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MALLIAVIN CALCULUS FOR STABLE PROCESSES ON HEISENBERG GROUP

BY

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Abstract. Smoothness of symmetric stable semigroups and some related semigroups of measures on the Heisenberg group is studied using Malliavin calculus for jump processes. If the Lévy measure of a symmetric stable semigroup is $\mathscr{C}^m$, then the semigroup is $\mathscr{C}^{2m-4}$. If the Lévy measure of a truncated stable semigroup is $\mathscr{C}^1$, then the semigroup is $\mathscr{C}^\infty$.

0. Introduction. Smoothness of stable semigroups of measures on homogeneous Lie groups was examined by analytical methods by Głowacki ([3], [4] in the case of the Heisenberg group, [5] generally) and recently by the second-named author [6], using Malliavin calculus for jump processes. In [5] and [6] it was proved that if the Lévy measure of a symmetric stable semigroup is smooth, then the semigroup itself has smooth densities.

In this paper* we examine smoothness of α-stable semigroups of measures $(\mu_t)_{t>0}$ on the Heisenberg group, with Lévy measure ν of class $\mathscr{C}^m$, $m < \infty$. In particular we show that if $\nu \in \mathscr{C}^m$, then $\mu_t \in \mathscr{C}^{2m-4}$ (if $\alpha > 1$, then $\mu_t \in C^{2m-3}$). This kind of implication may not be obtained by applying inequalities of Sobolev type.

We prove also the smoothness of a symmetric semigroup of measures $(\mu_t)_{t>0}$ with the Lévy measure of class $\mathscr{C}^1$. We assume that the Lévy measure of $(\mu_t)_{t>0}$ has a density of class $\mathscr{C}^1$ with compact support, coinciding on a neighbourhood of 0 with the density of a stable Lévy measure. We call such semigroups *truncated stable*. Truncated stable processes appear in some problems of stochastic analysis even in the case of Euclidean spaces (cf. [8]).

Our results are new and do not follow from the analytical methods of Głowacki. They are proved by using methods of Malliavin calculus. In particular, following Bismut [1] we avoid iteration of integration by parts

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on the same interval as it was made in [6]. The Markov property of the corresponding stochastic process is highly exploited here. However, the noncommutativity requires a subtle stochastic analysis using properties of the adjoint representation and martingale methods.

For simplicity, we present our results in the case of the Heisenberg group $\boldsymbol{H}$. The proofs can be generalized immediately to the case of nilpotent homogeneous Lie groups of order 2. In the case of higher order the computations become complicated but they are still feasible.

This work is an extension of [6]. We use the same notation and we repeat some fragments of the proofs of [6]. Theorem 3 of this paper generalizes the final Corollary 5.7 of [6] but its proof is much simpler and direct and it provides more effective estimates of L_1-norms of derivatives of considered measures.

Section 1 has a preliminary character. In Section 2 we present some rather technical lemmas concerning stochastic integrals on $\boldsymbol{H}$ and convergence on the Skorohod space $D_{\boldsymbol{H}}$. Section 3 contains the result concerning truncated stable semigroups. In Section 4 we extend this result to stable semigroups and to some other semigroups with Lévy measure of noncompact support.

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1. Preliminaries. In this paper we consider the Heisenberg group $\boldsymbol{H} \cong \boldsymbol{R}^3$ with the group product

$$(\sigma_1, \sigma_2, \sigma_3) \circ (\tau_1, \tau_2, \tau_3) = (\sigma_1+\tau_1, \sigma_2+\tau_2, \sigma_3+\tau_3+\sigma_1\tau_2)$$

and the dilations

$$t(\sigma_1, \sigma_2, \sigma_3) = (t\sigma_1, t\sigma_2, t^2\sigma_3), \qquad t > 0.$$

We denote by $0 = (0, 0, 0)$ the identity of $\boldsymbol{H}$.

A homogeneous basis of the Lie algebra $\mathfrak{h}$ of $\boldsymbol{H}$ is given by

(1) $$X = \frac{\partial}{\partial x}, \qquad Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \qquad Z = \frac{\partial}{\partial z},$$

where $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ are usual partial derivatives on $\boldsymbol{H}$.

The adjoint representation on $\mathfrak{h}$ acts as follows:

(2) $$\mathrm{Ad}_\sigma X = X - y(\sigma)Z, \qquad \mathrm{Ad}_\sigma Y = Y + x(\sigma)Z, \qquad \mathrm{Ad}_\sigma Z = Z,$$

where for $\sigma = (\sigma_1, \sigma_2, \sigma_3)$

(3) $$x(\sigma) = \sigma_1, \qquad y(\sigma) = \sigma_2.$$

Remark that the mappings x, y defined in (3) are homogeneous of order 1 and additive with respect to the group product.

If ϱ is a complete left-invariant metric generating the topology of $\boldsymbol{H}$ we write

$$\|\sigma\| = \varrho(0, \sigma).$$

$\|\cdot\|$ is a pseudonorm on $\boldsymbol{H}$.

A continuous semigroup $(\mu_t)_{t>0}$ of measures on $\boldsymbol{H}$ is said to be *stable* (with exponent α, $0 < \alpha < 2$) if for every $B \in \mathscr{B}_{\boldsymbol{H}}$ and $t > 0$

$$\mu_t(B) = \mu_1(t^{-1/\alpha} B).$$

For the properties of stable semigroups on homogeneous Lie groups see [6], Section 2.

We will say that $(\mu_t)_{t>0}$ is a *truncated stable* semigroup of measures on $\boldsymbol{H}$ if it does not have the Gaussian component and if there exist a stable semigroup (μ_t') on $\boldsymbol{H}$ with Lévy measure ν' and a nonnegative function $h \in \mathscr{C}_c(\boldsymbol{H})$ equal to 1 on a neighbourhood of 0 and bounded by 1 such that the Lévy measure ν of (μ_t) is given by

$$\nu = h\nu'. \tag{4}$$

2. Properties of some stochastic integrals. Throughout this section $\{z_t\}_{t>0}$ is a symmetric stochastic process on $\boldsymbol{H}$ with homogeneous independent increments and sample paths in the Skorohod space $D_{\boldsymbol{H}}(\boldsymbol{R}^+)$. We suppose that the generator of the corresponding semigroup of measures does not have the Gaussian part and that the Lévy measure ν of $\{z_t\}_{t>0}$ has a compact support.

Identically as in [6] we denote by $N(T, \varepsilon)(z)$ the number of jumps of $z \in D_{\boldsymbol{H}}$ such that $\|\Delta z\| > \varepsilon$, up to moment T, and by $S_1^{(\varepsilon)}, \ldots, S_{N(T,\varepsilon)}^{(\varepsilon)}$ the consecutive moments of these jumps. We define

$$z_t^{(\varepsilon)} = \Delta z_{S_1^{(\varepsilon)}} \ldots \Delta z_{S_{N(t,\varepsilon)}^{(\varepsilon)}}$$

to be the process with the Lévy measure $\nu|_{\{\|\cdot\| > \varepsilon\}}$. We write $\mathbf{S}^{(\varepsilon)}$ for the stochastic integral with respect to the process $\{z_t^{(\varepsilon)}\}$.

In the first two lemmas we define stochastic integrals of the form $\mathbf{S}_{s \leqslant T} x$ and $\mathbf{S}_{s \leqslant T} y$ with x, y as in (3) and examine their properties. The definition of these integrals, using convergence in L^2, is more general than the definition of stochastic integrals appearing in [6]. This is caused by a more subtle kind of analysis used in this paper.

Two next lemmas concern the continuity and convergence on the Skorohod space $D_{\boldsymbol{H}}[0, T]$ of some functionals of the process $\{z_t\}$, appearing in the sequel.

The results of this section are true on any nilpotent Lie group (in place of x, y one may take any global coordinate function).

LEMMA 1. (a) *The sequence* $\mathbf{S}_{s \leqslant T}^{(\varepsilon)} x$ *converges when* $\varepsilon \to 0$ *in* L^2. *We define*

$$\mathbf{S}_{s \leqslant T} x = \lim_{\varepsilon \to 0} \mathbf{S}_{s \leqslant T}^{(\varepsilon)} x.$$

Then for each sequence $\varepsilon_n \downarrow 0$ *we have* $\mathbf{S}^{(\varepsilon_n)}_{s \leqslant T} x \to \mathbf{S}_{s \leqslant T} x$ *almost everywhere.*

(b) $\mathbf{S}_{s \leqslant T} x \in L^p$ *for all* $p \geqslant 1$.

(c) $\lim_{\varepsilon \to 0} \mathbf{S}^{(\varepsilon)}_{s \leqslant T} x = \mathbf{S}_{s \leqslant T} x$ *in* L^p *for all* $p \geqslant 1$.

(d) $\lim_{\varepsilon \to 0} (\mathbf{S}^{(\varepsilon)}_{s \leqslant T} x)^m = \mathbf{S}^m_{s \leqslant T} x$ *in* L^p *for all* $p \geqslant 1$ *and* $m \in N$.

In place of x one may put the function y.

Proof. We have

$$\mathbf{E}\,|\mathbf{S}^{(\varepsilon_1)}_{s \leqslant T} x - \mathbf{S}^{(\varepsilon_2)}_{s \leqslant T} x|^2 = \mathbf{E}\Big(\sum_{\substack{s \leqslant T \\ \varepsilon_1 \leqslant \|\Delta z_s\| < \varepsilon_2}} x(\Delta z_s)\Big)^2$$

$$= \mathbf{E}\Big(\sum_{\substack{s \leqslant T \\ \varepsilon_1 \leqslant \|\Delta z_s\| < \varepsilon_2}} x^2(\Delta z_s)\Big) + 2\mathbf{E}\Big(\sum_{\substack{s_1 < s_2 \leqslant T \\ \varepsilon_1 \leqslant \|\Delta z_{s_i}\| < \varepsilon_2}} x(\Delta z_{s_1})\, x(\Delta z_{s_2})\Big).$$

By Theorem 1.4 of [6] the first expected value in the above formula equals

$$T \int_{\{\varepsilon_1 \leqslant \|x\| < \varepsilon_2\}} x^2\, d\nu(x)$$

and tends to 0 if $\varepsilon_1, \varepsilon_2 \to 0$. By independence and symmetry of jumps of $\{z_t\}$ the second expected value equals

$$\sum \mathbf{E} x(\Delta z_{s_1})\, \mathbf{E} x(\Delta z_{s_2}) = 0.$$

For each sequence $\varepsilon_n \downarrow 0$ the sequence $\mathbf{S}^{(\varepsilon_n)}_{s \leqslant T} x$ may be represented as a series of independent random variables $\mathbf{S}^{(\varepsilon_{n+1})}_{s \leqslant T} x - \mathbf{S}^{(\varepsilon_n)}_{s \leqslant T} x$. Convergence in L^2 of this sequence implies convergence almost everywhere. This proves (a). Arguing similarly as in the proof of Lemma 4.3 in [6] we get, for every $m \in N$ and $\varepsilon_n \downarrow 0$,

$$\sum_{n=1}^{\infty} \mathbf{E}\,|\mathbf{S}^{(\varepsilon_{n+1})}_{s \leqslant T} x - \mathbf{S}^{(\varepsilon_n)}_{s \leqslant T} x|^{2m} \leqslant C_m \mathbf{E}\,(\mathbf{S}_{s \leqslant T} x^2)^m.$$

Thus, by Theorem 1.4 (c) in [6], the sequence $\mathbf{S}^{(\varepsilon)}_{s \leqslant T} x$ converges in L^{2m} and (b) and (c) follow. To prove (d), one uses the Hölder inequality and (c). We omit the details. ∎

In the following lemmas we fix an $\varepsilon_0 > 0$ and consider a continuous function Ψ on H such that $\operatorname{supp} \Psi \subset \{\|\cdot\| > \varepsilon_0\}$. We denote by $\sigma_j^{(\varepsilon)}$ the product of successive jumps of the trajectory, greater than ε, following $\Delta z_{S_j^{(\varepsilon_0)}}$, up to moment T.

LEMMA 2. (a) *The sums* $x(\sigma_j^{(\varepsilon)}) = \mathbf{S}^{(\varepsilon)}_{S_j^{(\varepsilon_0)} \wedge T < s \leqslant T} x$ *converge in* L^2 *when* $\varepsilon \to 0$. *We define*

$$\mathbf{S}^T_{S_j^{(\varepsilon_0)} \wedge T} x = \lim_{\varepsilon \to 0} x(\sigma_j^{(\varepsilon)}).$$

For each sequence $\varepsilon_n \downarrow 0$ *the convergence holds almost everywhere.*

(b) *Let us put*

$$G(\{z_t\}_{t\leqslant T}) = \sum_{j=1}^{N(T,\varepsilon_0)} \Psi(\Delta z_{S_j^{(\varepsilon_0)}})\, \mathbf{S}^T_{S_j^{(\varepsilon_0)}\wedge T}\, x.$$

Then $G(\{z_t^{(\varepsilon)}\}) \to G(\{z_t\})$ *in* L^2 *and for every* $\varepsilon_n \downarrow 0$ *almost everywhere.*

Proof. (a) We have

$$(5)\qquad x(\sigma_j^{(\varepsilon)}) = \mathbf{S}^{(\varepsilon)}_{s\leqslant T}\, x - \mathbf{S}^{(\varepsilon)}_{s\leqslant S_j^{(\varepsilon_0)}\wedge T}\, x,$$

so by Lemma 1 (a) it is enough to prove the convergence of $\mathbf{S}^{(\varepsilon)}_{s\leqslant S_j^{(\varepsilon_0)}\wedge T}\, x$. We proceed as in the proof of Lemma 1 (a). In order to prove that

$$(6)\qquad \mathbf{E}\Big(\sum_{\substack{s_1 < s_2 \leqslant S_j^{(\varepsilon_0)}\wedge T \\ \varepsilon_1 \leqslant \|\Delta z_{s_i}\| < \varepsilon_2}} x(\Delta z_{s_1})\, x(\Delta z_{s_2})\Big) = 0$$

we show that the sum under the expected value is a martingale if we replace $S_j^{(\varepsilon_0)} \wedge T$ by $t \leqslant T$, so by the optional sampling theorem it is also a martingale indexed by j.

(b) The convergence almost everywhere for $\varepsilon_n \downarrow 0$ is obvious by using (a). Now, by the Schwarz inequality and (5),

$$\mathbf{E}|G(\{z_t^{(\varepsilon_1)}\}) - G(\{z_t^{(\varepsilon_2)}\})|^2 \leqslant (\mathbf{E}\mathbf{S}^2_{s\leqslant T}|\Psi|)^{1/2} \big(\mathbf{E}\sup_j |x(\sigma_j^{(\varepsilon_1)}) - x(\sigma_j^{(\varepsilon_2)})|^2\big)^{1/2}$$

$$\leqslant C\Big(\mathbf{E}|\mathbf{S}^{(\varepsilon_1)}_{s\leqslant T}\, x - \mathbf{S}^{(\varepsilon_2)}_{s\leqslant T}\, x|^2 + \mathbf{E}\sup_j \Big| \sum_{\substack{s\leqslant S_j^{(\varepsilon_0)}\wedge T \\ \varepsilon_1 \leqslant \|\Delta z_s\| < \varepsilon_2}} x(\Delta z_s)\Big|^2\Big)^{1/2}.$$

The first integral in this estimation tends to 0 if $\varepsilon_1, \varepsilon_2 \to 0$ in virtue of Lemma 1 (a). By the Doob–Kolmogoroff inequality and symmetry and independence of jumps we have

$$\mathbf{E}\sup_j \Big| \sum_{\substack{s\leqslant S_j^{(\varepsilon_0)}\wedge T \\ \varepsilon_1 \leqslant \|\Delta z_s\| < \varepsilon_2}} x(\Delta z_s)\Big|^2 \leqslant 4T \int_{\varepsilon_1 \leqslant \|z\| < \varepsilon_2} x^2\, d\nu(x) \to 0$$

if $\varepsilon_1, \varepsilon_2 \to 0$. ■

The following convergence lemma will be very useful in estimations after the integration by parts in Section 3.

LEMMA 3. *Suppose that*

(i) Φ *is a bounded continuous function on* $D_H[0, T]$ *such that* $\Phi(\{z_t^{(\varepsilon)}\}) = \Phi(\{z_t\})$ *for* $\varepsilon \leqslant \varepsilon_0$;

(ii) F *is a mapping defined on the trajectories of* $\{z_t\}_{t\leqslant T}$ *and* $\{z_t^{(\varepsilon)}\}_{t\leqslant T}$, $\varepsilon > 0$, *such that* $F(\{z_t^{(\varepsilon)}\}) \to F(\{z_t\})$, $\varepsilon \to 0$, *in* L^1 *and almost everywhere for all sequences* $\varepsilon_n \downarrow 0$;

(iii) *for all* $\varepsilon > 0$, $F(\{z_t^{(\varepsilon)}\}) = F_\varepsilon(\{z_t\})$, *where* F_ε *is a function on* $D_H[0, T]$ *continuous almost everywhere with respect to the distribution of* $\{z_t\}_{t \leqslant T}$;

(iv) $f \in \mathscr{C}_c(H)$.

Then for $\varepsilon \to 0$

$$\mathbf{E}[\Phi(\{z_t^{(\varepsilon)}\}) F(\{z_t^{(\varepsilon)}\}) f(\{z_T^{(\varepsilon)}\})] \to \mathbf{E}[\Phi(\{z_t\}) F(\{z_t\}) f(z_T)]. \tag{7}$$

Proof. It suffices to consider a fixed sequence $\varepsilon_n \downarrow 0$. We will write ε in place of ε_n. Let φ_m be a nonnegative continuous function on $\boldsymbol{R}$ such that $\varphi_m(x) = 1$ for $|x| \leqslant m$, $\varphi_m(x) = 0$ for $|x| > m+1$ and $\varphi \leqslant 1$. Then we put $F^{(m)} = F\varphi_m(F)$ and $F_\varepsilon^{(m)} = F_\varepsilon \varphi_m(F_\varepsilon)$. First we reduce the proof to the case of F bounded. We have

$$\begin{aligned} (8) \quad & |\mathbf{E}[\Phi F(\{z_t^{(\varepsilon)}\}) f(z_T^{(\varepsilon)})] - \mathbf{E}[\Phi F(\{z_t\}) f(z_T)]| \\ & \leqslant |\mathbf{E}[\Phi F(\{z_t^{(\varepsilon)}\}) f(z_T^{(\varepsilon)})] - \mathbf{E}[\Phi F^{(m)}(\{z_t^{(\varepsilon)}\}) f(z_T^{(\varepsilon)})]| \\ & \quad + |\mathbf{E}[\Phi F^{(m)}(\{z_t^{(\varepsilon)}\}) f(z_T^{(\varepsilon)})] - \mathbf{E}[\Phi F^{(m)}(\{z_t\}) f(z_T)]| \\ & \quad + |\mathbf{E}[\Phi F^{(m)}(\{z_t\}) f(z_T)] - \mathbf{E}[\Phi F(\{z_t\}) f(z_T)]|. \end{aligned}$$

Denote the terms of the right-hand side of (8) by J_1, J_2, J_3. Then

$$J_1 \leqslant \|\Phi\|_\infty \|f\|_\infty \mathbf{E}|F(\{z_t^{(\varepsilon)}\}) - F^{(m)}(\{z_t^{(\varepsilon)}\})| \leqslant C \int\limits_{\{|F(\{z_t^{(\varepsilon)}\})| > m\}} |F(\{z_t^{(\varepsilon)}\})|\, dP.$$

By (ii) the variables $F(\{z_t^{(\varepsilon)}\})$ are uniformly integrable with respect to ε. Thus $J_1 \to 0$ if $m \to \infty$, uniformly in ε. Next

$$J_3 \leqslant \|\Phi\|_\infty \|f\|_\infty \mathbf{E}|F^{(m)}(\{z_t\}) - F(\{z_t\})| \to 0$$

when $m \to \infty$ since $|F^{(m)}| \leqslant |F|$ and $F(\{z_t\})$ is integrable by (ii). Thus it suffices to show that for m fixed $J_2 \to 0$ if $\varepsilon \to 0$. We approximate $F^{(m)}$ by continuous functionals $F_\delta^{(m)}$:

$$\begin{aligned} (9) \quad J_2 & \leqslant |\mathbf{E}[\Phi F^{(m)}(\{z_t^{(\varepsilon)}\}) f(z_T^{(\varepsilon)})] - \mathbf{E}[\Phi F_\delta^{(m)}(\{z_t^{(\varepsilon)}\}) f(z_T^{(\varepsilon)})]| \\ & \quad + |\mathbf{E}[\Phi(\{z_t^{(\varepsilon)}\}) F_\delta^{(m)}(\{z_t^{(\varepsilon)}\}) f(z_T^{(\varepsilon)})] - \mathbf{E}[\Phi F_\delta^{(m)}(\{z_t\}) f(z_T)]| \\ & \quad + |\mathbf{E}[\Phi F_\delta^{(m)}(\{z_t\}) f(z_T)] - \mathbf{E}[\Phi F^{(m)}(\{z_t\}) f(z_T)]|. \end{aligned}$$

Denote the terms on the right-hand side of (9) by K_1, K_2, K_3. We have

$$K_1 \leqslant \|\Phi\|_\infty \|f\|_\infty \mathbf{E}|F_\delta^{(m)}(\{z_t^{(\varepsilon)}\}) - F^{(m)}(\{z_t^{(\varepsilon)}\})],$$

$$K_3 \leqslant \|\Phi\|_\infty \|f\|_\infty \mathbf{E}|F_\delta^{(m)}(\{z_t\}) - F^{(m)}(\{z_t\})|.$$

Remark that

$$F_\delta(\{z_t^{(\varepsilon)}\}) = \begin{cases} F(\{z_t^{(\varepsilon)}\}) & \text{for } \varepsilon \geqslant \delta, \\ F_\delta(\{z_t\}) & \text{for } \varepsilon < \delta. \end{cases}$$

Thus $K_1 = 0$ if $\varepsilon \geqslant \delta$. For $\varepsilon < \delta$

$$(10) \qquad |F_\delta^{(m)}(\{z_t^{(\varepsilon)}\}) - F^{(m)}(\{z_t^{(\varepsilon)}\})| \leqslant |F_\delta^{(m)}(\{z_t\}) - F^{(m)}(\{z_t\})| + |F_\varepsilon^{(m)}(\{z_t\}) - F^{(m)}(\{z_t\})|.$$

Let $\eta > 0$. There exists $\delta_0 > 0$ such that for all $\delta \leqslant \delta_0$ one has

$$\mathbf{E}|F_\delta^{(m)}(\{z_t\}) - F^{(m)}(\{z_t\})| < \eta \|\Phi\|_\infty^{-1} \|f\|_\infty^{-1}.$$

By (10), for $\delta \leqslant \delta_0$ and all ε we get the estimates $K_1 < 2\eta$ and $K_3 < \eta$.

For $\delta \leqslant \delta_0$ fixed, Theorem 1.1 in [6], the a.e. continuity and boundedness of Φ, $F_\delta^{(m)}$ and $z \mapsto f(z(T))$ imply $K_2 \to 0$ when $\varepsilon \to 0$. Thus for ε sufficiently small we have $J_2 \leqslant 4\eta$. This completes the proof. ∎

Now we will prove that condition (iii) of Lemma 3 is satisfied for some mappings appearing in the integration by parts in Section 3.

LEMMA 4. *Suppose that the Lévy measure of the process $\{z_t\}$ is absolutely continuous with respect to the Haar measure on* $\boldsymbol{H}$.

(a) *Let φ be continuous on $\boldsymbol{H}$. For all $\delta > 0$ the mapping*

$$\varphi_\delta(z) = \sum_{\substack{s \leqslant T \\ \|\Delta z_s\| > \delta}} \varphi(\Delta z_s), \qquad z \in D_{\boldsymbol{H}}[0, T],$$

is continuous almost everywhere with respect to the distribution of $\{z_t\}_{t \leqslant T}$ on $D_{\boldsymbol{H}}[0, T]$.

(b) *In the notation of Lemma 2 (b), the mapping*

$$G_\delta(z) = \sum_{j=1}^{N(T,\varepsilon_0)(z)} \Psi(\Delta z_{S_j^{(\varepsilon_0)}}) x(\sigma_j^{(\delta)}(z))$$

with $\delta < \varepsilon_0$ is continuous almost everywhere with respect to the distribution of $\{z_t\}$ on $D_{\boldsymbol{H}}[0, T]$.

Proof. First note that for any $\delta > 0$, by the absolute continuity of the Lévy measure of $\{z_t\}$, we have $P\{\|\Delta z_s\| = \delta \text{ for some } s \leqslant T\} = 0$. Hence almost all trajectories of $\{z_t\}_{t \leqslant T}$ do not have jumps of norm equal to δ.

Define the Skorohod distance

$$d(z, w) = \inf\{\varepsilon > 0 \mid \text{there exists } \lambda \in \Lambda \text{ such that } \sup_{0 \leqslant t \leqslant T} \varrho(z(t), w\lambda(t)) < \varepsilon,\ \|\mathrm{id} - \lambda\|_\infty < \varepsilon\},$$

where $\Lambda = \{f\colon [0, T] \mapsto [0, T] \text{ continuous and strictly increasing}\}$ (cf. [2]). If $z \in D_{\boldsymbol{H}}[0, T]$ does not have any jumps $\|\Delta z_s\| = \delta$, then $\|\Delta z\| < \delta - \alpha$ or

$\|\Delta z\| > \delta+\alpha$ for an α, $0<\alpha<\delta$. Then if $d(z, w) < \alpha/2$, we have

$$\|\Delta z(t)\| > \delta \Leftrightarrow \|\Delta w\lambda(t)\| > \delta, \tag{11}$$

where $\lambda \in \Lambda$ is such that $\sup_{t \leqslant T}\|z^{-1}(t)w\lambda(t)\| < \alpha/2$. Observe that $\varphi_\delta(w\lambda) = \varphi_\delta(w)$. The jumps of z greater than δ are fixed and the function φ is continuous. Therefore, for any $\varepsilon > 0$, if w is sufficiently near to z, then $|\varphi_\delta(z)-\varphi_\delta(w)| < \varepsilon$. This proves (a).

To prove (b) we fix $z \in D_H$ such that $\|\Delta z_s\| \neq \delta$ and $\|\Delta z_s\| \neq \varepsilon_0$ for $s \leqslant T$. There exists $0<\alpha<\delta$ such that

$$\|\Delta z_s\| \in (0, \delta-\alpha) \cup (\delta+\alpha, \varepsilon_0-\alpha) \cup (\varepsilon_0+\alpha, \infty) \quad \text{for } s \leqslant T.$$

By (11), if $d(z, w) < \alpha/2$, then

$$\|\Delta z(t)\| > \delta \Leftrightarrow \|\Delta w\lambda(t)\| > \delta \quad \text{and} \quad \|\Delta z(t)\| > \varepsilon_0 \Leftrightarrow \|\Delta w\lambda(t)\| > \varepsilon_0.$$

Observing that $G_\delta(w) = G_\delta(w\lambda)$ and denoting by $t_1, \ldots, t_n$ the moments of jumps of z greater than ε_0 we have

$$\begin{aligned} |G_\delta(z)-G_\delta(w)| &\leqslant \sum_j \Big| \sum_{\substack{s>t_j \\ \|\Delta z_s\|>\delta}} x(\Delta z_s)\Big| \big|\Psi(\Delta z_{t_j}) - \Psi\big(\Delta(w\lambda)_{t_j}\big)\big| \\ &\quad + \sum_j \Big| \sum_{\substack{s>t_j \\ \|\Delta z_s\|>\delta}} x(\Delta z_s) - x\big(\Delta(w\lambda)_s\big)\Big| \big|\Psi\big(\Delta(w\lambda)_{t_j}\big)\big| \\ &\leqslant \max_{j \leqslant n} \Big| \sum_{\substack{s>t_j \\ \|\Delta z_s\|>\delta}} x(\Delta z_s)\Big| \sum_{j=1}^{n} \big|\Psi(\Delta z_{t_j}) - \Psi\big(\Delta(w\lambda)_{t_j}\big)\big| \\ &\quad + \max_{j \leqslant n} \Big| \sum_{\substack{s>t_j \\ \|\Delta z_s\|>\delta}} \big[x(\Delta z_s) - x\big(\Delta(w\lambda)_s\big)\big]\Big| \sum_{j=1}^{n} \big|\Psi\big(\Delta(w\lambda)_{t_j}\big)\big|. \end{aligned} \tag{12}$$

The first term on the right-hand side of (12) tends to zero when $w \to z$ by continuity of Ψ. The continuity in z of the mappings

$$w \mapsto \sum_{s \leqslant T} |\Psi(\Delta w_s)|, \quad w \mapsto \sum_{\substack{t_j<s<T \\ \|\Delta w_s\|>\delta}} x(\Delta w_s), \; j = 1, \ldots, n,$$

following from part (a) of the lemma implies the vanishing of the second term in (12) when $w \to z$. Thus $G_\delta(w) \to G_\delta(z)$ if $w \to z$ in the Skorohod topology. ∎

3. Smoothness of a truncated stable semigroup. In this section we prove smoothness of truncated stable semigroups with Lévy measure of class $\mathscr{C}^1$. In [6] this result was obtained under the assumption $\mathscr{C}^\infty$ on the Lévy measure. Still under this assumption it was generalized for stable semigroups

(Proposition 3.3 in [6]). We present some generalizations of our result in Section 4.

Throughout this section $\{z_t\}_{t\geq 0}$ is a truncated symmetric stable process with Lévy measure ν of class $\mathscr{C}^1$, i.e.

$$\nu(dx) = g(x)\,dx, \qquad g\in\mathscr{C}_c^1(\boldsymbol{H}),$$

and the function h in (4) is $\mathscr{C}^1$ on $\boldsymbol{H}$.

The notation in this section is identical to those of [6]. All constants are denoted by C.

THEOREM 1. *For all $l_1, l_2\in N$ there exists $C>0$ such that*

$$|\mathbf{E}[\mathbf{S}^{l_1}_{s\leq T}x\,\mathbf{S}^{l_2}_{s\leq T}y\,Xf(z_T)]|\leq C\|f\|_\infty$$

for every function $f\in\mathscr{C}_c^\infty(\boldsymbol{H})$. In place of X one may insert Y or Z.

Proof. The idea of the proof is similar to that of Theorem 4.9 in [6]. We consider functions ϱ and u as in Section 4 (a) of [6]. In particular, $\operatorname{supp} u\subset\{\|\cdot\|>\varepsilon_0\}$ for an ε_0 fixed. First we integrate by parts on the jumps of the process $\{z_t^{(\varepsilon_0)}\}$. For $\varepsilon<\varepsilon_0$ we have

$$\mathbf{E}[\varrho_\eta(\mathbf{S}_{s\leq T}u)\,x^{l_1}(z_T^{(\varepsilon)})\,y^{l_2}(z_T^{(\varepsilon)})\,X f(z_T^{(\varepsilon)})] = \sum_{n=1}^{\infty} P\{N(T,\varepsilon_0)=n\}$$

$$\times\sum_{j=1}^{n}\mathbf{E}\Big[\int\frac{\varrho_\eta(K_j+u(z))}{K_j+u(z)}u(z)\Big(\sum_{i(i\neq j)}x(\Delta z_{S_i^{(\varepsilon)}})+x(z)\Big)^{l_1}$$

$$\times\Big(\sum_{i(i\neq j)}y(\Delta z_{S_i^{(\varepsilon)}})+y(z)\Big)^{l_2}Xf(\Delta z_{S_1^{(\varepsilon)}}\dots\Delta z_{S_j^{(\varepsilon)}})\frac{g(z)}{\nu\{\|\cdot\|>\varepsilon\}}dz\Big]$$

$$= -\mathbf{E}\Big[\Phi_1\Big(\sum_{j=1}^{N(T,\varepsilon_0)}u(\mathrm{Ad}_{\sigma_j^{(\varepsilon)}}X)\,u(\Delta z_{S_j^{(\varepsilon)}})\Big)(\mathbf{S}^{(\varepsilon)}_{s\leq T}x)^{l_1}(\mathbf{S}^{(\varepsilon)}_{s\leq T}y)^{l_2}f(z_T^{(\varepsilon)})\Big]$$

$$-\mathbf{E}\Big[\Phi_0\Big(\sum_{j=1}^{N(T,\varepsilon_0)}\frac{\mathrm{Ad}_{\sigma_j^{(\varepsilon_0)}}X(ug)}{g}(\Delta z_{S_j^{(\varepsilon_0)}})\Big)(\mathbf{S}^{(\varepsilon)}_{s\leq T}x)^{l_1}(\mathbf{S}^{(\varepsilon)}_{s\leq T}y)^{l_2}f(z_T^{(\varepsilon)})\Big]$$

$$-\mathbf{E}[l_1\varrho_\eta(\mathbf{S}_{s\leq T}u)(\mathbf{S}^{(\varepsilon)}_{s\leq T}x)^{l_1-1}(\mathbf{S}^{(\varepsilon)}_{s\leq T}y)^{l_2}f(z_T^{(\varepsilon)})],$$

where $K_j=\sum_{i\neq j}u(\Delta z_{S_i^{(\varepsilon_0)}})$, and

$$\Phi_0=\frac{\varrho_\eta(\mathbf{S}_{s\leq T}u)}{\mathbf{S}_{s\leq T}u},\qquad \Phi_1=\frac{\varrho'_\eta(\mathbf{S}_{s\leq T}u)}{\mathbf{S}_{s\leq T}u}-\frac{\varrho_\eta(\mathbf{S}_{s\leq T}u)}{\mathbf{S}^2_{s\leq T}u}$$

and the last term is zero if $l_1=0$.

Using the formulas (1), (2) for the adjoint representation we get the following form of the above expression:

$$(13)\quad -\mathbf{E}[\Phi_1 \mathbf{S}_{s\leqslant T}\Psi_1^{(X)}(\mathbf{S}^{(\varepsilon)}_{s\leqslant T}x)^{l_1}(\mathbf{S}^{(\varepsilon)}_{s\leqslant T}y)^{l_2}f(z_T^{(\varepsilon)})]$$
$$-\mathbf{E}[\Phi_0 \mathbf{S}_{s\leqslant T}\Psi_2^{(X)}(\mathbf{S}^{(\varepsilon)}_{s\leqslant T}x)^{l_1}(\mathbf{S}^{(\varepsilon)}_{s\leqslant T}y)^{l_2}f(z_T^{(\varepsilon)})]$$
$$+\mathbf{E}\Big[\Phi_1\Big(\sum_{j=1}^{N(T,\varepsilon_0)}\Psi_1^{(Z)}(\Delta z_{S_j^{(\varepsilon_0)}})\mathbf{S}^{(\varepsilon)}_{S_j^{(\varepsilon_0)}\wedge T}y\Big)(\mathbf{S}^{(\varepsilon)}_{s\leqslant T}x)^{l_1}(\mathbf{S}^{(\varepsilon)}_{s\leqslant T}y)^{l_2}f(z_T^{(\varepsilon)})\Big]$$
$$+\mathbf{E}\Big[\Phi_0\Big(\sum_{j=1}^{N(T,\varepsilon_0)}\Psi_2^{(Z)}(\Delta z_{S_j^{(\varepsilon_0)}})\mathbf{S}^{(\varepsilon)}_{S_j^{(\varepsilon_0)}\wedge T}y\Big)(\mathbf{S}^{(\varepsilon)}_{s\leqslant T}x)^{l_1}(\mathbf{S}^{(\varepsilon)}_{s\leqslant T}y)^{l_2}f(z_T^{(\varepsilon)})\Big]$$
$$-\mathbf{E}[l_1\varrho_\eta(\mathbf{S}_{s\leqslant T}u)(\mathbf{S}^{(\varepsilon)}_{s\leqslant T}x)^{l_1-1}(\mathbf{S}^{(\varepsilon)}_{s\leqslant T}y)^{l_2}f(z_T^{(\varepsilon)})],$$

where $\Psi_1^{(W)} = uWu$, $\Psi_2^{(W)} = W(ug)/g$ for any $W\in\mathfrak{h}$.

In order to estimate (13) we apply repeatedly the Schwarz inequality. In the estimation of $\mathbf{E}\sup_{j=1,\dots,N(T,\varepsilon_0)}|\mathbf{S}^{(\varepsilon)}_{S_j^{(\varepsilon_0)}\wedge T}y|^4$ we use the fact that $(\mathbf{S}^{(\varepsilon)}_{s\leqslant S_j^{(\varepsilon_0)}\wedge T}y)$ is a martingale, the Doob-Kolmogoroff inequality and the property $\mathbf{E}(\mathbf{S}^{(\varepsilon)}_{S_j\wedge T}y)^4 \leqslant C\mathbf{E}(\mathbf{S}^{(\varepsilon)}_{S_j\wedge T}y^2)^2$, obtained in a similar way to that of (6). Finally, we use Lemma 4.3 of [6] and for every $\varepsilon > 0$ we get the estimation

$$(14)\quad |\mathbf{E}[\varrho_\eta(\mathbf{S}_{s\leqslant T}u)x^{l_1}(z_T^{(\varepsilon)})y^{l_2}(z_T^{(\varepsilon)})Xf(z_T^{(\varepsilon)})]|$$
$$\leqslant C\|f\|_\infty\{(\mathbf{E}\mathbf{S}^{2l_1}_{s\leqslant T}r^2\,\mathbf{E}\mathbf{S}^{2l_2}_{s\leqslant T}r^2)^{1/4}\{(\mathbf{E}[\Phi_1^2\mathbf{S}^2_{s\leqslant T}\Psi_1^{(X)}])^{1/2}$$
$$+(\mathbf{E}[\Phi_0^2\mathbf{S}^2_{s\leqslant T}\Psi_2^{(X)}])^{1/2}\}$$
$$+(\mathbf{E}\mathbf{S}^2_{s\leqslant T}r^2)^{1/4}(\mathbf{E}\mathbf{S}^{4l_1}_{s\leqslant T}r^2\,\mathbf{E}\mathbf{S}^{4l_2}_{s\leqslant T}r^2)^{1/8}\{(\mathbf{E}[\Phi_1^2\mathbf{S}^2_{s\leqslant T}|\Psi_1^{(Z)}|])^{1/2}$$
$$+(\mathbf{E}[\Phi_0^2\mathbf{S}^2_{s\leqslant T}|\Psi_2^{(Z)}|])^{1/2}\}+(\mathbf{E}\mathbf{S}^{l_1-1}_{s\leqslant T}r^2\,\mathbf{E}\mathbf{S}^{l_2}_{s\leqslant T}r^2)^{1/2}\},$$

where r denotes a homogeneous norm on $\boldsymbol{H}$ (see [6]) and the constant C depends only on l_1 and l_2.

Next we let $\varepsilon\to 0$ applying Lemmas 3, 1 (d) and 4 (a) to the left-hand side of (14). To complete the proof we repeat steps (c) and (d) of the proof of Theorem 4.9 in [6]. ∎

THEOREM 2. *Let $\{z_t\}_{t\geqslant 0}$ be a truncated stable process on $\boldsymbol{H}$ with Lévy measure of class $\mathscr{C}^1$, and V be a left-invariant second order differential operator on $\boldsymbol{H}$. Then for every $T>0$ there exists $C>0$ such that*

$$|\mathbf{E}[Vf(z_T)]| \leqslant C\|f\|_\infty \quad \textit{for all } f\in\mathscr{C}_c^\infty(\boldsymbol{H}).$$

Proof. It suffices to prove the theorem for $V = YX$. The method is identical for all the superpositions of the operators X, Y, Z.

As usual we start with the process $\{z_t^{(\varepsilon)}\}$ for an $\varepsilon > 0$. We consider

$$\mathbf{E}[\varrho_\eta(\mathbf{S}_{s\leqslant T/2}u)YXf(z_T^{(\varepsilon)})]$$

and we perform one integration by parts in this integral. We get

$$\begin{aligned}
&\mathbf{E}\,[\varrho_\eta(\mathbf{S}_{s\leqslant T/2}u)\,YXf(z_T^{(\varepsilon)})] \\
&\quad = -\mathbf{E}\,[\Phi_1(\mathbf{S}_{s\leqslant T/2}\,u)\,\mathbf{S}_{s\leqslant T/2}\,\Psi_1^{(Y)}\,Xf(z_T^{(\varepsilon)}) \\
&\qquad + \Phi_0(\mathbf{S}_{s\leqslant T/2}u)\,\mathbf{S}_{s\leqslant T/2}\,\Psi_2^{(Y)}\,Xf(z_T^{(\varepsilon)})] \\
&\qquad - \mathbf{E}\,[\Phi_1(\mathbf{S}_{s\leqslant T/2}u)\{\sum_{j=1}^{N(T/2,\varepsilon_0)} x(\sigma_j^{(\varepsilon)})\,\Psi_1^{(Z)}(\Delta z_{S_j^{(\varepsilon_0)}})\}\,Xf(z_T^{(\varepsilon)})] \\
&\qquad - \mathbf{E}\,[\Phi_0(\mathbf{S}_{s\leqslant T/2}u)\{\sum_{j=1}^{N(T/2,\varepsilon_0)} x(\sigma_j^{(\varepsilon)})\,\Psi_2^{(Z)}(\Delta z_{S_j^{(\varepsilon_0)}})\}\,Xf(z_T^{(\varepsilon)})],
\end{aligned}$$

where Φ_i and Ψ_j are the same as before.

Now we let $\varepsilon\to 0$ by using Lemmas 3, 2 (b) and 4 (b). We apply the conditional expected value $\mathbf{E}\,[\cdot|\mathscr{F}_{T/2}]$ (cf. [1]) to the right-hand side of the obtained formula. Writing

$$\mathbf{S}^T_{S_j^{(\varepsilon_0)}}x = \mathbf{S}^{T/2}_{S_j^{(\varepsilon_0)}}x + \mathbf{S}^T_{T/2}x$$

and using the basic properties of conditional expectations and the Markov property of $\{z_t\}$ we get

$$\begin{aligned}
(15)\quad &\mathbf{E}\,[\varrho_\eta(\mathbf{S}_{s\leqslant T/2}u)\,YXf(z_T)] \\
&\quad = -\mathbf{E}\,[\{\Phi_1(\mathbf{S}_{s\leqslant T/2}u)\,\mathbf{S}_{s\leqslant T/2}\,\Psi_1^{(Y)} \\
&\qquad + \Phi_0(\mathbf{S}_{s\leqslant T/2}\,u)\mathbf{S}_{s\leqslant T/2}\,\Psi_2^{(Y)}\}\,\mathbf{E}^{z_{T/2}(\omega)}[Xf(z_{T/2})]] \\
&\qquad - \mathbf{E}\,[\Phi_1(\mathbf{S}_{s\leqslant T/2}u)\{\sum_{j=1}^{N(T/2,\varepsilon_0)} \Psi_1^{(Z)}(\Delta z_{S_j^{(\varepsilon_0)}})\,\mathbf{S}^{T/2}_{S_j^{(\varepsilon_0)}}x\}\,\mathbf{E}^{z_{T/2}(\omega)}[Xf(z_{T/2})]] \\
&\qquad - \mathbf{E}\,[\Phi_1(\mathbf{S}_{s\leqslant T/2}u)\,\mathbf{S}_{s\leqslant T/2}\,\Psi_1^{(Z)}\,\mathbf{E}^{z_{T/2}(\omega)}[\mathbf{S}_{s\leqslant T/2}x\,Xf(z_{T/2})]] \\
&\qquad - \mathbf{E}\,[\Phi_0(\mathbf{S}_{s\leqslant T/2}u)\{\sum_{j=1}^{N(T/2,\varepsilon_0)} \Psi_2^{(Z)}(\Delta z_{S_j^{(\varepsilon_0)}})\,\mathbf{S}^{T/2}_{S_j^{(\varepsilon_0)}}x\}\,\mathbf{E}^{z_{T/2}(\omega)}[Xf(z_{T/2})]] \\
&\qquad - \mathbf{E}\,[\Phi_1(\mathbf{S}_{s\leqslant T/2}u)\,\mathbf{S}_{s\leqslant T/2}\,\Psi_2^{(Z)}\mathbf{E}^{z_{T/2}(\omega)}[\mathbf{S}_{s\leqslant T/2}x\,Xf(z_{T/2})]].
\end{aligned}$$

Next we estimate (15) using Theorem 1 and the equalities

$$\mathbf{E}^\sigma[Xf(z_{T/2})] = \mathbf{E}\,[X(f\circ l_\sigma)(z_{T/2})],$$

$$\mathbf{E}^\sigma[\mathbf{S}_{s\leqslant T/2}x\,Xf(z_{T/2})] = \mathbf{E}\,[\mathbf{S}_{s\leqslant T/2}x\,X(f\circ l_\sigma)(z_{T/2})]$$

for all $\sigma\in H$. The fact that $\{\mathbf{S}_{s\leqslant S_j^{(\varepsilon_0)}\wedge T/2}\,x\}_j$ is an L^2-martingale, the Doob –Kolmogoroff inequality and an argument similar to that of the proof of (6) imply

$$\mathbf{E}\sup_j|\mathbf{S}_{s\leqslant S_j^{(\varepsilon_0)}\wedge T/2}\,x|^2 \leqslant 4\sup_j \mathbf{E}\mathbf{S}^2_{s\leqslant S_j^{(\varepsilon_0)}\wedge T/2}\,x \leqslant 4\,\mathbf{E}\mathbf{S}_{s\leqslant T/2}x^2,$$

$$\mathbf{E}\sup_j|\mathbf{S}^{T/2}_{S_j^{(\varepsilon_0)}}x|^2 = C\,\mathbf{E}\mathbf{S}_{s\leqslant T/2}x^2.$$

By the Schwarz inequality we obtain finally the estimation

$$
\begin{aligned}
&|\mathbf{E}[\varrho_\eta(\mathbf{S}_{s\leqslant T/2}u)\,YXf(z_T)]|\\
&\leqslant C\|f\|_\infty\{\mathbf{E}|\Phi_1(\mathbf{S}_{s\leqslant T/2}u)\,\mathbf{S}_{s\leqslant T/2}\,\Psi_1^{(Y)}|+\mathbf{E}|\Phi_0(\mathbf{S}_{s\leqslant T/2}u)\,\mathbf{S}_{s\leqslant T/2}\,\Psi_2^{(Y)}|\\
&\quad+\mathbf{E}[\Phi_1^2(\mathbf{S}_{s\leqslant T/2}u)\,\mathbf{S}^2_{s\leqslant T/2}|\,\Psi_1^{(Z)}|]+\mathbf{E}[\Phi_0^2(\mathbf{S}_{s\leqslant T/2}u)\,\mathbf{S}^2_{s\leqslant T/2}|\,\Psi_2^{(Z)}|]\}.
\end{aligned}
$$

The rest of the proof is identical to the proof of Theorems 4.7 and 4.9 in [6].

THEOREM 3. *If $\{z_t\}$ is a truncated stable process on $\boldsymbol{H}$ with Lévy measure of class $\mathscr{C}^1$, and V is a left-invariant differential operator of order n on $\boldsymbol{H}$, then for every $T>0$ there exists $C>0$ such that*

$$(16)\qquad |\mathbf{E}[Vf(z_t)]|\leqslant C\|f\|_\infty$$

for all $f\in\mathscr{C}_c^\infty(\boldsymbol{H})$.

Proof. It is enough to consider the case $V=X^n$. Then $Vf=X(X^{n-1}f)$. Formula (15) shows that in order to have (16) we must have an analogous estimate of $|\mathbf{E}[X^{n-1}f(z_{T/2})]|$ and $|\mathbf{E}[\mathbf{S}_{s\leqslant T/2}\,yX^{n-1}f(z_{T/2})]|$. By (13) we see that

$$
\begin{aligned}
(17)\quad &\mathbf{E}[\varrho_\eta(\mathbf{S}_{s\leqslant T/4}u)\,\mathbf{S}^{(\varepsilon)}_{s\leqslant T/2}\,yX^{n-1}f(z^{(\varepsilon)}_{T/2})]\\
&=-\mathbf{E}[(\Phi_1(T/4)\,\mathbf{S}_{s\leqslant T/4}\,\Psi_1^{(X)}+\Phi_0(T/4)\,\mathbf{S}_{s\leqslant T/4}\,\Psi_2^{(X)})\,\mathbf{S}^{(\varepsilon)}_{s\leqslant T/2}\,yX^{n-2}f(z^{(\varepsilon)}_{T/2})]\\
&\quad+\mathbf{E}[\Phi_1(T/4)\{\sum_{j=1}^{N(T/4,\varepsilon_0)}\Psi_1^{(Z)}(\Delta z_{S_j^{(\varepsilon_0)}})\,y(\sigma_j^{(\varepsilon)})\}\,\mathbf{S}^{(\varepsilon)}_{s\leqslant T/2}\,yX^{n-2}f(z^{(\varepsilon)}_{T/2})]\\
&\quad+\mathbf{E}[\Phi_0(T/4)\{\sum_{j=1}^{N(T/4,\varepsilon_0)}\Psi_2^{(Z)}(\Delta z_{S_j^{(\varepsilon_0)}})\,y(\sigma_j^{(\varepsilon)})\}\,\mathbf{S}^{(\varepsilon)}_{s\leqslant T/2}\,yX^{n-2}f(z^{(\varepsilon)}_{T/2})].
\end{aligned}
$$

Now we let $\varepsilon\to 0$ and by Lemmas 1–4 we get (17) not depending on ε. Applying $\mathbf{E}[\cdot|\mathscr{F}_{T/4}]$ we see that in order to get an estimation for

$$|\mathbf{E}[\mathbf{S}_{s\leqslant T/2}\,yX^{n-1}f(z_{T/2})]|\quad\text{and}\quad|\mathbf{E}X^nf(z_T)|$$

it is sufficient to have estimations for

$$|\mathbf{E}X^{n-2}f(z_{T/4})|,\quad|\mathbf{E}[\mathbf{S}_{s\leqslant T/4}\,yX^{n-2}f(z_{T/4})]|\quad\text{and}\quad|\mathbf{E}[\mathbf{S}^2_{s\leqslant T/4}\,yX^{n-2}f(z_{T/4})]|.$$

Repeating this procedure $n-1$ times we reduce the proof of (16) to estimations of

$$\mathbf{E}Xf(z_{T/2^{n-1}}),\ \mathbf{E}[\mathbf{S}_{s\leqslant T/2^{n-1}}\,yXf(z_{T/2^{n-1}})],\ \ldots,\ \mathbf{E}[\mathbf{S}^{n-1}_{s\leqslant T/2^{n-1}}\,yXf(z_{T/2^{n-1}})].$$

These estimations are given by Theorem 1. ∎

LEMMA 3.2 in [6] and Theorem 3 imply

COROLLARY 1. *If (μ_t) is a truncated stable semigroup of symmetric measures on $\boldsymbol{H}$ with Lévy measure of class $\mathscr{C}^1$, then μ_t have smooth densities on $\boldsymbol{H}$.* ∎

COROLLARY 2. *Denote by C_t the constant in* (16) *corresponding to z_t and V fixed. Then*

$$\sup_{\delta < t < M} C_t < \infty \quad \textit{for any } 0 < \delta < M.$$

Proof. We put

$$c_{ij}(t) = \mathbb{E}\left[|\mathbf{S}_{s \leqslant t}^{-(i+1)} u|\, \mathbf{S}_{s \leqslant t} |\Psi_j|\right], \qquad d_{ij}(t) = \left(\mathbb{E}\left[|\mathbf{S}_{s \leqslant t}^{-2(i+1)} u|\, \mathbf{S}_{s \leqslant t}^2 |\Psi_j|\right]\right)^{1/2}$$

and

$$m(t) = \max(c_{11}, c_{02}, d_{11}, d_{02}, 1).$$

Then the procedure of estimation of $|\mathbb{E}[Vf(z_t)]|$ throughout the proofs of Theorems 1–3 shows that

$$|\mathbb{E}[Vf(z_t)]| \leqslant C(n)\, m(t/2) \ldots m(t/2^{n-2})\, m^2(t/2^{n-1}) I_n \|f\|_\infty, \tag{18}$$

where I_n is a linear combination of products of integrals of the form $(\mathbb{E}\mathbf{S}_{s \leqslant t/2^k}^l r^2)^q$. Formula (18), the Schwarz inequality and the fact that $\mathbf{S}_{s \leqslant t}^{-1} u \in L^p$ and $\mathbf{S}_{s \leqslant t} |\Psi_j| \in L^p$ for all $p \geqslant 1$ imply the statement of the corollary. ∎

Remark. Formula (18) provides a different estimation of the constants in (16) from that in Theorem 5.6 of [6]. Here the constants in estimation of higher order are obtained from constants in estimation of order 1 by multiplication. In [6] the constants of higher order were expressed by expected values of complicated random variables.

4. Smoothness of semigroups with Lévy measure of noncompact support. In this section we present results concerning stable and more general semigroups of measures, without the assumption that the support of Lévy measure is compact.

Theorem 4 presents a relationship between the classes of smoothness of the Lévy measure ν and of the measures in a symmetric stable semigroup $\{\mu_t\}$ that is rather surprising from the analytical point of view: μ_t are more smooth than ν if ν is sufficiently smooth. Theorem 5 asserts smoothness of a semigroup $\{\mu_t\}$ with Lévy measure ν of class $\mathscr{C}^1$ under some asymptotic conditions on ν in infinity.

THEOREM 4. *Let $\{\mu_t\}$ be an α-stable symmetric semigroup of measures on $\boldsymbol{H}$ with Lévy measure ν. If ν is of class $\mathscr{C}^m$ for an $m \geqslant 2$, then μ_t have densities of class $\mathscr{C}^{2m-4}$ and in the case $\alpha > 1$ of class $\mathscr{C}^{2m-3}$.*

Proof. As in Section 3 of [6] we take a symmetric function $h \in \mathscr{C}_c^\infty(H)$ such that $0 \leqslant h \leqslant 1$ and $h = 1$ on a neighbourhood of 0. We denote by $\{\tilde{\mu}_t\}$ the semigroup of measures with Lévy measure $\tilde{\nu} = h\nu$ and we put $k = \nu - \tilde{\nu}$. We write $\bar{\mu}_t = e^{\|k\|_1 t}\tilde{\mu}_t$. By Corollary 1 the measures $\tilde{\mu}_t$ and $\bar{\mu}_t$ have smooth densities.

By a perturbation formula (see [9]), for any $f \in \mathscr{C}_c^\infty(H)$

$$\langle \mu_t, f\rangle = \langle \bar{\mu}_t, f\rangle + \int_0^t \langle \mu_s * k * \bar{\mu}_{t-s}, f\rangle\, ds. \tag{19}$$

In the first part of the proof we estimate $|\langle \mu_t, Vf\rangle|$ for homogeneous V of order $n \leqslant m$, proceeding in exactly the same way as in the proof of Proposition 3.3 in [6], i.e. differentiating n times the function k under the integral in (19). Corollary 2 and the final estimation in the proof of Proposition 3.3 in [6] show that there exist constants C_t such that

$$|\langle \mu_t, Vf\rangle| \leqslant C_t \|f\|_\infty, \quad t > 0, \tag{20}$$

for all $f \in \mathscr{C}_c^\infty(H)$ and

$$\sup_{\delta < t < M} C_t < \infty \tag{21}$$

for any $0 < \delta < M$.

Now consider V of order $m+1$. Suppose that V is of the form $V = XV_0$. By (19) we have

$$\begin{aligned}|\langle \mu_t, Vf\rangle| \leqslant |\langle \bar{\mu}_t, Vf\rangle| + \Big|\int_0^{t/2} \langle \mu_s * k * V\bar{\mu}_{t-s}, f\rangle\, ds\Big| \\ + \Big|\int_{t/2}^{t} \langle XV_0(\mu_s * k * \bar{\mu}_{t-s}), f\rangle\, ds\Big|.\end{aligned} \tag{22}$$

Formula (2) for the adjoint representation implies that

$$V_0(\mu_s * k * \bar{\mu}_{t-s}) = \sum_W \mu_s * Wk * a_W\, \bar{\mu}_{t-s}, \tag{23}$$

where W are homogeneous of order m, their number is finite and the functions a_W are of the form $a_W(z) = x(z)^{l_1}\, y(z)^{l_2}$ with $l_1 + l_2 \leqslant m$. Distributions $a_W\bar{\mu}_{t-s}$ are finite measures [7]. Applying to (23) the formula

$$X(\mu * \varphi * \gamma) = X\mu * \varphi * \gamma - Z\mu * y\varphi * \gamma - Z\mu * \varphi * y\gamma \tag{24}$$

with μ_s in place of μ (by (20), $X\mu_s$ and $Z\mu_s$ are finite measures), Wk for φ and $a_W\bar{\mu}_{t-s}$ for γ we estimate the second integral in (22) by

$$\begin{aligned}\frac{t}{2}\|f\|_\infty \sum_W \Big\{ \sup_{t/2 \leqslant s \leqslant t} \|X\mu_s\|_1 \|Wk\|_1 \sup_{0 \leqslant s \leqslant t/2} \int |a_W|\, d\bar{\mu}_s + \sup_{t/2 \leqslant s \leqslant t} \|Z\mu_s\|_1 \\ \times \Big(\|yWk\|_1 \sup_{0 \leqslant s \leqslant t/2} \int |a_W|\, d\bar{\mu}_s + \|Wk\|_1 \sup_{0 \leqslant s \leqslant t/2} \int |ya_W|\, d\bar{\mu}_s\Big)\Big\}.\end{aligned}$$

Using (21), the integrability of yWk and Corollary 2 we obtain properties (20) and (21) for the operator V.

Iterating (24) and arguing in the same way as for $n = m+1$ we see that (20) and (21) hold for V of order $n \leqslant 2m$ due to integrability of $x^l y^{m-l} Wk$, where W is homogeneous of order m and $l \leqslant m$. If $\alpha > 1$, the function $x^l y^{m+1-l} Wk$, $l \leqslant m+1$, is still integrable, so (20) is true for V of order $2m+1$. Since $\boldsymbol{H} \cong \boldsymbol{R}^3$, it follows that the densities of μ_t are of class $\mathscr{C}^{2m-4}$ and, in the case $\alpha > 1$, of class $\mathscr{C}^{2m-3}$ (see [10]). ∎

In the following theorem we consider a more general symmetric semigroup of measures $\{\mu_t\}$ related to stable semigroups on $\boldsymbol{H}$. We suppose that its Lévy measure ν is given by (4) with $h \in \mathscr{C}^1(\boldsymbol{H})$ of support not necessarily compact. We decompose $\nu = \tilde{\nu} + k$ as in the proof of Theorem 4.

THEOREM 5. *Suppose that the Lévy measure of the semigroup $\{\mu_t\}$ is of class $\mathscr{C}^1$ and that the functions $x^{l_1} y^{l_2} Dk$ are integrable for $D = X, Y, Z$ and $l_1, l_2 \in \boldsymbol{N}$. Then the semigroup $\{\mu_t\}$ has smooth densities.*

Proof. The idea of the proof is the same as in Theorem 4. By assumption one may differentiate k only once. One shows (20) and (21) by induction with respect to the order of V. In particular, in order to get (20) and (21) for V of order n one uses the integrability of $x^{l_1} y^{l_2} Dk$ for $l_1 + l_2 < n$. We omit the details of the proof. ∎

Remark. The integrability condition in Theorem 5 is not satisfied for stable semigroups of measures. It holds for example for semigroups with Lévy measure decreasing rapidly in infinity. Nevertheless Theorem 4 supports the hypothesis that stable semigroups on $\boldsymbol{H}$ with Lévy measure of class $\mathscr{C}^1$ have smooth densities.

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