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STEREOLOGICAL FORMULAS FOR MANIFOLD PROCESSES

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Abstract. Stationary and isotropic q-dimensional random manifolds in R^n are considered. Formulas are given which allow to determine the expected q-dimensional volume of the random manifold per unit volume of R^n by measurements of the intersection with a hyperplane or other manifold.

1. Introduction. Let Φ be a stationary and isotropic q-dimensional random manifold in R^n, where R denotes the real axis and n is fixed $(0 \leqslant q \leqslant n)$. A suitable definition of a random manifold or a manifold process is given in section 2. We are interested in the problem of determining the expected q-dimensional volume of Φ per unit volume of R^n by the measurement of the intersection of Φ with an r-dimensional manifold Ψ. The problem is solved in section 3 for the case where Ψ is fixed (non-random) and bounded, in section 4 for the case where Ψ is a flat, and in section 5 for the case where Ψ is a stationary and isotropic random manifold. The results in sections 4 and 5 are consequences of the result in section 3. The key for the solution of our problems is a theorem of Poincaré type given in [3]. Geometrical notation as "manifold piecewise smooth of class C^1", "special motion", and "kinematic measure" are used in the same sense as in [3].

If some convexity conditions are satisfied, the results of sections 3 and 4 have already been contained in [2] (formulas (7.4) and (7.6)). The results of the paper are also connected with [4].

2. Manifold processes. Denote by σ_q the volume measure for q-dimensional rectifiable manifolds in R^n $(0 \leqslant q \leqslant n)$. Then σ_n is the Lebesgue measure in R^n and $\sigma_0(A)$ equals the number of elements in the set A. Let $\mathscr{A}_q$ be a family of subsets $\varphi \subset R^n$ with the property that for any ball $B \subset R^n$ the intersection $\varphi \cap B$ is a q-dimensional manifold piecewise smooth of class C^1 and $\sigma_q(\varphi \cap B) < \infty$. Denote by $\mathfrak{R}_n$ the Borel σ-algebra in R^n. Let $\mathfrak{A}_q$ be the σ-algebra in $\mathscr{A}_q$ generated by all functions $\varphi \to \sigma_q(\varphi \cap C)$ $(C \in \mathfrak{R}_n)$.

By a *q-dimensional manifold process* we mean a random variable Φ with range $[\mathscr{A}_q, \mathfrak{A}_q]$. Its distribution is a probability measure on $[\mathscr{A}_q, \mathfrak{A}_q]$. Examples are given by: flat processes (line processes in the special case $q = 1$), processes of boundaries of special random closed sets (cf. [2]), random fibrefields in the sense of Ambartzumian [1] ($q = 1$, e.g. line-segment processes), point processes ($q = 0$). Let M be the set of special motions $\mathfrak{m}$ of R^n, $\mathfrak{M}$ the usual σ-algebra in M, and $\varkappa$ the kinematic measure on $[M, \mathfrak{M}]$.

A q-dimensional manifold process Φ is called *stationary and isotropic* if the process $\mathfrak{m}\Phi$ has for all $\mathfrak{m} \in M$ the same distribution as Φ. If P is the distribution of Φ, this condition is equivalent to

$$\int P(d\varphi) f(\mathfrak{m}\varphi) = \int P(d\varphi) f(\varphi) \qquad (\mathfrak{m} \in M;\ f\colon \mathscr{A}_q \to [0, \infty),\ \mathfrak{A}_q\text{-measurable}). \tag{2.1}$$

Simple examples of stationary and isotropic manifold processes are homogeneous Poisson point processes ($q = 0$) and unions of hyperspheres with constant radius whose centres form a homogeneous Poisson process ($q = n-1$).

Suppose Φ is a q-dimensional stationary and isotropic manifold process and $\vartheta \in \mathscr{A}_n$, $\sigma_n(\vartheta) = 1$; then it is easily seen that $\mathrm{E}\sigma_q(\Phi \cap \vartheta) = J_\Phi$ does not depend on the special choice of ϑ. The value J_Φ is called the *intensity* of Φ. If C_n denotes the unit cube $[0, 1]^n$ in R^n, we have $J_\Phi = \mathrm{E}\sigma_q(\Phi \cap C_n)$.

3. Intersection with manifolds. Denote by O_m the surface area of the m-dimensional unit sphere:

$$O_m = \frac{2\pi^{(m+1)/2}}{\Gamma((m+1)/2)} \qquad (m = 0, 1, 2, \ldots).$$

In [3], p. 259, the following theorem of Poincaré type is mentioned:

THEOREM 3.1. *Let M^q and M^r be q- and r-dimensional manifolds in R^n piecewise smooth of class C^1 ($q, r = 0, \ldots, n$; $q+r \geqslant n$). Then*

$$\int \varkappa(d\mathfrak{m}) \sigma_{q+r-n}(M^q \cap \mathfrak{m}M^r) = \frac{O_n \ldots O_1 O_{q+r-n}}{O_q O_r} \sigma_q(M^q) \sigma_r(M^r).$$

Putting

$$c(n, q, r) = \frac{O_n O_{q+r-n}}{O_q O_r}$$

or, equivalently,

$$c(n, q, r) = \frac{\Gamma((q+1)/2)\, \Gamma((r+1)/2)}{\Gamma((n+1)/2)\, \Gamma((r+q-n+1)/2)}, \tag{3.1}$$

we can prove the following

THEOREM 3.2. *If Φ is a stationary and isotropic q-dimensional manifold process with distribution P and if $\psi \in \mathscr{A}_r$ $(r, q = 0, \ldots, n;\ r+q \geqslant n)$, then*

$$E\sigma_{q+r-n}(\Phi \cap \psi) = c(n, q, r)\sigma_r(\psi) J_\Phi \ (^1)$$

or, equivalently,

$$\int P(d\varphi)\sigma_{q+r-n}(\varphi \cap \psi) = c(n, q, r)\sigma_r(\psi) \int P(d\varphi)\sigma_q(\varphi \cap C_n). \tag{3.2}$$

Proof. Putting

$$a(n, q, r) = \frac{O_n \ldots O_1 O_{q+r-n}}{O_q O_r},$$

we obtain, according to theorem 3.1,

$$\begin{aligned} c(n, q, r) \int P(d\varphi)\sigma_q(\varphi \cap C_n)\sigma_r(\psi) \\ = [a(n, q+r-n, n)]^{-1} \int P(d\varphi) \int \varkappa(d\mathfrak{m})\sigma_{q+r-n}(\varphi \cap C_n \cap \mathfrak{m}\psi). \end{aligned} \tag{3.3}$$

By Fubini's theorem, the stationarity and isotropy of P (formula (2.1)) and σ_{q+r-n} we obtain

$$\begin{aligned} \int P(d\varphi) \int \varkappa(d\mathfrak{m})\sigma_{q+r-n}(\varphi \cap C_n \cap \mathfrak{m}\psi) \\ = \int \varkappa(d\mathfrak{m}) \int P(d\varphi)\sigma_{q+r-n}(\mathfrak{m}\varphi \cap \mathfrak{m}\psi \cap C_n) \\ = \int P(d\varphi) \int \varkappa(d\mathfrak{m})\sigma_{q+r-n}(\varphi \cap \psi \cap \mathfrak{m}^{-1} C_n). \end{aligned} \tag{3.4}$$

Using theorem 3.1 and substituting $q+r-n$ for q and n for r we obtain

$$\begin{aligned} \int P(d\varphi) \int \varkappa(d\mathfrak{m})\sigma_{q+r-n}(\varphi \cap \psi \cap \mathfrak{m}^{-1} C_n) \\ = a(n, q+r-n) \int P(d\varphi)\sigma_{q+r-n}(\varphi \cap \psi). \end{aligned} \tag{3.5}$$

Equation (3.2) follows now from (3.3)-(3.5).

4. Intersection with flats. Let Φ be a stationary and isotropic q-dimensional manifold process with distribution P and let L_r be an r-dimensional flat (r-flat) $(q, r = 0, \ldots, n;\ q+r \geqslant n)$. We are interested in the process $\Phi \cap L_r$. Because of the stationarity and isotropy it is sufficient to consider the special case $L_r = R^r \subset R^n$ (²). The intersection $\Phi \cap R^r$ is almost surely a $(q+r-n)$-dimensional manifold process in R^r invariant under all special motions of R^r. Its intensity (as of a process in R^r) will be denoted by $S(\Phi, r)$:

$$S(\Phi, r) = \int P(d\varphi)\sigma_{q+r-n}(\varphi \cap R^r \cap C_r).$$

(¹) $\Phi \cap \psi$ is $(q+r-n)$-dimensional almost surely.

(²) R^r is identified with $\{(x_1, \ldots, x_n) \in R^n\colon x_{r+1} = \ldots = x_n = 0\}$.

By theorem 3.2 we have

$$S(\Phi, r) = c(n, q, r) J_\Phi, \tag{4.1}$$

where J_Φ is the intensity of Φ and $c(n, q, r)$ is given by (3.1).

Examples:

n	q	r	$c(n, q, r)$
2	1	1	$2/\pi$
3	2	2	$\pi/4$
3	2	1	$1/2$
3	1	2	$1/2$
m	m	s	1

The last case corresponds to the usual stereological formulas.

5. Intersection of manifold processes. If Φ and Ψ are independent stationary and isotropic manifold processes of dimensions q and r, respectively, then $\Phi \cap \Psi$ with probability one is an $(r+q-n)$-dimensional manifold process. We are interested in its intensity $J_{\Phi \cap \Psi}$. Let P be the distribution of Φ and Q the distribution of Ψ. We have

$$J_{\Phi \cap \Psi} = \int Q(d\psi) \int P(d\varphi)\, \sigma_{q+r-n}(\varphi \cap \psi \cap C_n).$$

By theorem 3.2 we obtain

$$\int P(d\varphi)\, \sigma_{q+r-n}(\varphi \cap \psi \cap C_n) = c(n, q, r)\, \sigma_r(\psi \cap C_n) J_\Phi.$$

Hence

$$J_{\Phi \cap \Psi} = c(n, q, r) J_\Phi \int Q(d\psi)\, \sigma_r(\psi \cap C_n).$$

Since $\int Q(d\psi)\, \sigma_r(\psi \cap C_n) = J_\Psi$, we have the final formula

$$J_{\Phi \cap \Psi} = c(n, q, r) J_\Phi J_\Psi.$$

In the special case where $n = 2$ and $q = r = 1$ it reduces to the following nice result:

The intersection of two independent stationary and isotropic random fibrefields in R^2 with intensities J_1 and J_2 is, with probability one, a point process in R^2 whose intensity equals $(2/\pi) J_1 J_2$.

Added in proof. The definition of $\mathscr{A}_q$ in section 2 must be completed by the assumption that each $\varphi \in \mathscr{A}_q$ is a closed subset of R^n. (Then a manifold process is a special random closed set.)

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