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ON MARTINGALE MEASURES FOR STOCHASTIC PROCESSES WITH DISCRETE TIME

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Abstract. Let $(X(t); t \in N^+)$ be a random sequence adopted to a filtration $(\mathscr{F}_t)$ in $(\Omega, \mathscr{F}, P)$ satisfying some natural assumption. If none of the events $(X(t+1) > X(t))$, $(X(t+1) < X(t))$ can be predicted, i.e. none contains some $A \in \mathscr{F}_t$, $P(A) > 0$, then $(X(t), \mathscr{F}_t)$ is a martingale for some probability P^* on $\mathscr{F}$. It is a version of the "fundamental theorem of option pricing".

1. Introduction. Let $X(t)$, $t \in \boldsymbol{R}$, be a stochastic process. If $X(t) = e^{mt+\sigma w(t)}$ with $w(t)$ being a Wiener process, then $X(t)$ becomes a martingale with respect to P^* being a probability equivalent to the original one P. This theory, initiated by Girsanov, has been very tempting and widely researched for the last 30 years (we only mention monographs [4] and [11]–[13]). As one of the most famous applications of the theory one should mention the Black–Scholes model describing a replication strategy for European options (see [1], [8], [10] and [12]).

In the so-called financial mathematics, many efforts were also devoted to the formulation of the so-called "no free lunch" condition which, in more general situations, guarantees the existence of a martingale measure P^* equivalent to the original probability P. The notion of free lunch is defined (in a non-effective way) by the use of some space of strategies $\Theta(t)$ being stochastic processes predictable for some filtration $(\mathscr{F}_t)$. The construction of the martingale measure P^* is obtained by some development of the Banach–Mazur theory of the separating of convex sets (cf. [3], [7]–[10] and [12]). Free lunch conditions look simpler for processes indexed by discrete finite times (cf. [2] and [6]).

In the paper we use one scalar stochastic process $X(t)$ which corresponds to the simplest case of one security. The strategy is described by our position $\Theta(t)$ in the security. We assume that all our outcomes and incomes are cumulated in a riskless bond.

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We propose a simple condition (analogous to that of Dalang–Morton–Willinger with zero interest rate [2]) which assures the existence of a martingale measure. This condition, later referred to as the *change of sign property*, states that

$$P\big((X(t) > X(s)) \cap A\big) > 0 \Leftrightarrow P\big((X(t) < X(s)) \cap A\big) > 0$$

for any $A \in \mathscr{F}_s$, $s < t$. Our arguments are rather classical. The required martingale measure P^* is obtained by the Kolmogorov extension theorem (see [4] and [13]). The main result is contained in Theorem 3.3.

2. Elementary examples. To explain the possibilities and restrictions appearing in constructing a martingale measure, let us consider some elementary examples.

2.1. Example. Suppose we are tossing a symmetric coin. Assume that $\omega = (\varepsilon_1, \varepsilon_2, \ldots)$ is a sequence of outcomes, $\varepsilon_i = 0$ or 1 depending on the result of the i-th toss. Let $\mathscr{F}_0 = \{\emptyset, \Omega\}$, $\mathscr{F}_i = \sigma(\varepsilon_1, \ldots, \varepsilon_i)$ (i.e., a σ-field generated by random variables $\varepsilon_1, \ldots, \varepsilon_i$) and $\mathscr{F} = \sigma(\varepsilon_1, \varepsilon_2, \ldots)$. Let $X(t) = \sum_{i=1}^{t} (\varepsilon_i - \beta)$ for some $\beta \in (0, 1)$. Then $X(t)$ is a martingale with respect to the sequence $(\mathscr{F}_n)$ for $\beta = \frac{1}{2}$. For $\beta \neq \frac{1}{2}$, $X(t)$ becomes a martingale if the original probability $P(\varepsilon_i = 0) = P(\varepsilon_i = 1) = \frac{1}{2}$ is replaced by $P^*(\varepsilon_i = 1) = \beta = 1 - P^*(\varepsilon_i = 0)$, which corresponds to an asymmetric coin. Moreover, P^* is uniquely determined. Thus each martingale measure P^* satisfies

$$P^*\left(\left\{\omega;\ \lim_{n\to\infty} \frac{1}{n}(\varepsilon_1 + \ldots + \varepsilon_n) = \beta\right\}\right) = 1$$

by the strong law of large numbers, while

$$P\left(\left\{\omega;\ \lim_{n\to\infty} \frac{1}{n}(\varepsilon_1 + \ldots + \varepsilon_n) = \frac{1}{2}\right\}\right) = 1.$$

Thus P and P^* are singular for $\beta \neq \frac{1}{2}$.

When $X(t)$ is indexed by an infinite set of t's, it is impossible to obtain a martingale measure P^* equivalent to P.

2.2. Example. As previously, we toss a coin obtaining outcomes $\omega = (\varepsilon_1, \varepsilon_2, \ldots)$. Let us put

$$\Omega^0 = \left\{\omega;\ \lim_{n\to\infty} \frac{1}{n}(\varepsilon_1 + \ldots + \varepsilon_n) = \frac{1}{2}\right\} \quad (\text{then } P(\Omega^0) = 1),$$

$$\mathscr{F}^0 = \{A \cap \Omega^0;\ A \in \mathscr{F} = \sigma(\varepsilon_1, \varepsilon_2, \ldots)\}, \quad X^0(t) = X(t)|_{\Omega^0}.$$

Since $P(\Omega^0) = 1$, the finite-dimensional distributions of the processes $X^0(t)$ and $X(t)$ are identical.

Suppose that there exists a martingale measure P_0^* on $(\Omega^0, \mathscr{F}^0)$ for the process $X^0(t)$. Then $P_0^*(\varepsilon_i = 1) = \beta = 1 - P_0^*(\varepsilon_i = 0)$ and, by the strong law of large numbers,

$$P_0^*(\Omega^0) = P_0^*\left\{\omega;\ \lim_{n\to\infty} \frac{1}{n}(\varepsilon_1 + \ldots + \varepsilon_n) = \frac{1}{2} \neq \beta\right\} = 0,$$

which is a contradiction.

It is worth noting that Ω^0 is not a closed set in the Tikhonov topology in $\Omega = \{0, 1\}^{N^+}$ (namely, $\overline{\Omega^0} = \Omega$). We shall show that the closure of the set of trajectories of the process is a natural support of a martingale measure P^*.

3. Main results. Let $Y(t)$, $t \in N^+$, be a stochastic process on a probability space $(\Omega, \mathscr{F}, P)$. By $Y(t)$ we also denote its canonical representation on the space $(\boldsymbol{R}^{N^+}, \sigma(\mathscr{C}), P_Y)$. Thus

1° $Y(t)(\omega) = \varepsilon_t$ for $\omega = (\varepsilon_1, \varepsilon_2, \ldots) \in \boldsymbol{R}^{N^+}$;

2° $\mathscr{C} = \bigcup_{n \in N^+} \mathscr{C}_n$;

3° $\mathscr{C}_n = \{\mathscr{C}_n(A^{(n)}); (A^{(n)}) \in B_{\boldsymbol{R}^n}\}$;

4° $\mathscr{C}_n(A^{(n)}) = \{(\varepsilon_1, \varepsilon_2, \ldots) \in \boldsymbol{R}^{N^+}; (\varepsilon_1, \ldots, \varepsilon_n) \in A^{(n)}\}$, $A^{(n)} \in B_{\boldsymbol{R}^n}$ (i.e., σ-fields of Borel sets in $\boldsymbol{R}^n$);

5° $P_Y(\mathscr{C}_n(A^{(n)})) = P_n(A^{(n)})$ for a finite-dimensional distribution

$$P_n(A^{(n)}) = P\big((Y(1), \ldots, Y(n)) \in A^{(n)}\big)$$

for $n \in N^+$. Obviously, the image $Y[\Omega]$ can be treated as a subspace of $\boldsymbol{R}^{N^+}$ (proper, in general).

We need some modification of the classical Kolmogorov theorem. To explain new elements precisely, we decided to formulate two self-interesting lemmas. The following exercise will be used. For any set $T \subset X$ and any family $\mathscr{R} \subset 2^X$, we have

$$\sigma(\mathscr{R} \cap T) = \sigma(\mathscr{R}) \cap T.$$

Obviously, $\mathscr{R} \cap T$ means $\{R \cap T; R \in \mathscr{R}\}$.

3.1. Lemma. *If, in the introduced notation 2°–4°, T is any set closed in $\boldsymbol{R}^{N^+}$ in the Tikhonov topology, P_n is a probability distribution on $\mathscr{C}_n \cap T$, $n \in N^+$, satisfying*

$$\text{(c)} \qquad P_{n+1}(\mathscr{C}_{n+1}(A_n \times R) \cap T) = P_n(\mathscr{C}_n(A_n) \cap T),$$

then there exists a uniquely defined probability measure P on $\sigma(\mathscr{C} \cap T) = T \cap \sigma(\mathscr{C})$ such that

$$P_n = P|_{\mathscr{C}_n \cap T}.$$

Proof. Let $\mathscr{T}_n = T \cap \mathscr{C}_n$, $\mathscr{T} = \bigcup_n (T \cap \mathscr{C}_n) = T \cap \mathscr{C}$. For $B \in \mathscr{T}$, taking any representation B in the form $B = T \cap \mathscr{C}_n(A_n)$, we can uniquely define the function

$$Q(B) = P_n(T \cap \mathscr{C}_n(A_n))$$

which is finitely-additive and normed on $\mathscr{T}$. It remains only to prove 'continuity'. Let $B_1 \supset B_2 \supset \ldots$, $Q(B_i) \geqslant \varepsilon > 0$. To use the classical Kolmogorov construction (see [4] and [13]), one has to show that $\bigcap B_i \neq \emptyset$. We consider

$$\tilde{p}_n(C_n) = P_n(T \cap C_n) \quad \text{for } C_n \in \mathscr{C}_n,$$

obtaining a consistent system of distributions on $\mathscr{C}_n$'s.

From the Kolmogorov lemma we infer that if $C_1 \supset C_2 \supset C_3 \supset \ldots$ and $\tilde{p}_n(C_n) \geqslant \varepsilon > 0$, then there exists $\omega \in \bigcap_i C_i$.

We put $C_n = \mathscr{C}_n(A_n \cap T_n)$ for projections

$$T_n = \{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n);\ (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \eta_{n+1}, \eta_{n+2}, \ldots) \in T \quad \text{for some } \eta_{n+1}, \eta_{n+2}, \ldots\},$$

assuming that $B_n = \mathscr{C}_n(A_n)$.

Let ω be as in the Kolmogorov lemma. Since T is closed, we have $\omega \in T$ and

$$\omega \in T \cap \mathscr{C}_n(A_n) = \bigcap_n B_n.$$

The lemma is proved.

3.2. LEMMA. *If, in the introduced notation,* $\mathscr{F} = \sigma(Y(1), Y(2), \ldots)$ *and* $Y[\Omega]$ *is closed in* R^{N^+} *in the Tikhonov topology, and if* $P_Y^*|_{\mathscr{C}_n} \sim P_Y|_{\mathscr{C}_n}$ *for some probability measure* P_Y^* *on* $\sigma(\mathscr{C})$, *then there exists a uniquely defined probability measure* P^* *on* $\mathscr{F}$ *satisfying*

$$(1) \qquad P_Y^*(A) = P^*(Y^{-1}[A]), \quad A \in \sigma(\mathscr{C}).$$

Proof. We put $T = Y[\Omega]$ and define $P_n(C_n \cap T) = P_Y^*(C_n)$ for $C_n \in \mathscr{C}_n$. Distributions P_n are well defined: if $C_n \cap T = C_n' \cap T$, then

$$(C_n \Delta C_n') \cap T = \emptyset;$$

it follows that $P_Y(C_n \Delta C_n') = 0$, so $P_n^*(C_n \Delta C_n') = 0$.

The condition of consistency (c) in Lemma 3.1 is obvious from the definition of P_n's. The probability measure P on $T \cap \sigma(\mathscr{C})$ exists by Lemma 3.1, and $P_n = P|_{\mathscr{C}_n \cap T}$.

The measure P^* that is being looked for can be defined by the formula

$$P^*(Y^{-1}(A)) = P(A \cap Y[\Omega]) \quad \text{for } A \in \sigma(\mathscr{C}).$$

The measure P on $\sigma(\mathscr{C}) \cap T$ corresponds to a measure $P_Y^{**}(A) = P(A \cap T)$ on $\sigma(\mathscr{C})$. But from $P_n = P|_{\mathscr{C}_n \cap T}$ we get $P_Y^{**}(C_n) = P_Y^*(C_n)$ for $C_n \in \mathscr{C}$. The uniqueness of the extension of a countable additive function completes the proof (cf. [5]).

Remark. Obviously, to prove Lemma 3.2, it is enough to show that $P_Y^* = 0$ for any $A \in \sigma(\mathscr{C})$ disjoint from $Y[\Omega]$, or that $A \cup Y[\Omega] \neq \emptyset$ when $P_Y^*(A) = \varepsilon > 0$. It seems natural to repeat Kolmogorov's arguments for decreasing cylinders $C_1 \supset C_2 \supset \ldots$ defined by projections of A,

$$C_n = \{(\varepsilon_1, \varepsilon_2, \ldots);\ (\varepsilon_1, \ldots, \varepsilon_n, \eta_{n+1}, \ldots) \in A \text{ for some } \eta_{n+1}, \eta_{n+2}, \ldots\}.$$

An element $\omega \in \bigcap_n C_n$ satisfies $\omega \in Y[\Omega]$ but it may happen that $\omega \notin A$. Lemma 3.2 cannot be obtained in such a way.

For a sequence of bounded random variables $(X(t),\ t\in N^+)$ on a probability space $(\Omega, \mathscr{F}, P)$, let

(2)
$$\mathscr{F} = \sigma\big(X(0), X(1), \ldots\big), \quad X(0) = 0,$$
$X[\Omega] = \{X(t)(\omega);\ \omega\in\Omega\}$ is a closed set in the Tikhonov topology in R^{N^+}.

Let us write $\mathscr{F}_t = \sigma\big(X(0), \ldots, X(t)\big)$, $t\in N$.

3.3. THEOREM. *Under assumption* (2) *the following conditions are equivalent*:

(i) $P\big(A\cap(X(t+1) > X(t))\big) > 0 \Leftrightarrow P\big(A\cap(X(t+1) < X(t))\big) > 0$ *for any* $t\in N$, $A\in\mathscr{F}_t$ *(the change of sign property)*;

(ii) *there exists a measure* P^* *on* $\mathscr{F}$ *for which* $\big(X(t)\big)$ *is a martingale with respect to* $(\mathscr{F}_t)$, *and* $P^*|_{\mathscr{F}_t} \sim P|_{\mathscr{F}_t}$, $t\in N$.

Proof. It is enough to prove that (i) implies (ii). Let us put $Y(t) = X(t)-X(t-1)$, $t = 1, 2, \ldots$ To use Lemma 3.1, we discuss, at first, a canonical representation $\big(R^N, \sigma(\mathscr{C}), P_Y\big)$ for the process $Y(t)$. There exists a measure P_Y^* on $\sigma(\mathscr{C})$ (cf. notation 3° and 2° at the beginning of Section 3) satisfying

$$E_{P_Y^*}^{\mathscr{C}_t} Y(t+1) = 0 \tag{3}$$

(for conditional expectation with respect to a σ-field $\mathscr{C}_t$ and a probability P_Y^*),

$$P_Y^*|_{\mathscr{C}_t} \sim P_Y|_{\mathscr{C}_t}, \quad t\in N. \tag{4}$$

To show this, we define by induction a sequence of probabilities $P(t)$ on $\sigma(\mathscr{C})$ satisfying

$$P(t+1)|_{\mathscr{C}_t} = P(t)|_{\mathscr{C}_t}, \tag{5}$$

$$E_{P(t)}^{\mathscr{C}_t} Y(t+1) = 0, \quad t\in N. \tag{6}$$

Let $P(0) = P_Y$. Define $\varphi_1(\omega) \equiv 1$ if $P_Y\big(Y(1) > 0\big) = 0$; otherwise

$$\varphi_1(\omega) = \begin{cases} x(\omega) & \text{for } Y(1)(\omega) > 0, \\ 1 & \text{for } Y(1)(\omega) = 0, \\ y(\omega) & \text{for } Y(1)(\omega) < 0 \end{cases}$$

with x, y uniquely determined by

$$x(\omega)E_{P_Y}^{\mathscr{C}_0} Y(1)^+ - y(\omega)E_{P_Y}^{\mathscr{C}_0} Y(1)^- = 0,$$

$$x(\omega)E_{P_Y}^{\mathscr{C}_0} 1_{(Y(1)>0)} + y(\omega)E_{P_Y}^{\mathscr{C}_0} 1_{(Y(1)<0)} = E_{P_Y}^{\mathscr{C}_0} 1_{(Y(1)\neq 0)}$$

with $\mathscr{C}_0 = \{\emptyset, R^{N^+}\}$, and $1_{(Y(1)>0)}(\omega) = 0$ or 1 when $\omega\notin\big(Y(1) > 0\big)$ or $\omega\in\big(Y(1) > 0\big)$. Then

$$P(0)|_{\mathscr{C}_0} = P(1)|_{\mathscr{C}_0},$$

$$E_{P(1)}^{\mathscr{C}_0} Y(1) = 0 \quad \text{for } dP(1)/dP_Y = \varphi_1.$$

Assume that $P(0), \ldots, P(n)$ are defined so that (5) and (6) are satisfied for $t = 0, \ldots, n-1$. Let $\varphi_{n+1}(\omega) \equiv 1$ if $P(n)\big(Y(n+1) > 0\big) = 0$; otherwise

$$\varphi_{n+1}(\omega) = \begin{cases} x(\omega) & \text{for } Y(n+1)(\omega) > 0, \\ 1 & \text{for } Y(n+1)(\omega) = 0, \\ y(\omega) & \text{for } Y(n+1)(\omega) < 0, \end{cases}$$

where $x(\omega)$ and $y(\omega)$ are uniquely determined almost everywhere on a set $\big(Y(n+1) \neq 0\big)$ by

$$x(\omega) E_{P(n)}^{\mathscr{C}_n} Y(n+1)^+ - y(\omega) E_{P(n)}^{\mathscr{C}_n} Y(n+1)^- = 0,$$

$$x(\omega) E_{P(n)}^{\mathscr{C}_n} 1_{(Y(n+1)>0)} + y(\omega) E_{P(n)}^{\mathscr{C}_n} 1_{(Y(n+1)<0)} = E_{P(n)}^{\mathscr{C}_n} 1_{(Y(n+1)\neq 0)}.$$

Then we obtain (5) and (6) with $t = n$ for $P(n+1)$ defined by

$$dP(n+1)/dP_Y = \varphi_{n+1}.$$

By the Kolmogorov extension theorem, the measure P_Y^*, $P_Y^*|_{\mathscr{C}(t)} = P(n)|_{\mathscr{C}(t)}$, is uniquely defined on $\sigma(\mathscr{C})$, and conditions (3) and (4) are satisfied.

Let us return to the space Ω. For bounded random variables $Y(t)$, assumption (2) implies that $Y[\Omega]$ is closed in $\boldsymbol{R}^N$. Then formula (1) defines a probability P^* on $\mathscr{F} = \sigma\big(Y(1), Y(2), \ldots\big)$ by virtue of Lemma 3.2. Equivalence (4) implies $P^*|_{\mathscr{F}_t} \sim P|_{\mathscr{F}_t}$ for $\mathscr{F}_t = \sigma\big(Y(0), \ldots, Y(t)\big)$ as $\mathscr{F}_t = Y^{-1}(\mathscr{C}_t)$. The equality $E_{P^*}^{\mathscr{F}_t} Y(t+1) = 0$ is a consequence of (3) by elementary changes of variables in integrals.

Obviously, $\big(X(t)\big) = \big(X(0) + Y(1) + \ldots + Y(t)\big)$ is a martingale with respect to P^*, and σ-fields $\sigma\big(X(0), \ldots, X(t)\big) = \sigma\big(Y(1), \ldots, Y(t)\big)$.

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