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HIGH CONDUCTIVITY LIMITS FOR REACTION-DIFFUSION EQUATIONS

BY

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Abstract. A reaction-diffusion equation on $[0, 1]^d$ with the heat conductivity $\kappa > 0$, a polynomial drift term, and an additive random perturbation is considered. It is shown that if κ tends to infinity, then the corresponding solutions of the equation converge to a process satisfying an ordinary Itô equation.

0. INTRODUCTION

This work is concerned with the asymptotic behaviour of the solutions of the stochastic reaction-diffusion equation

$$\frac{\partial X_t(\xi)}{\partial t} = \frac{\kappa}{2} \Delta X_t(\xi) + f(X_t(\xi)) + \frac{\partial}{\partial t} W_Q(t, \xi), \qquad \xi \in O, \; t \geqslant 0, \tag{0.1}$$

on the d-dimensional cube $O =]0, 1[^d$, as κ tends to infinity. In equation (0.1), Δ is the Laplace operator with the Neumann boundary conditions, W_Q stands for a Brownian motion with the covariance operator Q, f is a real function, and κ is a positive constant. We will show that, under some conditions on the function f, the limit exists and can be identified with a solution Y to a stochastic ordinary differential equation. So if the heat conductivity κ is sufficiently large, the solution to the stochastic reaction-diffusion equation (0.1) can be regarded as a solution of an ordinary Itô equation which is much easier to investigate. We allow rather general class of non-linear functions f including polynomials with negative leading coefficients, frequently used in applications (see e.g. [1]).

The same asymptotic problem was considered by Funaki [4]. He studied the stochastic heat equation

$$\frac{\partial X_t(\xi)}{\partial t} = \frac{\kappa}{2} \frac{\partial^2}{\partial \xi^2} X_t(\xi) + f(X_t(\xi)) + g(X_t(\xi)) \frac{\partial}{\partial t} W_I(t, \xi), \qquad \xi \in [0, 1], \; t \geqslant 0,$$

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on the interval $]0, 1[$. Funaki required that $d = 1$ and that functions f and g satisfy Lipschitz conditions. We investigate the case of general dimensions d and do not assume the Lipschitz condition on f. With the initial value considered to be continuous in space variable, we treat the equation in the space of continuous functions and derive the convergence of its solutions in that space almost surely. In [4] a considerably weaker convergence is obtained (see [4], Theorem 3.1). On the other hand, we deduce the convergence of sequences of solutions indexed by $\{\kappa_n\}_{n=1}^{\infty}$, satisfying a certain condition, where Funaki does not restrict the coefficient κ in the equation. We assume also that the random fluctuations do not depend on the solution, setting g constant. The case of non-constant diffusion function g and general dimension d cannot be handled at the moment with the techniques of the paper.

The paper is organized as follows. In Section 1 we give a precise definition of the solution to equation (0.1), and introduce the equation for the limit process. We state conditions under which these equations have solutions. The proof of the existence results can be deduced from recent works (see [2], [3] and [5]). For complete considerations we refer to our preprint [6]. In Section 2 we show that solutions to equations (0.1) are bounded, with respect to κ, in an appropriate sense. For this we need an estimate of stochastic convolution as well as a formula which links the norms of the solutions directly to the norm of the stochastic convolution (see Lemmas 2.1 and 2.4). We show also that the generalized solution of (0.1) depends continuously on the initial condition uniformly in κ. Finally, Section 3 is devoted to the proof of the main result.

1. FORMULATION OF THE MAIN RESULT

The reaction-diffusion equation we deal with is the following:

$$
\begin{aligned}
&\frac{\partial X_t(\xi)}{\partial t} = \frac{\kappa}{2}\Delta X_t(\xi) + f\left(X_t(\xi)\right) + \frac{\partial}{\partial t} W_Q(t, \xi), \qquad \xi \in O,\ t \geqslant 0,\\
&X_0(\xi) = x(\xi),\ \xi \in O, \qquad \frac{\partial}{\partial \nu} X_t(\xi) = 0,\ \xi \in \partial O,
\end{aligned}
\tag{1.1}
$$

where $O =]0, 1[^d$ $(d \geqslant 1)$, and $\nu = \nu(\xi)$ is the outer normal to ∂O at $\xi \in \partial O$. The coefficient κ of the Laplace operator Δ is a positive number. By W_Q we denote the Brownian motion with covariance operator Q on the probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ with the right continuous filtration $\{\mathscr{F}_t\}_{t \geqslant 0}$.

We consider the following condition on f, the non-linear part of the equation:

(A.1) The function f: $\boldsymbol{R} \to \boldsymbol{R}$ is of the form $f = f_0 + f_1$, where f_0 is a decreasing polynomial of degree $\gamma > 1$ and f_1 is Lipschitz continuous with the Lipschitz constant K.

It is important to note that all polynomials of odd degrees and with negative leading coefficients satisfy (A.1).

We will interpret equation (1.1) as an evolution equation

$$dX = (\kappa AX + F(X))\,dt + dW, \quad X(0) = x, \tag{1.2}$$

on the space $E = C(\overline{O})$ of continuous functions on $\overline{O}$, and on the Hilbert space $H = L^2(O)$. To cover the Neumann boundary condition we define the operator A on H by

$$D(A) = \left\{x \in H^2(O):\ \frac{\partial x}{\partial \nu}(\xi) = 0,\ \xi \in \partial O\right\}, \quad Ax = \frac{\Delta}{2}x,\ x \in D(A),$$

where ν denotes the outer normal to ∂O, and $H^2(O)$ is the Sobolev space $W^{2,2}(O)$. The operator A is a non-positive self-adjoint generator of an analytic semigroup $S(t)$, $t \geqslant 0$, on H. We denote the orthogonal eigenvectors and the eigenvalues of A by $\{e_i\}_{i=0}^{\infty}$ and $\{-\lambda_i\}_{i=0}^{\infty}$, respectively. λ_0 is zero, and $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$, e_0 is the constant function 1. The orthogonal projection of H into its subspace spanned by e_0 (the space of constant functions) will be denoted by P_1, and the orthogonal projection into the subspace spanned by $\{e_i\}_{i=1}^{\infty}$ by $P_1^{\perp}$.

Define F: $E \to E$ to be the *Nemitskii operator* corresponding to f,

$$F(x)(\xi) = f(x(\xi)), \quad x \in E,\ \xi \in O,$$

and $\{W(t)\}_{t \geqslant 0}$ the *Q-Wiener process* on H which is defined by

$$W(t) = \sum_{i=0}^{\infty} \sqrt{\gamma_i}\, e_i \beta_i(t),$$

where $\{\beta_i\}_{i=0}^{\infty}$ is a sequence of independent real-valued $\mathscr{F}_t$-adapted standard Wiener processes, and $\{\gamma_i\}_{i=0}^{\infty}$ are the eigenvalues of the operator Q.

We fix the Hilbert space H and the Banach space E:

$$H = L^2(O), \quad E = C(\overline{O}),$$

throughout the paper. The scalar product and the norm in H are denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|_H$, whereas the duality form on $E \times E^*$ and the norm in E are denoted by $(\cdot, \cdot)$ and $|\cdot|_E$, respectively.

Further we assume

(A.2) For the covariance operator Q on H we have

(i) $$Qe_i = \gamma_i e_i, \quad \gamma_i > 0, \quad i = 0, 1, 2, \ldots,$$

(ii) $$\sum_{i=1}^{\infty} \frac{\gamma_i}{\lambda_i^{1-\alpha}} < \infty \quad \text{for some } \alpha \in]0, 1[,$$

where $\{e_i\}_{i=0}^{\infty}$ and $\{-\lambda_i\}_{i=0}^{\infty}$ are eigenvectors and eigenvalues of A, respectively, as introduced before.

Remark 1.1. Denote the part of A on $P_1^{\perp} H$ by A_0. Then the operator Q,

$$Qx = \begin{cases} x & \text{if } x \in P_1 H, \\ (-A_0)^{-\beta} x & \text{if } x \in P_1^{\perp} H, \end{cases}$$

satisfies (A.2) if and only if $\beta > d/2-1$. In particular, in the case $d = 1$, Q may be the identity operator and $W(t)$ a cylindrical Wiener process, the case considered by Funaki [4].

By a *mild solution* of (1.1) (or (1.2)) we mean the predictable process X taking values in the domain of F (i.e. $C(\overline{O})$) and satisfying the following integral equation:

$$X(t) = S(\kappa t)\,x + \int_0^t S(\kappa(t-s))\,F(X(s))\,ds + \int_0^t S(\kappa(t-s))\,dW(s). \tag{1.3}$$

An H-valued process $\tilde{X}$ is a *generalized solution* of (1.2) if for an arbitrary sequence $\{x_n\} \subset E$ such that $\lim_{n\to\infty} |x-x_n|_H = 0$ the corresponding sequence of mild solutions $\{X_n\}$ converges to $\tilde{X}$ in $C([0, T]; H)$ P-a.s. for any interval $[0, T]$.

Existence and uniqueness of a mild and generalized solution to equation (1.2) for arbitrary $\kappa > 0$ could be find in literature (see e.g. [2], [3] and [5]). For a self-contained presentation of these results see [6].

Now consider the following scalar equation:

$$dY(t) = f(Y(t))\,dt + \sqrt{\gamma_0}\,d\beta_0(t), \qquad Y(0) = \hat{x}, \tag{1.4}$$

where

$$\hat{x} = \int_O x(\sigma)\,d\sigma,$$

f and x are the same as in equation (1.1), and γ_0 and $\beta_0(t)$ the same as in the expansion of $W(t) = \sum_{i=0}^{\infty} \sqrt{\gamma_i}\,\beta_i(t)\,e_i$.

Existence and uniqueness of a non-exploding solution to equation (1.4) can be derived for the drift term f satisfying (A.1). For the proof see [6]. This equation could be considered as the projection of equation (1.2) (as an equation on H) into the subspace $P_1 H$, that is

$$dY(t) = P_1 F(Y(t))\,dt + \sqrt{\gamma_0}\,d\beta_0(t), \qquad Y(0) = P_1 x. \tag{1.5}$$

Our main result is the following theorem:

THEOREM 1.2. *Assume that* (A.1) *and* (A.2) *hold and* $\{\kappa_n\}_{n=1}^{\infty}$ *is every sequence satisfying* $\sum_{n=1}^{\infty} \kappa_n^{-\beta} < \infty$ *for a constant* $\beta > 0$. *Then:*

(i) *for every initial value* $x \in E$ *the mild solutions* $X^{(\kappa_n)}(\cdot, x)$ *of equations* (1.2) *converge to the solution* $Y(\cdot, \hat{x})$ *of equation* (1.4) *in the sense that*

$$P\Big(\lim_{n\to\infty} |X^{(\kappa_n)}(t, x) - Y(t, \hat{x})|_E = 0 \quad \text{uniformly in } t \text{ on each compact subset of }]0, \infty[\Big) = 1;$$

(ii) *for every* $x \in H$ *and the generalized solutions* $\tilde{X}^{(\kappa_n)}(\cdot, x)$ *of* (1.2), *one has*

$$P\Big(\lim_{n\to\infty} |\tilde{X}^{(\kappa_n)}(t, x) - Y(t, \hat{x})|_H = 0 \quad \text{uniformly in } t \text{ on each compact subset of }]0, \infty[\Big) = 1.$$

We would like to state that considering the sequence of parameters $\{\kappa_n\}_{n=1}^{\infty}$ satisfying the condition mentioned in the theorem is imposed by the method of the proof for which we do not know any alternatives in the case of non-Lipschitz f. We will use Borel–Cantelli's lemma to derive some pathwise estimates for a stochastic convolution from the momential ones (see Corollary 2.2). To use this lemma it is necessary to work with a countable set of κ's, namely $\{\kappa_n\}_{n=1}^{\infty}$, which should satisfy also the mentioned condition as the weakest one. However, in the case where f is a Lipschitz function, it is possible to proceed in the moment form and there is no need to restrict the coefficient κ, as can be observed by following the proof of Theorem 2.1 in Section 3.

2. PRELIMINARY RESULTS

2.1. Main estimates. Let $U(t)$, $t \geqslant 0$, be the semigroup generated by A_0, the part of A in $P_1^{\perp} H$. The following lemma holds:

LEMMA 2.1. *Assume* (A.2) *holds. Let* $\kappa \geqslant 1$. *Then the convolution*

$$W_{A_0}^{(\kappa)}(t) = \int_0^t U\big(\kappa(t-s)\big)\, dW(s), \qquad t \geqslant 0,$$

has an E-continuous version. Moreover, given $T > 0$, *for every* $r > 2(d+1)/\alpha$ *there exists a constant* $C_r = C_r(T) > 0$, *such that*

$$E\big[\sup_{t\in[0,T]} |W_{A_0}^{(\kappa)}(t)|_E^{2r}\big] \leqslant C_r \left(\frac{1}{\kappa^{1-\alpha}} \sum_{i=1}^{\infty} \frac{\gamma_i}{\lambda_i^{1-\alpha}}\right)^r.$$

A proof of the lemma for the case $\kappa = 1$ could be found in [3] (Theorem 5.2.9). For general $\kappa \geqslant 1$ the proof differs only in detail and is stated in [6].

Here we would like to define a notation. For every $\beta > 0$ we denote by $\mathscr{K}(\beta)$ the set of all increasing sequences $\{\kappa_n\}_{n=1}^{\infty}$ satisfying $\kappa_1 \geqslant 1$ and $\sum_{n=1}^{\infty} \kappa_n^{-\beta} < \infty$.

COROLLARY 2.2. *Assume* (A.2) *is satisfied, the sequence* $\{\kappa_n\}_{n=1}^{\infty}$ *belongs to* $\mathscr{K}(\beta)$ *for a constant* $\beta > 0$, *and* $T > 0$. *Then for every* $p > 1/(1-\alpha)$ *there exists a set* $\mathscr{G}$ *of probability* 1 *with the property that for all* $\omega \in \mathscr{G}$ *there is* $n_0(\omega) \in N$ *such that*

$$\sup_{t\in[0,T]} |W_{A_0}^{(\kappa_n)}(t, \omega)|_E \leqslant \kappa_n^{-1/2p}, \qquad n \geqslant n_0(\omega).$$

Proof. Let $p > 0$ be such that $1/p < 1-\alpha$ ($p > 1/(1-\alpha)$), and choose

$$r > \max\left(\frac{\beta}{1-\alpha-1/p}, \frac{2(d+1)}{\alpha}\right).$$

Then for every $\kappa \geqslant 1$

$$P\big(\sup_{t\in[0,T]} |W_{A_0}^{(\kappa)}(t)|_E > \kappa^{-1/2p}\big) \leqslant \kappa^{r/p} E\big[\sup_{t\in[0,T]} |W_{A_0}^{(\kappa)}(t)|_E^{2r}\big]$$

and, by Lemma 2.1,

$$P\Big(\sup_{t\in[0,T]} |W_{A_0}^{(\kappa)}(t)|_E > \kappa^{-1/2p}\Big) \leqslant \kappa^{r/p}\, C_r \left(\frac{1}{\kappa^{1-\alpha}} \sum_{i=1}^{\infty} \frac{\gamma_i}{\lambda_i^{1-\alpha}}\right)^r$$

$$= C_r \frac{1}{\kappa^{r(1-\alpha-1/p)}} \left(\sum_{i=1}^{\infty} \frac{\gamma_i}{\lambda_i^{1-\alpha}}\right)^r \leqslant C_r \frac{1}{\kappa^{\beta}} \left(\sum_{i=1}^{\infty} \frac{\gamma_i}{\lambda_i^{1-\alpha}}\right)^r.$$

Writing this latter inequality for the sequence $\{\kappa_n\}_{n=1}^{\infty}$ in $\mathscr{K}(\beta)$, we obtain

$$\sum_{n=1}^{\infty} P\Big(\sup_{t\in[0,T]} |W_0^{(\kappa_n)}(t)|_E > \kappa_n^{-1/2p}\Big) < \infty.$$

Now Borel–Cantelli's lemma implies

$$P\Big(\sup_{t\in[0,T]} |W_{A_0}^{(\kappa_n)}(t)|_E > \kappa_n^{-1/2p} \text{ for infinitely many } n\in N\Big) = 0.$$

That is, there exists a set $\mathscr{G}$ of probability 1, and for every $\omega\in\mathscr{G}$ there is $n_0(\omega)\in N$ such that

$$\sup_{t\in[0,T]} |W_{A_0}^{(\kappa_n)}(t,\omega)|_E \leqslant \kappa_n^{-1/2p}, \qquad n \geqslant n_0(\omega). \blacksquare$$

Remark 2.3. The convolution

$$W_A^{(\kappa)}(t) = \int_0^t S\big(\kappa(t-s)\big)\, dW(s), \qquad t \geqslant 0,$$

can be rewritten as

$$W_A^{(\kappa)}(t) = \sqrt{\gamma_0}\, \beta_0(t)\, e_0 + W_{A_0}^{(\kappa)}(t), \qquad t \geqslant 0.$$

Thus $W_A^{(\kappa)}(t)$, $t \geqslant 0$, has also an E-continuous version. By the same hypothesis as in Corollary 2.2 and for the same set $\mathscr{G}$, $\omega\in\mathscr{G}$, and $n_0(\omega)\in N$

$$\sup_{t\in[0,T]} |W_A^{(\kappa_n)}(t,\omega)|_E \leqslant \sqrt{\gamma_0} \sup_{t\in[0,T]} |\beta_0(t,\omega)| + \kappa_n^{-1/2p}, \qquad n \geqslant n_0(\omega).$$

LEMMA 2.4. *Assume that (A.1) is satisfied and $\kappa \geqslant 1$. Then for every $x\in E$ and $z\in C([0,\infty[;E)$ there exists a unique mild solution for the following equation:*

$$du = \big(\kappa Au + F(u+z)\big)\, dt, \qquad u(0) = x. \tag{2.1}$$

Moreover, there exists $C > 0$ independent of κ such that for all $t \geqslant 0$

$$|u(t)|_E \leqslant e^{Ct}\Big(|x|_E + C\int_0^t \big(1+|z(s)|_E^{\gamma}\big)\, ds\Big).$$

Lemma 2.4 is the main part of existence and uniqueness theorems for equation (1.2) and could be found in references stated before, especially we refer the reader to [2], Proposition 3.2. The complete proof is stated in [6].

THEOREM 2.5. *Assume that* (A.1) *and* (A.2) *are satisfied, the sequence* $\{\kappa_n\}_{n=1}^{\infty}$ *belongs to* $\mathscr{K}(\beta)$ *for a constant* $\beta > 0$, *and* $T > 0$. *Then for arbitrary* $x \in E$ *the unique mild solution* $X^{(\kappa)}(t, x)$, $t \geqslant 0$, *of equation* (1.2) *has an E-continuous version. Moreover,*

$$\sup_{n \in N} \sup_{t \in [0,T]} |X^{(\kappa_n)}(t, x)|_E < \infty \quad a.s.$$

Proof. Fix $x \in E$, and for $\kappa \geqslant 1$ set

$$u^{(\kappa)}(t) = X^{(\kappa)}(t, x) - W_A^{(\kappa)}(t), \qquad t \geqslant 0.$$

Then $u^{(\kappa)}$ is the mild solution of (2.1) for $z = W_A^{(\kappa)}$, which by the previous lemma exists, is in $C([0, \infty[; E)$ and

$$|u^{(\kappa)}(t)|_E \leqslant e^{Ct}\Big(|x|_E + C\int_0^t \big(1 + |W_A^{(\kappa)}(s)|_E^{\gamma}\big)\,ds\Big), \qquad t \geqslant 0.$$

Consequently,

$$|X^{(\kappa)}(t, x)|_E \leqslant e^{Ct}\Big(|x|_E + C\int_0^t \big(1 + |W_A^{(\kappa)}(s)|_E^{\gamma}\big)\,ds\Big) + |W_A^{(\kappa)}(t)|_E, \qquad t \geqslant 0,$$

and

$$\sup_{t \in [0,T]} |X^{(\kappa)}(t, x)|_E \leqslant e^{CT}\Big(|x|_E + CT + CT \sup_{t \in [0,T]} |W_A^{(\kappa)}(t)|_E^{\gamma}\Big) + \sup_{t \in [0,T]} |W_A^{(\kappa)}(t)|_E. \tag{2.2}$$

Now take $p > 1/(1-\alpha)$ arbitrary and let $\omega \in \mathscr{G}$, the set mentioned in Corollary 2.2. Then by Remark 2.3 there exists $M_1(\omega) > 0$ such that

$$\sup_{n \geqslant n_0(\omega)} \sup_{t \in [0,T]} |W_A^{(\kappa_n)}(t, \omega)|_E \leqslant M_1(\omega).$$

Also, as $W_A^{(\kappa)}(t, \omega)$ is continuous in t, setting

$$M_2(\omega) = \sup_{n < n_0(\omega)} \sup_{t \in [0,T]} |W_A^{(\kappa_n)}(t, \omega)|_E,$$

and

$$M(\omega) = \max\big(M_1(\omega), M_2(\omega)\big),$$

we obtain

$$\sup_{n \in N} \sup_{t \in [0,T]} |W_A^{(\kappa_n)}(t, \omega)|_E \leqslant M(\omega).$$

This last estimate together with inequality (2.2) implies the assertion of the theorem. ∎

The following theorem will be concluded from a lemma analogous to Lemma 2.4 for equation (1.4).

THEOREM 2.6. *Assume that* (A.1) *is satisfied and* $T > 0$. *Then for an arbitrary initial value* $a \in \boldsymbol{R}$ *the solution* $Y(t, a)$, $t \geqslant 0$, *of equation* (1.4) *has a continuous version which satisfies*

$$\sup_{t \in [0,T]} |Y(t, a)| < \infty \quad a.s.$$

2.2. The Nemitskii operator. The following lemma summarizes the properties of the Nemitskii operator F which will be used throughout.

LEMMA 2.7. *Assume that* (A.1) *holds. Then*

(i) *for every* $x, y \in E$,

$$\langle F(x)-F(y), x-y\rangle \leqslant K|x-y|_H^2;$$

(ii) *there exists* $C>0$ *such that for every* $x, y \in E$

$$|F(x)-F(y)|_E \leqslant C|x-y|_E(1+|x|_E^{\gamma-1}+|y|_E^{\gamma-1}).$$

Proof. (i) Let $\sigma, \eta \in \boldsymbol{R}$ be arbitrary. The decreasing function f_0 satisfies

$$(f_0(\sigma)-f_0(\eta))(\sigma-\eta) \leqslant 0,$$

and for the Lipschitz function f_1 with the Lipschitz constant $K>0$ we have

$$(f_1(\sigma)-f_1(\eta))(\sigma-\eta) \leqslant |f_1(\sigma)-f_1(\eta)|\,|\sigma-\eta| \leqslant K|\sigma-\eta|^2.$$

Therefore f satisfies

$$(f(\sigma)-f(\eta))(\sigma-\eta) \leqslant K|\sigma-\eta|^2,$$

from which part (i) follows.

(ii) For the polynomial part f_0 of f there exists a positive constant C_1 such that

$$|f_0(\sigma)-f_0(\eta)| \leqslant C_1|\sigma-\eta|(1+|\sigma|^{\gamma-1}+|\eta|^{\gamma-1}), \qquad \sigma, \eta \in \boldsymbol{R},$$

and for the Lipschitz part f_1 of f

$$|f_1(\sigma)-f_1(\eta)| \leqslant K|\sigma-\eta|, \qquad \sigma, \eta \in \boldsymbol{R}.$$

Consequently,

$$|f(\sigma)-f(\eta)| \leqslant C|\sigma-\eta|(1+|\sigma|^{\gamma-1}+|\eta|^{\gamma-1}), \qquad \sigma, \eta \in \boldsymbol{R},$$

where $C>0$ is also a constant. From this latter inequality we obtain part (ii) ∎

2.3. Continuous dependence on initial data. Concerning the continuity of the generalized solution of equation (1.2) with respect to the initial value, we have the following result uniform in κ.

THEOREM 2.8. *Assume* (A.1) *and* (A.2) *hold. Then for arbitrary initial values* $x, y \in H$ *and all* $\kappa \geqslant 1$, *the generalized solution of* (1.2) *satisfies*

$$|\tilde{X}^{(\kappa)}(t, x)-\tilde{X}^{(\kappa)}(t, y)|_H \leqslant e^{Kt}|x-y|_H \text{ a.s.}, \qquad t \geqslant 0.$$

Proof. By the definition of the generalized solution it is enough to prove the theorem for $x, y \in E$ and for the corresponding mild solutions. Let $X^{(\kappa)}(t, x)$ and $X^{(\kappa)}(t, y)$ be two solutions of (1.2) corresponding to initial values $x, y \in E$. Then, going if necessary to smooth approximations of the solutions, we can assume that $X^{(\kappa)}(t, x)-X^{(\kappa)}(t, y)$ satisfies the following problem strongly:

$$du/dt = \kappa Au(t)+F(X^{(\kappa)}(t, x))-F(X^{(\kappa)}(t, y)), \qquad u(0) = x-y.$$

We have

$$\frac{1}{2}\frac{d}{dt}|X^{(\kappa)}(t, x)-X^{(\kappa)}(t, y)|_H^2$$
$$= \kappa\langle A(X^{(\kappa)}(t, x)-X^{(\kappa)}(t, y)), X^{(\kappa)}(t, x)-X^{(\kappa)}(t, y)\rangle$$
$$+\langle F(X^{(\kappa)}(t, x))-F(X^{(\kappa)}(t, y)), X^{(\kappa)}(t, x)-X^{(\kappa)}(t, y)\rangle,$$

which, by the non-positivity of A and Lemma 2.7 (i), implies

$$\frac{1}{2}\frac{d}{dt}|X^{(\kappa)}(t, x)-X^{(\kappa)}(t, y)|_H^2 \leqslant K|X^{(\kappa)}(t, x)-X^{(\kappa)}(t, y)|_H^2,$$

and, by the Gronwall lemma,

$$|X^{(\kappa)}(t, x)-X^{(\kappa)}(t, y)|_H \leqslant e^{Kt}|x-y|_H, \quad t \geqslant 0. \blacksquare$$

A similar result holds for the solution of equation (1.4).

THEOREM 2.9. *Assume that* (A.1) *is satisfied. Then for arbitrary initial values* $a, b \in \boldsymbol{R}$ *the solution of equation* (1.4) *satisfies*

$$|Y(t, a)-Y(t, b)| \leqslant e^{Kt}|a-b| \text{ a.s.}, \quad t \geqslant 0.$$

3. PROOF OF THE MAIN RESULT

We start with the following basic lemma:

LEMMA 3.1. *Assume that* (A.1) *and* (A.2) *are satisfied and the sequence* $\{\kappa_n\}_{n=1}^{\infty}$ *belongs to* $\mathscr{K}(\beta)$ *for a constant* $\beta > 0$. *Further, suppose* $T > 0$ *and* $x \in E$. *Let* $p > 1/(1-\alpha)$ *be arbitrary and* $\mathscr{G}$ *the corresponding set as mentioned in Corollary* 2.2. *Then for every* $\omega \in \mathscr{G}$ *there exists* $n_0(\omega) \in N$ *and* $C = C(\omega) > 0$ *such that*

$$|X^{(\kappa_n)}(t, x, \omega)-Y(t, \hat{x}, \omega)|_E \leqslant C(\exp\{-\lambda_1\kappa_n t\}+\kappa_n^{-1})|x|_E+C\kappa_n^{-1/2p},$$
$$n \geqslant n_0(\omega), \ t \in [0, T].$$

Proof. Let $x \in E$ and $\omega \in \mathscr{G}$ be fixed, and denote the mild solution of (1.2) and the solution of (1.4) by $X^{(\kappa)}(t)$ and $Y(t)$, respectively. Let $t \in [0, T]$. We recall that $X^{(\kappa)}(t)$ satisfies the following integral equation:

$$X^{(\kappa)}(t) = S(\kappa t)x+\int_0^t S(\kappa(t-s))F(X^{(\kappa)}(s))ds+\int_0^t S(\kappa(t-s))dW(s),$$

which, since λ_0 is zero and, consequently, $S(\kappa t) = P_1 \oplus P_1^{\perp} S(\kappa t)$, can be written as

$$X^{(\kappa)}(t) = P_1 x+\int_0^t P_1 F(X^{(\kappa)}(s))ds+\sqrt{\gamma_0}\beta_0(t)+P_1^{\perp}X^{(\kappa)}(t).$$

By (1.5) we have

$$Y(t) = P_1 x+\int_0^t P_1 F(Y(s))ds+\sqrt{\gamma_0}\beta_0(t),$$

whence

$$X^{(\kappa)}(t)-Y(t)=\int_0^t P_1\left[F\left(X^{(\kappa)}(s)\right)-F\left(Y(s)\right)\right]ds+P_1^{\perp}X^{(\kappa)}(t),$$

and

$$(3.1)\qquad |X^{(\kappa)}(t)-Y(t)|_E\leqslant\Big|\int_0^t P_1\left[F\left(X^{(\kappa)}(s)\right)-F\left(Y(s)\right)\right]ds\Big|_E+|P_1^{\perp}X^{(\kappa)}(t)|_E.$$

We start with the second term in the right-hand side of (3.1). We have

$$P_1^{\perp}X^{(\kappa)}(t)=U(\kappa t)x+\int_0^t U\left(\kappa(t-s)\right)F\left(X^{(\kappa)}(s)\right)ds+\int_0^t U\left(\kappa(t-s)\right)dW(s),$$

where $U(t)$, $t\geqslant 0$, is the semigroup generated by A_0, as mentioned before. Consequently,

$$(3.2)\qquad |P_1^{\perp}X^{(\kappa)}(t)|_E\leqslant|U(\kappa t)x|_E+\Big|\int_0^t U\left(\kappa(t-s)\right)F\left(X^{(\kappa)}(s)\right)ds\Big|_E+\Big|\int_0^t U\left(\kappa(t-s)\right)dW(s)\Big|_E.$$

First, we obtain

$$|U(\kappa t)x|_E\leqslant\|U(\kappa t)\|_{L(E)}|x|_E\leqslant\exp\{-\lambda_1\kappa t\}|x|_E.$$

Next, for a constant $C>0$,

$$\begin{aligned}\Big|\int_0^t U\left(\kappa(t-s)\right)F\left(X^{(\kappa)}(s)\right)ds\Big|_E&\leqslant\int_0^t\left|U\left(\kappa(t-s)\right)F\left(X^{(\kappa)}(s)\right)\right|_E ds\\&\leqslant C\int_0^t\exp\{-\lambda_1\kappa s\}\left(1+|X^{(\kappa)}(s)|_E^{\gamma}\right)ds\\&\leqslant C\sup_{s\in[0,t]}\left(1+|X^{(\kappa)}(s)|_E^{\gamma}\right)\int_0^t\exp\{-\lambda_1\kappa s\}ds\\&\leqslant\sup_{s\in[0,t]}\left(1+|X^{(\kappa)}(s)|_E^{\gamma}\right)\frac{C}{\lambda_1\kappa}.\end{aligned}$$

Setting

$$M=M(T)=\frac{C}{\lambda_1}\sup_{n\in N}\sup_{t\in[0,T]}\left(1+|X^{(\kappa_n)}(t)|_E^{\gamma}\right),$$

which is a finite number by Theorem 2.5, we get

$$\Big|\int_0^t U\left(\kappa_n(t-s)\right)F\left(X^{(\kappa_n)}(s)\right)ds\Big|_E\leqslant M\kappa_n^{-1}.$$

Finally, for the last term of (3.2) by Corollary 2.2 we have

$$\Big|\int_0^t U\left(\kappa_n(t-s)\right)dW(s)\Big|_E\leqslant\kappa_n^{-1/2p},\qquad n\geqslant n_0(\omega).$$

Summing up

$$(3.3)\qquad |P_1^{\perp} X^{(\kappa_n)}(t)|_E \leqslant \exp\{-\lambda_1 \kappa_n t\}\,|x|_E + M\kappa_n^{-1} + \kappa_n^{-1/2p}, \qquad n \geqslant n_0(\omega).$$

For the first term in the right-hand side of (3.1) we have

$$\begin{aligned}\Big|\int_0^t P_1\big[F(X^{(\kappa)}(s)) - F(Y(s))\big]\,ds\Big|_E &\leqslant \int_0^t \big|P_1\big[F(X^{(\kappa)}(s)) - F(Y(s))\big]\big|_E\,ds\\ &\leqslant \int_0^t \big|F(X^{(\kappa)}(s)) - F(Y(s))\big|_E\,ds,\end{aligned}$$

and, by Lemma 2.7 (ii), the last term in the inequality may be estimated by

$$\begin{aligned}&C\int_0^t |X^{(\kappa)}(s) - Y(s)|_E\big(1 + |X^{(\kappa)}(s)|_E^{\gamma-1} + |Y(s)|_E^{\gamma-1}\big)\,ds\\ &\qquad\leqslant C\sup_{s\in[0,t]}\big(1 + |X^{(\kappa)}(s)|_E^{\gamma-1} + |Y(s)|_E^{\gamma-1}\big)\int_0^t |X^{(\kappa)}(s) - Y(s)|_E\,ds.\end{aligned}$$

Setting

$$L = L(T) = C\sup_{n\in N}\sup_{t\in[0,T]}\big(1 + |X^{(\kappa_n)}(t)|_E^{\gamma-1} + |Y(t)|_E^{\gamma-1}\big),$$

which is a finite number by Theorems 2.5 and 2.6, we obtain

$$(3.4)\qquad \Big|\int_0^t P_1\big[F(X^{(\kappa_n)}(s)) - F(Y(s))\big]\,ds\Big|_E \leqslant L\int_0^t |X^{(\kappa_n)}(s) + Y(s)|_E\,ds.$$

Combining (3.1), (3.3) and (3.4), for $n \geqslant n_0(\omega)$ we have

$$\begin{aligned}&|X^{(\kappa_n)}(t) - Y(t)|_E\\ &\qquad\leqslant L\int_0^t |X^{(\kappa_n)}(s) - Y(s)|_E\,ds + \exp\{-\lambda_1 \kappa_n t\}\,|x|_E + M\kappa_n^{-1} + \kappa_n^{-1/2p}.\end{aligned}$$

Since $\kappa_n^{-1} \leqslant \kappa_n^{-1/2p}$ for all $n \in N$, for $D = D(T) = M + 1$ we obtain

$$\begin{aligned}&|X^{(\kappa_n)}(t) - Y(t)|_E\\ &\quad\leqslant L\int_0^t |X^{(\kappa_n)}(s) - Y(s)|_E\,ds + \exp\{-\lambda_1 \kappa_n t\}\,|x|_E + D\kappa_n^{-1/2p}, \qquad n \geqslant n_0(\omega).\end{aligned}$$

By the Gronwall inequality, we get

$$\begin{aligned}&|X^{(\kappa_n)}(t) - Y(t)|_E\\ &\leqslant \left(\exp\{-\lambda_1 \kappa_n t\} + \frac{e^{LT}}{\lambda_1}\kappa_n^{-1}\right)|x|_E + D\kappa_n^{-1/2p} + \frac{e^{LT}}{L}D\kappa_n^{-1/2p}, \qquad n \geqslant n_0(\omega),\end{aligned}$$

or, for a suitable $C = C(T) > 0$,

$$|X^{(\kappa_n)}(t) - Y(t)|_E \leqslant C\big(\exp\{-\lambda_1 \kappa_n t\} + \kappa_n^{-1}\big)|x|_E + C\kappa_n^{-1/2p}, \qquad n \geqslant n_0(\omega). \quad \blacksquare$$

Remark 3.2. The assertion of the lemma holds also in the H-norm. That is,

$$\sup_{t\in[0,T]} |X^{(\kappa_n)}(t, x, \omega) - Y(t, \hat{x}, \omega)|_H \leqslant C(\exp\{-\lambda_1 \kappa_n t\} + \kappa_n^{-1})|x|_H + C\kappa_n^{-1/2p}, \quad n \geqslant n_0(\omega).$$

Notice that in the left-hand side of the inequalities in the previous lemma the E-norms may always be replaced by H-norms as the latters are smaller, and also

$$|U(\kappa t)x|_H^2 = \sum_{i=1}^{\infty} \langle U(\kappa t)x, e_i\rangle^2 = \sum_{i=1}^{\infty} \exp\{-2\lambda_i \kappa t\} \langle x, e_i\rangle^2 \leqslant \exp\{2\lambda_1 \kappa t\}|x|_H^2, \quad t\in[0, T].$$

Now we can complete the proof of the theorem.

(i) Let $0 < T_1 < T_2 < \infty$ be arbitrary. Then, by Lemma 3.1, for every $\omega \in \mathcal{G} = \mathcal{G}_{T_2}$ we have

$$\sup_{t\in[T_1,T_2]} |X^{(\kappa_n)}(t, x, \omega) - Y(t, \hat{x}, \omega)|_E \leqslant C(\exp\{-\lambda_1 \kappa_n T_1\} + \kappa_n^{-1})|x|_E + C\kappa_n^{-1/2p}, \quad n \geqslant n_0(\omega).$$

This inequality implies

$$\lim_{n\to\infty} \sup_{t\in[T_1,T_2]} |X^{(\kappa_n)}(t, x) - Y(t, \hat{x})|_E = 0 \text{ a.s.},$$

which, as T_1 and T_2 are arbitrary, states part (i) of the theorem.

(ii) Let $0 < T_1 < T_2 < \infty$ be arbitrary. Fix $x \in H$ and $\omega \in \mathcal{G} = \mathcal{G}_{T_2}$. For every $y \in E$ and $t \in [T_1, T_2]$ we may write

$$|\tilde{X}^{(\kappa)}(t, x) - Y(t, \hat{x})|_H \leqslant |\tilde{X}^{(\kappa)}(t, x) - X^{(\kappa)}(t, y)|_H + |X^{(\kappa)}(t, y) - Y(t, \hat{y})|_H + |Y(t, \hat{y}) - Y(t, \hat{x})|_H.$$

Now suppose $\varepsilon > 0$ is given and let $y_\varepsilon \in E$ be such that

$$|x - y_\varepsilon|_H < \frac{\varepsilon}{2}\exp\{-KT_2\}.$$

Then, using Theorem 2.8, Remark 3.2, and Theorem 2.9, we obtain

$$\begin{aligned} |\tilde{X}^{(\kappa_n)}(t, x, \omega) - Y(t, \hat{x}, \omega)|_H &\leqslant e^{Kt}|x - y_\varepsilon|_H + C_\varepsilon(\exp\{-\lambda_1 \kappa_n t\} + \kappa_n^{-1})|y_\varepsilon|_H \\ &\quad + C_\varepsilon \kappa_n^{-1/2p} + e^{Kt}|\hat{x} - \hat{y}_\varepsilon| \\ &\leqslant 2e^{Kt}|x - y_\varepsilon|_H + C_\varepsilon(\exp\{-\lambda_1 \kappa_n t\} + \kappa_n^{-1})|y_\varepsilon|_H \\ &\quad + C_\varepsilon \kappa_n^{-1/2p} \end{aligned}$$

for $t \in [T_1, T_2]$ and all $n \geqslant n_0(\omega)$, where C_ε is the constant C in Remark 3.2 for y_ε. By the choice of y_ε we have

$$(3.5) \qquad \sup_{t \in [T_1, T_2]} |\tilde{X}^{(\kappa_n)}(t, x, \omega) - Y(t, \hat{x}, \omega)|_H \leqslant \varepsilon + C_\varepsilon (\exp\{-\lambda_1 \kappa_n T_1\} + \kappa_n^{-1}) |y_\varepsilon|_H + C_\varepsilon \kappa_n^{-1/2p}$$

for all $n \geqslant n_0(\omega)$. That is

$$\lim_{n \to \infty} \sup_{t \in [T_1, T_2]} |\tilde{X}^{(\kappa_n)}(t, x) - Y(t, \hat{x})|_H \leqslant \varepsilon \text{ a.s.}$$

As $\varepsilon > 0$ and T_1 and T_2 are arbitrary, we obtain the desired result. ∎

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