, -\alpha)}{(1+|y|)^{d+\alpha}} E^x_v[e_q(\zeta)]\,|u(y)|\,dv\,dy$$

$$\leqslant c_1 E^x e_q(\tau_U) + c_2 c_3^{-d-\alpha} \int_U G_U(x, v) \mathscr{A}(d, -\alpha) E_v^x [e_q(\zeta)] \, dv$$

$$\leqslant c_1 E^x e_q(\tau_U) + \frac{c_2 c_3^{-d-\alpha} \operatorname{diam}(V)^{d+\alpha}}{m(V \setminus U)} \int_{V \setminus U} \int_U G_U(x, v) \frac{\mathscr{A}(d, -\alpha)}{|y-v|^{d+\alpha}} E_v^x [e_q(\zeta)] \, dv \, dy$$

$$= c_4 E^x e_q(\tau_U) < \infty,$$

where $c_4 = \left(c_1 + c_2 c_3^{-d-\alpha} \operatorname{diam}(V)^{d+\alpha}/m(V \setminus U)\right)$. Let

$$v(y) = u(y) + c_4 E^y e_q(\tau_U) + E^y [e_q(\tau_U) |u(X_{\tau_U})|], \quad y \in \boldsymbol{R}^d.$$

Note that $v \geqslant 0$ on $\boldsymbol{R}^d$ and v is q-harmonic on U_1 and U_2. By Lemma 4.4, v is q-harmonic on $U = U_1 \cup U_2$, therefore so is u. This completes the proof. ∎

5. GREEN POTENTIALS

Let $\alpha \in (0, 2)$, $d \in \boldsymbol{N}$, and $D \subseteq \boldsymbol{R}^d$ be open. As usual, by G_D we denote the Green operator for D and our symmetric α-stable process X_t:

$$G_D f(x) = E^x \int_0^{\tau_D} f(X_t) \, dt,$$

whenever it makes sense, e.g., if $f \in \mathscr{B}_+(\boldsymbol{R}^d)$ or the expectation is absolutely convergent (say $f \in \mathscr{B}(\boldsymbol{R}^d)$ is bounded and $G_D 1(x) = E^x \tau_D < \infty$).

The following simple result states an important global integrability property of the Green potentials $G_D f$. Recall that $\alpha < 2$.

LEMMA 5.1. *Assume that $f \in \mathscr{B}_+(\boldsymbol{R}^d)$ and $G_D f(x) < \infty$ for some $x \in D$. Then*

$$\int_{\boldsymbol{R}^d} \frac{G_D f(x)}{(1+|x|)^{d+\alpha}} \, dx < \infty. \tag{5.1}$$

Proof. Assume that $x_0 \in D$ is such that $G_D f(x_0) < \infty$. Let $B = B(x_0, r)$, where $0 < r < \operatorname{dist}(x_0, D^c)$. We have

$$\infty > G_D f(x_0) = E^{x_0} \int_0^{\tau_D} f(X_t) \, dt \geqslant E^{x_0} \int_{\tau_B}^{\tau_D} f(X_t) \, dt$$

$$= E^{x_0} E^{X_{\tau_B}} \int_0^{\tau_D} f(X_t) \, dt = \int_{B^c} G_D f(y) \, \omega_B^{x_0}(dy).$$

By (2.2) we easily conclude that $G_D f(y) < \infty$ a.e., $G_D f$ is locally integrable on $\boldsymbol{R}^d$ and, finally, that (5.1) holds. ∎

Remark 5.2. Clearly, $G_D \geqslant G_B$ if B is an open ball, $B \subseteq D$. Under the assumptions of Lemma 5.1, by the fact that for every $\varepsilon > 0$

$$\inf \{ G_B(x, y) \colon \operatorname{dist}(x, B^c) > \varepsilon, \operatorname{dist}(y, B^c) > \varepsilon \} > 0$$

(see (2.3)), we obtain $f \in L^1_{\text{loc}}(D)$.

Recall that for any function u satisfying (4.7) we can define $\tilde{\Delta}^{\alpha/2} u$ as the distribution given by $(\tilde{\Delta}^{\alpha/2} u, \phi) = (u, \Delta^{\alpha/2} \phi)$, $\phi \in C_c^\infty(\mathbf{R}^d)$; see [6] for a detailed exposition.

The following result is an extension of Proposition 3.13 in [6].

LEMMA 5.3. *Assume that* $f \in \mathscr{B}(\mathbf{R}^d)$ *and* $G_D|f|(x) < \infty$ *for some* $x \in D$. *Then*

$$\tilde{\Delta}^{\alpha/2} G_D f = -f \ (distr.) \text{ on } D. \tag{5.2}$$

Proof. Note that the right-hand side of (5.2) is a well-defined distribution on D by Remark 5.2. By Lemma 5.1 the same holds true for the left-hand side of (5.2).

It is enough to prove (5.2) for $f \in \mathscr{B}_+(\mathbf{R}^d)$. We assume that $f \in \mathscr{B}_+(\mathbf{R}^d)$ and $B \subseteq D$ is open and bounded. We have, for $x \in \mathbf{R}^d$,

$$G_D f(x) = G_B f(x) + E^x G_D f(X_{\tau_B}).$$

Since $G_D f(x) < \infty$ a.e., we see in particular that $E^x G_D f(X_{\tau_B})$ is regular α-harmonic in B (see Theorem 4.1). Since $\tilde{\Delta}^{\alpha/2}$ annihilates α-harmonic functions, to prove (5.2) we may and do assume that D is bounded and $f \in L^1(D)$ (see Remark 5.2). The validity of (5.2) in this case was proved in Proposition 3.13 of [6] for $d > \alpha$. In what follows we essentially repeat arguments given there but we treat all $d \in \mathbf{N}$, $\alpha \in (0, 2)$. As usual, K_α denotes the Riesz kernel (or compensated kernel if $\alpha \geqslant d = 1$) in $\mathbf{R}^d$. We have

$$G_D(x, y) = K_\alpha(x, y) - E^x K_\alpha(X_{\tau_D}, y), \quad x, y \in D; \tag{5.3}$$

see [2]. As usual, $G_D(x, y) = \int_0^\infty p_D(t, x, y)$ is the *Green function* of D defined by means of the transition densities for the process killed at τ_D. (Note that, if $\alpha > d = 1$, (5.3) may fail to hold for some *unbounded* sets D). Assume that f is bounded. For $x \in D$ we obtain

$$\begin{aligned} G_D f(x) &= \int_D G_D(x, y) f(y)\, dy \\ &= \int_D K_\alpha(x, y) f(y)\, dy - \int_D E^x K_\alpha(X_{\tau_D}, y) f(y)\, dy \\ &= \int_D K_\alpha(x, y) f(y)\, dy - E^x \Big[\int_D K_\alpha(X_{\tau_D}, y) f(y)\, dy\Big]. \end{aligned}$$

The application of (5.3) and Fubini's theorem are justified since the integrals are absolutely convergent.

The last term above is (finite) regular α-harmonic in D and, to prove (5.2), we only recall that for $\phi \in C_c^\infty(\mathbf{R}^d)$, $K_\alpha \Delta^{\alpha/2} \phi = -\phi$ (pointwise). This is well known and can be obtained by means of the Fourier transform; see also [15], Lemma 1.11. With this result we obtain, for $\phi \in C_c^\infty(D)$,

$$(\tilde{\Delta}^{\alpha/2} G_D f, \phi) = (\tilde{\Delta}^{\alpha/2} K_\alpha \mathbf{1}_D f, \phi)$$

$$= \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} K_\alpha(x, y) \mathbf{1}_D(y) f(y) \, dy \, \Delta^{\alpha/2} \phi(x) \, dx$$

$$= - \int_{\mathbf{R}^d} \mathbf{1}_D(y) f(y) \phi(y) \, dy = (-f, \phi),$$

where we use symmetry $K_\alpha(x, y) = K_\alpha(y, x)$. For general $f \in L^1(D)$, the result follows by an approximation argument. ■

We consider $s_D(x) = G_D \mathbf{1}(x) = E^x \tau_D$ for $x \in \mathbf{R}^d$. By the definition of τ_D, $s_D(x) = 0$ on D^c. For $B = B(x_0, r) \subset \mathbf{R}^d$, $|x - x_0| < r$ and $|y - x_0| > r$, by (2.2) and (2.14) we obtain

$$C_\alpha^d \left[\frac{r^2 - |x - x_0|^2}{|y - x_0|^2 - r^2} \right]^{\alpha/2} |x - y|^{-d} |y|^{d+\alpha} = P_r(x - x_0, y - x_0) |y|^{d+\alpha}$$

$$= \int_B \frac{\mathscr{A}(d, -\alpha) G_B(x, v)}{|v - y|^{d+\alpha}} |y|^{d+\alpha} \, dv.$$

By letting $|y| \to \infty$ we obtain

$$E^x \tau_B = \int_B G_B(x, v) \, dv = \frac{C_\alpha^d}{\mathscr{A}(d, -\alpha)} [r^2 - |x - x_0|^2]^{\alpha/2}, \qquad x \in B(x_0, r). \tag{5.4}$$

We have $C_\alpha^d / \mathscr{A}(d, -\alpha) = \Gamma(d/2) / [2^\alpha \Gamma((d+\alpha)/2) \Gamma(1 + \alpha/2)]$. For arbitrary open D and $B = B(x_0, r) \subset D$, the strong Markov property yields

$$s_D(x) = E^x \int_0^{\tau_B} \mathbf{1}_D(X_t) \, dt + E^x \int_{\tau_B}^{\tau_D} \mathbf{1}_D(X_t) \, dt$$

$$= \frac{C_\alpha^d}{\mathscr{A}(d, -\alpha)} [r^2 - |x - x_0|^2]^{\alpha/2} + E^x s_D(X_{\tau_{B(x_0, r)}}), \qquad x \in B.$$

It follows in particular that $s_D \in C^\infty(D)$ provided that $s_D(x) < \infty$ for some $x \in D$.

To investigate the behavior of s_D at ∂D we recall that $\tau'_D = \inf\{t > 0 : X_t \notin D\} \geqslant \tau_D$ and we define $s'_D(x) = E^x \tau'_D \geqslant s_D(x)$ ($x \in \mathbf{R}^d$). Recall that P^x-a.s. we have $\tau_D = \tau'_D$ (and so $s_D(x) = s'_D(x)$) except for those $x \in \partial D$ which are irregular for D, the set of such points x being a polar set for X_t. The following semicontinuity property of s'_D is remarkable because we essentially put no boundedness restrictions on D; its analogue for the Brownian motion is false in this generality.

LEMMA 5.4. *Let $0 < \alpha < 2$. If $s'_D(x) < \infty$ for some $x \in D$, then s'_D is upper semicontinuous in $\mathbf{R}^d$.*

Proof. Recall that polar sets are of Lebesgue measure zero. Thus, by Lemma 5.1,

$$\int_{\mathbf{R}^d} \frac{s'_D(x)}{(1+|x|)^{d+\alpha}} dx < \infty. \tag{5.5}$$

We now verify the upper semicontinuity of s'_D. Let $x \in \mathbf{R}^d$. We have P^x-a.s.

$$\tau'_D \circ \theta_t + t \downarrow \tau'_D \quad \text{as } t \downarrow 0. \tag{5.6}$$

For $t > 0$ let $p_t(x, y)$ be the density function of X_t under P^x. By scaling, $p_t(x, y) \doteq p_t(y-x) = t^{-d/\alpha} p_1(t^{-1/\alpha}(y-x))$. Furthermore, there is a constant $c = c(d, \alpha)$ such that

$$p_1(z) \leqslant c(1+|z|)^{-d-\alpha}, \quad z \in \mathbf{R}^d.$$

In consequence, for every bounded set $B \subset \mathbf{R}^d$ and $t > 0$ there is a constant $c_1 = c_1(B, \alpha, t)$ such that

$$p_t(x, y) \leqslant c_1(1+|y|)^{-d-\alpha}, \quad y \in \mathbf{R}^d,\ x \in B. \tag{5.7}$$

By (5.5), (5.7), the Markov property, (5.6) and the bounded convergence theorem we obtain for $t > 0$

$$\begin{aligned}\infty > \int_{\mathbf{R}^d} p_t(x, y) s'_D(y)\,dy + t &= E^x E^{X_t} \tau'_D + t \\ &= E^x(\tau'_D \circ \theta_t + t) \downarrow E^x \tau'_D \quad \text{as } t \downarrow 0\ (x \in \mathbf{R}^d).\end{aligned}$$

We only need to prove that the function $x \mapsto \int_{\mathbf{R}^d} p_t(x, y) s'_D(y)$ is continuous on $\mathbf{R}^d$ for each $t > 0$. But $p_t(x, y)$ is continuous in x and we can use the bounded convergence theorem, (5.5) and (5.7) to complete the proof. ∎

By Lemma 5.4 and the fact that $s_D \leqslant s'_D$ we obtain the following result:

COROLLARY 5.5. *If $s_D(x) < \infty$ for some $x \in D$, then s is locally bounded in $\mathbf{R}^d$ and $\lim_{y \to x} s(y) = 0$ for every $x \in \partial D$ which is regular for D.*

6. SCHRÖDINGER OPERATOR

We first make the following simple comparison. Recall that $\tilde{\Delta}^{\alpha/2} f = 0$ (distr.) on D if and only if (after a modification on a subset of D of Lebesgue measure zero) f is α-harmonic in (open) $D \subseteq \mathbf{R}^d$, see [6]. Furthermore, let $q \in \mathcal{J}^\alpha_{\text{loc}}(D)$. Assume that u is q-harmonic in D. Then, by (4.5) in the proof of Theorem 4.1 and Lemma 5.3,

$$\tilde{\Delta}^{\alpha/2} u + qu = 0 \text{ (distr.) on } S$$

on every ball S such that $\bar{S} \subset D$; thus

$$\tilde{\Delta}^{\alpha/2} u + qu = 0 \text{ (distr.) on } D. \tag{6.1}$$

Whether or not the converse implication is true depends on q and D; we treat the problem in this section.

PROPOSITION 6.1. *Let $D \subseteq \mathbf{R}^d$ be open and $q \in \mathscr{J}^{\alpha}_{\text{loc}}(D)$. Assume that*

$$\tilde{\Delta}^{\alpha/2} u + qu = 0 \text{ (distr.) on } D. \tag{6.2}$$

Then for every bounded U with the exterior cone property such that $\bar{U} \subset D$ we have

$$u(x) = E^x u(X_{\tau_U}) + G_U(qu)(x) \text{ a.e.}, \tag{6.3}$$

and the right-hand side of (6.3) *is continuous on U.*

Proof. We essentially repeat the arguments of [6] (but see Remark 6.2 below).

The implicit assumptions for (6.2) are that u satisfies (4.7) and $qu \in L^1_{\text{loc}}(D)$. In particular, we have u, $qu \in L^1(U)$. We note that (6.2) is unaffected if u is changed on a set of Lebesgue measure zero. The observation, however, does not contradict (6.3): because the P^x-distribution of X_{τ_U}, i.e. the α-harmonic measure $\omega^x_U(\cdot)$, is absolutely continuous with respect to the Lebesgue measure on U^c ([3], Lemma 6), $E^x u(X_{\tau_U})$ is unaffected by such a change of u.

Let V be open, bounded and such that $\bar{V} \subset D$. We can define

$$h^V(x) = u(x) - G_V(qu)(x) \quad \text{for a.e. } x \in \mathbf{R}^d, \tag{6.4}$$

where the Green potential is absolutely convergent a.e. since $qu \in L^1(V)$. By Lemma 5.3 we have

$$\tilde{\Delta}^{\alpha/2} h^V = -qu + qu = 0 \text{ (distr.) on } V,$$

therefore, after a modification on a set of Lebesgue measure zero in V, h^V becomes α-harmonic in V ([6], Theorem 3.12). So a modified version of h^V will be denoted by $\tilde{h}^V$.

Let U be open, $\bar{U} \subset V$ and assume that U has the exterior cone property. For every $x \in \mathbf{R}^d$ for which $G_V(|qu|)(x) < \infty$, thus a.e., we have

$$G_V(qu)(x) = G_U(qu)(x) + E^x[G_V(qu)](X_{\tau_U}).$$

Similarly as before, we define a.e.

$$h^U(x) = u(x) - G_U(qu)(x),$$

and we have a.e.

$$h^U(x) - \tilde{h}^V(x) = G_V(qu)(x) - G_U(qu)(x) = E^x[G_V(qu)](X_{\tau_U}),$$

where, for $x \in U$, we used the above-mentioned result on $\omega_U^x(\cdot)$. By the same result, (6.4) and α-harmonicity of $\widetilde{h^V}$ we obtain a.e.

$$h^U(x) - \widetilde{h^V}(x) = E^x[u(X_{\tau_U}) - \widetilde{h^V}(X_{\tau_U})] = E^x u(X_{\tau_U}) - \widetilde{h^V}(x).$$

We thus have $h^U(x) = E^x u(X_{\tau_U})$ for a.e. $x \in U$, or

$$\widetilde{h^U}(x) = E^x u(X_{\tau_U}), \qquad x \in \boldsymbol{R}^d,$$

where $\widetilde{h^U}$ denotes the equivalent version of h^U which is α-harmonic in U. Therefore $\widetilde{h^U}$ is *regular* α-harmonic on U. Hence, we have

$$u(x) = E^x u(X_{\tau_U}) + G_U(qu)(x) \text{ a.e.} \tag{6.5}$$

In particular, let U be a ball $B = B(x_0, r)$ such that $\bar{B} \subset V$ and $0 < r < r_0$, where $r_0 = r_0(q\mathbf{1}_V, \alpha)$ is the constant from Lemma 3.4 with $\varepsilon = 1/2$ there. We define for $x \in \boldsymbol{R}^d$

$$h(x) = \widetilde{h^B}(x) = E^x u(X_{\tau_B}), \qquad H(x) = E^x |u(X_{\tau_B})|,$$
$$f(x) = E^x[e_q(\tau_B) u(X_{\tau_B})], \qquad F(x) = E^x[e_q(\tau_B) |u(X_{\tau_B})|].$$

The above functions are regular α-harmonic or regular q-harmonic on B, respectively, and, by Lemma 3.5 with $\varepsilon = 1/2$, we have

$$\tfrac{1}{2} H(x) \leqslant F(x) \leqslant 2H(x), \qquad x \in \boldsymbol{R}^d,$$

which justifies the absolute convergence of the expectation defining f. By [6], Theorem 5.3, for every $x \in \boldsymbol{R}^d$ we have

$$f(x) = E^x f(X_{\tau_B}) + G_B(qf)(x) = h(x) + G_B(qf)(x),$$

where the Green potential is absolutely convergent. By (6.5) we obtain

$$f(x) - u(x) = G_B\big(q(f-u)\big)(x) \text{ a.e.} \tag{6.6}$$

Since the Green potential is absolutely convergent a.e., by Lemma 5.1 and the Fubini–Tonelli theorem we have

$$\int_B G_B|q(f-u)|(x)\,dx = \int_B \big|q(x)\big(f(x) - u(x)\big)\big|\, s_B(x)\,dx < \infty, \tag{6.7}$$

where $s_B(x) = E^x \tau_B = \int_B G_B(x, y)\,dy$, $x \in \boldsymbol{R}^d$. Let $R(x) = f(x) - u(x)$, $x \in \boldsymbol{R}^d$. Using (6.6), symmetry of G_B and (3.15) we obtain

$$\int_B |q(x) R(x)|\, s_B(x)\,dx \leqslant \int_B |q(x)|\, G_B(|qR|)(x)\, s_B(x)\,dx$$
$$= \int_B \int_B \int_B |q(x)|\, G_B(x, y)\, |q(y) R(y)|\, G_B(x, v)\,dv\,dy\,dx$$

$$= \int_B \int_B |q(y) R(y)| G_B(v, y) \int_B \frac{G_B(v, x) G_B(x, y)}{G_B(v, y)} |q(x)| \, dx \, dv \, dy$$

$$\leqslant \tfrac{1}{2} \int_B |q(y) R(y)| s_B(y) \, dy.$$

By (6.7) it follows that

$$q(f-u) = 0 \text{ a.e.}; \tag{6.8}$$

thus $u = f$ a.e. by (6.6). In particular, by Theorem 4.1, u is essentially continuous in B, hence in D. Furthermore, u is locally essentially bounded on D, and consequently $qu \in \mathcal{J}^{\alpha}_{\text{loc}}(D)$, which yields the continuity of the right-hand side of (6.3). The proof is complete. ■

Remark 6.2. The proof of Proposition 6.1 is based on arguments reproduced in [6], Theorem 5.5, after [10], Theorem 5.21, but here we use the integrability condition (6.7) to obtain (6.8), rather than the condition

$$\int_B |q(x)(f(x) - u(x))| \, dx < \infty, \tag{6.9}$$

which was used tacitly in [6] and [10]. As the integrability of qf on B is in some doubt at the considered stage of proof, the present modification of the proof is necessary. A similar modification may be applied in the case of Theorem 5.21 in [10].

Remark 6.3. Under the assumptions and with the notation of Proposition 6.1 above, there is a function $\tilde{u}$ continuous on D, such that $\tilde{u} = u$ on D^c and $\tilde{u} = u$ a.e. on D. The function clearly satisfies $\tilde{\Delta}^{\alpha/2} \tilde{u} + q\tilde{u} = 0$ in the sense of distributions on D. By the proof of Proposition 6.1, for every $x \in D$ there is some positive $r < \operatorname{dist}(x, D^c)$ such that for $B = B(x, r)$ we have

$$\tilde{u}(y) = E^y [e_q(\tau_B) \tilde{u}(X_{\tau_B})], \qquad y \in \mathbf{R}^d,$$

i.e. $\tilde{u}$ is (regular) q-harmonic on B.

By Remark 6.3, Lemmas 4.4 and 4.5 and the usual compactness argument we obtain the following result:

THEOREM 6.4. *Let $D \subseteq \mathbf{R}^d$ be open and let $q \in \mathcal{J}^{\alpha}_{\text{loc}}(D)$. Assume that $u \in \mathcal{B}(\mathbf{R}^d)$ satisfies*

$$\tilde{\Delta}^{\alpha/2} u + qu = 0 \ (\textit{distr.}) \ \textit{on} \ D.$$

If u is nonnegative or open bounded subsets of D are gaugeable, then, after a modification on a subset of D of Lebesgue measure zero, u is q-harmonic in D.

Remark 6.5. A part of the above result was proved before by means of the Conditional Gauge Theorem in [6], Theorem 5.5. Note that for $q \leqslant 0$ the gaugeability assumption is always satisfied and, in particular, Theorem 6.4

extends the analogue of the Weyl lemma for $\tilde{\Delta}^{\alpha/2}$ given in [6], Theorem 3.12. Note, however, that the present extension employs substantially the above-mentioned result (for $q = 0$).

We are in a position to give an explicit example of gauge functions based on $s_D(x) = E^x \tau_D$ for a broad class of domains. Similar examples based on more general Green potentials are left for the reader.

PROPOSITION 6.6. *Assume that* $s_D(x) < \infty$ *for some* $x \in D$. *Let* $a > 0$ *and define*

$$u(x) = 1 + a s_D(x), \quad q(x) = [s_D(x) + 1/a]^{-1}, \quad x \in \mathbf{R}^d. \tag{6.10}$$

Then we have

$$E^x e_q(\tau_D) \leqslant u(x), \quad x \in \mathbf{R}^d. \tag{6.11}$$

If also D is Green-bounded, then

$$E^x e_q(\tau_D) = u(x), \quad x \in \mathbf{R}^d, \tag{6.12}$$

and, in particular,

$$1 = \inf_{x \in \mathbf{R}^d} E^x e_q(\tau_D) \leqslant \sup_{x \in \mathbf{R}^d} E^x e_q(\tau_D) < \infty. \tag{6.13}$$

Proof. Note that q is bounded, in particular $q \in \mathscr{J}^\alpha$. By Lemma 5.3

$$\tilde{\Delta}^{\alpha/2} u = -a = -a(1 + a s_D)/(1 + a s_D) = -uq \text{ (distr.) on } D.$$

Since $u \geqslant 0$, by Theorem 6.4, u is q-harmonic in D, i.e. for every open bounded set $B \subset \bar{B} \subset D$ we have $u(x) = E^x [e_q(\tau_B) u(X_{\tau_B})]$, $x \in B$. We consider an increasing sequence $\{B_n\}_{n=1}^\infty$ of such sets with $\bigcup_{n=1}^\infty B_n = D$. By Fatou's lemma

$$\begin{aligned} u(x) &= \lim_{n\to\infty} E^x [e_q(\tau_{B_n}) u(X_{\tau_{B_n}})] \\ &\geqslant E^x [\liminf_{n\to\infty} e_q(\tau_{B_n}) u(X_{\tau_{B_n}})], \quad x \in \mathbf{R}^d. \end{aligned} \tag{6.14}$$

Let $n \to \infty$. Clearly, $e_q(\tau_{B_n}) \to e_q(\tau_D)$ since $\tau_D < \infty$ (a.s.). Also, by quasi left-continuity, $X_{\tau_{B_n}} \to X_{\tau_D}$ a.s. By continuity of s_D at regular points of D^c (Corollary 5.5) we obtain $u(X_{\tau_{B_n}}) \to u(X_{\tau_D})$ a.s. Thus $\infty > u(x) \geqslant E^x e_q(\tau_D)$, $x \in \mathbf{R}^d$, which is (6.11).

Assume that D is Green-bounded. By (6.11), the relation (6.13) holds for such D. To prove (6.12), note that

$$E^x [e_q(\tau_D) | \mathscr{F}_{\tau_{B_n}}] = e_q(\tau_{B_n}) E^{X_{\tau_{B_n}}} e_q(\tau_D) \geqslant e_q(\tau_{B_n}), \quad n \in \mathbf{N},\ x \in \mathbf{R}^d.$$

In particular, $\{e_q(\tau_{B_n})\}_{n \in \mathbf{N}}$ is uniformly P^x-integrable (every $x \in \mathbf{R}^d$). By boundedness of u, the same holds true for $\{e_q(\tau_{B_n}) u(X_{\tau_{B_n}})\}_{n \in \mathbf{N}}$. Invoking (6.14) we obtain (6.12). ∎

Example 6.7. By the above we see that, given $r > 0$, $B(0, r) \subset \mathbf{R}^d$ is gaugeable for $q(x) = [C_\alpha^d (r^2 - |x|^2)^{\alpha/2} / \mathscr{A}(d, -\alpha) + \varepsilon]^{-1}$, $x \in B(0, r)$, with any $\varepsilon > 0$. In particular, $(B(0, r), cr^{-\alpha})$ is gaugeable for every $c < \mathscr{A}(d, -\alpha)/C_\alpha^d$.

Example 6.8. We consider the following basic but less explicit example. Let $D \subset \mathbf{R}^d$ be bounded and regular, and define $q \equiv \lambda_0$, $u = \phi_0$, where $1/\lambda_0 > 0$ is the greatest eigenvalue of G_D (in, say, $L^2(\mathbf{R}^d)$), and $\phi_0 \in C_0(D)$ is the corresponding eigenvector, which is known to be positive by a choice (see, e.g., [14]). We thus have

$$\phi_0(x) = G_D(\lambda_0 \phi_0)(x), \qquad x \in \mathbf{R}^d.$$

By Theorem 6.4, ϕ_0 is q-harmonic in D. Clearly, it is not regular q-harmonic in D. Note that (D, q) is not gaugeable (see, e.g., the statement of the Gauge Theorem in [6]), however, by Theorem 4.2 it can be easily verified that (B, q) is gaugeable for every open set $B \subset D$ such that $D \backslash B$ is not polar.

7. FUNDAMENTAL EXPECTATION

In this section we investigate the behavior of the gauge function of the sets $D = (-\infty, y) \subseteq \mathbf{R}^1$. A parallel theory can be developed for sets (x, ∞) (cf. [10] for $\alpha = 2$). We always assume that $q \in \mathscr{J}_{\text{loc}}^\alpha$. Let $y \in \mathbf{R}^1$. Denote $\tau_{(-\infty, y)}$ by τ_y. We have $\tau_y < \infty$ P^x-a.s. Therefore, $\int_0^{\tau_y} q(X_s)\, ds$ is well-defined P^x-a.s.

We define the *fundamental expectation* as

$$u(x, y) = E^x e_q(\tau_y). \tag{7.1}$$

If $x \geqslant y$, then obviously $u(x, y) = 1$.

Suitable examples of functions $u(x, y)$ are provided at the end of the last section.

We begin our investigations with stating the following important consequence of Theorem 4.1:

THEOREM 7.1. *Assume that $q \in \mathscr{J}_{\text{loc}}^\alpha$. If $x, y \in \mathbf{R}^1$, then $u(x, y) > 0$. If $x < y$ and $u(x, y) < \infty$, then $u(w, y) < \infty$ for every $w < y$, and $u(\cdot, y)$ is a continuous regular q-harmonic function on $(-\infty, y)$. Moreover, for every w, v such that $w < v < y$ the following holds*:

$$u(x, y) = G_{(w,v)} q u(\cdot, y)(x) + E^x u(X_{\tau_{(w,v)}}, y).$$

Applying Theorem 7.1, we obtain

LEMMA 7.2. *For $x < y < z$ the following holds*:

$$\Big(1 \wedge \inf_{w \in [y,z)} u(w, z)\Big) u(x, y) \leqslant u(x, z) \leqslant \Big(1 \vee \sup_{w \in [y,z)} u(w, z)\Big) u(x, y). \tag{7.2}$$

Proof. By Theorem 7.1 we have

$$\begin{aligned} u(x, z) &= E^x[e_q(\tau_y)u(X_{\tau_y}, z)] \\ &= E^x[\tau_y < \tau_z;\, e_q(\tau_y)u(X_{\tau_y}, z)] + E^x[\tau_y = \tau_z;\, e_q(\tau_y)] \\ &\geqslant \inf_{w\in[y,z)} u(w, z)\, E^x[\tau_y < \tau_z;\, e_q(\tau_y)] + E^x[\tau_y = \tau_z;\, e_q(\tau_y)] \\ &\geqslant \big(1 \wedge \inf_{w\in[y,z)} u(w, z)\big)\, u(x, y). \end{aligned}$$

The proof of the right-hand side inequality is similar and is omitted. ■

LEMMA 7.3. *Let $y < z$ and let $y, z \to x_0$. Then*

$$\liminf \inf_{w\in[y,z)} u(w, z) \geqslant 1. \tag{7.3}$$

Proof. Using the assumption $q \in \mathscr{J}^{\alpha}_{\mathrm{loc}}$ and a conditional version of Khasminski's lemma, for a given $\varepsilon > 0$ we find $x < x_0$ such that for all z, $x < z < x_0 + (x_0 - x)$, the following holds (see Lemma 3.5):

$$e^{-\varepsilon} \leqslant \inf_{v,w\in(x,z)} E^w_v e_q(\zeta) \leqslant \sup_{v,w\in(x,z)} E^w_v e_q(\zeta) \leqslant (1-\varepsilon)^{-1}, \tag{7.4}$$

where E^w_v denotes the expectation with respect to the process conditioned by $G_{(x,y)}(\cdot, v)$ and, as usual, $\zeta = \tau_{(x,z)\setminus\{v\}}$ if $\alpha \leqslant 1$ and $\zeta = \tau_{(x,z)}$ otherwise.

This, q-harmonicity of $u(\cdot, z)$ and the formula (2.15) yield

$$\begin{aligned} u(w, z) &= E^w[e_q(\tau_{(x,z)})u(X_{\tau_{(x,z)}}, z)] \geqslant e^{-\varepsilon} E^w u(X_{\tau_{(x,z)}}, z) \\ &\geqslant e^{-\varepsilon} E^w[\tau_{(x,z)} = \tau_z;\, u(X_{\tau_{(x,z)}}, z)] = e^{-\varepsilon} P^w\{\tau_{(x,z)} = \tau_z\}. \end{aligned}$$

Let now $x < y \leqslant w < z$. The following direct calculation, using Corollary 1 from [2], gives

$$\begin{aligned} P^w\{\tau_{(x,z)} = \tau_z\} &= P^w\{X_{\tau_{(x,z)}} \geqslant z\} = (z-x)^{1-\alpha} \frac{\Gamma(\alpha)}{\Gamma(\alpha/2)^2} \int_x^w \frac{dv}{((v-x)(z-v))^{1-\alpha/2}} \\ &\geqslant (z-x)^{1-\alpha} \frac{\Gamma(\alpha)}{\Gamma(\alpha/2)^2} \int_x^y \frac{dv}{((v-x)(z-v))^{1-\alpha/2}} \\ &= \frac{\Gamma(\alpha)}{\Gamma(\alpha/2)^2} \int_0^{(y-x)/(z-x)} \frac{dv}{(v(1-v))^{1-\alpha/2}}. \end{aligned}$$

As $y, z \to x_0$, the last expression converges to

$$\frac{\Gamma(\alpha)}{\Gamma(\alpha/2)^2} \int_0^1 \frac{dv}{(v(1-v))^{1-\alpha/2}} = 1.$$

This completes the proof. ■

COROLLARY 7.4. *If x, y, z are such that $x < y < z$ and $u(x, z) < \infty$, then $u(x, y) < \infty$.*

Proof. By the above lemma together with Theorem 7.1 we obtain $\inf_{w\in[y,z)} u(w, z) > 0$, which, in turn, by virtue of Lemma 7.2, gives the conclusion of the corollary. ∎

Now, we prove that the set $\{y\in \boldsymbol{R}^1;\ u(x, y) < \infty \text{ if } x < y\}$ is either empty or equal to $(-\infty, a)$ (possibly with $a = \infty$).

THEOREM 7.5. *Let $q\in \mathscr{J}^{\alpha}_{\mathrm{loc}}$ and let $u(x, y) < \infty$ for a fixed $y\in \boldsymbol{R}^1$ and $x < y$. Then $u(x, z) < \infty$ whenever $z > y$ and z is close enough to y.*

Proof. Assume that for some $x < y$ we have $u(x, y) < \infty$. Let $z > y$ be fixed. We may and do assume that x and z are so close that (7.4) holds with $\varepsilon = 1/2$.

Define $S_1 = \tau_{(x,z)}$ and $S_2 = \tau_{(x,z)} + \tau_y \circ \theta_{\tau_{(x,z)}}$ if $X_{\tau_{(x,z)}} \leqslant x$ or $S_2 = \tau_{(x,z)}$ if $X_{\tau_{(x,z)}} \geqslant z$. Furthermore, for $n = 1, 2, \ldots$, we put inductively

$$S_{2n+1} = \begin{cases} S_{2n} + \tau_{(x,z)} \circ \theta_{S_{2n}} & \text{if } X_{S_{2n}} \in [y, z), \\ S_{2n} & \text{if } X_{S_{2n}} \notin [y, z); \end{cases}$$

$$S_{2n+2} = \begin{cases} S_{2n+1} + \tau_y \circ \theta_{S_{2n+1}} & \text{if } X_{S_{2n+1}} \leqslant x, \\ S_{2n+1} & \text{if } X_{S_{2n+1}} \geqslant z. \end{cases}$$

Next, for $n \geqslant 1$ we estimate the following expressions:

$$\begin{aligned} (7.5) \qquad & E^y[S_{2n} < \tau_z;\ e_q(S_{2n})] \\ & \quad = E^y\big[S_{2n} < \tau_z;\ e_q(S_{2n-1}) E^{X_{S_{2n-1}}}[\tau_y < \tau_z;\ e_q(\tau_y)]\big] \\ & \quad \leqslant E^y[S_{2n-1} < \tau_z;\ e_q(S_{2n-1}) u(X_{S_{2n-1}}, y)], \end{aligned}$$

$$\begin{aligned} (7.6) \qquad & E^y[S_{2n+1} < \tau_z;\ e_q(S_{2n+1}) u(X_{S_{2n+1}}, y)] \\ & = E^y\big[S_{2n} < \tau_z;\ e_q(S_{2n}) E^{X_{S_{2n}}}[\tau_{(x,z)} < \tau_z;\ e_q(\tau_{(x,z)}) u(X_{\tau_{(x,z)}}, y)]\big] \\ & \leqslant \sup_{u,w\in(x,z)} E^w_u[e_q(\zeta)] \sup_{w\in[y,z)} E^w[\tau_{(x,z)} < \tau_z; u(X_{\tau_{(x,z)}}, y)] E^y[S_{2n} < \tau_z;\ e_q(S_{2n})]. \end{aligned}$$

We now estimate the expression

$$E^w[\tau_{(x,z)} < \tau_z;\ u(X_{\tau_{(x,z)}}, y)],$$

applying the formula for the Poisson kernel for intervals. Here we have $x < y \leqslant w < z$. We obtain

$$\begin{aligned} & E^w[\tau_{(x,z)} < \tau_z;\ u(X_{\tau_{(x,z)}}, y)] \\ & \qquad = \frac{\sin(\alpha\pi/2)}{\pi} \int_{-\infty}^{x} \left(\frac{(z-w)(w-x)}{(z-v)(x-v)}\right)^{\alpha/2} \frac{u(v, y)\,dv}{w-v} \end{aligned}$$

$$\leqslant \frac{\sin(\alpha\pi/2)}{\pi} \int_{-\infty}^{x} \left(\frac{(z-y)(z-x)}{(y-v)(x-v)}\right)^{\alpha/2} \frac{u(v, y)\, 2\, dv}{x+y-2v}$$
$$= \left(\frac{(z-y)(z-x)}{(y-x)^2}\right)^{\alpha/2} 2^{\alpha} E^{(x+y)/2} [\tau_{(x,y)} < \tau_y;\, u(X_{\tau_{(x,y)}}, y)].$$

Now, applying (7.4) with $\varepsilon = 1/2$ and Theorem 7.1 we obtain

$$E^{(x+y)/2} [\tau_{(x,y)} < \tau_y;\, u(X_{\tau_{(x,y)}}, y)] \leqslant 2E^{(x+y)/2} [e_q(\tau_{(x,y)})\, u(X_{\tau_{(x,y)}}, y)]$$
$$= 2u((x+y)/2, y) < \infty.$$

Hence

$$\sup_{w \in [y,z)} E^w [\tau_{(x,z)} < \tau_z;\, u(X_{\tau_{(x,z)}}, y)] \tag{7.7}$$

converges to 0 as $z \downarrow y$.

Therefore we may and will assume that $z > y$ is such that (7.7) is less than 1/8.

Let $N = \min\{n \geqslant 1;\, X_{S_n} \geqslant z\}$. We obtain

$$\begin{aligned} E^y [N = 2n+1;\, e_q(\tau_z)] &= E^y [S_{2n} < \tau_z,\, S_{2n+1} = \tau_z;\, e_q(\tau_z)] \\ &= E^y [S_{2n} < \tau_z;\, e_q(S_{2n})\, E^{X_{S_{2n}}} [\tau_{(x,z)} = \tau_z;\, e_q(\tau_z)]] \\ &\leqslant E^y [S_{2n} < \tau_z;\, e_q(S_{2n})\, E^{X_{S_{2n}}} [e_q(\tau_{(x,z)})]] \\ &\leqslant \sup_{w \in [y,z)} E^w [e_q(\tau_{(x,z)})]\, E^y [S_{2n} < \tau_z;\, e_q(S_{2n})]. \end{aligned} \tag{7.8}$$

Analogously,

$$\begin{aligned} E^y [N = 2n+2;\, e_q(\tau_z)] &= E^y [S_{2n+1} < \tau_z,\, S_{2n+2} = \tau_z;\, e_q(\tau_z)] \\ &= E^y [S_{2n+1} < \tau_z;\, e_q(S_{2n+1})\, E^{X_{S_{2n+1}}} [\tau_z = \tau_y;\, e_q(\tau_z)]] \\ &\leqslant E^y [S_{2n+1} < \tau_z;\, e_q(S_{2n+1})\, u(X_{S_{2n+1}}, y)]. \end{aligned} \tag{7.9}$$

By the recurrence of X_t, $N < \infty$ a.s. By the estimates (7.8), (7.9), the choice of z in (7.7) and by the estimates (7.5) and (7.6) we finally obtain

$$u(y, z) = E^y e_q(\tau_z) = \sum_{n=1}^{\infty} E^y [N = n;\, e_q(\tau_z)]$$
$$= E^y [\tau_{(x,z)} = \tau_z;\, e_q(\tau_{(x,z)})] + \sum_{n=1}^{\infty} E^y [N = 2n+1;\, e_q(\tau_z)]$$
$$+ \sum_{n=1}^{\infty} E^y [N = 2n;\, e_q(\tau_z)] \leqslant 2 + \sum_{n=1}^{\infty} 2^{-n} + \sum_{n=1}^{\infty} 2^{-n} < \infty.$$

The proof is complete. ■

LEMMA 7.6. *Assume that $u(v, x_0) < \infty$ for fixed x_0 and $v < x_0$. If $y < z$ and $y, z \to x_0$, then*

$$\limsup \sup_{w \in [y,z)} u(w, z) \leqslant 1. \tag{7.10}$$

Proof. Given $\varepsilon > 0$, we choose, as in the proof of Lemma 7.3, $x < x_0$ such that for all z, $x < z < x_0 + (x_0 - x)$, (7.4) holds. By the proof of the above-mentioned lemma it follows that it is enough to show that if $x < y \leqslant w < z < x_0 + (x_0 - x)$, then

$$E^w[\tau_{(x,z)} < \tau_z;\ u(X_{\tau_{(x,z)}}, z)] \to 0$$

whenever $y, z \to x_0$, uniformly with respect to $w \in [y, z)$. Let us put $z' = x_0 + (x_0 - x)$. By virtue of Theorem 7.5 we may and do assume that z' is so close to x_0 that $u(v, z') < \infty$ for $v < z'$. We assume, further, that $(x + x_0)/2 < y \leqslant w < z < z'$. Then for $v \leqslant x$ we obtain

$$\frac{1}{z-v} \leqslant \frac{2}{x_0 - v} \quad \text{and} \quad \frac{1}{w-v} \leqslant \frac{2}{x + x_0 - 2v}.$$

By Lemma 7.2 we get

$$u(v, z) \leqslant \frac{u(v, z')}{\inf_{r \in [z,z')} u(r, z')} \leqslant \frac{u(v, z')}{\inf_{r \in (x,z')} u(r, z')} = C u(v, z')$$

for $v \leqslant x$. Then

$$\begin{aligned}
E^w&[\tau_{(x,z)} < \tau_z;\ u(X_{\tau_{(x,z)}}, z)] \\
&= \frac{\sin(\alpha\pi/2)}{\pi} \int_{-\infty}^{x} \left(\frac{(z-w)(w-x)}{(z-v)(x-v)}\right)^{\alpha/2} \frac{u(v, z)\,dv}{w-v} \\
&\leqslant 2^{\alpha/2} C \frac{\sin(\alpha\pi/2)}{\pi} \int_{-\infty}^{x} \left(\frac{(z-y)(z-x)}{(x_0-v)(x-v)}\right)^{\alpha/2} \frac{u(v, z')\,2\,dv}{x + x_0 - 2v} \\
&= 2^{3\alpha/2} C \left(\frac{(z-y)(z-x)}{(x_0-x)^2}\right)^{\alpha/2} E^{(x_0+x)/2}[\tau_{(x,x_0)} < \tau_{x_0};\ u(X_{\tau_{(x,x_0)}}, z')] \to 0
\end{aligned}$$

as $y, z \to x_0$. This completes the proof of the lemma. ∎

THEOREM 7.7. *Let $x_0 \leqslant y_0$ and $u(v, y_0) < \infty$ for $v < y_0$. Then the function $u(\cdot, \cdot)$ is continuous at (x_0, y_0).*

Proof. Let $u(v, y_0) < \infty$ for $v < y_0$ and $x, y \to x_0 \leqslant y_0$ with $x < y$. We consider three cases.

Case 1. Assume that $x_0 = y_0$. Then we have

$$\inf_{w \in [x,y)} u(w, y) \leqslant u(x, y) \leqslant \sup_{w \in [x,y)} u(w, y).$$

The application of Lemmas 7.3 and 7.6 ends the proof of this case.

Case 2. Assume that $y_0 < y$. By Lemma 7.2 we obtain

$$1 \wedge \inf_{w \in [y_0, y)} u(w, y) \leqslant \frac{u(x, y)}{u(x, y_0)} \leqslant 1 \vee \sup_{w \in [y_0, y)} u(w, y).$$

Again, the application of Lemmas 7.3 and 7.6 gives

$$\lim \frac{u(x, y)}{u(x, y_0)} = 1.$$

The continuity of $u(\cdot, y_0)$ at x_0 ends the proof of this case.

Case 3. In this case we assume that $y < y_0$. Then we obtain, as above,

$$1 \wedge \inf_{w \in [y, y_0)} u(w, y_0) \leqslant \frac{u(x, y_0)}{u(x, y)} \leqslant 1 \vee \sup_{w \in [y, y_0)} u(w, y_0).$$

The remaining arguments are the same as in the previous case and are omitted. ∎

8. KELVIN TRANSFORM

In this section we describe the action of the Kelvin transform on a q-harmonic function. We use the description to reduce some problems concerning q-harmonic functions on unbounded domains to the case of bounded domains.

As before, we fix $d \in \boldsymbol{N}$ and $\alpha \in (0, 2]$. In this section by the *Kelvin transform* we understand the mapping $T: \boldsymbol{R}^d \setminus \{0\} \mapsto \boldsymbol{R}^d \setminus \{0\}$ given by $Tx = x/|x|^2$. Note that $T^2 = \mathrm{id}_{\boldsymbol{R}^d \setminus \{0\}}$. For a function $f: \boldsymbol{R}^d \mapsto \boldsymbol{R} \cup \{-\infty, \infty\}$, the Kelvin transform Tf is defined by

$$(8.1) \qquad \begin{aligned} Tf(x) &= |x|^{\alpha-d} f(Tx) = |x|^{\alpha-d} f(x/|x|^2), \qquad x \neq 0, \\ Tf(0) &= 0, \end{aligned}$$

the latter being a rather useful convention, introduced here for convenience. If ν is a Radon measure on $\boldsymbol{R}^d \setminus \{0\}$, then we define a measure $\tilde{\nu}$ on $\boldsymbol{R}^d \setminus \{0\}$ by $\tilde{\nu}(dTx) = |x|^{\alpha-d} \nu(dx)$, which is to mean that for, e.g., $\phi \in C_c(\boldsymbol{R}^d \setminus \{0\})$

$$(8.2) \qquad \int_{\boldsymbol{R}^d \setminus \{0\}} \phi(y) \tilde{\nu}(dy) = \int_{\boldsymbol{R}^d \setminus \{0\}} \phi(Tx) |x|^{\alpha-d} \nu(dx) = \int_{\boldsymbol{R}^d \setminus \{0\}} T\phi(x) \nu(dx);$$

cf. [15]. Note that $\tilde{\tilde{\nu}} = \nu$. If $\nu(dx) = g(x)\,dx$, i.e. ν is absolutely continuous with respect to the Lebesgue measure, then by change of variables we have

$$(8.3) \qquad \tilde{\nu}(dx) = |x|^{-2\alpha} Tg(x)\,dx \quad \text{on } \boldsymbol{R}^d \setminus \{0\}.$$

LEMMA 8.1. *For every ball $B \subset \boldsymbol{R}^d$ such that* $\operatorname{dist}(0, B) > 0$ *and all $x, y \in \boldsymbol{R}^d$ we have*

$$(8.4) \qquad T_x T_y G_B(x, y) = G_{TB}(x, y).$$

The subscripts in (8.4) mark the variables with respect to which the Kelvin transform acts.

Proof of Lemma 8.1. Let $B = B(Q, r)$, where $Q \in R^d$ and $0 < r < |Q|$. Recall that $TB = B(S, \varrho)$, where

$$S = Q/(|Q|^2 - r^2), \qquad \varrho = r/(|Q|^2 - r^2). \tag{8.5}$$

By regularity of B and TB and by the convention (8.1) we may and do assume that $x, y \in TB \subset R^d \backslash \{0\}$. Then (8.4) is equivalent to

$$|x|^{\alpha-d} |y|^{\alpha-d} G_B(Tx, Ty) = G_{TB}(x, y). \tag{8.6}$$

To prove (8.6) we use (3.16) and (2.3):

$$\begin{aligned} LHS &= |x|^{\alpha-d} |y|^{\alpha-d} G_B(Tx, Ty) \\ &= |x|^{\alpha-d} |y|^{\alpha-d} r^{\alpha-d} G\big((Tx-Q)/r, (Ty-Q)/r\big) \\ &= |x|^{\alpha-d} |y|^{\alpha-d} |Tx-Ty|^{\alpha-d} \mathscr{B}^d_\alpha I^d_\alpha\big(w\big((Tx-Q)/r, (Ty-Q)/r\big)\big) \\ &= \mathscr{B}^d_\alpha |x-y|^{\alpha-d} I^d_\alpha\big(w\big((Tx-Q)/r, (Ty-Q)/r\big)\big), \end{aligned}$$

where we used the fact that

$$|Tx-Ty|\,|x|\,|y| = |x-y|. \tag{8.7}$$

We also have

$$\begin{aligned} RHS &= G_{TB}(x, y) = G_{B(S,\varrho)}(x, y) = \varrho^{\alpha-d} G\big((x-S)/\varrho, (y-S)/\varrho\big) \\ &= \mathscr{B}^d_\alpha |x-y|^{\alpha-d} I^d_\alpha\big(w\big((x-S)/\varrho, (y-S)/\varrho\big)\big), \end{aligned}$$

so we only need to verify that

$$W = w\big((Tx-Q)/r, (Ty-Q)/r\big)/w\big((x-S)/\varrho, (y-S)/\varrho\big) = 1.$$

We have by (8.7)

$$\begin{aligned} W &= \frac{1-(|Tx-Q|/r)^2}{1-(|x-S|/\varrho)^2}\,\frac{1-(|Ty-Q|/r)^2}{1-(|y-S|/\varrho)^2}\left(\frac{|x-y|}{|Tx-Ty|}\right)^2 \frac{r^2}{\varrho^2} \\ &= \frac{1-(|Tx-Q|/r)^2}{1-(|x-S|/\varrho)^2}|x|^2(|Q|^2-r^2)\,\frac{1-(|Ty-Q|/r)^2}{1-(|y-S|/\varrho)^2}|y|^2(|Q|^2-r^2) \\ &= J(x)\,J(y). \end{aligned}$$

By (8.5) and (8.7) we obtain

$$\begin{aligned} J(x) &= \frac{(r^2|x|^2 - |Tx-Q|^2|x|^2)(|Q|^2-r^2)}{r^2 - |x-S|^2(|Q|^2-r^2)^2} \\ &= \frac{(r^2|x|^2 - |x-TQ|^2|Q|^2)(|Q|^2-r^2)}{r^2 - [|x|^2 - 2(x, Q)/(|Q|^2-r^2) + |Q|^2/(|Q|^2-r^2)^2](|Q|^2-r^2)^2} \\ &= \frac{(r^2|x|^2 - [|x|^2 - 2(x, Q)/|Q|^2 + 1/|Q|^2]|Q|^2)(|Q|^2-r^2)}{r^2 - |x|^2(|Q|^2-r^2)^2 + 2(x, Q)(|Q|^2-r^2) - |Q|^2} \end{aligned}$$

$$(8.8)\qquad = \frac{(r^2|x|^2-|Q|^2|x|^2+2(x,Q)-1)(|Q|^2-r^2)}{(r^2-|Q|^2)|x|^2(|Q|^2-r^2)+2(x,Q)(|Q|^2-r^2)-(|Q|^2-r^2)}$$

$$(8.9)\qquad = 1.$$

Analogously, $J(y) = 1$, thus $W = 1$, and the proof is complete. ∎

The next result is obtained by a similar explicit calculation, so we omit the proof; the reader may also consult [4]. (The case $\alpha = 2$ may be obtained, e.g., by a limiting procedure, see [15].)

LEMMA 8.2. *For every ball $B \subset R^d$ such that* $\operatorname{dist}(0, B) > 0$ *we have for* $x \neq 0$

$$(8.10)\qquad \tilde{\omega}^x_{TB} = |x|^{\alpha-d}\,\omega^{Tx}_B = T\omega^{(x)}_B.$$

For clarity we note that, for $x \in (TB)^c\setminus\{0\}$, (8.10) is trivial; we then have $\tilde{\omega}^x_{TB} = \tilde{\delta}_x = |x|^{\alpha-d}\,\delta_{Tx}$ and $\omega^{Tx}_B = \delta_{Tx}$.

We remark that very general versions of Lemmas 8.1 and 8.2 in fact hold true but are not needed here. Proofs of the generalizations may be obtained from the authors.

The main result of the section is the following generalization of Lemma 8 in [4].

THEOREM 8.3. *Let $d \in N$, $\alpha \in (0, 2)$. Let $D \subseteq R^d$ be open and $q \in \mathcal{J}^{\alpha}_{\mathrm{loc}}(D)$. Let $u \in \mathcal{B}(R^d)$ be such that*

$$(\tilde{\Delta}^{\alpha/2} + q)\,u = 0 \ (distr.) \text{ on } D.$$

Then

$$(8.11)\qquad (\tilde{\Delta}^{\alpha/2} + \varrho)\,Tu = 0 \ (distr.) \text{ on } T(D\setminus\{0\})$$

with $\varrho(y) = |y|^{-2\alpha} q(y/|y|^2)$.

Proof. We may and do assume that u is continuous, see Remark 6.3. If $Q \in D\setminus\{0\}$ and $0 < r < \operatorname{dist}(Q, D^c \cup \{0\})$, then, by Proposition 6.1 and the continuity of u, we have for $B = B(Q, r)$

$$u(x) = E^x u(X_{\tau_B}) + G_B(qu)(x), \qquad x \in R^d.$$

Note that for $x \neq 0$ we have, by (8.2) and Lemma 8.2,

$$E^x\,Tu(X_{\tau_{TB}}) = \int_{R^d} Tu(y)\,\omega^x_{TB}(dy) = \int_{R^d} u(y)\,\tilde{\omega}^x_{TB}(dy)$$
$$= |x|^{\alpha-d}\int_{R^d} u(y)\,\omega^{Tx}_B(dy) = TE^{(x)}u(X_{\tau_B}).$$

By (8.2), (8.3) and Lemma 8.1 we then have for $x \neq 0$

$$T[G_B(qu)](x) = |x|^{\alpha-d} G_B(qu)(Tx) = |x|^{\alpha-d}\int_{R^d} G_B(Tx, y)\,q(y)\,u(y)\,dy$$
$$= |x|^{\alpha-d}\int_{R^d} G_B(Tx, Ty)\,|y|^{\alpha-d}|y|^{-2\alpha}|y|^{\alpha-d} q(Ty)\,u(Ty)\,dy$$
$$= \int_{TB} G_{TB}(x, y)\,|y|^{-2\alpha} q(Ty)\,Tu(y)\,dy = G_{TB}(\varrho Tu)(x),$$

the integrals being absolutely convergent. It follows that

$$Tu(x) = E^x\, Tu(X_{\tau_{TB}}) + G_{TB}(\varrho Tu)(x), \qquad x \neq 0.$$

Therefore, by Lemma 5.3,

$$\tilde{\Delta}^{\alpha/2}\, Tu + \varrho Tu = 0 \text{ (distr.) on } TB.$$

From this local result, (8.11) follows. ∎

The potential-theoretic counterpart of Theorem 8.3 is the following

THEOREM 8.4. *Let $d \in N$, $\alpha \in (0, 2)$. Let $D \subseteq R^d$ be open and $q \in \mathscr{J}^{\alpha}_{\text{loc}}(D)$. If $w \in \mathscr{B}(R^d)$ is q-harmonic in D, then Tw is ϱ-harmonic in $T(D\backslash\{0\})$, where $\varrho(y) = |y|^{-2\alpha} q(y/|y|^2)$.*

Proof. If $w = 0$ on D and $w = 0$ a.e. on D^c, then Tw is trivially ϱ-harmonic in $T(D\backslash\{0\})$.

Otherwise, let U be open bounded with $\bar{U} \subset D\backslash\{0\}$. By Lemma 4.3 the gauge function $u(x) = E^x e_q(\tau_U)$ is positive and finite for every $x \in R^d$. The same being true for Tu (except for $x = 0$), by Theorem 6.4 and Lemma 4.3 we see that (TV, ϱ) is gaugeable for every open V precompact in U. By Theorems 6.4 and 8.3, Tw is ϱ-harmonic in V, hence in D. ∎

Remark 8.5. If $w = 0$ on D, then under the assumptions of Theorem 8.4, it follows that $w = 0$ a.e. on D^c. Indeed, by (6.1), $\tilde{\Delta}^{\alpha/2} w = 0$ (distr.) on D, thus w is α-harmonic on D and vanishes in D. By the important uniqueness result of [8], $w = 0$ a.e. on D^c. We do not use this observation in our development and state it only for completeness.

The analogues of Theorems 8.3 and 8.4 also hold true for $\alpha = 2$ but a detailed verification is left for the interested reader.

The following example is an analogue of Example 4 in [10]. It illustrates Section 7 and indicates how to use the Kelvin transform to investigate q-harmonic functions on "large" domains.

EXAMPLE 8.6. Let $\alpha \in (0, 2)$, $D = (-1, 0) \subset R^1$ and let $\varrho(x) = c|x|^{-\beta}$, $x \in D$, where $\beta \in R$ and $c > 0$. Note that $\varrho \in \mathscr{J}^{\alpha}(D)$ if and only if $\beta < \alpha \wedge 1$, which we assume in what follows. For small $c > 0$ we have $\|G_D \varrho\|_\infty < 1$. Thus, by Khasminski's lemma there is $c_0 = c_0(\alpha, \beta, \gamma) > 0$ such that (D, ϱ) is gaugeable if $c < c_0$. For convenience we further assume that c_0 is so small that the conditional gauge function is bounded by 2; see Remark 3.6.

Let $f(x) = |x|^{\alpha-1}$, $x \in R^1$. If $\alpha \neq 1$, then, up to a constant, f is the potential kernel (compensated potential kernel if $\alpha > 1$) of X_t. Thus, regardless of $\alpha \in (0, 2)$, f is α-harmonic in $R^1\backslash\{0\}$. Clearly, it is not *regular* α-harmonic in $R^1\backslash\{0\}$. However, it follows that

(8.12) $$\int_{D^c} f(y)\, \omega_D^x(dy) = f(x), \qquad x \in D,$$

i.e. f is regular α-harmonic in D. The equality in (8.12) may be verified by considering intervals $D_n = (-1, -1/n)$, $n = 2, 3, \ldots$, applying (2.1) and (2.2) and letting $n \to \infty$.

Let $u(x) = E^x[e_\varrho(\tau_D) f(X_{\tau_D})]$, $x \in \mathbf{R}^1$. By our assumptions we have

$$u(x) = f(x),\ x \in D^c; \quad f(x) \leqslant u(x) \leqslant 2f(x),\ x \in D. \tag{8.13}$$

Clearly, u is regular ϱ-harmonic in D. By Lemma 8.4, $Tu(x) = |x|^{\alpha-1} u(1/x)$ is q-harmonic in $TD = (-\infty, -1)$ with $q(x) = c|x|^{\beta-2\alpha}$, $x \in TD$. Note that $q\mathbf{1}_{TD} \in \mathcal{J}^\alpha$ if and only if $\beta \leqslant 2\alpha$, which does not restrict our previous range of β. By (8.13) we have $Tu(x) = Tf(x) = |x|^{\alpha-1}|x|^{1-\alpha} = 1$, $x \in (-1, 0) \cup (0, \infty)$ and $1 \leqslant Tu(x) \leqslant 2$, $x \in (-\infty, -1)$. Let $B_n = (-n, -1-1/n)$, $n = 2, 3, \ldots$ Observe that by quasi-left-continuity of X_t it follows that, for every $x \in \mathbf{R}^1$, $P^x\{\tau_{B_n} = \tau_{TD}\} \to 1$ as $n \to \infty$, because, for $x \in D$, $P^x\{X_{\tau_{TD}} = -1\} = 0$ (see, e.g., [16]). By Fatou's lemma and bounded convergence we easily obtain

$$\begin{aligned} Tu(x) &= \lim_{n\to\infty} E^x[e_q(\tau_{B_n}) Tu(X_{\tau_{B_n}})] = E^x[\lim_{n\to\infty} e_q(\tau_{B_n}) Tu(X_{\tau_{B_n}})] \\ &= E^x e_q(\tau_{TD}), \qquad x \in TD. \end{aligned}$$

We see that the gauge function for $((-\infty, -1), c|x|^{\beta-2\alpha})$ is bounded on $\mathbf{R}^1$ provided $\beta < \alpha \wedge 1$ and $c > 0$ is small enough.

The critical rate $(\alpha \wedge 1) - 2\alpha$ of decay of q at infinity in Example 8.6 is not optimal (cf. [10], Example 9.4, for the case $\alpha = 2$). Better results require weakening of the defining conditions for the Kato class $\mathcal{J}^\alpha(D)$ by taking into account the asymptotics of G_D at ∂D. However, such an extension is beyond the scope of the paper.

REFERENCES

[1] R. M. Blumenthal and R. K. Getoor, *Markov Processes and Their Potential Theory*, Pure Appl. Math., Academic Press, New York 1968.

[2] R. M. Blumenthal, R. K. Getoor and D. B. Ray, *On the distribution of first hits for the symmetric stable processes*, Trans. Amer. Math. Soc. 99 (1961), pp. 540–554.

[3] K. Bogdan, *The boundary Harnack principle for the fractional Laplacian*, Studia Math. 123 (1) (1997), pp. 43–80.

[4] K. Bogdan, *Representation of α-harmonic functions in Lipschitz domains*, Hiroshima Math. J. 29 (1999), pp. 227–243.

[5] K. Bogdan, *Sharp estimates for the Green function in Lipschitz domains*, J. Math. Anal. Appl. 243 (2000), pp. 326–337.

[6] K. Bogdan and T. Byczkowski, *Potential theory for the α-stable Schrödinger operator on bounded Lipschitz domains*, Studia Math. 133 (1) (1999), pp. 53–92.

[7] Z.-Q. Chen, *Multidimensional symmetric stable processes*, Korean J. Comput. Appl. Math. 6 (2) (1999), pp. 227–266.

[8] Z.-Q. Chen and R. Song, *Martin boundary and integral representation for harmonic functions of symmetric stable processes*, J. Funct. Anal. 159 (1) (1998), pp. 267–294.

[9] K. L. Chung, *Green's function for a ball*, in: *Seminar on Stochastic Processes, 1986*, Progr. Probab. Statist. 13, Birkhäuser, Boston 1987, pp. 1–13.
[10] K. L. Chung and Z. Zhao, *From Brownian motion to Schrödinger's equation*, Springer, New York 1995.
[11] M. Cranston, E. Fabes and Z. Zhao, *Conditional gauge and potential theory for the Schrödinger operator*, Trans. Amer. Math. Soc. 307 (1) (1988), pp. 171–194.
[12] N. Ikeda and S. Watanabe, *On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes*, J. Math. Kyoto Univ. 2 (1) (1962), pp. 79–95.
[13] T. Kulczycki, *Properties of Green function of symmetric stable processes*, Probab. Math. Statist. 17 (2) (1997), pp. 339–364.
[14] T. Kulczycki, *Intrinsic ultracontractivity for symmetric stable processes*, Bull. Polish Acad. Sci. Math. 46 (3) (1998), pp. 325–334.
[15] N. S. Landkof, *Foundations of Modern Potential Theory*, Springer, New York 1972.
[16] D. B. Ray, *Stable processes with an absorbing barrier*, Trans. Amer. Math. Soc. 89 (1958), pp. 16–24.
[17] Z. Zhao, *A probabilistic principle and generalized Schrödinger perturbation*, J. Funct. Anal. 101 (1) (1991), pp. 162–176.

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