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ON SEQUENTIAL ESTIMATION OF PARAMETERS OF CONTINUOUS GAUSSIAN MARKOV PROCESSES

BY

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Abstract. Assuming that the mean function of a continuous Gaussian Markov process y is of the form $m(t) = \theta\varphi(t)+\psi(t)$, we give admissible, minimax and minimum variance unbiased sequential plans for estimation of θ. For a parameter of the covariance function of y, parallel results are presented.

1. Introduction. Recently a number of authors have studied various estimators of parameters of stochastic processes and nonasymptotic optimal properties of such estimators. In particular, Arató [1] and Hajek [7] have investigated nonsequential minimum variance unbiased estimators for parameters of Gaussian processes. Novikov [18] has compared sequential and nonsequential methods of estimation for a shift parameter of a diffusion Gaussian process. Dvoretzky et al. [5] have shown that, for the Poisson process, the negative-binomial process, the gamma process and the Wiener process, fixed-time sequential plans are minimax if the weighted quadratic loss function is used. Magiera [14] has extended these results of Dvoretzky et al. to a class of processes which contains all the processes considered in [5].

In this paper we consider a continuous Gaussian Markov process $y = (y(t), t \geqslant 0)$ with mean $m(t)$ and covariance $K(t, s)$ and we assume that $m(t) = \theta\varphi(t)+\psi(t)$, where $\varphi(t)$ and $\psi(t)$ are known, while θ is unknown. If $K(t, s)$ is known, we consider the problem of sequential estimation of θ, and if

$$K(t, s) = \exp\left\{\int_s^t (\alpha p(u)+q(u))\,du\right\} K(s, s),$$

where $p(t)$ and $q(t)$ are known, we estimate α. Comparing the sequential plans, the usual quadratic loss function and the quadratic loss function plus the cost function will be used. Admissible, minimax and minimum variance unbiased sequential plans for estimation of θ and α will be given.

2. Absolute continuity of measures. Throughout the paper we assume that the derivatives $m'(t)$ and $K'(t,t)$ exist for all t $(0 \leqslant t < \infty)$. Moreover, we assume that

$$K_1(t) = \lim_{h \downarrow 0} \frac{K(t+h, t) - K(t,t)}{h}$$

exists for all t $(0 \leqslant t < \infty)$.

Let

$$A(t) = K_1(t) K^+(t,t), \quad B(t) = K'(t,t) - 2K_1(t),$$
$$a(t) = m'(t) - A(t)m(t),$$

where $K^+ = K^{-1}$ for $K \neq 0$ and $K^+ = 0$ for $K = 0$. Assume that

$$\int_0^t \big(|a(u)| + |A(u)| + B(u)\big)\, du < \infty$$

for all t $(0 \leqslant t < \infty)$. Let $\{F_t\}$ be the family of the σ-fields generated by random variables $\{y(s)\colon s \leqslant t\}$. Under the assumptions above there exists a Wiener process $w = \big(w(t), F_t\big)$ such that

$$y(t) = y(0) + \int_0^t \big(a(u) + A(u)y(u)\big)\, du + \int_0^t B^{1/2}(u)\, dw(u) \tag{2.1}$$

(see [17]). Consequently, the process y is a semimartingale with a Gaussian martingale component. This fact will be useful when considering the absolute continuity.

Let C be the space of all continuous functions $c\colon [0, \infty) \to \boldsymbol{R}$, where $\boldsymbol{R}$ is the set of real numbers, and let $\mathscr{B}$ denote the σ-field of Borel subsets of C relative to the topology of uniform convergence on compact subsets. Moreover, let C_t be the subspace of the space C of continuous functions which are constant on the interval (t, ∞) and let $\mathscr{B}_t = \mathscr{B} \cap C_t$.

A function $\tau\colon C \to [0, \infty]$ is said to be a *stopping time* if $\{c\colon \tau(c) \leqslant t\} \in \mathscr{B}_t$ for every $t \geqslant 0$.

Now we define a new Gaussian process

$$V^x(t) = x + \int_0^t B^{1/2}(u)\, dw(u), \quad x \in \boldsymbol{R}.$$

Let μ_y and μ_{Vx} be the measures induced by y and V^x, respectively. Moreover, let ν be a measure defined by

$$\nu(\cdot) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu_{Vx}(\cdot) \exp\left\{-\frac{1}{2}x^2\right\} dx$$

and let $\mu = \frac{1}{2}(\mu_y + \nu)$. Denote by $\mathscr{B}(\mu)$ the completion of $\mathscr{B}$ with respect to μ, and by $\mathscr{B}_t(\mu)$ the σ-field generated by $\mathscr{B}_t$ and all A from $\mathscr{B}(\mu)$ such that $\mu(A) = 0$. Finally, denote by $\mu_{\tau,y}$, $\mu_{\tau,Vx}$ and ν_τ the restrictions of μ_y, μ_{Vx} and ν to the σ-field $\mathscr{B}_\tau(\mu)$. With this notation we have $\mu_{\infty,y} = \mu_y$, $\nu_\infty = \nu$ and $\mathscr{B}_\infty(\mu) = \mathscr{B}(\mu)$. In the sequel, $\mu \ll \nu$, $\mu \sim \nu$ and $\mu \perp \nu$ mean that the measures μ and ν are absolutely continuous, mutually absolutely continuous (equivalent) and singular, respectively.

THEOREM 2.1. (i) *If*

$$\int_0^\infty \left[(m'(u))^2 + A^2(u)K(u,u)\right] B^+(u)\,du < \infty,$$

then $\mu_{\tau,y} \ll \nu_\tau$ *for every stopping time* τ.

(ii) *If for all* t $(0 \leqslant t < \infty)$

$$\int_0^t \left[(m'(u))^2 + A^2(u)K(u,u)\right] B^+(u)\,du < \infty$$

and

$$\int_0^\infty \left[(m'(u))^2 + A^2(u)K(u,u)\right] B^+(u)\,du = \infty,$$

then

$$\mu_y(\tau < \infty) = 1 \quad \textit{iff} \quad \mu_{\tau,y} \ll \nu_\tau$$

and

$$\mu_y(\tau = \infty) = 1 \quad \textit{iff} \quad \mu_{\tau,y} \perp \nu_\tau.$$

(iii) *If* $\mu_{\tau,y} \ll \nu_\tau$, *then the density function is given by*

$$\frac{d\mu_{\tau,y}}{d\nu_\tau}(y) = \frac{1}{\sqrt{K(0,0)}} \exp\left\{\frac{1}{2}\left[(y(0))^2 - \frac{(y(0)-m(0))^2}{K(0,0)}\right]\right\} \times$$

$$\times \exp\left\{\int_0^\tau (a(u)+A(u)y(u))B^+(u)\,dy(u) - \frac{1}{2}\int_0^\tau (a(u)+A(u)y(u))^2 B^+(u)\,du\right\}.$$

We give a proof of this theorem in the Appendix. The usefulness of Theorem 2.1 in our considerations is illustrated in the following

Example 1. Assume, in addition, that y is stationary. Then $m(t) = m$ and $K(t,s) = \sigma^2 \exp\{-\beta|t-s|\}$, where $\sigma^2 > 0$ and $\beta > 0$. In this particular case we have $a(t) = \beta m$, $A(t) = -\beta$, $B(t) = 2\sigma^2\beta$ and, consequently,

$$y(t) = y(0) + \int_0^t (\beta m - \beta y(u))\,du + \sqrt{2\sigma^2\beta}\, w(t),$$

where $w(t)$ is a Wiener process. Theorem 2.1 implies that $\mu_y(\tau < \infty) = 1$ iff $\mu_{\tau,y} \ll \nu_\tau$ and that

$$\frac{d\mu_{\tau,y}}{d\nu_\tau}(y) = \frac{1}{\sigma}\exp\left\{\frac{1}{2}\left[(y(0))^2 - \frac{(y(0)-m)^2}{\sigma^2}\right]\right\}\times$$
$$\times\exp\left\{\frac{1}{2\sigma^2\beta}\left[\int_0^\tau (\beta m - \beta y(u))\,dy(u) - \frac{1}{2}\int_0^\tau (\beta m - \beta y(u))^2\,du\right]\right\}$$

is the density function.

3. Estimation of θ. Recall that the mean function is of the form $m(t) = \theta\varphi(t) + \psi(t)$. We assume that the derivatives φ' and ψ' exist and that for $0 \leqslant t < \infty$

$$\int_0^t \left[(\varphi'(u))^2 + (\psi'(u))^2 + A^2(u)K(u,u)\right]B^+(u)\,du < \infty.$$

We consider the family $\{\mu_y^\theta\colon \theta\in\Theta \subset \boldsymbol{R}\}$ of Gaussian Markov measures with the mean function $m(t)$ and the covariance operator $K(t,s)$. For each $\theta\in\Theta$ let $\mu_{\tau,y}^\theta$ be the restriction of the measure μ_y^θ to the σ-field $\mathscr{B}_\tau$. The index θ indicates that the distribution μ_y^θ of y depends upon $\theta\in\Theta$, where Θ is an open interval on the real line.

Theorem 2.1 asserts that if $\mu_y^\theta(\tau < \infty) = 1$ for all $\theta\in\Theta$, then $\mu_{\tau,y}^\theta \ll \nu_\tau$ for all $\theta\in\Theta$ and the density function is given by

$$\frac{d\mu_{\tau,y}^\theta}{d\nu_\tau}(y) = S(\tau,y)\exp\left\{-\frac{1}{2}\theta^2 u(\tau) + \theta\lambda(\tau,y)\right\}, \tag{3.1}$$

where

$$u(\tau) = \varphi^2(0)K^+(0,0) + \int_0^\tau (\varphi'(u) - A(u)\varphi(u))^2 B^+(u)\,du, \tag{3.2}$$

$$\lambda(\tau,y) = (y(0)-\psi(0))\varphi(0)K^+(0,0) + \tag{3.3}$$
$$+\int_0^\tau (\varphi'(u) - A(u)\varphi(u))B^+(u)\left[dy(u) - (\psi'(u) + A(u)(y(u)-\psi(u)))\,du\right],$$

$$S(\tau,y) = \frac{1}{\sqrt{K(0,0)}}\exp\left\{\frac{1}{2}\left[(y(0))^2 - \frac{(y(0)-\psi(0))^2}{K(0,0)}\right]\right\}\times \tag{3.4}$$
$$\times\exp\left\{\int_0^\tau (\psi'(u) + A(u)(y(u)-\psi(u)))B^+(u)\,dy(u) - \right.$$
$$\left. -\frac{1}{2}\int_0^\tau (\psi'(u) + A(u)(y(u)-\psi(u)))^2 B^+(u)\,du\right\}.$$

Having an explicit formula for the density function we may use the maximum likelihood method to study sequential plans for estimation of θ.

Let τ be a stopping time with respect to $\{\mathscr{B}_t\}$. A function f: $[0, \infty] \times C \to \boldsymbol{R}$ is called an *estimator* of θ if $f(\tau(\cdot), \cdot)$ is $\mathscr{B}_\tau$-measurable for every τ. A pair $\delta = (\tau, f)$, where τ is a stopping time and f is an estimator, is called a *sequential plan.*

We restrict our considerations to a loss function $L(\theta, \delta) = (f-\theta)^2 + H(\tau)$, where H is a cost function. We assume that $H(t)$ is nonnegative, lower semicontinuous and such that

$$\lim_{t\to\infty} H(t) = \infty .$$

Let $\mathscr{D}$ denote the set of all sequential plans $\delta = (\tau, f)$ which have a finite risk function

$$R(\theta, \delta) = \mathrm{E}_\theta\, [(f-\theta)^2 + H(\tau)] \quad \text{for all } \theta \in \Theta, \tag{3.5}$$

where the expectation is taken with respect to μ_y^θ.

A sequential plan $\delta^* = (\tau^*, f^*)$ is said to be *minimax* if

$$\sup_{\theta\in\Theta} R(\theta, \delta^*) = \inf_{\delta\in\mathscr{D}} \sup_{\theta\in\Theta} R(\theta, \delta).$$

Suppose that a prior probability distribution $\Pi(\theta)d\theta$ of θ is given. The integral

$$r(\Pi, \delta) = \int_\Theta R(\theta, \delta)\Pi(\theta)\,d\theta$$

is called the *Bayesian risk* of δ, provided it exists.

A sequential plan $\delta^* = (\tau^*, f^*)$ is said to be *Bayes* with respect to Π if

$$r(\Pi, \delta^*) = \inf_{\delta\in\mathscr{D}} r(\Pi, \delta).$$

First we consider the case $\Theta = \boldsymbol{R}$. In view of (3.1) the maximum likelihood estimator of θ is given by

$$\hat{\theta}_\tau = \frac{\lambda(\tau, y)}{u(\tau)}.$$

In latter considerations we use the fact that $\hat{\theta}_\tau$ is a limit of Bayes estimators. To prove this we introduce a sequence of normal prior distributions with densities

$$\Pi_n(\theta) = \sqrt{\frac{u_n}{2\pi}} \exp\left\{-u_n \frac{\theta^2}{2}\right\}.$$

According to (3.1) the density function of the posterior probability distribution is given by

$$\Pi_n(\theta\,|\,y) = \frac{\exp\{-(\theta^2/2)(u(\tau)+u_n)+\theta\lambda(\tau, y)\}}{\int_{-\infty}^{\infty} \exp\{-(\theta^2/2)(u(\tau)+u_n)+\theta\lambda(\tau, y)\}\,d\theta}.$$

Using simple calculations we obtain

$$\Pi_n(\theta|y) = \sqrt{\frac{u(\tau)+u_n}{2\pi}}\exp\left\{-\frac{u(\tau)+u_n}{2}\left(\theta-\frac{\lambda(\tau, y)}{u(\tau)+u_n}\right)^2\right\}.$$

Since

$$r(\Pi_n, f\,|\,y) = \int_{-\infty}^{\infty} (f-\theta)^2\,\Pi_n(\theta\,|\,y)\,d\theta$$

attains its minimum value at

$$\hat{\theta}_\tau^n = \int_{-\infty}^{\infty} \theta\Pi_n(\theta\,|\,y)\,d\theta,$$

the Bayes estimator with respect to Π_n is given by

$$\hat{\theta}_\tau^n = \frac{\lambda(\tau, y)}{u(\tau)+u_n}.$$

Clearly, if $\lim_{n\to\infty} u_n = 0$, then

$$\lim_{n\to\infty} \hat{\theta}_\tau^n = \hat{\theta}_\tau.$$

A simple calculation shows that the posterior risk of the estimator $\hat{\theta}_\tau^n$ is equal to

$$r(\Pi_n, \hat{\theta}_\tau^n\,|\,y) = \frac{1}{u(\tau)+u_n}.$$

Now we proceed to sequential estimation of θ.

Since

$$\inf_{\delta\in\mathcal{S}} r(\Pi_n, \delta) = \inf_{\tau} \mathrm{E}_\theta\left(\frac{1}{u(\tau)+u_n}+H(\tau)\right),$$

the problem of finding the Bayes sequential plan reduces to the problem of minimizing

$$\mathrm{E}_\theta\left(\frac{1}{u(\tau)+u_n}+H(\tau)\right)$$

with respect to τ. It is clear that a fixed-time sequential plan $\delta_n = (T_n, \hat{\theta}^n_{T_n})$, where T_n is determined by

$$\frac{1}{u(T_n)+u_n}+H(T_n) = \inf_T\left(\frac{1}{u(T)+u_n}+H(T)\right),$$

is Bayes with respect to Π_n.

Now let $\delta_0 = (T_0, \theta_{T_0})$ be a fixed-time sequential plan with

$$\hat{\theta}_{T_0} = \frac{\lambda(T_0, y)}{u(T_0)}$$

and with T_0 determined by

$$\frac{1}{u(T_0)}+H(T_0) = \inf_T\left(\frac{1}{u(T)}+H(T)\right).$$

THEOREM 3.1. *The plan $\delta_0 = (T_0, \hat{\theta}_{T_0})$ is minimax. Moreover, $\hat{\theta}_{T_0}$ is normally distributed with mean value θ and variance $1/u(T_0)$.*

Proof. Using (2.1), (3.2) and (3.3) we get

$$\lambda(T_0, y)-\theta u(T_0) = \big(y(0)-\theta\varphi(0)-\psi(0)\big)\varphi(0)K^+(0,0)+ \\ + \int_0^{T_0} \big(\varphi'(u)-A(u)\varphi(u)\big)\big(B^{1/2}(u)\big)^+\,dw(u).$$

It is clear that

$$E_\theta\big(y(0)-\theta\varphi(0)-\psi(0)\big)\varphi(0)K^+(0,0)\int_0^{T_0}\big(\varphi'(u)-A(u)\varphi(u)\big)\big(B^{1/2}(u)\big)^+\,dw(u) = 0.$$

Thus the assertion concerning the distribution of $\hat{\theta}_{T_0}$ holds.

A simple calculation shows that

$$\sup_{\theta\in\Theta} R(\theta,\delta_0) = \inf_{\delta\in\mathcal{D}} r(\Pi_n,\delta)+\frac{1}{u(T_0)}+H(T_0)-\frac{1}{u(T_n)+u_n}-H(T_n)$$

$$\leqslant \inf_{\delta\in\mathcal{D}}\sup_{\theta\in\Theta} R(\theta,\delta)+\frac{1}{u(T_0)}+H(T_0)-\frac{1}{u(T_n)+u_n}-H(T_n).$$

Moreover, if $\lim_{n\to\infty} u_n = 0$, then

$$\lim_{n\to\infty}\left(\frac{1}{u(T_n)+u_n}+H(T_n)\right) = \frac{1}{u(T_0)}+H(T_0).$$

Hence

$$\sup_{\theta\in\Theta} R(\theta,\delta_0) \leqslant \inf_{\delta\in\mathcal{D}}\sup_{\theta\in\Theta} R(\theta,\delta),$$

and δ_0 is minimax.

Example 2. Let y be defined as in Example 1. The following results may be easily deduced from Theorems 2.1 and 3.1.

For all $\theta \in \Theta$ we have $\mu_y^\theta(\tau < \infty) = 1$ iff $\mu_{\tau,y} \ll \nu_\tau$.

The density function is given by (3.1), where

$$u(\tau) = \frac{2+\beta\tau}{2\sigma^2} \quad \text{and} \quad \lambda(\tau, y) = \frac{y(0)+y(\tau)+\beta \int_0^\tau y(u)\,du}{2\sigma^2}.$$

The maximum likelihood estimator of θ is given by

$$\hat{\theta}_\tau = \frac{y(0)+y(\tau)+\beta \int_0^\tau y(u)\,du}{2+\beta\tau}.$$

The fixed-time sequential plan $\delta_0 = (T_0, \hat{\theta}_{T_0})$, where T_0 is determined by

$$\frac{2\sigma^2}{2+\beta T_0} + H(T_0) = \inf_T \left[\frac{2\sigma^2}{2+\beta T} + H(T)\right],$$

is minimax.

The problem of estimation of θ, for a stationary Gaussian Markov process, has been also considered by Różański [20].

All the results above have been derived under the assumption that the risk of δ is given by formula (3.5). Now we consider sequential estimation of θ assuming that the cost H of observations is not taken into account, i.e. that the risk of δ is given by

$$\tilde{R}(\theta, \delta) = E_\theta (f-\theta)^2.$$

Clearly, in this case it is necessary to impose additional restrictions on the stopping times considered. Otherwise, the optimal stopping time τ would be equal to $+\infty$ with probability 1.

Let $\mathscr{D}(T)$ denote the set of all sequential plans $\delta = (\tau, f)$ for which $\tilde{R}(\theta, \delta)$ is finite and $E_\theta u(\tau) \leqslant u(T)$ holds for all $\theta \in \Theta$.

If the function $(\varphi'(u) - A(u)\varphi(u))^2 B^+(u)$ is nonincreasing and if $E_\theta \tau \leqslant T$, then $E_\theta u(\tau) \leqslant u(T)$.

A sequential plan $\delta_1 = (\tau_1, f_1)$ is said to be *better* than $\delta_2 = (\tau_2, f_2)$ if

$$\tilde{R}(\theta, \delta_1) \leqslant \tilde{R}(\theta, \delta_2)$$

for all θ and a strict inequality holds for at least one $\theta \in \Theta$.

A sequential plan $\delta \in \mathscr{D}(T)$ is said to be *admissible* in $\mathscr{D}(T)$ if there is no other plan in $\mathscr{D}(T)$ which is better than δ.

A sequential plan δ^* is said to be *minimax* if

$$\sup_{\theta\in\Theta} \tilde{R}(\theta, \delta^*) = \inf_{\delta\in\mathscr{D}(T)} \sup_{\theta\in\Theta} \tilde{R}(\theta, \delta).$$

We say that $b(\theta) = \mathrm{E}_\theta(f-\theta)$ is the *bias function* of $\delta = (\tau, f)$. If $b(\theta) = 0$, then $\delta = (\tau, f)$ is said to be *unbiased*.

A sequential plan $\delta = (\tau, f)$ is said to be *best unbiased* if it is unbiased and if $\tilde{R}(\theta, \delta') \geqslant \tilde{R}(\theta, \delta)$ for all $\theta\in\Theta$ and for all unbiased sequential plans δ' in $\mathscr{D}(T)$.

We prove that $\delta_T = (T, \hat{\theta}_T)\in\mathscr{D}(T)$ is admissible and minimax. To establish this we need the following lemma which can be considered as an analogue to the classical Cramér-Rao inequality:

LEMMA 3.1. *If* $\delta = (\tau, f)$ *is a sequential plan and if*

$$\int_{\theta_1}^{\theta_2} \mathrm{E}_\theta\big(f^2+u(\tau)\big)\,d\theta < \infty \quad \textit{for } \theta_1 < \theta_2, \theta_1, \theta_2\in\Theta,$$

then

$$\text{(3.6)}\quad \mathrm{E}_{\theta_2} f - \mathrm{E}_{\theta_1} f = \int_{\theta_1}^{\theta_2} \mathrm{E}_\theta f\,\Big\{\big(y(0)-\theta\varphi(0)-\psi(0)\big)\varphi(0)K^+(0,0) + \int_0^\tau \big(\varphi'(t)-A(t)\varphi(t)\big)\big(B^{1/2}(t)\big)^+\,dw(t)\Big\}\,d\theta.$$

Moreover, if

$$\int_{\theta_1}^{\theta_2} \mathrm{E}_\theta u(\tau)\,d\theta > 0,$$

then

$$\text{(3.7)}\quad \int_{\theta_1}^{\theta_2} \tilde{R}(\theta, \delta)\,d\theta \geqslant \int_{\theta_1}^{\theta_2} b^2(\theta)\,d\theta + \frac{\big(\theta_2-\theta_1+b(\theta_2)-b(\theta_1)\big)^2}{\int_{\theta_1}^{\theta_2} \mathrm{E}_\theta u(\tau)\,d\theta}.$$

Proof. Note that

$$\mathrm{E}_{\theta_2} f - \mathrm{E}_{\theta_1} f = \int_C f(c^x) \int_{\theta_1}^{\theta_2} \frac{d}{d\theta} \frac{1}{\sqrt{K(0,0)}} \exp\left\{-\frac{\big(x-\theta\varphi(0)-\psi(0)\big)^2}{2K(0,0)}\right\} \frac{d\mu^\theta_{\tau,yx}}{d\mu_{\tau,Vx}}(c^x)\,d\theta\,d\nu_\tau(x, c^x),$$

where

$$d\nu_\tau(x, c^x) = \frac{1}{\sqrt{2\pi}}\,d\mu_{\tau,Vx}(c^x)\exp\left\{-\frac{x^2}{2}\right\}dx.$$

By (3.1) we have

$$\int_{\theta_1}^{\theta_2}\int_{C}\left|f(c^x)\frac{d}{d\theta}\frac{1}{\sqrt{K(0,0)}}\exp\left\{-\frac{(x-\theta\varphi(0)-\psi(0))^2}{2K(0,0)}\right\}\frac{d\mu^0_{\tau,yx}}{d\mu_{\tau,vx}}(c^x)\right|dv_\tau(x,c^x)\,d\theta$$

$$= \int_{\theta_1}^{\theta_2} E_\theta\Big|f\Big\{(y(0)-\theta\varphi(0)-\psi(0))\varphi(0)K^+(0,0)+$$

$$+\int_0^\tau(\varphi'(t)-A(t)\varphi(t))(B^{1/2}(t))^+dw(t)\Big\}\Big|\,d\theta$$

$$\leqslant \Big(\int_{\theta_1}^{\theta_2}E_\theta f^2\,d\theta\Big)^{1/2}\Big(\int_{\theta_1}^{\theta_2}E_\theta u(\tau)\,d\theta\Big)^{1/2}<\infty.$$

Now Fubini's theorem yields (3.6). To complete the proof note that

$$(E_{\theta_2}f-E_{\theta_1}f)^2 \leqslant \int_{\theta_1}^{\theta_2}E_\theta(f-E_\theta f)^2\,d\theta\int_{\theta_1}^{\theta_2}E_\theta u(\tau)\,d\theta$$

$$= \Big\{\int_{\theta_1}^{\theta_2}E_\theta(f-\theta)^2\,d\theta-\int_{\theta_1}^{\theta_2}(E'_\theta(f-\theta))^2\,d\theta\Big\}\int_{\theta_1}^{\theta_2}E_\theta u(\tau)\,d\theta.$$

THEOREM 3.2. *If $\Theta = \boldsymbol{R}$, then $\delta_T = (T, \hat{\theta}_T)$ is an admissible and minimax estimator of θ.*

Proof. Suppose that δ_T is not admissible. Then there exists a sequential plan $\delta = (\tau, f)$ such that

$$\tilde{R}(\theta,\delta) \leqslant \tilde{R}(\theta,\delta_T) = \frac{1}{u(T)}$$

with a strict inequality for at least one θ. Since

$$\sup_{\theta_1\leqslant\theta\leqslant\theta_2} E_\theta u(\tau) \leqslant u(T) < \infty,$$

the assumption of Lemma 3.1 is fulfilled. Hence, according to (3.7),

$$\frac{1}{u(T)} \geqslant \sup_{\theta_1\leqslant\theta\leqslant\theta_2}\tilde{R}(\theta,\delta) \geqslant \frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\tilde{R}(\theta,\delta)\,d\theta \tag{3.8}$$

$$\geqslant \frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}b^2(\theta)\,d\theta+\frac{\left(1+\dfrac{b(\theta_2)-b(\theta_1)}{\theta_2-\theta_1}\right)^2}{u(T)}.$$

Now we show that $b(\theta)\equiv 0$ is the only function satisfying this inequality. The function $b(\theta)$ is nonincreasing because

$$\left(1+\frac{b(\theta_2)-b(\theta_1)}{\theta_2-\theta_1}\right)^2 \leqslant 1.$$

Moreover, $b(\theta)$ is bounded. To prove this we consider first the case $b(\theta) \geqslant 0$. Then

$$b^2(\theta_2) \leqslant \frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2} b^2(\theta)\,d\theta \leqslant \frac{1}{u(T)} \quad \text{for every } \theta_1 \leqslant \theta_2.$$

Similarly, $b\theta)$ is bounded when $b(\theta) \leqslant 0$. Since $b(\theta)$ is nonincreasing, there exists at most one value θ_0 such that $b(\theta) \geqslant 0$ for $\theta \leqslant \theta_0$ and $b(\theta) < 0$ for $\theta \geqslant \theta_0$. Considering these two intervals separately, we establish easily that $b(\theta)$ is bounded.

Note that there exists a sequence $\{\theta_n\}$ such that

$$\lim_{n\to\infty} \theta_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{b(\theta_n)-b(\theta_{n-1})}{\theta_n-\theta_{n-1}} = 0.$$

Suppose, on the contrary, that there exist $\varepsilon > 0$ and θ^* such that for every $\theta_2 \geqslant \theta_1 \geqslant \theta^*$

$$\frac{b(\theta_2)-b(\theta_1)}{\theta_2-\theta_1} < -\varepsilon.$$

Then for every $\theta > \theta_1$ we have $b(\theta) < -\varepsilon(\theta-\theta_1)+b(\theta_1)$, which shows that $b(\theta)$ cannot be bounded.

Substituting $\{\theta_n\}$ into (3.8) we have

$$\frac{1}{\theta_n-\theta_{n-1}}\int_{\theta_{n-1}}^{\theta_n} b^2(\theta)\,d\theta \leqslant \frac{1}{u(T)}\,\frac{b(\theta_{n-1})-b(\theta_n)}{\theta_n-\theta_{n-1}}\left(2+\frac{b(\theta_n)-b(\theta_{n-1})}{\theta_n-\theta_{n-1}}\right).$$

Moreover, for sufficiently large n (such that $\theta_{n-1} > \theta^*$)

$$\min\left(b^2(\theta_{n-1}), b^2(\theta_n)\right) \leqslant \frac{1}{\theta_n-\theta_{n-1}}\int_{\theta_{n-1}}^{\theta_n} b^2(\theta)\,d\theta,$$

so that

$$\lim_{n\to\infty} b(\theta_n) = 0.$$

Similarly we can prove that there exists a sequence $\{\tilde{\theta}_n\}$ such that

$$\lim_{n\to\infty} \tilde{\theta}_n = -\infty \quad \text{and} \quad \lim_{n\to\infty} b(\tilde{\theta}_n) = 0.$$

Since $b(\theta)$ is nonincreasing and $b(\theta)\to 0$ as $\theta\to\pm\infty$, we infer that $b(\theta) \equiv 0$.

In view of (3.8) it is clear that

$$\sup_{\theta_1\leqslant\theta\leqslant\theta_2} \tilde{R}(\theta,\delta) = \frac{1}{u(T)} \quad \text{for every } \theta_1 < \theta_2.$$

This implies that $\tilde{R}(\theta, \delta) = 1/u(T)$ for all θ. Thus δ_T is admissible.

To prove that δ_T is minimax we use the fact that δ_T has a constant risk. Indeed, suppose that δ_T is not minimax. Then there exists a sequential plan $\delta = (\tau, f)$ such that

$$\sup_{\theta} \tilde{R}(\theta, \delta) < \sup_{\theta} \tilde{R}(\theta, \delta_T) = \frac{1}{u(T)},$$

which implies that $\tilde{R}(\theta, \delta) < \tilde{R}(\theta, \delta_T)$ for all θ. This shows that δ_T is not admissible.

It is interesting to note that in case where the parameter space is truncated δ_T is minimax but not admissible. For example, δ_T is worse than $\delta_T^* = (T, \max(\theta_0, \hat{\theta}_T))$ when $\Theta = (\theta_0, \infty)$.

THEOREM 3.3. *If* $\Theta = (\theta_0, \infty)$, *then* $\delta_T = (T, \hat{\theta}_T)$ *is minimax.*

Proof. Suppose that δ_T is not minimax. Then there exists a plan $\delta = (\tau, f)$ such that

$$\sup_{\theta \geqslant \theta_0} \tilde{R}(\theta, \delta) < \frac{1}{u(T)}.$$

Hence $\tilde{R}(\theta, \delta) \leqslant 1/u(T) - \varepsilon$ for all $\theta \geqslant \theta_0$ and some $\varepsilon > 0$. It is easy to see that $b(\theta)$ is bounded.

Since

$$\left(1 + \frac{b(\theta_2) - b(\theta_1)}{\theta_2 - \theta_1}\right)^2 \leqslant 1 - \varepsilon u(T),$$

after simple calculations we infer that the inequality

$$\frac{b(\theta_2) - b(\theta_1)}{\theta_2 - \theta_1} < -\varepsilon \frac{u(T)}{1 + \sqrt{1 - \varepsilon u(T)}}$$

holds for every $\theta_2 > \theta_1 > \theta_0$. This implies that $b(\theta)$ is unbounded, which is a contradiction. Thus δ_T is minimax.

Now we assume that the parameter space Θ is an open interval on the real line and consider best unbiased sequential plans for θ. As mentioned earlier, δ_T is unbiased.

THEOREM 3.4. *The plan* $\delta_T = (T, \hat{\theta}_T)$ *is best among all unbiased plans in* $\mathscr{D}(T)$.

This assertion follows in a straightforward way from (3.8).

Example 3. From Theorem 3.1 it follows that the plan $\delta_T = (T, \hat{\theta}_T)$, where

$$\hat{\theta}_T = \frac{y(0) + y(T) + \beta \int_0^T y(u)\, du}{2 + \beta T},$$

is admissible and minimax in the class of plans $\delta = (\tau, f)$ which satisfy the following two conditions: $E_\theta f^2 < \infty$, $E_\theta \tau \leqslant T$, $\theta \in \boldsymbol{R}$. If the parameter space is truncated, δ_T is minimax but not admissible. Finally, Theorem 3.4 shows that δ_T is a best unbiased plan.

4. Estimation of α. Note that the covariance operator $K(t, s)$ of the stochastic process y defined in Section 1 is equal to

$$K(t, s) = \exp\Big\{\int_s^t A(u)\,du\Big\} K(s, s),$$

where

$$K(s, s) = \exp\Big\{2\int_0^s A(u)\,du\Big\}\Big\{K(0, 0) + \int_0^s \exp\Big\{-2\int_0^u A(v)\,dv\Big\} B(u)\,du\Big\}.$$

In this section we assume that $A(t) = \alpha p(t) + q(t)$ and consider the problem of estimation of the parameter α. We assume that α ranges over an open interval Ω on the real line. Functions p and q are known and such that

$$\int_0^t \big[(m'(u))^2 + \big(p^2(u) + q^2(u)\big) K(u, u)\big] B^+(u)\,du < \infty \qquad \text{for } t < \infty$$

and

$$\int_0^\infty p^2(u) K(u, u) B^+(u)\,du = \infty.$$

Since we use here the same methods as in the case of estimation of θ, we omit the proofs.

Consider the stopping time

$$\tau_T = \tau_T(y) = \inf\{t\colon\, Z(t, y) > T\},$$

where

$$Z(t, y) = \int_0^t p^2(u)\big(y(u) - m(u)\big)^2 B^+(u)\,du.$$

It is easy to see that τ_T is nondecreasing with respect to T,

$$\mu_y^\alpha(\tau_T < \infty) = 1, \qquad \alpha \in \Omega,$$

for all $T < \infty$, and

$$\mu_y^\alpha\big(\lim_{T\to\infty} \tau_T = \infty\big) = 1, \qquad \alpha \in \Omega.$$

Consider the sequential plan

$$\varrho_T = \Big(\tau_T, \frac{1}{T}\eta(\tau_T, y)\Big),$$

where

$$\eta(\tau_T, y) = \int_0^{\tau_T} p(u)\big(y(u)-m(u)\big)B^+(u)\Big[dy(u)-\big(\dot{m}'(u)+q(u)\big(y(u)-m(u)\big)\big)du\Big].$$

The estimator $T^{-1}\eta(\tau_T, y)$ has a normal distribution with expectation α and variance $1/T$. The risk function including the cost term is now of the form

$$\mathring{R}(\alpha, \delta) = E_\alpha[(f-\alpha)^2 + H(Z(\tau, y))],$$

where H is defined as in Section 3. Assuming that $\Omega = \boldsymbol{R}$, the following theorem can be established:

THEOREM 4.1. *The plan ϱ_T, where T is determined by*

$$\frac{1}{T}+H(T) = \inf_{t\geqslant 0}\left(\frac{1}{t}+H(t)\right),$$

is minimax.

If the risk function $R(\alpha, \delta) = E_\alpha(f-\alpha)^2$ does not take into account the cost of observations, one can establish, using arguments similar to those in Section 3, the following results.

Let $\mathscr{D}(T)$ denote the set of all sequential plans $\delta = (\tau, f)$ for which $\tilde{R}(\alpha, \delta)$ is finite and $E_\alpha Z(\tau, y) \leqslant T$ holds for all $\alpha \in \Omega$.

THEOREM 4.2. (i) *If $\Omega = \boldsymbol{R}$, then ϱ_T is admissible, minimax and best unbiased in $\mathscr{D}(T)$.*

(ii) *If the parameter space is truncated, say $\Omega = (\alpha_0, \infty)$, then ϱ_T is minimax and best unbiased in $\mathscr{D}(T)$.*

(iii) *If Ω is an open interval, then ϱ_T is best among all unbiased plans in $\mathscr{D}(T)$.*

As already mentioned, the optimal stopping time τ_T is finite. The following result can be established by using some ideas of Wognik (Theorem 17.7 in [12]) and Musiela [15].

THEOREM 4.3. *If*

$$0 < \inf_t \frac{|p(t)|}{B(t)} = a, \quad \sup_t \frac{|p(t)|}{B(t)} = b < \infty,$$

$$0 < \inf_t B(t) = c, \quad \sup_t \frac{|q(t)|}{B(t)} = d < \infty,$$

then for every $n = 1, 2, \ldots$ there exist constants a_n, b_n and c_n depending only upon a, b, c and d such that

$$E_\alpha \tau_T^n \leqslant (a_n|\alpha|^n + b_n)T^n + c_n T^{n/2}.$$

Appendix. The proof of Theorem 2.1 is divided into 5 steps.

1. First define a new process

$$y^x(t) = \exp\{\int_0^t A(u)\,du\}\{x+\int_0^t \exp\{-\int_0^u A(v)\,dv\}\,a(u)\,du+$$
$$+\int_0^t \exp\{-\int_0^u A(v)\,dv\}\,B^{1/2}(u)\,dw(u)\}, \qquad x\in \boldsymbol{R}.$$

According to (2.1) the Ito formula yields

$$y^x(t)-\mathrm{E}y^x(t) = y(t)-\exp\{\int_0^t A(u)\,du\}\,y(0)-$$
$$-\exp\{\int_0^t A(u)\,du\}\int_0^t \exp\{-\int_0^u A(v)\,dv\}\,a(u)\,du.$$

Therefore, it is obvious that

$$\mathrm{E}\big(y^x(t)-\mathrm{E}y^x(t)\big)\big(y(0)-m(0)\big) = K(t,0)-\exp\{\int_0^t A(u)\,du\}\,K(0,0).$$

Moreover, since y is a Gaussian Markov process, we have

$$K'(t,0) = A(t)K(t,0).$$

Thus $y^x(t)$ and $y(0)$ are independent for all x and t.

2. It is known that $\mu_{t,y^x} \sim \mu_{t,\gamma^x}$, $t\in[0,\infty]$, if and only if

$$P\Big(\int_0^t \big(a(u)+A(u)\,y^x(u)\big)^2 B^+(u)\,du < \infty\Big) = 1.$$

The zero-one law for the Gaussian measures and some calculations show that the measures μ_{t,y^x} and μ_{t,γ^x} are equivalent if and only if

$$\int_0^t \big(m'(u)+(x-m(0))A(u)\exp\{\int_0^u A(v)\,dv\}\big)^2 B^+(u)\,du+$$
$$+\int_0^t A^2(u)\big(K(u,u)-K^2(u,0)K^+(0,0)\big)B^+(u)\,du < \infty.$$

3. Let L be defined by

$$L(t) = \int_0^t \big((m'(u))^2+A^2(u)K(u,u)\big)B^+(u)\,du, \qquad t\in[0,\infty],$$

and let $l(x) = m(0)+xK(0,0)$. Taking into account Step 2 we easily infer that $\mu_{t,y^{l(x)}} \sim \mu_{t,\gamma^{l(x)}}$ for every x if and only if $L(t) < \infty$.

4. It is not difficult to prove that if $L(\infty) < \infty$, then $\mu_{\tau,yl(x)} \ll \mu_{\tau,Vl(x)}$ for all τ. Moreover, if $L(t) < \infty$ for all $t < \infty$ and if $L(\infty) = \infty$, then

$$\mu_{yl(x)}(\tau < \infty) = 1 \quad \text{iff} \quad \mu_{\tau,yl(x)} \ll \mu_{\tau,Vl(x)},$$
$$\mu_{yl(x)}(\tau = \infty) = 1 \quad \text{iff} \quad \mu_{\tau,yl(x)} \perp \mu_{\tau,Vl(x)}.$$

5. Finally, according to Step 1, we have

$$\begin{aligned}\mu_{\tau,y}(\cdot) &= \frac{1}{\sqrt{2\pi K(0,0)}} \int_{-\infty}^{\infty} \mu_{\tau,y}(\cdot\,|\,c(0) = x) \exp\left\{-\frac{(x-m(0))^2}{2K(0,0)}\right\} dx \\ &= \frac{1}{\sqrt{2\pi K(0,0)}} \int_{-\infty}^{\infty} \mu_{\tau,y^x}(\cdot) \exp\left\{-\frac{(x-m(0))^2}{2K(0,0)}\right\} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu_{\tau,yl(x)}(\cdot) \exp\left\{-\frac{x^2}{2}\right\} dx.\end{aligned}$$

This combined with Step 4 provides the result.

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