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EQUILIBRIUM AND ENERGY

BY

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Abstract. In this paper* it is shown that the equilibrium measure ν for a compact K in potential theory can be related with a unique invariant measure π for a discrete time Markov process by the formula $\pi(dy) = \varphi(y)\nu(dy)$. The chain has the transition function $L(x, A)$, where L is the last-exit kernel in [1]. For a general non-symmetric potential density u the modified energy $I(\lambda) = \iint \lambda(dx)u(x, y)\varphi(y)^{-1}\lambda(dy)$ and the Gauss quadratic $G(\lambda) = I(\lambda)-2\lambda(K)$ are introduced. Then G is minimized by π among all signed measures λ on K of finite modified energy, provided I is positive. This includes the classical symmetric case of Newtonian and M. Riesz potentials as a special case. The modification corresponds to a time change for the underlying Markov process. The positivity of I is established for a class of signed measures associated with continuous additive functionals in the sense of Revuz.

Introduction. In electrostatics, the equilibrium charge on a conductor minimizes the potential energy. Gauss showed this but assumed the existence of a minimum, which assumption became known as the Dirichlet principle. The method was extended by Frostman to M. Riesz potentials. More generally, a theory of energy has been developed for symmetric potential kernels (see, e.g., [6]). From quite another direction the existence of the equilibrium measure was established in [1] by modern methods of Markov processes, together with its probabilistic significance in terms of a last-exit distribution. The question arises whether such a measure minimizes the corresponding energy. For a symmetric kernel this was answered in the affirmative in [2]

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under further superfluous conditions, but a simpler proof due to J. R. Baxter is contained in Section 3. Here we prove a general minimization result for a *modified* energy which corresponds to a time change of the process. The significance of this is not clear yet, but it contains the classical symmetric case without modification. Since energy concepts in the non-symmetric case have been little studied, we hope the results presented in the sequel may spur on further research in this direction. Let us add that although the term "equilibrium" is often used to connote a "steady state" in physics, namely a stationary or invariant distribution, the association of the electrostatic equilibrium measure with the invariant distribution of a simple Markov chain described in Section 1 is apparently new. What is its physical significance?

1. Let U be the potential kernel of a Markov process and let u be its density with respect to some reference measure m such that

$$U(x, A) = \int_A u(x, y)\, m(dy)$$

for each Borel set A. Let K be a compact set and suppose there exists a measure ν with support in K such that

$$1 = \int_K u(x, y)\, \nu(dy) \quad \text{for all } x \in K. \tag{1.1}$$

Then ν is called the *equilibrium measure* for K (it is unique under general conditions). We introduce the kernel L as

$$L(x, A) = \int_A u(x, y)\, \nu(dy), \qquad x \in K,\ A \in \mathfrak{B}(K), \tag{1.2}$$

where $\mathfrak{B}(K)$ is the Borel field of K. Then (1.1) takes the form

$$L(x, K) = 1 \quad \text{for all } x \in K. \tag{1.3}$$

Thus L is strictly Markovian and a (discrete time) Markovian chain with state space K may be constructed from L.

Under the basic assumptions of [1] and [3], the measure ν exists for each compact K provided all points of K are regular for K. In general, the constant 1 in (1.1) is to be replaced by $P_K 1$, the hitting probability of K. Furthermore, the method shows that L is the "last-exit kernel". Although we are particularly interested in the setting of [1] and [3], we shall here simply assume the validity of (1.1) without reference to the specific conditions under which it is derived. Other conditions on the function u needed for further development will be added as we proceed.

The following proposition is due to Mamoru Kanda, who improved an easier, less satisfactory condition:

PROPOSITION 1. *If for each fixed y the function $u(\cdot, y)$ is lower semi-continuous, then the set of functions $\{u(x, \cdot), x \in K\}$ is uniformly integrable with respect to v.*

Proof. Let $0 \leqslant f \leqslant 1$ and let f be Borelian. Then by (1.3) we have

$$\int_K u(x, y) f(y) v(dy) + \int_K u(x, y)[1-f(y)] v(dy) = 1 \quad \text{for all } x.$$

Both terms on the left-hand side are lower semi-continuous, hence both are continuous in x. Let $A_n \subset K$ and $v(A_n) \downarrow 0$. Then, by Dini's theorem,

$$\int_{A_n} u(x, y) v(dy) \downarrow 0$$

uniformly for $x \in K$. This together with (1.3) establishes the asserted uniform integrability.

PROPOSITION 2. *Under the condition of Proposition 1 there exists a unique probability measure π on K such that $\pi = \pi L$, namely,*

$$\pi(A) = \int_K \pi(dx) L(x, A), \quad A \in \mathfrak{B}(K). \tag{1.4}$$

π is absolutely continuous with respect to v.

Proof. The existence follows at once from an old theorem due to Doblin since uniform integrability is much stronger than his hypothesis (D) (see [5], p. 192). Alternatively, we can apply Schauder's fixed point theorem as follows. Consider the class $M(K)$ of probability measures on K. This is convex and compact with respect to the vague topology. The kernel L in (1.2) induces a mapping $\lambda \to \lambda L$ of K into K, where

$$\lambda L(\cdot) = \int \lambda(dx) L(x, \cdot). \tag{1.5}$$

If $\lambda_n \to \lambda$ vaguely, then for each $f \in C(K)$ we have

$$\lambda_n L(f) = \int \lambda_n(dx) L(x, f) \to \int \lambda(dx) L(x, f)$$

because $x \to L(x, f)$ is continuous, as shown in the proof of Proposition 1 (even for a bounded Borelian f). Thus there exists a fixed point under the mapping which is the π in (1.4).

If we put

$$\varphi(y) = \int \pi(dx) u(x, y). \tag{1.6}$$

then

$$\pi(dy) = \varphi(y) v(dy). \tag{1.7}$$

Substituting back in (1.6), we obtain

$$\varphi(y) = \int v(dx) \varphi(x) u(x, y). \tag{1.8}$$

To prove the uniqueness, suppose that π_1 is another probability measure such that $\pi_1 = \pi_1 L$, and put $\mu = \pi - \pi_1$. Using (1.7), (1.8) and their analogues for π_1, we have

$$\varphi(y) - \varphi_1(y) = \int \nu(dx)\,[\varphi(x) - \varphi_1(x)]\,u(x, y),$$

and so

$$|\varphi(y) - \varphi_1(y)| \leqslant \int \nu(dx)\,|\varphi(x) - \varphi_1(x)|\,u(x, y). \tag{1.9}$$

But the integrals with respect to ν of the two members of (1.9) are equal by (1.3). Together with (1.9) this forces $\varphi - \varphi_1$ to be of a constant sign ν-a.e. Since $\int \varphi d\nu = 1 = \int \varphi_1 d\nu$, it follows that $\varphi = \varphi_1$ ν-a.e. and, therefore, $\pi = \pi_1$. This completes the proof.

Under the assumptions of [1], u is strictly positive, hence the function φ defined by (1.6) is also strictly positive. We shall assume this from now on in the general context. Put

$$u_\varphi(x, y) = \frac{u(x, y)}{\varphi(y)}. \tag{1.10}$$

Then (1.6) may be written as

$$\int \pi(dx)\,u_\varphi(x, y) = 1, \quad y \in K, \tag{1.11}$$

whereas, in view of (1.7), (1.3) may be written in the form

$$\int u_\varphi(x, y)\,\pi(dy) = 1, \quad x \in K. \tag{1.12}$$

We call u the *modified potential density* (relative to K) and record the next result as follows:

PROPOSITION 3. *There are a Borel function $\varphi > 0$ and a probability measure π on K satisfying* (1.11) *and* (1.12).

Let us tell the probabilistic origin of the symmetry exhibited in (1.11) and (1.12). Since π is an invariant probability measure for the Markovian kernel L, a *reverse kernel* in the most elementary (and classical) sense is given by

$$\hat{L}(y, dx) = \frac{\pi(dx)\,L(x, dy)}{\pi(dy)} = \pi(dx)\,u_\varphi(x, y).$$

Thus (1.11) states that $\hat{L}(y, \cdot)$ is a probability measure for each y. Now it is trivial that π is also an invariant measure for the Markovian kernel $\hat{L}$, namely, $\pi\hat{L} = \pi$. Written out this is just (1.12).

As an immediate application of (1.11), we mention the following extension of a familiar result in potential theory:

COROLLARY. *If μ is any measure such that*

$$\int_K u_\varphi(x, y)\,\mu(dy) \leqslant 1 \quad \textit{for all } x \in K,$$

then $\mu(K) \leqslant 1$.

Proof. Fubini's theorem yields

$$1 = \pi(K) \geqslant \int_K \pi(dx) \int_K u_\varphi(x, y)\mu(dy) \geqslant \int_K 1\mu(dy) = \mu(K).$$

Thus π fulfills the physicist's concept of the equilibrium distribution on K with respect to the modified potential u_φ, as well as the probabilist's one which is (1.12). We proceed to strengthen this analogy by considerations of energy.

2. Let λ be a signed finite measure on K; namely, $\lambda = \lambda^+ - \lambda^-$, where λ^+ and λ^- are measures on K such that $|\lambda| = \lambda^+ + \lambda^-$ is a finite measure. We denote this class of signed measures by $S(K)$. For $\lambda_1 \in S(K)$ and $\lambda_2 \in S(K)$, we define the *mutual energy* of λ_1 and λ_2 relative to u_φ by

$$\langle \lambda_1, \lambda_2 \rangle_\varphi = \iint \lambda_1(dx) u_\varphi(x, y) \lambda_2(dy) \tag{2.1}$$

with the stipulation that $\langle |\lambda_1|, |\lambda_2| \rangle_\varphi < \infty$; otherwise, the quantity in (2.1) is not defined and will not be written. For $\lambda \in S(K)$ we write

$$I_\varphi(\lambda) = \langle \lambda, \lambda \rangle_\varphi.$$

If $\lambda \in S(K)$ and $I_\varphi(|\lambda|) < \infty$, we call $I_\varphi(\lambda)$ the *energy* of λ and write $\lambda \in \mathscr{E}_\varphi$. The subclass of probability measures in $\mathscr{E}_\varphi$ will be denoted by $\mathscr{E}_\varphi^0$. Next, for $\lambda \in \mathscr{E}_\varphi$ we put

$$G_\varphi(\lambda) = I_\varphi(\lambda) - 2\lambda(1),$$

where, of course, $\lambda(1)$ may also be denoted by $\lambda(K)$. Then G_φ is the *Gauss quadratic*. It follows from (1.11) and (1.12) that

$$I_\varphi(\pi) = 1 \quad \text{and} \quad G_\varphi(\pi) = -1.$$

From here on the subscript φ will be omitted from these symbols except in u_φ, when there is no risk of confusion.

PROPOSITION 4. *If $\lambda \in \mathscr{E}$, then $\lambda + \pi \in \mathscr{E}$ and*

$$G(\lambda + \pi) = I(\lambda) + G(\pi) = I(\lambda) - 1. \tag{2.2}$$

Proof. We have

$$I(|\lambda + \pi|) \leqslant I(|\lambda| + \pi) = I(|\lambda|) + I(\pi) + \langle |\lambda|, \pi \rangle + \langle \pi, |\lambda| \rangle.$$

Now, by (1.12) and Fubini's theorem

$$\langle |\lambda|, \pi \rangle = \int |\lambda|(dx) \int u_\varphi(x, y)\pi(dy) = |\lambda|(1) < \infty;$$

similarly, by (1.11),

$$\langle \pi, |\lambda| \rangle = \int \left[\int \pi(dx) u_\varphi(x, y) \right] |\lambda|(dy) = |\lambda|(1) < \infty.$$

Thus the same calculations yield

$$I(\lambda+\pi) = I(\lambda)+I(\pi)+\langle\lambda, \pi\rangle+\langle\pi, \lambda\rangle = I(\lambda)+I(\pi)+2\lambda(1)$$

and (2.2) follows at once.

COROLLARY 1. *If λ is a probability measure on K such that $\lambda-\pi\in\mathscr{E}$, then $I(\lambda) = I(\pi)+I(\lambda-\pi)$.*

We say that I satisfies the *positivity principle* iff

$$I(\lambda) \geqslant 0 \quad \text{for every } \lambda\in\mathscr{E}_\varphi; \tag{2.3}$$

we say that I satisfies the *energy principle* iff (2.3) is true and, moreover, $I(\lambda) = 0$ implies $\lambda^+ \equiv \lambda^- \equiv 0$.

COROLLARY 2. *If I satisfies the positivity principle, then we have*

$$G(\pi) \leqslant G(\lambda) \quad \textit{for every } \lambda\in\mathscr{E}_\varphi, \tag{2.4}$$

$$I(\pi) \leqslant I(\lambda) \quad \textit{for every } \lambda\in\mathscr{E}^0_\varphi. \tag{2.5}$$

If I satisfies the energy principle, then π is the unique member of $\mathscr{E}_\varphi$ for which (2.4) *is true; and it is also the unique member of $\mathscr{E}^0_\varphi$ for which* (2.5) *is true.*

When u is symmetric, v is invariant for L because

$$\int v(dx)u(x, y) = 1, \qquad y\in K;$$

and we may normalize v by putting

$$\pi = \frac{v}{v(1)}.$$

In this case $\varphi \equiv 1$, and if we use 1 as the subscript to indicate this case, we have

$$\langle\lambda_1, \lambda_2\rangle_1 = \frac{1}{v(1)}\iint \lambda_1(dx)u(x, y)\lambda_2(dy)$$

with the original potential density u. This is the classical situation and Corollary 2 contains the theorems on the minimization of energy by the equilibrium measure as given in the literature. For the Newtonian and Marcel Riesz potentials the energy principle is satisfied (see, e.g., [6]). Indeed, in the general symmetric case satisfying the energy principle it can be shown that the two ways of minimization in (2.4) and (2.5), respectively, are equivalent. We have not been able to trace the source of this fact, but the proof is standard.

A true symmetrization of u_φ may be considered by putting

$$\tilde{u}_\varphi(x, y) = \frac{1}{2}\left[\frac{u(x, y)}{\varphi(y)}+\frac{u(y, x)}{\varphi(x)}\right]. \tag{2.6}$$

Then (1.11) and (1.12) continue to hold when u_φ is replaced there by $\tilde{u}_\varphi$. Since $\tilde{u}_\varphi$ is symmetric, the well-known methods of potential theory apply. It will be seen in Section 4 that not only u_φ but also its transpose $u(y, x)/\varphi(x)$ is the potential density of a process, namely the dual process (see, e.g., [4]). But the operation of addition or averaging in (2.6) may not have a useful interpretation for probability theory, so we shall let it pass here.

3. Recall that the kernel L in (1.2) may be defined by

$$Lf(x) = \int u_\varphi(x, y) f(y) \pi(dy), \quad f \in \mathfrak{B}_+.$$

Its dual is defined by

$$\hat{L}f(y) = \int \pi(dx) f(x) u_\varphi(x, y), \quad f \in \mathfrak{B}_+.$$

Note that

$$\int (\hat{L}f) g d\pi = \int f(Lg) d\pi = \langle f \cdot \pi, g \cdot \pi \rangle,$$

where $f \cdot \pi$ is the measure $f(y)\pi(dy)$ and the notation above omits the subscript φ as before. Both L and $\hat{L}$ are contractions on $\mathscr{L}^p(\pi)$ for $1 \leqslant p \leqslant \infty$. To see this let $\int |f|^p d\pi < \infty$; then, since $|Lf|^p \leqslant L(|f|^p)$ and $\pi L = \pi$, we obtain

$$\int |Lf|^p d\pi \leqslant \int L|f|^p d\pi = \int |f|^p d\pi. \tag{3.1}$$

Note that it is sufficient that L be submarkovian and π subinvariant, $\pi L \leqslant \pi$, for (3.1) to hold. Similarly for $\hat{L}$.

Let λ be any probability measure on K, and put

$$g(y) = \int \lambda(dx) u_\varphi(x, y).$$

The measure λL is given by

$$\lambda L(dy) = \int \lambda(dx) L(x, dy) = g(y) \pi(dy).$$

An induction shows that, for $n \geqslant 1$,

$$\lambda L^n(dy) = \hat{L}^{n-1} g(y) \pi(dy).$$

We can now calculate, for $n \geqslant 1$ and $m \geqslant 1$,

$$\langle \lambda L^n, \lambda L^m \rangle = \iint \hat{L}^{n-1} g(x) \pi(dx) u_\varphi(x, y) \hat{L}^{m-1} g(y) \pi(dy) = \int \hat{L}^n g \cdot \hat{L}^{m-1} g d\pi.$$

This is also valid for $n = 0$ and $m \geqslant 1$ with, of course, $\hat{L}^0 g = g$. In particular,

$$\langle \lambda, \lambda L \rangle = \int g^2 d\pi, \tag{3.2}$$

$$\langle \lambda L, \lambda L \rangle = \int \hat{L}g \cdot g d\pi, \tag{3.3}$$

$$\langle \lambda L, \lambda L^2 \rangle = \int (\hat{L}g)^2 d\pi, \tag{3.4}$$

$$\langle \lambda L^2, \lambda L \rangle = \int \hat{L}^2 g \cdot g d\pi = \int \hat{L}g \cdot Lg d\pi. \tag{3.5}$$

PROPOSITION 5. *The three quantities in* (3.3)-(3.5) *are all dominated by that in* (3.2). *In particular,* $\langle \lambda L^n, \lambda L^{n+1} \rangle$ *decreases as n increases.*

Proof. We have

$$\int \hat{L}g \cdot g d\pi \leqslant \left[\int (\hat{L}g)^2 \, d\pi \int g^2 d\pi\right]^{1/2} \leqslant \int g^2 \, d\pi$$

by the Cauchy-Schwarz inequality followed by (3.1) for $\hat{L}$ with $p = 2$. The rest is similar. Having shown that $\langle \lambda L, \lambda L^2 \rangle \leqslant \langle \lambda, \lambda L \rangle$ for any λ, an iteration establishes the last assertion in Proposition 5.

In the symmetry case, it is trivial that the positivity principle implies the Cauchy-Schwarz inequality for mutual energy as follows:

$$\langle \lambda_1, \lambda_2 \rangle^2 \leqslant \langle \lambda_1, \lambda_1 \rangle \langle \lambda_2, \lambda_2 \rangle. \tag{3.6}$$

In general it is not clear when this is valid.

COROLLARY. *If I satisfies the Cauchy-Schwarz inequality* (3.6), *then*

(*) $\langle \lambda L^n, \lambda L^n \rangle$ *decreases as n increases for* $n \geqslant 0$.

Proof. We have $\langle \lambda, \lambda L \rangle^2 \leqslant \langle \lambda, \lambda \rangle \langle \lambda L, \lambda L \rangle$. Together with $\langle \lambda L, \lambda L \rangle \leqslant \langle \lambda, \lambda L \rangle$, as given in Proposition 5, we obtain $\langle \lambda L, \lambda L \rangle \leqslant \langle \lambda, \lambda \rangle$. This implies (*) upon iteration.

For a symmetric u (i.e., $\varphi \equiv 1$), Proposition 5 is due to J. R. Baxter and the Corollary answers a conjecture by J. B. Walsh. Under stronger conditions on u (see, e.g., [5]), so that $\lambda L^n \to \pi$ as $n \to \infty$, it appears that (*) should imply the limiting relation $\langle \pi, \pi \rangle \leqslant \langle \lambda, \lambda \rangle$. But this implication is actually false in general because u is not bounded. On the other hand, the last written inequality has been proved directly in Corollary 2 to Proposition 4.

4. It is common knowledge that if u is the potential density of the Markov process $\{X_t, t \geqslant 0\}$, then the modified u_φ defined in (1.11) is that of another Markov process $\{Y_t, t \geqslant 0\}$ obtained from X by a random time change. More precisely, let

$$\tau(t) = \int_0^t \frac{1}{\varphi(X_s)} \, ds$$

defined for each sample path. Under the assumptions of [1], $\{X_t\}$ is a transient Hunt process and φ is lower semi-continuous and $\varphi > 0$. Thus, for each t, $\varphi(X_s)$ is bounded away from zero for $0 \leqslant s \leqslant t$. It follows that $\tau(t) < \infty$ for each t, almost surely. Thus $\tau(t)$ is continuous non-decreasing in t, and so has a right continuous, strictly increasing inverse τ^{-1}. Define Y by

$$Y_t = X_{\tau^{-1}(t)}, \quad t \geqslant 0.$$

Then Y is a right continuous strong Markov process in $[0, \infty)$. For every positive measurable f we have

$$\int_0^\infty f(Y_t)\,dt = \int_0^\infty f(X_t)\frac{1}{\varphi(X_t)}\,dt.$$

Hence, if we write U_X and U_Y for the potential kernel of X and Y, respectively, we obtain

$$U_Y f = \int u_\varphi(x, y) f(y) m(dy).$$

Thus Y has the potential density $u_\varphi(x, y) m(dy)$.

We now discuss the positivity principle for I_φ in the context of additive functionals. Consider a natural increasing additive functional $\{A_t, t > 0\}$. Its potential $U_A 1$ has a representation $U\mu_A$, by Theorem 2, Corollary 2 of [3]. This measure can be shown to be the Revuz measure associated with A (see [7]). For positive measurable f we write $f_K = f 1_K$; then we have

$$\text{(4.1)}\qquad E^x\{\int_0^\infty f(X_{s-}) 1_K(X_{s-})\,dA_s\} = U_A f_K(x) = \int_K u(x, y) f(y) \mu_A(dy).$$

Taking $f = 1/\varphi$, where φ is the function in (1.7), and the difference of two such functionals, we obtain in obvious notation the formula

$$E^x\left\{\int_0^\infty \frac{1_K(X_{s-})}{\varphi(X_{s-})}\,d(A_s^+ - A_s^-)\right\} = \int_K u_\varphi(x, y) \mu_A(dy),$$

where $A = A^+ - A^-$. Putting

$$dB_s = \frac{1_K(X_{s-})}{\varphi(X_{s-})}\,dA_s,$$

we have

$$h(x) \stackrel{\text{def}}{=} E^x\{B_\infty\} = \int_K u_\varphi(x, y) \mu_A(dy).$$

It follows by a familiar calculation that

$$E^x\{B_\infty^2\} = E^x\{\int_0^\infty [\int_{[t,\infty)} dB_s + \int_{(t,\infty)} dB_s]\,dB_t\} = E^x\{\int_0^\infty [h(X_t)_- + h(X_t)]\,dB_t\},$$

where

$$h(X_t)_- = \lim_{s \uparrow\uparrow t} h(X_s).$$

Since h is the difference of two excessive functions, the limit exists and is equal to $h(X_t)$ except for a countable set of t (depending on the path). Hence, if B is a continuous additive functional, then the above is equal to

$$2E^x\{\int_0^\infty h(X_t)\,dB_t\} = 2U_B h(x) = 2\int_K u_\varphi(x, y) h(y) \mu_A(dy).$$

Thus we obtain

$$\mathrm{E}^x\{B_\infty^2\} = 2\int_K u_\varphi(x, y)\int_K u_\varphi(y, z)\,\mu_A(dz)\,\mu_A(dy),$$

which shows that the iterated integral above has a value greater than or equal to 0. Now, if we integrate it with respect to π and use (1.12), the result is

$$I_\varphi(\mu_A) = \int_K\int_K \mu_A(dy)\,u_\varphi(y, z)\,\mu_A(dz) \geqslant 0.$$

We have proved the following

PROPOSITION 6. *For every measure μ_A associated with a continuous additive functional A of the process X as in* (4.1), *we have $I_\varphi(\mu_A|_K) \geqslant 0$, where $\mu_A|_K$ is the restriction of μ_A to the compact K.*

The minimization results of (2.4) and (2.5) therefore hold true for this class of measures. The question whether $I_\varphi(\mu_A) = 0$ implies $\mu_A \equiv 0$ seems more difficult and remains to be investigated.

Added in proof. S. Orey informed us of the following complement to Proposition 2:

If an invariant probability measure π exists as in (1.4), *then the Markov chain associated with the kernel L is ν-recurrent and, for each x,* $\lim_n L^{(n)}(x, A) = \pi(A)$.

The proof is simple.

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