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VERTICES OF DEGREE ONE IN A RANDOM SPHERE OF INFLUENCE GRAPH

BY

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Abstract. The sphere of influence graph of the set of vertices in R^d is constructed by identifying the nearest neighbour of each vertex, centering a ball at each vertex so that its nearest neighbour lies on the boundary, and joining two vertices by an edge if and only if their balls intersect. We determine the expectation and variance of the number of vertices of degree one in the random sphere of influence graph.

1. Introduction. Let $N = (X, \|\cdot\|)$ be a d-dimensional normed vector space, $d \geqslant 2$. Denote by $B(\boldsymbol{a}, r)$ the open ball with center $\boldsymbol{a}$ and radius $r \geqslant 0$. The volume of this ball is γr^d, where $\gamma = \gamma(N)$ depends on the space. Let $A \subset X$ be a finite set of at least two points. For each point $\boldsymbol{a} \in A$ let $r(\boldsymbol{a})$ be the closest distance to any other point in the set, i.e., $B(\boldsymbol{a}, r(\boldsymbol{a}))$ is the largest empty ball centered at $\boldsymbol{a}$. The *sphere of influence graph* of A (written as SIG(A) or SIG) is the intersection graph $L(\{B(\boldsymbol{a}, r(\boldsymbol{a})):\ \boldsymbol{a} \in A\})$, i.e., its vertex set is A with $\boldsymbol{x}$ and $\boldsymbol{y}$ in A adjacent if and only if their open balls have nonempty intersection, which means that

$$r(\boldsymbol{x}) + r(\boldsymbol{y}) \geqslant \|\boldsymbol{x} - \boldsymbol{y}\|.$$

Sphere of influence graphs was first introduced in 1980's by G. Toussaint. It is known that on Euclidean plane SIG always has a vertex of degree at most 18 (Füredi and Loeb [3]). So such SIG on n vertices has at most $18n$ edges. It is conjectured that for the Euclidean plane SIG cannot have more than $9n$ edges.

Recent research has focused on finding the expectation and variance of the number of edges in a *random sphere of influence graph*. Let R be an open, bounded, convex region in the d-dimensional normed metric space N. Choose the points $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\} = A$ randomly and independently of R with even distribution. Form the corresponding sphere of influence graph and denote it by RSIG(A) or, shortly, RSIG.

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Let $E(n, E^d)$ denote the expected number of edges in RSIG on n vertices in Euclidean space. Dwyer [1] showed that

$$(0.32)\,2^d < \lim_{n\to\infty} \frac{E(n, E^d)}{n} < (0.72)\,2^d.$$

Fűredi [2] generalized and improved this result showing that

$$E(n, N) = C(d)\,n + o(n), \quad \text{where } \frac{\pi}{8}2^d < C(d) < \left(1+\frac{1}{2d}\right)\frac{\pi}{8}2^d.$$

In [4] Hitczenko et al. proved that variance of the number of edges in a random sphere of influence graph built on n vertices which are uniformly distributed over the unit cube in R^d grows linearly with n, i.e.

$$c_d\, n \leqslant \operatorname{Var}(e_n) \leqslant C_d\, n,$$

where c_d and C_d are absolute positive constants.

This paper concentrates on finding the expectation and variance of the number of vertices of degree one in a random sphere of influence graph. Let X_1 be the number of vertices of degree one in RSIG.

THEOREM 1. *Let X_1 be the number of vertices of degree one in RSIG. Then*

$$E(X_1) = n\exp\big(K(d, 1)-1\big), \tag{1}$$

$$\operatorname{Var}(X_1) = n^2\exp\big(\tfrac{1}{3}(4K(d, 2)-1)\big), \tag{2}$$

where

$$K(d, l) = -l^{-1/d}\left(1+\pi\frac{d-1}{d}\sum_{k=1}^{d-1}\binom{d-2}{k-1}\frac{l^{-k/d}}{\sin(k/d)\,\pi}\right). \ ■$$

2. Preliminaries. The following lemma will be helpful in next sections.

LEMMA 2. *For l, d, n integer and positive,*

$$I = \int_0^1\int_0^t \gamma d^2\exp(-\gamma l r^d n)\exp\big(-\gamma(t-r)^d n\big)\,\gamma(tr)^{d-1}\,dr\,dt$$
$$= l^{-1/d}\left(1+\pi\frac{d-1}{d}\sum_{k=1}^{d-1}\binom{d-2}{k-1}\frac{l^{-k/d}}{\sin(k/d)\,\pi}\right).$$

Proof. In I substitute first $\gamma l r^d n = ay$ and $\gamma(t-r)^d n = a$, i.e.,

$$r = (ay/\gamma l n)^{1/d} \quad\text{and}\quad t = (a/\gamma n)^{1/d}\big(1+(y/l)^{1/d}\big).$$

The Jacobian is equal to

$$(\gamma n)^{-2/d}\, l^{-1/d}\, d^{-2}\, a^{-1+2/d}\, y^{-1+y^{1/d}}.$$

Consequently, we obtain

(3) $$I = n^{-2}\, l^{-1/d} \int_0^\infty \int_0^\infty \exp(-ay)\exp(-a)\, a\left(1+(y/l)^{1/d}\right)^{d-1} da\, dy.$$

As the antiderivative of xe^{-cx} is $(-1/c)xe^{-cx}-(1/c^2)e^{-cx}$ we get

(4) $$\int_0^\infty x\exp(-xc) = \frac{1}{c^2}.$$

Applying (4) in (3) we obtain

$$\int_0^\infty \exp(-ay)\exp(-a)\, a da = \frac{1}{(1+y)^2}.$$

Thus we only have to calculate the integral

$$I = n^{-2}\, l^{-1/d} \int_0^\infty \frac{\left(1+(y/l)^{1/d}\right)^{d-1}}{(1+y)^2} dy.$$

Standard application of residue theory enables us to calculate

$$\int_0^\infty \frac{z^\alpha}{(1+z)^2} dz = \frac{\pi\alpha}{\sin \alpha\pi}, \qquad \text{where } |\alpha| < 1.$$

Thus we have

$$I = n^{-2}\, l^{-1/d}\left(1+\sum_{k=1}^{d-1} \binom{d-1}{k} \frac{l^{-k/d}\,\pi(k/d)}{\sin(k/d)\pi}\right).$$

Using the identity

$$\frac{k}{d}\binom{d-1}{k} = \frac{d-1}{d}\binom{d-2}{k-1}$$

we obtain

$$I = n^{-2}\, l^{-1/d}\left(1+\pi\frac{d-1}{d}\sum_{k=1}^{d-1} \binom{d-2}{k-1} \frac{l^{-k/d}}{\sin(k/d)\pi}\right). \blacksquare$$

3. The expected number of vertices of degree one. Consider the sphere of influence graph G generated by the set A. Points $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_i$ are such that:

$B(\boldsymbol{a}_1, \|\boldsymbol{a}_2-\boldsymbol{a}_1\|)$ contains only $\boldsymbol{a}_1$ from the elements of A, i.e., $\boldsymbol{a}_2$ is the nearest neighbour;

$B(\boldsymbol{a}_i, \|\boldsymbol{a}_i-\boldsymbol{a}_1\|-\|\boldsymbol{a}_2-\boldsymbol{a}_1\|)$ contains at least one from the points $A-\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_i\}$; in other words, there exists a vertex $\boldsymbol{a}_j$ such that $B(\boldsymbol{a}_i, \|\boldsymbol{a}_j-\boldsymbol{a}_i\|)$ is empty and its radius is at most $\|\boldsymbol{a}_i-\boldsymbol{a}_1\|-\|\boldsymbol{a}_2-\boldsymbol{a}_1\|$.

Such a configuration of three points guarantees that the point $\boldsymbol{a}_1$ will be either adjacent to $\boldsymbol{a}_2$ or to $\boldsymbol{a}_i$. We choose first adjacent points $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ such that the degree of vertex a_1 in RSIG(A) equals one. So the remaining $n-3$ points cannot be adjacent to a_1. The expected number of points of degree one in SIG(A) equals

$$E(X_1) = nP(A_1\overline{A_3})P(A_1\overline{A_4})\ldots P(A_1\overline{A_n}),$$

where $P(A_1\overline{A_i})$, $3 \leqslant i \leqslant n$, is the probability that a_1, a_2 and a_i form the above--mentioned ordered triple.

Let $r, s, t, \varepsilon > 0$ be reals such that

$$r < \|a_1-a_2\| < r+\varepsilon, \qquad t < \|a_1-a_i\| < t+\varepsilon, \qquad s < \|a_i-a_j\| < s+\varepsilon.$$

After fixing a_1 the probability of the distribution of a_1 and a_i is exactly

$$\frac{V\big((B(a_i, s+\varepsilon)-B(a_i, s))\cap R\big)}{V(R)} \times \frac{V\big((B(a_1, r+\varepsilon)-B(a_1, r))\cap R\big)}{V(R)}.$$

The probability that given balls do not intersect is exactly

$$\left(1-\frac{V(B_1\cap R)}{V(R)}\right)^{n-2} \times \left(1-\frac{V(B_2)\cap R}{V(R)}\right)^{n-2},$$

where B_1 is the ball $B(a_1, r)$, B_2 is the ball $B(a_i, s)$, and $V\big(B(a, r)\big)$ denotes volume of the ball with center A and radius r. The product of the above two probabilities is equal to

$$(\gamma dr^{d-1})(\gamma ds^{d-1})(1-\gamma r^d)^{n-2}(1-\gamma s^d)^{n-2}.$$

This term can be approximated as

$$(\gamma dr^{d-1})(\gamma ds^{d-1})\exp(-\gamma r^d n)\exp(-\gamma s^d n).$$

Assume that given vertices a_1 and a_i are at distance t. Then

$$\begin{aligned}
P\big(B(a_1, r)\cap B(a_i, s) = \varnothing \mid \|a_1-a_i\| = t\big)
&= \int_0^t\int_0^t n^2 \exp(-\gamma r^d n)\exp(-\gamma s^d n)\,\gamma dr^{d-1}\,\gamma ds^{d-1}\,ds\,dr \\
&= \int_0^t n\gamma dr^{d-1}\exp(-\gamma r^d n)\big(1-\exp(-\gamma(t-r)^d n)\big)\,dr \\
&= \int_0^t n\gamma dr^{d-1}\exp(-\gamma r^d n)\,dr - \int_0^t n\gamma dr^{d-1}\exp(-\gamma r^d n)\exp\big(-\gamma(t-r)^d n\big)\,dr \\
&= 1-\exp(-\gamma t^d n) - \int_0^t n\gamma dr^{d-1}\exp(-\gamma r^d n)\exp\big(-\gamma(t-r)^d n\big)\,dr \\
&= h(n, t, d).
\end{aligned}$$

Consider the probability $P(A_1\overline{A_i})$:

$$P(A_1\overline{A_i}) \leqslant \int_0^1 dt^{d-1}\, h(n, t, d)\, dt$$
$$= \int_0^1 \gamma dt^{d-1}\left(1-\exp(-\gamma t^d n)\right) dt - I_1 = 1-\frac{1}{n}-I_1,$$

where

$$I_1 = \int_0^1 \int_0^t n\gamma dr^{d-1}\, \gamma dt^{d-1} \exp(-\gamma r^d n) \exp(-\gamma (t-r)^d n)\, dr\, dt$$

and, by Lemma 2 with $l = 1$,

$$I_1 = n^{-1}\left(1+\pi \frac{d-1}{d} \sum_{k=1}^{d-1} \binom{d-2}{k-1} \frac{1}{\sin(k/d)\pi}\right).$$

As the sum of the binomial coefficients is exactly 2^{d-2}, we infer that I_1 is at least $(\pi/8)\, 2^{d-1}(1-1/d)$. With a little more careful calculations (using, e.g., the Taylor series of $1/\sin(\pi j/d)$ for $j \sim d/2$) we get

$$I_1 = n^{-1}\, 2^{d-1} \frac{\pi}{8}\left(1+\frac{(\pi^2/8)-1}{d}+O(1/d^2)\right).$$

Therefore, the expected number of vertices of degree one is

$$E(X_1) = n\left[1-n^{-1}\, 2^{d-1}\frac{\pi}{8}\left(1+\frac{(\pi^2/8)-1}{d}+O(1/d^2)\right)-n^{-1}\right]^{n-2}$$
$$= n \exp\left[-2^{d-1}\frac{\pi}{8}\left(1+\frac{(\pi^2/8)-1}{d}+O(1/d^2)\right)-1\right].$$

4. The variance of vertices of degree one. We now estimate the second factorial moment $E_2(X_1)$ of the number of vertices of degree one. The degree of two given vertices, say $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, is one. Let us consider the probability that there is no edge between $\boldsymbol{a}_1$ and any other vertex and no edge between $\boldsymbol{a}_2$ and any other vertex. Call this quantity v. Now, v will be separated into two parts, namely:

$$v = v_1 + v_2,$$

where v_1 means that $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are each other's nearest neighbours, and v_2 means that $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ have distinct nearest neighbours and their spheres do not intersect.

We first consider v_1. Then $\|\boldsymbol{a}_1-\boldsymbol{a}_2\| = r$ and any of the remaining vertices lies on the surface area of two intersecting balls $B(\boldsymbol{a}_1, t)$ and $B(\boldsymbol{a}_2, t)$.

Let $h_1(n, t, d)$ be the probability that there is no edge between $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ and any other vertex. We have

$$h_1(n, t, d) = \int_0^t \int_0^{t-r} n^2 \gamma d r^{d-1} \gamma d s^{d-1} \exp(-2\gamma r^d n) \exp(-\gamma s^d n)\, ds\, dr$$

$$= \int_0^t n\gamma d r^{d-1} \exp(-2\gamma r^d n)\, dr - \int_0^t n\gamma d r^{d-1} \exp(-2\gamma r^d n) \exp\left(-\gamma (t-r)^d n\right) dr$$

$$= \left(\tfrac{1}{2} - \tfrac{1}{2}\exp(-2\gamma t^d n)\right) - \int_0^t n\gamma d r^{d-1} \exp(-2\gamma r^d n) \exp\left(-\gamma (t-r)^d n\right) dr.$$

Let $P(A_1 A_2 \overline{A_i})$ be the probability that $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are each other's nearest neighbour and $\boldsymbol{a}_i$ is neither adjacent to $\boldsymbol{a}_1$ nor to $\boldsymbol{a}_2$. We have

$$P(A_1 A_2 \overline{A_i}) \leqslant \int_0^1 \tfrac{4}{3} d t^{d-1} h_1(n, t, d)\, dt$$

$$= \int_0^1 \left(\tfrac{2}{3} - \tfrac{2}{3}\exp(-2\gamma t^d n)\right) dt$$

$$- \int_0^1 \int_0^t \tfrac{4}{3} d\gamma t^{d-1} n\gamma d r^{d-1} \exp(-2\gamma r^d n) \exp\left(-\gamma (t-r)^d n\right) dr\, dt$$

$$= \frac{2}{3} - \frac{1}{3n} - I_2,$$

where

$$I_2 = \frac{4}{3} \int_0^1 \int_0^t n\gamma d r^{d-1} \gamma d t^{d-1} \exp(-2\gamma r^d n) \exp\left(-\gamma (t-r)^d n\right) dr\, dt.$$

By Lemma 2 with $l = 2$ we get

$$I_2 = n^{-1} 2^{-1/d} \left(1 + \pi \frac{d-1}{d} \sum_{k=1}^{d-1} \binom{d-2}{k-1} \frac{2^{-k/d}}{\sin(k/d)\pi}\right).$$

Thus we have

$$v_1 = n^2 \left(P(A_1 A_2 \overline{A_i})\right)^n$$

$$= n^2 \left[\frac{2}{3} - \frac{1}{3n}\left(1 + 2^{(2d-1)/d}\left(1 + \pi \frac{d-1}{d} \sum_{k=1}^{d-1} \binom{d-2}{k-1} \frac{2^{-k/d}}{\sin(k/d)\pi}\right)\right)\right]^n$$

$$= n^2 \exp\left[-\frac{1}{3}\left(1 + 2^{(2d-1)/d}\left(1 + \pi \frac{d-1}{d} \sum_{k=1}^{d-1} \binom{d-2}{k-1} \frac{2^{-k/d}}{\sin(k/d)\pi}\right)\right)\right].$$

Now let us look at v_2. Balls centered at $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ with radii r_1 and r_2, respectively, do not intersect either with each other or with any other balls.

Without loss of generality we may assume that $r_1 > r_2$. With $n \to \infty$ we have

$$v_2 = \int_0^1 \int_0^{t_1} \int_0^{t_1 - r_1} n^2 \exp(-\gamma r_1^d n) \exp(-\gamma s^d n)\, d\gamma r^{d-1} (\gamma d)^3 (st_1 r_1)^{d-1}\, ds dr_1 dt_1$$

$$\times \int_0^1 \int_0^{t_2} \int_0^{t_2 - r_2} n^2 \exp(-\gamma r_2^d n) \exp(-\gamma s^d n) (\gamma d)^3 (sr_2 t_2)^{d-1}\, ds dr_2 dt_2$$

$$= n^2 \exp\left[-2^d \frac{\pi}{8} \left(1 + \frac{(\pi^2/8) - 1}{d} + O(1/d^2) \right) - 2 \right].$$

Since

$$\operatorname{Var}(X) = E_2(X) - (E(X))^2$$

and in our case $E_2(X_1) = v_1 + v_2$, we infer finally that the variance of the number of vertices of degree one equals

$$\operatorname{Var}(X_1) = n^2 \exp\left[-\frac{1}{3} \left(1 + 2^{(2d-1)/d} \left(1 + \pi \frac{d-1}{d} \sum_{k=1}^{d-1} \binom{d-2}{k-1} \frac{2^{-k/d}}{\sin(k/d)\pi} \right) \right) \right],$$

which completes the proof.

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