PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 11, Fasc. 2 (1991), pp. 291–304

MATHEMATICAL EXPECTATION AND MARTINGALES OF RANDOM SUBSETS OF A METRIC SPACE

BY

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Abstract. Let F be a closed, bounded, non-empty random subset of a metric space (X, ϱ). For some class of metric spaces we define in terms of the metric ϱ (developing an idea of S. Doss) mathematical expectation and conditional mathematical expectation of F. We then consider martingales of random subsets of a metric space and prove theorems of convergence for such martingales.

0. Introduction and preliminaries. S. Doss has introduced in [3] a concept of mathematical expectation of a random variable with values in a metric space (see also [1] and [4]). This and other concepts of mathematical expectation were studied by M. Fréchet in [5] and [6].

In this paper we develop an idea of S. Doss and investigate notions of mathematical expectation (Section 1), conditional mathematical expectation (Section 2) and martingale (Section 3) of random subsets of a metric space.

Results of this paper were announced in [8] and [9].

Let (X, ϱ) be a metric space. By $(\hat{X}, \hat{\varrho})$ we denote the metric space of closed, bounded and non-empty subsets of X, equipped with the Hausdorff metric $\hat{\varrho}$ defined as

$$\hat{\varrho}(F, F') = \max\{\sup_{x\in F}\varrho(x, F'), \sup_{x'\in F'}\varrho(x', F)\},$$

where $\varrho(x, F) = \inf\{\varrho(x, y):\ y\in F\}$ for $x\in X$ and $F\in\hat{X}$.

We put

$$F = \operatorname{Lim}_n F_n \quad \text{iff} \quad \lim_n \hat{\varrho}(F_n, F) = 0.$$

For $x\in X$ and $F\in\hat{X}$, we set

$$\delta(x, F) = \sup\{\varrho(x, y):\ y\in F\}.$$

A metric space (X, ϱ) is called *finitely compact* iff every closed bounded subset of X is compact. Let us note the following known

PROPOSITION 0.1 ([11], Proposition 1.2.5). *Let (X, ϱ) be a finitely compact metric space and let $\{F_n\}_{n=1}^{\infty}$ be a sequence of set elements of $\hat{X}$ such that $\bigcup_{n=1}^{\infty} F_n$ is a bounded subset of X. Suppose there exists a dense set $D \subset X$ such that for every $x \in D$ the limit $\lim_n \varrho(x, F_n)$ exists and is finite. Then the sequence $\{F_n\}_{n=1}^{\infty}$ converges in $(\hat{X}, \hat{\varrho})$.*

Let $(\Omega, \mathscr{A}, P)$ be a probability space. An event $A \in \mathscr{A}$ is called *negligible* iff $P(A) = 0$. For a collection $\mathscr{B}$ of subsets of Ω we denote by $\sigma(\mathscr{B})$ the σ-field generated by $\mathscr{B}$.

A Borel measurable map $F\colon \Omega \to \hat{X}$ is called an *$\hat{X}$-valued random set* (r.s.) and a Borel measurable map $f\colon \Omega \to X$ is called an *X-valued random variable* (r.v.). We shall frequently identify a random variable f with a random set $\{f\}$. An r.s. is called *scalarly integrable* iff

$$\int_{\Omega} \delta(x, F(\omega))\, dP(\omega) < \infty \qquad \text{for every } x \in X.$$

Throughout this paper $(\Omega, \mathscr{A}, P)$ will be a fixed complete, non-atomic probability space and all random sets will be defined on $(\Omega, \mathscr{A}, P)$.

1. Mathematical expectation.

DEFINITION 1.1. Let (X, ϱ) be a metric space and F an $\hat{X}$-valued random set. The set $E[F]$ defined as

$$E[F] = \{a \in X\colon\ \varrho(x, a) \leqslant \int_{\Omega} \delta(x, F(\omega))\, dP(\omega) < \infty,\ \forall x \in X\}$$

is called a *mathematical expectation* of F.

For every $\hat{X}$-valued r.s. F the set $E[F]$ is evidently closed. If F is scalarly integrable, then the set $E[F]$ is also bounded.

We shall state now the condition imposed on a metric space (X, ϱ) in order that for every X-valued r.s. F the set $E[F]$ is non-empty.

DEFINITION 1.2. A metric space (X, ϱ) is called *convex in the sense of Doss* (or *D-convex*) iff for any two elements $x_1, x_2 \in X$ there exists an element $a \in X$ such that

$$\varrho(x, a) \leqslant \tfrac{1}{2}[\varrho(x, x_1) + \varrho(x, x_2)], \qquad \forall x \in X.$$

Remark 1.1. It is easily checked that every D-convex metric space is metrically convex in the sense of Menger (see [2], Definition 14.1) but not conversely (e.g., a circle in the Euclidean plane with an arc metric).

Remark 1.2. In ([7], Section 8) the authors have proved that the hyperbolic plane (of Lobochevski) equipped with the geodesic metric is a D-convex metric space (it can be proved that any simply connected Riemannian manifold of non-positive curvature equipped with geodesic metric is a D-convex metric space).

Remark 1.3. Suppose $(Y, \|\cdot\|)$ is a Banach space and $\varrho(x, y) = \|x-y\|$ for $x, y \in Y$. Then for every Y-valued Bochner-integrable random variable the Bochner integral $\int_\Omega f(\omega)\,dP(\omega) \in E[f]$.

Doss has proved in ([3], Théorème 1) that

$$E[f] = \{\int_\Omega f\,dP\} \quad \text{if } \dim Y = 1.$$

In ([7], Theorem 1) the authors have proved (answering the question of Fréchet [6]) that $E[f] = \{\int_\Omega f\,dP\}$ for any two-valued random variable f.

Remark 1.4. Suppose X is a closed, bounded, convex subset of a Banach space Y. Then the metric space (X, ϱ) is D-convex and the Bochner integral $\int_\Omega f\,dP \in E[f]$ for any Bochner-integrable X-valued random variable f.

The following example shows that the Bochner integral is not necessarily the only element of $E[f]$.

EXAMPLE 1.1. Let

$$X = \{[\alpha_1, \alpha_2]:\ \alpha_1 \geqslant 0,\ \alpha_2 \geqslant 0,\ \alpha_1+\alpha_2 \leqslant 1\}$$

and

$$\varrho([\alpha_1, \alpha_2], [\beta_1, \beta_2]) = |\alpha_1-\beta_1|+|\alpha_2-\beta_2|.$$

Let f be an X-valued r.v. satisfying

$$P(f = [1, 0]) = P(f = [0, 1]) = \tfrac{1}{2}.$$

One checks easily that $E[f] = \{[\alpha_1, \alpha_2] \in X:\ \alpha_1 = \alpha_2\}$.

THEOREM 1.1. *Let (X, ϱ) be a finitely compact metric space. Then for every $\hat{X}$-valued random set F the set $E[F]$ is non-empty iff (X, ϱ) is a D-convex metric space.*

Proof. The necessity of D-convexity of a metric space (X, ϱ) is evident, since "D-convexity" means precisely that for every X-valued r.v. f satisfying $P(f = x_1) = P(f = x_2) = \frac{1}{2}$, one has $E[f] \neq \emptyset$.

We shall prove now that if a metric space (X, ϱ) is D-convex, then for every $\hat{X}$-valued r.s. F the set $E[F]$ is non-empty.

If a random set F is not scalarly integrable, then $E[F] = X \neq \emptyset$.

If F is a scalarly integrable r.s., then any measurable selection f of F (which always exists by [10]) is a scalarly integrable r.v. and $E[f] \subset E[F]$.

It is thus sufficient to prove that if a metric space (X, ϱ) is D-convex, then for every scalarly integrable X-valued r.v. f the set $E[f]$ is non-empty.

This will be proved in several steps.

(1°) If f is an X-valued r.v. with $\operatorname{card} f(\Omega) \leqslant 2$, then $E[f] \neq \emptyset$.

We have to prove that for every $x_1, x_2 \in X$ and any $p \in [0, 1]$ there is an element $a \in X$ such that

$$\varrho(x, a) \leqslant p\varrho(x, x_1) + (1-p)\varrho(x, x_2), \quad \forall x \in X.$$

We shall prove this first for dyadic rationals $p = k/2^n$ ($k = 1, \ldots, 2^n$; $n = 1, 2, \ldots$). We shall proceed by induction; for $n = 1$ our statement is true by the definition of D-convexity of a metric space (X, ϱ). If for some $n \geqslant 1$ and $1 \leqslant k, l \leqslant 2^n$ one has

$$\varrho(x, a) \leqslant \frac{k}{2^n}\varrho(x, x_1) + \left(1 - \frac{k}{2^n}\right)\varrho(x, x_2), \quad \forall x \in X,$$

and

$$\varrho(x, b) \leqslant \frac{l}{2^n}\varrho(x, x_1) + \left(1 - \frac{l}{2^n}\right)\varrho(x, x_2), \quad \forall x \in X,$$

then there exists $c \in X$ such that

$$\varrho(x, c) \leqslant \frac{1}{2}[\varrho(x, a) + \varrho(x, b)] = \frac{k+l}{2^{n+1}}\varrho(x, x_1) + \left(1 - \frac{k+l}{2^{n+1}}\right)\varrho(x, x_2),$$

which completes the induction.

Let $p \in [0, 1]$ be arbitrary and let $\{p_n\}_{n=1}^{\infty}$ be a sequence of dyadic rationals converging to p. For every $n = 1, 2, \ldots$ there are elements $a_n \in X$ such that

$$\varrho(x, a_n) \leqslant p_n\varrho(x, x_1) + (1-p_n)\varrho(x, x_2), \quad \forall x \in X.$$

Since the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded and (X, ϱ) is finitely compact, we can extract from $\{a_n\}_{n=1}^{\infty}$ a subsequence $\{a_{k_n}\}_{n=1}^{\infty}$ converging to some $a \in X$. One evidently has

$$\varrho(x, a) \leqslant p\varrho(x, x_1) + (1-p)\varrho(x, x_2), \quad \forall x \in X.$$

(2°) If f is an X-valued r.v. with $f(\Omega)$ finite, then $E[f] \neq \emptyset$.

We shall proceed by induction. By (1°), our statement is true if $\operatorname{card} f(\Omega) \leqslant 2$.

Let $f(\Omega) = \{x_1, x_2, \ldots, x_n, x_{n+1}\}$ and $P(f = x_i) = p_i$ for $i = 1, 2, \ldots, n+1$. Let us consider the r.v. distributed as follows:

$$P(g = x_i) = p_i / \sum_{j=1}^{n} p_j \quad \text{for } i = 1, 2, \ldots, n.$$

Supposing that (2°) is true for an n-valued r.v. g we have $E[g] \neq \emptyset$. Let $a \in E[g]$ and let us consider the r.v. h distributed as follows:

$$P(h = a) = \sum_{j=1}^{n} p_j, \quad P(h = x_{n+1}) = p_{n+1}.$$

Then by (1°) we infer that $E[h] \neq \emptyset$. It is easily checked that $E[h] \subset E[f]$, and thus $E[f] \neq \emptyset$.

(3°) If f is an X-valued scalarly integrable r.v. with $f(\Omega)$ countable, then $E[f] \neq \emptyset$.

Let $f(\Omega) = \{x_1, x_2, \ldots\}$ and $P(f = x_i) = p_i$ for $i = 1, 2, \ldots$ By (2°), for every $n = 1, 2, \ldots$ there are elements $a_n \in X$ such that

$$\varrho(x, a_n) \leqslant \sum_{i=1}^{n} \varrho(x, x_i) q_i^n, \quad \forall x \in X,$$

where

$$q_i^n = p_i / \sum_{j=1}^{n} p_j, \quad i = 1, 2, \ldots, n.$$

For every $x \in X$ we have

$$\lim_n \sum_{i=1}^{n} \varrho(x, x_i) q_i^n = \sum_{i=1}^{\infty} \varrho(x, x_i) p_i < \infty.$$

This implies that the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded, and since the metric space (X, ϱ) is finitely compact, we can extract from $\{a_n\}_{n=1}^{\infty}$ a subsequence $\{a_{k_n}\}_{n=1}^{\infty}$ convergent to some $a \in X$. Then for every $x \in X$ we have

$$\varrho(x, a) = \lim_n \varrho(x, a_{k_n}) \leqslant \lim_n \sum_{i=1}^{k_n} \varrho(x, x_i) q^{k_n} = \sum_{i=1}^{\infty} \varrho(x, x_i) p_i,$$

which means that $a \in E[f]$.

(4°) If f is an arbitrary scalarly integrable X-valued r.v., then $E[f] \neq \emptyset$.

Since the metric space (X, ϱ) is separable, for every $n = 1, 2, \ldots$ there exists an X-valued r.v. f_n such that $f_n(\Omega)$ is countable and

$$\varrho(f_n(\omega), f(\omega)) \leqslant 1/n, \quad \forall \omega \in \Omega,$$

which implies that

$$\varrho(x, f_n(\omega)) \leqslant \varrho(x, f(\omega)) + 1/n, \quad \forall \omega \in \Omega,\ \forall x \in X,\ \forall n \geqslant 1.$$

By (3°) there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of elements of X such that

$$\varrho(x, a_n) \leqslant \int_\Omega \varrho(x, f_n)\, dP \leqslant \int_\Omega \varrho(x, f)\, dP + 1/n, \quad \forall x \in X,\ \forall n \geqslant 1.$$

Thus the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded and we can extract from it a subsequence $\{a_{k_n}\}_{n=1}^{\infty}$ convergent to some $a \in X$. Since

$$\varrho(x, f_n(\omega)) \leqslant \varrho(x, f(\omega)) + 1/n \quad \text{for } x \in X,\ \omega \in \Omega,\ n = 1, 2, \ldots,$$

by Lebesgue's bounded convergence theorem we have

$$\varrho(x, a) = \lim_n \varrho(x, a_{k_n}) \leqslant \lim_n \int_\Omega \varrho(x, f_n)\, dP = \int_\Omega \varrho(x, f)\, dP, \quad \forall x \in X,$$

which means that $a \in E[f]$.

2. Conditional mathematical expectation. Throughout this section we shall assume that (X, ϱ) is a finitely compact, D-convex metric space and F is an $\hat{X}$-valued scalarly integrable random set.

Suppose $\mathscr{F}$ is a finite subfield of $\mathscr{A}$ with non-negligible atoms (throughout this paper we shall always assume that finite subfields of $\mathscr{A}$ have non-negligible atoms). Let us define the following random set:

$$E^{\mathscr{F}}[F](\omega) = E[F \mid A] \quad \text{for } \omega \in A, \text{ an atom of } \mathscr{F},$$

where

$$E[F \mid A] = \left\{ a \in X \colon \varrho(x, a) \leqslant \frac{1}{P(A)} \int_{\Omega} \delta(x, F)\, dP, \ \forall x \in X \right\}.$$

LEMMA 2.1. *Let $\{\mathscr{F}_n\}_{n=1}^{\infty}$ be an increasing sequence of finite subfields of $\mathscr{A}$. Then*:

(1°) $\bigcup_{n=1}^{\infty} E^{\mathscr{F}_n}[F](\omega)$ *is a bounded subset of X for almost every $\omega \in \Omega$.*

(2°) *For every $x \in X$ the sequence of reals $\{\varrho(x, E^{\mathscr{F}_n}[F](\omega))\}_{n=1}^{\infty}$ converges to a finite limit for almost every $\omega \in \Omega$.*

Proof. (1°) Let x be some element of X. For every $\omega \in \Omega$ and every $a \in E^{\mathscr{F}_n}[F](\omega)$ we have

$$\varrho(x, a) \leqslant \frac{1}{P(A_n)} \int_{A_n} \delta(x, F)\, dP, \quad \text{where } \omega \in A_n, \text{ an atom of } \mathscr{F}_n.$$

The real martingale

$$\left\{ \frac{1}{P(A_n)} \int_{A_n} \delta(x, F)\, dP, \ \mathscr{F}_n \right\}_{n=1}^{\infty}$$

converges almost surely to a finite limit ([12], Proposition II-2-11). Thus for almost every $\omega \in \Omega$

$$\sup_n \sup_{a \in F_n} \varrho(x, a) < \infty, \quad \text{where } F_n = E^{\mathscr{F}_n}[F](\omega),$$

which proves that the set $\bigcup_{n=1}^{\infty} E^{\mathscr{F}_n}[F](\omega)$ is bounded for almost every $\omega \in \Omega$.

(2°) Let x be some element of X. Denote by $\{\xi_n\}_{n=1}^{\infty}$ the sequence of real random variables defined as

$$\xi_n(\omega) = \varrho(x, E^{\mathscr{F}_n}[F](\omega)) \quad \text{for } \omega \in \Omega \ (n = 1, 2, \ldots).$$

It is sufficient to prove that $\{\xi_n, \mathscr{F}_n\}_{n=1}^{\infty}$ is a submartingale satisfying Doob's condition ([12], Theorem IV.1.2):

$$\sup_n \int_{\Omega} \xi_n\, dP < \infty.$$

For every A, an atom of $\mathscr{F}_n$, we have

$$\int_A \xi_n dP = \int_A \varrho(x, E^{\mathscr{F}_n}[F])\, dP = P(A)\varrho(x, E[F \mid A]) \leqslant \int_A \delta(x, F)\, dP.$$

Hence

$$\int_\Omega \xi_n dP \leqslant \int_\Omega \delta(x, F)\, dP < \infty$$

and Doob's condition is satisfied.

For every $n = 1, 2, \ldots, \xi_n$ is evidently $\mathscr{F}_n$-measurable. Thus we have to check that for every $n = 1, 2, \ldots$ and every atom A of $\mathscr{F}_n$ the inequality

$$\int_A \xi_n dP \leqslant \int_A \xi_{n+1} dP$$

holds, that is

$$\text{(2.1)} \qquad P(A)\varrho(x, E[F \mid A]) \leqslant \int_A \varrho(x, E^{\mathscr{F}_{n+1}}[F])\, dP \qquad (n = 1, 2, \ldots).$$

Let $A_1, \ldots, A_k$ be (disjoint) atoms of $\mathscr{F}_{n+1}$ such that

$$A = \bigcup_{i=1}^{k} A_i.$$

Since every set $E[F \mid A_i]$ is non-empty and compact, we can find the elements $a_i \in E[F \mid A_i]$ such that

$$\varrho(x, a_i) = \varrho(x, E[F \mid A_i]) \quad \text{for } i = 1, \ldots, k.$$

Let g be an X-valued r.v. distributed as follows:

$$P(g = a_i) = \frac{P(A_i)}{P(A)} \quad \text{for } i = 1, \ldots, k.$$

It is easily checked that $E[g] \subset E[F \mid A]$. Hence for every $a \in E[g]$ we have

$$\varrho(x, E[F \mid A]) \leqslant \varrho(x, a).$$

Thus, taking an arbitrary element $a \in E[g]$, we obtain

$$P(A)\varrho(x, E[F \mid A]) \leqslant P(A)\varrho(x, a) \leqslant P(A) \sum_{i=1}^{k} \frac{P(A_i)}{P(A)} \varrho(x, a_i)$$

$$= \sum_{i=1}^{k} P(A_i)\varrho(x, E[F \mid A_i]) = \int_A \varrho(x, E^{\mathscr{F}_{n+1}}[F])\, dP,$$

which proves (2.1) and completes the proof of the lemma.

THEOREM 2.1. *Let $\{\mathscr{F}_n\}_{n=1}^{\infty}$ be an increasing sequence of finite subfields of $\mathscr{A}$. Then the sequence $\{E^{\mathscr{F}_n}[F]\}_{n=1}^{\infty}$ of $\hat{X}$-valued random sets converges almost surely.*

Proof. Let D be a countable dense set in (X, ϱ). By Lemma 2.1 there exists a negligible event N such that for every $\omega \in \Omega \setminus N$ the set $\bigcup_{n=1}^{\infty} E^{\mathscr{F}_n}[F](\omega)$ is bounded in X and $\lim_n \varrho(x, E^{\mathscr{F}_n}[F](\omega))$ exists and is finite for every $x \in D$. Thus by Lemma 0.1 it follows that the sequence $\{E^{\mathscr{F}_n}[F](\omega)\}_{n=1}^{\infty}$ is convergent in $(\hat{X}, \hat{\varrho})$ for every $\omega \in \Omega \setminus N$.

We shall now define conditional mathematical expectation of F relative to an arbitrary sub-σ-field $\mathscr{B}$ of $\mathscr{A}$.

Let L_0 be a space of (equivalence classes of) $\hat{X}$-valued random sets equipped with topology of convergence in probability with respect to the Hausdorff metric $\hat{\varrho}$ in $\hat{X}$. This topology is metrizable by the metric:

$$\hat{\varrho}_0(F, F') = \inf\{\varepsilon > 0:\ P(\hat{\varrho}(F, F') > \varepsilon) < \varepsilon\}$$

and the metric space $(L_0, \hat{\varrho}_0)$ is complete.

Let us remark that, as in the real case, almost sure convergence of F_n to F implies that $\hat{\varrho}_0(F_n, F) \to 0$ as $n \to \infty$.

Let $\mathscr{B}$ be an arbitrary (not necessarily finite) sub-σ-field of $\mathscr{A}$ and let $\mathscr{F}(\mathscr{B})$ be the collection of all finite subfields of $\mathscr{B}$ downward directed by inclusion. Theorem 2.1 states that, for any increasing sequence $\{\mathscr{F}_n\}_{n=1}^{\infty}$ of elements in the directed set $\mathscr{F}(\mathscr{B})$, the sequence $\{E^{\mathscr{F}_n}[F]\}_{n=1}^{\infty}$ converges in a complete metric space $(L_0, \hat{\varrho}_0)$. This implies ([12], Lemma V-1-1) that the net $\{E^{\mathscr{F}}[F]\}_{\mathscr{F} \in \mathscr{F}(\mathscr{B})}$ is convergent in $(L_0, \hat{\varrho}_0)$.

DEFINITION 2.1. Any random set from the equivalence class $\lim_{\mathscr{F} \in \mathscr{F}(\mathscr{B})} E^{\mathscr{F}}[F]$ is called a *(version of the) conditional mathematical expectation of F relative to $\mathscr{B}$.*

We shall prove now the following metric analogous of a theorem of P. Lévy.

THEOREM 2.2. *Let $\{\mathscr{B}_n\}_{n=1}^{\infty}$ be an increasing sequence of countably generated sub-σ-fields of $\mathscr{A}$. Then the sequence $\{E^{\mathscr{B}_n}[F]\}_{n=1}^{\infty}$ of $\hat{X}$-valued random sets converges almost surely and*

$$\operatorname*{Lim}_n E^{\mathscr{B}_n}[F] \subset E^{\mathscr{B}_\infty}[F] \ \text{a.s.}, \qquad \text{where } \mathscr{B}_\infty = \sigma\Big(\bigcup_{n=1}^{\infty} \mathscr{B}_n\Big).$$

Before proving Theorem 2.2 let us state two lemmas.

LEMMA 2.2. *Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of non-negligible events in $\mathscr{A}$ such that $P(A_n \Delta A_0) \to 0$ as $n \to \infty$ (Δ stands for symmetric difference of sets). If a sequence $a_n \in E[F \mid A_n]$ $(n = 1, 2, \ldots)$ is convergent, then $\lim_n a_n \in E[F \mid A_0]$.*

Proof. For every $n = 1, 2, \ldots$

$$\varrho(x, a_n) \leqslant \frac{1}{P(A_n)} \int_{A_n} \delta(x, F)\, dP, \qquad \forall x \in X.$$

Hence, for every $x \in X$,

$$\varrho(x, \lim_n a_n) = \lim_n \varrho(x, a_n) \leqslant \frac{1}{P(A_0)} \int_{A_0} \delta(x, F)\, dP,$$

which means that $\lim_n a_n \in E[F \mid A_0]$.

LEMMA 2.3. *Let $\{\mathscr{F}_n\}_{n=1}^{\infty}$ be an increasing sequence of finite subfields of $\mathscr{A}$. Then*

$$\operatorname*{Lim}_n E^{\mathscr{F}_n}[F] \subset E^{\mathscr{F}_\infty}[F] \text{ a.s.,} \quad \text{where } \mathscr{F}_\infty = \sigma\Big(\bigcup_{n=1}^{\infty} \mathscr{F}_n\Big).$$

Proof. Since $E^{\mathscr{F}_\infty}[F]$ is a limit in $(L_0, \hat{\varrho}_0)$ of a net $\{E^{\mathscr{F}}[F]\}_{\mathscr{F} \in \mathscr{F}(\mathscr{F}_\infty)}$, there exists an increasing sequence $\{\mathscr{F}_n'\}_{n=1}^{\infty}$ of finite subfields of $\mathscr{F}_\infty$ such that

$$\text{(2.2)} \qquad \lim_n \hat{\varrho}_0(E^{\mathscr{F}_n'}[F], E^{\mathscr{F}_\infty}[F]) = 0 \quad \text{and} \quad \mathscr{F}_n \subset \mathscr{F}_n' \quad \text{for } n = 1, 2, \ldots$$

By Theorem 2.1, both sequences $\{E^{\mathscr{F}_n}[F]\}_{n=1}^{\infty}$ and $\{E^{\mathscr{F}_n'}[F]\}_{n=1}^{\infty}$ converge almost surely. By Egoroff's theorem (which is just as valid for r.v.'s with values in a metric space as for real r.v.'s) and by the density of $\bigcup_{n=1}^{\infty} \mathscr{F}_n$ in $\mathscr{F}_\infty$ we infer that for every $\varepsilon > 0$ there is a positive integer $n(\varepsilon)$ and a set $B_\varepsilon \in \mathscr{F}_{n(\varepsilon)}$ such that $P(B_\varepsilon) > 1-\varepsilon$ and both sequences $\{E^{\mathscr{F}_n}[F]\}_{n=1}^{\infty}$ and $\{E^{\mathscr{F}_n'}[F]\}_{n=1}^{\infty}$ converge uniformly on B_ε. Thus there exists a subsequence $\{p_n\}_{n=1}^{\infty}$ of positive integers with $p_1 \geqslant n(\varepsilon)$ and such that

$$\text{(2.3)} \qquad \hat{\varrho}_0\big(E^{\mathscr{F}_{p_i}}[F](\omega), E^{\mathscr{F}_{p_j}}[F](\omega)\big) \leqslant 1/n, \quad \forall \omega \in B_\varepsilon,\ \forall n \geqslant 1,\ \forall i, j \geqslant n.$$

Let us fix arbitrary $\omega_0 \in B_\varepsilon$. We shall prove that

$$\text{(2.4)} \qquad \operatorname*{Lim}_n E^{\mathscr{F}_n}[F](\omega_0) \subset \operatorname*{Lim}_n E^{\mathscr{F}_n'}[F](\omega_0).$$

If $x \in \operatorname*{Lim}_n E^{\mathscr{F}_n}[F](\omega_0)$, then there exists a subsequence $\{p_{r_n}\}_{n=1}^{\infty}$ of $\{p_n\}_{n=1}^{\infty}$ and a sequence $\{x_n\}_{n=1}^{\infty}$ of elements $x_n \in E^{\mathscr{F}_{p_{r_n}}}[F](\omega_0)$ $(n = 1, 2, \ldots)$ such that

$$x = \lim_n x_n.$$

For every $n = 1, 2, \ldots$ and every $\omega \in B_\varepsilon$ let us denote by $A_n(\omega)$ and $A_n'(\omega)$ the atoms of $\mathscr{F}_{p_{r_n}}$ and $\mathscr{F}_{p_{r_n}}'$, respectively, such that $\omega \in A_n(\omega)$ and $\omega \in A_n'(\omega)$.

Since $\mathscr{F}_\infty$ is generated by the $\mathscr{F}_n$'s, for every $n = 1, 2, \ldots$ there is a sequence of sets $\{B_{n,m}\}_{m=1}^{\infty}$ such that $B_{n,m} \in \mathscr{F}_{p_{r_{n+m}}}$ for $m = 1, 2, \ldots$ and

$$\lim_m P(A'_n(\omega_0) \Delta B_{n,m}) = 0.$$

Since, by (2.2), $A'_n(\omega_0) \subset A_n(\omega_0)$, we can and shall assume that $B_{n,m} \subset A_{n,m}(\omega_0)$ for $n, m = 1, 2, \ldots$ From (2.3) we have

$$\varrho(x_n, E[F \mid A_j(\omega_0)]) \leqslant 1/n, \quad \forall n \geqslant 1, \ \forall j \geqslant n. \tag{2.5}$$

Since $A_j(\omega) = A_j(\omega_0)$ for every $\omega \in A_j(\omega_0)$ $(j = 1, 2, \ldots)$, from (2.5) we obtain

$$\varrho(x_n, E[F \mid A_j(\omega)]) \leqslant 1/n, \quad \forall n \geqslant 1, \ \forall j \geqslant n, \ \forall \omega \in A_j(\omega_0). \tag{2.6}$$

Since $\mathscr{F}_k$ $(k = 1, 2, \ldots)$ are finite subfields of $\mathscr{A}$, there are finite sets of indices $I = I(n, m)$ $(n, m = 1, 2, \ldots)$ such that

$$B_{n,m} = \bigcup_{i \in I} A_{n+m}(\omega_i) \quad \text{and} \quad A_{n+m}(\omega_i) \cap A_{n+m}(\omega_l) = \emptyset \quad \text{for } i \neq l, \ i, l \in I.$$

By (2.6) there exist elements $a^i_{n,m} \in E[F \mid A_{n+m}(\omega_i)]$ such that $\varrho(x_n, a^i_{n,m}) \leqslant 1/n$. Since the metric space (X, ϱ) is D-convex, there exist elements $b_{n,m} \in X$ $(n, m = 1, 2, \ldots)$ such that

$$\varrho(x, b_{n,m}) \leqslant \sum_{i \in I} \frac{P(A_{n+m}(\omega_i))}{P(B_{n,m})} \varrho(x, a^i_{n,m}), \quad \forall x \in X.$$

It is easily checked that

$$b_{n,m} \in E[F \mid B_{n,m}] \quad \text{and} \quad \varrho(x_n, b_{n,m}) \leqslant 1/n \quad (n, m = 1, 2, \ldots). \tag{2.7}$$

Since (X, ϱ) is finitely compact, for every $n = 1, 2, \ldots$ the bounded sequence $\{b_{n,m}\}_{m=1}^{\infty}$ contains a convergent subsequence $\{b_{n,k_m}\}_{m=1}^{\infty}$. Put

$$b_n = \lim_m b_{n,k_m} \quad (n = 1, 2, \ldots).$$

Since for $n = 1, 2, \ldots$ we have $P(B_{n,k_m} \Delta A'_n(\omega_0)) \to 0$ as $m \to \infty$, we infer from (2.7) and Lemma 2.1 that

$$b_n \in E[F \mid A'_n(\omega_0)] \quad \text{and} \quad \varrho(x_n, b_n) \leqslant 1/n \quad \text{for } n = 1, 2, \ldots \tag{2.8}$$

Thus

$$x = \lim_n x_n = \lim_n b_n \in \operatorname{Lim}_n E[F \mid A'_n(\omega_0)] = \operatorname{Lim}_n E^{\mathscr{F}'_n}[F](\omega_0),$$

which proves (2.4) and completes the proof of Lemma 2.3.

Proof of Theorem 2.2. Since each $\mathscr{B}_n$ is countably generated, for each $n = 1, 2, \ldots$ there exists an increasing sequence $\{\mathscr{F}_{n,m}\}_{m=1}^{\infty}$ of finite subfields of $\mathscr{B}_n$ which generates $\mathscr{B}_n$. Since each $E^{\mathscr{B}_n}[F]$ is a limit in L_0 of a net

$\{E^{\mathscr{F}}[F]\}_{\mathscr{F}\in\mathscr{F}(\mathscr{B}_n)}$ and $\mathscr{B}_n \subset \mathscr{B}_{n+1}$, we can and shall assume that

$$\lim_m \hat{\varrho}_0(E^{\mathscr{F}_{n,m}}[F], E^{\mathscr{B}_n}[F]) = 0 \quad \text{and} \quad \mathscr{F}_{n,m} \subset \mathscr{F}_{n+1,m} \quad \text{for } n, m = 1, 2, \ldots$$

Let us fix an arbitrary $\varepsilon > 0$. By Egoroff's theorem there are sets $B_n \in \mathscr{B}_n$ such that $P(B_n) > 1-\varepsilon/2^n$ $(n = 1, 2, \ldots)$ and

$$\operatorname{Lim}_m E^{\mathscr{F}_{n,m}}[F] = E^{\mathscr{B}_n}[F] \text{ uniformly on } B_n \quad (n = 1, 2, \ldots).$$

Thus there exists a subsequence $\{m_n\}$ of positive integers such that

$$\sup_{\omega\in B} \hat{\varrho}_0\big(E^{\mathscr{F}_{n,m_n}}[F](\omega), E^{\mathscr{B}_n}[F](\omega)\big) \to 0 \quad \text{as } n\to\infty, \tag{2.9}$$

where

$$B = \bigcap_{n=1}^{\infty} B_n.$$

Defining $\mathscr{F}_n = \mathscr{F}_{n,m_n}$ we obtain an increasing sequence $\{\mathscr{F}_n\}_{n=1}^{\infty}$ of finite subfields of $\mathscr{B}_\infty$ which generates $\mathscr{B}_\infty$. By Lemma 2.3 we have

$$\operatorname{Lim}_n E^{\mathscr{F}_n}[F] \subset E^{\mathscr{B}_\infty}[F] \text{ a.s.}$$

Let $B_\varepsilon \in \mathscr{B}_\infty$, $B_\varepsilon \subset \mathscr{B}$, be a set with $P(B_\varepsilon) > 1-2\varepsilon$ and such that $E^{\mathscr{F}_n}[F]$ converges uniformly on B_ε. It follows from (2.9) that $E^{\mathscr{B}_n}[F]$ converges uniformly on B_ε and

$$\operatorname{Lim}_n E^{\mathscr{B}_n}[F](\omega) = \operatorname{Lim}_n E^{\mathscr{F}_n}[F](\omega) \subset B^{\mathscr{B}_\infty}[F](\omega), \quad \forall \omega \in B_\varepsilon,$$

which completes the proof since $P(B_\varepsilon) > 1-2\varepsilon$ and $\varepsilon > 0$ was chosen arbitrarily.

3. Martingales. Throughout this section (X, ϱ) is a finitely compact, D-convex metric space and all random sets take values in $\hat{X}$.

If F is a random set and $\mathscr{B}$ a sub-σ-field of $\mathscr{A}$, then we denote by $S(F; \mathscr{B})$ the collection of all $\mathscr{B}$-measurable selections of F.

Definition 3.1. Let $\{\mathscr{B}_n\}_{n=1}^{\infty}$ be an increasing sequence of sub-σ-fields of $\mathscr{A}$ and $\{F_n\}_{n=1}^{\infty}$ a sequence of scalarly integrable, $\mathscr{B}_n$-measurable random sets. We say that $\{F_n, \mathscr{B}_n\}_{n=1}^{\infty}$ is a *martingale* iff

$$E^{\mathscr{B}_n}[f] \subset F_n \text{ a.s.} \quad \text{for every } f \in S(F_{n+1}; \mathscr{B}_{n+1}),\ n = 1, 2, \ldots$$

Lemma 3.1. *Let F be a scalarly integrable random set, $\mathscr{B}$ a sub-σ-field of $\mathscr{A}$ and $\{\mathscr{F}_m\}_{m=1}^{\infty}$ an increasing sequence of finite subfields of $\mathscr{B}$ such that*

$$E^{\mathscr{B}}[F] = \operatorname{Lim}_m E^{\mathscr{F}_m}[F] \text{ a.s.}$$

Then for every $x \in X$ there exists a negligible event N such that for every $\omega \in \Omega \setminus N$ and $a \in E^{\mathscr{B}}[F](\omega)$ we have

$$\varrho(x, a) \leqslant E^{\mathscr{F}_\infty}[\delta(x, F)](\omega), \quad \text{where } \mathscr{F}_\infty = \sigma\Big(\bigcup_{m=1}^{\infty} \mathscr{F}_m\Big).$$

Proof. Let N' be a negligible event such that

$$E^{\mathscr{B}}[F](\omega) = \operatorname*{Lim}_m E^{\mathscr{F}_m}[F](\omega), \quad \forall \omega \in \Omega \setminus N'.$$

Let $a \in E^{\mathscr{B}}[F](\omega)$ for some $\omega \in \Omega \setminus N'$. Thus there exists a sequence $\{a_m\}_{m=1}^{\infty}$ of elements of X converging to a and such that $a_m \in E^{\mathscr{F}_m}[F](\omega)$ $(m = 1, 2, \ldots)$, which means that for every $x \in X$ the following inequality holds:

$$\varrho(x, a_m) \leqslant \frac{1}{P(A_m)} \int_{A_m} \delta(x, F), \quad \text{where } \omega \in A_m, \text{ an atom of } \mathscr{F}_m.$$

The real martingale

$$\left\{\frac{1}{P(A_m)} \int_{A_m} \delta(x, F)\, dP, \mathscr{F}_m\right\}_{m=1}^{\infty}$$

converges to $E^{\mathscr{F}_\infty}[\delta(x, F)]$ outside some negligible event N'' ([12], Proposition II.2.11). Thus for every $x \in X$ there is a negligible event $N = N' \cup N''$such that

$$\varrho(x, a) = \lim_m \varrho(x, a_m) \leqslant E^{\mathscr{F}_\infty}[\delta(x, F)](\omega), \quad \forall \omega \in \Omega \setminus N.$$

THEOREM 3.1. *Let F be a scalarly integrable random set and $\{\mathscr{B}_n\}_{n=1}^{\infty}$ an increasing sequence of sub-σ-fields of $\mathscr{A}$. Then $\{E^{\mathscr{B}_n}[F], \mathscr{B}_n\}_{n=1}^{\infty}$ is a martingale.*

Proof. Let $n \geqslant 1$ be fixed and $f \in S(E^{\mathscr{B}_{n+1}}[F]; \mathscr{B}_{n+1})$. Let $\{F_m^n\}_{m=1}^{\infty}$ and $\{\mathscr{F}_m^{n+1}\}_{m=1}^{\infty}$ be two increasing sequences of finite subfields of $\mathscr{B}_n$ and $\mathscr{B}_{n+1}$, respectively, such that $\mathscr{F}_m^n \subset \mathscr{F}_m^{n+1}$ for $m = 1, 2, \ldots$ and satisfying

$$E^{\mathscr{B}_n}[f](\omega) = \operatorname*{Lim}_m E^{\mathscr{F}_m^n}[f](\omega), \quad E^{\mathscr{B}_n}[F](\omega) = \operatorname*{Lim}_m E^{\mathscr{F}_m^n}[F],$$

$$E^{\mathscr{B}_{n+1}}[F] = \operatorname*{Lim}_m E^{\mathscr{F}_m^{n+1}}[F](\omega), \quad \forall \omega \in \Omega \setminus N,$$

for some negligible event N.

Let $a \in E^{\mathscr{B}_n}[f](\omega)$ for some $\omega \in \Omega \setminus N$. There is then a sequence $\{a_m\}_{m=1}^{\infty}$ of elements of X converging to a such that $a_m \in E^{\mathscr{F}_m^n}[f](\omega)$ for $m = 1, 2, \ldots$, which means that for every $x \in X$ the following inequality holds:

$$\varrho(x, a_m) \leqslant \frac{1}{P(A_m)} \int_{A_m} \varrho(x, f)\, dP, \quad \text{where } \omega \in A_m, \text{ an atom of } \mathscr{F}_m^n.$$

Since $f(\omega)\in E^{\mathscr{B}_{n+1}}[F](\omega)$ for every $\omega\in\Omega$, by Lemma 3.1 we have

$$\varrho(x, a_m) \leqslant \frac{1}{P(A_m)} \int_{A_m} E^{\mathscr{F}_\infty^{n+1}}[\delta(x, F)]\, dP, \qquad \forall x\in X,$$

where

$$\mathscr{F}_\infty^{n+1} = \sigma\Big(\bigcup_{m=1}^{\infty} \mathscr{F}_m^{n+1}\Big).$$

But $\mathscr{F}_m^n \subset \mathscr{F}_\infty^{n+1}$ for $m = 1, 2, \ldots$, and thus we have

$$\varrho(x, a_m) \leqslant \frac{1}{P(A_m)} \int_{A_m} \delta(x, F)\, dP, \qquad \forall x\in X,\ m \geqslant 1,$$

which means that $a_m \in E^{\mathscr{F}_m^n}[F](\omega)$ for $m = 1, 2, \ldots$, and thus

$$a = \lim_m a_m \in E^{\mathscr{B}_n}[F](\omega).$$

The theorem is proved.

THEOREM 3.2. *Let* $\{F_n, \mathscr{B}_n\}_{n=1}^{\infty}$ *be a martingale and suppose that*:

(a) *The set* $\bigcup_{n=1}^{\infty} F_n(\omega)$ *is a bounded subset of* X *for almost every* $\omega\in\Omega$.

(b) $\sup \int_{\Omega} \varrho(x, F_n)\, dP < \infty$, $\forall x\in X$.

(c) *The* σ-*fields* $\mathscr{B}_n$ *are countably generated for* $n = 1, 2, \ldots$

Then the sequence $\{F_n\}_{n=1}^{\infty}$ *of random sets converges almost surely.*

Proof. We shall show first that $\{\varrho(x, F_n), \mathscr{B}_n\}_{n=1}^{\infty}$ is a (real) submartingale for every $x\in X$.

Let $n \geqslant 1$ be fixed and let $f\in S(F_{n+1}; \mathscr{B}_{n+1})$ satisfy

$$\varrho(x, f(\omega)) = \varrho(x, F_{n+1}(\omega)), \qquad \forall \omega\in\Omega.$$

Since $\{F_n, \mathscr{B}_n\}_{n=1}^{\infty}$ is a martingale, for every $x\in X$ we have

$$\varrho(x, F_n) \leqslant \varrho(x, E^{\mathscr{B}_n}[f]) \text{ a.s.}$$

Thus for every $A\in\mathscr{B}_n$ we have

$$\int_A \varrho(x, F_n)\, dP \leqslant \int_A \varrho(x, E^{\mathscr{B}_n}[f])\, dP.$$

Since the σ-field $\mathscr{B}_n$ is countably generated, from Lemma 3.1 we obtain

$$\int_A \varrho(x, E^{\mathscr{B}_n}[f])\, dP \leqslant \int_A E^{\mathscr{B}_n}[\varrho(x, f)]\, dP.$$

But

$$\int_A E^{\mathscr{B}_n}[\varrho(x,f)]\,dP = \int_A \varrho(x,f)\,dP = \int_A \varrho(x, F_{n+1})\,dP,$$

which proves that $\{\varrho(x, F_n), \mathscr{B}_n\}_{n=1}^{\infty}$ is a submartingale.

By (b), the submartingale $\{\varrho(x, F_n), \mathscr{B}_n\}_{n=1}^{\infty}$ converges almost surely ([12], Theorem IV.1.2) for every $x \in X$. Let D be a countable dense subset of X. There exists a negligible event N such that for every $\omega \in \Omega \backslash N$ the set $\bigcup_{n=1}^{\infty} F_n(\omega)$ is a bounded subset of X and the sequence of reals $\{\varrho(x, F_n(\omega))\}_{n=1}^{\infty}$ converges for every $x \in D$. Hence, by Proposition 0.1, the sequence $\{F_n(\omega)\}_{n=1}^{\infty}$ converges in $(\hat{X}, \hat{\varrho})$ for every $\omega \in \Omega \backslash N$.

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Received on 2.5.1988