A review of Jakub Gogolok's doctoral thesis
"On model theory of fields with operators"

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This dissertation is a significant contribution to the model theory of fields with operators, especially in positive characteristic. It contains strong, interesting, and attractive results. It is written very clearly (my many questions and suggestions below, notwithstanding), and exhibits a remarkably high degree of mathematical maturity. It is my opinion that it not only meets the standards of a PhD in mathematics at the best universities, but that it should be nominated for a prestigious doctoral dissertation award.

Let me summarise the main contributions. Gogolok introduces a new formalism for studying fields with operators, what he calls $\mathcal{B}$-operators where $\mathcal{B}$ is a co-ordinate algebra scheme over a field. This elegantly generalises the formalism that Thomas Scanlon and I proposed some 15 years ago, and hence includes as special cases the classical (and motivating) examples of derivations and automorphisms. In Chapter 2 Gogolok studies the basic theory of the algebra schemes, showing that in characteristic zero it coincides with our older formalism (Corollary 2.18), describing exactly when it does so in positive characteristic (Proposition 2.20), and also classifying them in general (Theorem 2.17). Furthermore, in this chapter, he introduces a natural notion of iterativity that both extends and is more natural than our formalism. He describes certain specific settings (nice pairs) where appropriate finiteness conditions are satisfied, and explains how many of the cases treated before fit this framework. In Chapter 3 he addresses the central problem of the existence of a model companion. Again he works in an elegantly general and versatile setting. The main theorem here (Theorems 3.13) proves the existence of model companions for interative $\mathcal{B}$-fields under two basic assumptions. He points out that almost all the model companion results thus far obtained about operators on fields in positive characteristic are thereby subsumed. He then proves that this model companion satisfies all the tameness properties one expects. The chapter ends with a number of special, somewhat ad hoc, but very interesting topics. The notions of pseudo algebraically closed, separably closed, and largeness are relativised to this setting of iterative $\mathcal{B}$-fields and are shown to be elementary, in Section 3.3, 3.4.1, and 3.4.2, respectively. A final section, 3.4.2, adresses the intriguing variant of differentially closed fields where one only asks for existential closedness in finitely generated field extension, something not studied before to the best of my knowledge, and shows by an ingeneous example that one does not in this way recover the usual theory.

Here are some comments, questions, and suggestions:

- p 17, l-3 It may be useful to mention right away that in characteristic zero $\mathcal{B}$-operators and $B$-operators will agree (Corollary 2.18).
- Not 2.16 Is $\Theta$ a morphism of $k$-algebra schemes?
- Prop 2.20(1) is an isomorphism of ...?
- Def 2.37 How does this compare with how iteration is done in the free case of [36], Section 6.1.
- p 38, l 1 Given that you say this is the "defining property", is saying the bijection is "natural in $R$" enough?

Rem 2.44 Do you in the end obtain an extension of the representability of Weil restrictions theorem to a more general setting than finite $K$-algebras? Maybe when $R$ is $K^e$. Also, it would be good to give a specific example of when this level of generality is needed.

p 40, l -7 This bijection preserves the additive group structure, right? Is that used?

Rem 2.47 Again, do you just need that the underlying schemes agree, or also that the additive group structure agrees?

Rem 2.50 It may be worth saying what this is in co-ordinates in a down-to-earth way.

Def 2.53 Here does it really not matter that $\tau^{\partial}V$ is not reduced? You should explain.

Lem 2.57 The statement is misleading. There are generic points (living in $\mathcal{B}$-field extensions) that are not $\sharp$-points. I think you either want to only claim the existence of such $a$, or make explicit the identification of $K(V)$ with $K(a)$ in the statement.

Sec 2.4.1 Omar and I worked out (in "The model companion of differential fields with free operators", JSL 2015) the case where one has several commuting derivations (char 0) and then additional generalised operators that commute with the derivations but are otherwise free. Does this fit into your formalism?

p 45, l -5 I don't understand what you mean by the last sentence of this paragraph. Maybe you are just referring to Section 6 of [36], so "automatic" iterativity in the free case. Or are you really referring to the general iterativity formalism developed in [35]?

Def 2.58 Maybe explain what the maps in the diagrams (for example $\mu F$) are.

Rem 2.62 Don't you eventually use more than this (say in the proofs of 2.68 and 2.69)? Namely that they are generated as field extensions by a single application of $\partial$.

p 47, l -8 It may be worth explaining this last direction a bit more. For example, in what sense are the "governed by comonads" as you say in the proof of 2.68.

Def 2.63 How does this relate to the general iterativity formalism developed in [35]?

p 48, l 8 I think you should spell out the iterativity condition in at least some of these examples of interest (such as commuting operators, for instance).

Ex 2.64 It might be worth saying something about the trivial case of a vacuous iterativity condition.

p 48, l -3 Maybe informally restate Proposition 2.34.

p 49, l 6 In (3) are you allowing any iterativity condition$\phi$ here? If so, how does that follow from Proposition 2.34 which does not mention iterativity. Or if you are assuming $\phi$ is vacuous here you should say so. In any case, you should explain how and why this example (3) gives a nice pair.

Def 2.67(3) This is of course a very important definition for you. Maybe give more details. I don't see what role $\mathcal{B}_l$ is playing, for example. And refer back to Definition 2.37 about how you are dealing with commuting operators. Also, do you need to assume here that $\mathrm{Fr}(\ker \pi_{\mathcal{B}}) = 0$?

p 53, par 2 It is not so clear what exactly you mean here. For example, it is hard to parse what the quantifiers are. Make **definable** a formal and explicit definition.

Ex 3.2(3) Give the argument.

Rem 3.15 You should explain why the "random" example you give is not a nice pair but still satisfies Assumptions 3.6 and 3.10.

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|---|---|
| Thm 3.16 | Why did assumption 3.10 disappear? Explain. |
| p 58, l 1 | I didn't find any "chart" in the Introduction. |
| Sec 3.4.3 | Is there any connection here with Seidenberg's embedding theorem? The differential field of germs of meromorphic functions seems like it could be close to satisfying this theory (though I guess not literally). |

Here are some typos:

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|---|---|
| p 16, l -4 | "it was shown" |
| p 17, l -3 | "we define them" |
| p 23, l 12 | "does not matter here" |
| p 25, l -1 | $\beta$ should be $\mathcal{B}$ |
| p 25, l 1 | "is an automorphism of $\mathcal{B}(k)$" |
| p 25, l 3 | $\beta$ should be $\mathcal{B}$ |
| Lem 2.29 | "is formally smooth" |
| p 30, footnote | "and it is" |
| Cor 2.30 | "is a $\mathcal{B}$-field" |
| Cor 2.31 | "is a $\mathcal{B}$-operator" |
| Lem 2.32 | "such that $p^n \geq e$" |
| p 35, l 9 | "two $\mathcal{B}$-ring extensions" |
| p 37, l -12 | "left-adjoint functor." |
| p 45, l -8 | I think $H \otimes_k K$ is more consistent notation. |
| Ex 2.61 | I think $B$ should $H$. |
| Def 2.67(3) | "a $G$-action" |
| p 52, l 8 | Do you mean Lemma 2.29 here? |
| p 53, l -9 | "As we will see later" |
| p 53, l -7 | "For the sake of brevity," |
| p 57, l 7 | "our proofs work" |

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