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PHD THESIS REVIEW OF "NORM ESTIMATES FOR RIESZ TRANSFORMS" BY MACIEJ KUCHARSKI

1. Assessment

This is an exceptionally strong phd thesis written by Maciej Kucharski. The content of the thesis is based in part on six papers by Kucharski, two of these papers (one is single-authored) have appeared in leading international peer-reviewed journals. The single-authored paper [1] has appeared in Colloquium Math. and two papers are joint with B. Wróbel ([3] and [4]), one has appeared in Math. Annalen. Two other papers [5] and [6] are preprints and joint with B. Wróbel and J. Zienkiewicz. And finally, the last paper [2] is a single-authored paper.

It is interesting to note that Maciej has two single-authored papers, the first one came prior to the following four joint papers, clearly demonstrating that Maciej developed independent research before embarking on collaborative work with two senior mathematicians. And even more interesting is that Maciej took what he learned from his collaborative work in [3], [4], [5] and [6] and returned to the topic of his first paper [3]; extended, improved and generalised the results in [4]. It is important for a young researcher not only to have an independent research programme, but also to have a track record of collaborating successfully with other research mathematicians.

All this is a remarkable acheivement for a phd candidate. By comparison, the thesis of one of my best students was based on one published paper and one preprint. He now holds the presitigious Hans Rademacher Postdoctoral Research Fellowship at the University of Pennsylvania. Maciej's accomplishment at the same stage is more impressive.

2. Review of the thesis

The thesis examines the L^p mapping properties of Riesz transforms, both classical Riesz transforms R_j, including those R_P defined with respect to spherical harmonics P, and a class of Riesz transforms related to Schrödinger operators $L = -\Delta + V$ defined in terms of general potentials V.

A main subtheme of the thesis is to strengthen known bounds to dimension-free bounds.

Regarding classical Riesz transforms R_j and R_P, basic bounds (both for the thesis and more generally) are the following maximal inequalities: for any spherical harmonic P of degree k and $1 < p < \infty$, the estimate

$$\|R_P^* f\|_{L^p(\mathbb{R}^d)} \ \leq C_{p,k,d} \, \|R_P f\|_{L^p(\mathbb{R}^d)} \tag{1}$$

holds. Here R_P^* denotes the maximal Riesz transform defined over all truncations of R_P. When $P(\xi) = c_j \xi_j / |\xi|^{d+1}$ (that is, when $R_P = R_j$ is the classical Riesz transform), this is due to Mateu and Verdera [10]. The general case is due to Mateu, Orobitg, Pérez and Verdera; see [8] and [9].

2.1. Dimension-free bounds for (1).

For the Riesz transforms R_j, Kucharski and Wróbel [3] established (1) for $p = 2$ with $C_{p,k,d} = 2 \cdot 10^8$, a dimension-free bound. This is a significant result, introducing new ideas which have been used in several works since. In particular they prove a factorisation $R_j^t = M^t R_j$ of the truncated Riesz transforms R_j^t defining the maximal function R_j^*, showing that the maximal operator $M^* f = \sup_{t>0} |M^t f|$ is bounded on all L^p and most importantly, the operator norm $\|M^*\|_{p-p}$ is equal to the optimal constant $C_{p,d}$ in (1) for $R_P = R_j$. Furthermore, using ideas going back to Bourgain and Stein, they reduce the estimate of M^* to an oscillation inequality which in turn is controlled by a Rademacher-Menshov-type inequality, an inequality that has played an important role in recent pointwise ergodic theorems. These ideas were developed in [5] and [6] where Kucharski, Wróbel and Zienkiewicz improve the general bound in (1) to a dimension-free bound which holds for all $1 < p < \infty$ and for all k. Some of these ideas were instrumental in the work of Liu, Melentijevic and Zhu [7] who improved the dimension-free bound in [6] where once can take $C = (2 + 1/\sqrt{2})^{2/p}$ in (1) when $p \geq 2$ and $R_P = R_j$.

These dimension-free bounds are also extended to the natural vector-valued setting in [5] and [6]. Here Maciej clearly demonstrates his mastery of the sophisticated complex method of rotations due to Iwaniec and Martin and how to adapt it in order to prove dimension-free bounds for vector-valued versions of (1). Maciej does a commendable job at explaining the differences between the real and complex method of rotations and why the real method cannot be used in the even case.

Criticism: The thesis does not make clear the timeline when these results were established. For instance, near the end of Section 1.3.1 and after the statement of the main result in [8] and [9], the thesis states "*Recently Liu, Melentijević and Zhu* *partially improved the results of Mateu, Orobitg, Pérez and Verdera to a dimension-free estimate in the case of first-order Riesz transforms and $p \in [2, \infty)$*". This is confusing to the reader and mis-leading, especiallly when the thesis discusses the results of [3] and [6] in Section 1.4. Furthermore, it would be helpful to highlight the main new ideas in [3] and how they were used in the subsequent work of [5], [6] and [7]. With respect to dimension-free $L^p, p \geq 2$ estimates for R_j, what is the main difference between the arguments in [6] and [7]? It would be helpful to make clear that the L^2 dimension-free bound in [3] came first, followed by [5] and [6] which generalised [3] to general L^p dimension-free bounds and to higher order Riesz transforms. After these works, the paper [7], using the factorisation $R_j^t = M^t R_j$ established in [3], gives a different argument to bound the maximal operator $M^* f = \sup_{t>0} |M^t f|$ proving $L^p, p \geq 2$ dimension-free bounds for R_j.

2.2. **Riesz transforms related to Schrödinger operators.** Let $L = (-1/2)\Delta + V$ be the Schrödinger equation defined with respect to a non-negative potential $V \in L^1_{loc}(\mathbb{R}^d)$. The classical Riesz transform associated to L is given by $\nabla L^{-1/2}$ and has been extensively studied. The Riesz transforms *related* to L studied in this thesis is given by $R_V^a f(x) = V^a(x) L^{-a} f(x)$ and one of the main differences from the classical Riesz tranform is that R_V^a is positivity preserving wheras $\nabla L^{-1/2}$ is not. The arguments in this thesis heavily rely on the fact that the Riesz transforms R_V^a are positivity preserving (it would enhance the thesis if there was a robust discussion why these more non-traditional Riesz transforms are being analysed – is it simply that the methods, based on the positivity preserving property, work for R_V^a and not for $\nabla L^{-1/2}$?).

Criticism: It is difficult for the reader at times to follow the argument as there are continual references in the literature to technical computations and results (often made in mid-sentence) which makes the arguments highly *non self-contained*. I realise that it is difficult to make arguments self-contained given the technical nature of the topics in the thesis however striving to make it *as self-contained as possible* is an important endeavour for any working mathematician.

This part of the thesis is divided into two sections, Section 4 and Section 5. Section 4 details the results in [4] where basic L^p mapping properties are established for R_V^a for general classes of potentials V. The issue of upgrading these bounds to dimension-free bounds for special classes of potentials is considered in Section 5 and is the content of Maciej's second single-authored paper [2].

A basic result in Section 4 establishes that R_V^a is bounded on $L^p(\mathbb{R}^d)$ for all $1 < p \leq 2$ and for all $0 \leq a \leq 1/p$. Furthermore, this holds for any non-negative $V \in L^1_{loc}(\mathbb{R}^d)$. This is established by an elegant interpolation argument, using Stein's interpolation theorem for analytic families of operators which reduce matters to showing R_V^1 is bounded on L^1 (well-known) and $R_V^{1/2}$ is bounded on L^2 which follows from $\|V^{1/2} f\|_{L^2} \leq \|L^{1/2} f\|_{L^2}$ on test functions f and this inequality in turn follows from the definition of $L^{1/2}$. The remaining part of Section 4 focusses on classes of potentials V for which R_V^a is bounded on L^∞ and bounded on L^1. The resulting analysis is a deep, sophisticated argument involving the clever use of the Feynman-Kac formula.

An interesting observation is made that R_V^a cannot be bounded on L^∞ whenever $a > 0$ and $V \in L^q(\mathbb{R}^d)$ and $q > d/2$.

Critisicm: It would be helpful for the reader to highlight at this point that to establish L^∞ bounds for R_V^a, the potential V necessarily needs to grow at infinity, given the above counterexample. This then gives context to Theorem 4.3.5 and Corollary 4.3.6. Furthermore, we already know that R_V^1 is bounded on L^1 for any $V \in L^1_{loc}(\mathbb{R}^d)$ (and R_V^a is bounded on $L^p, p \in (1,2]$ for any $0 \leq a \leq 1/p$ – this is Theorem 4.1.5). How does this square with Theorem 4.4.8? Does the potential V necessarily have to be unbounded for R_V^a to be bounded on L^1 when $0 < a < 1$?

The final section, Section 5, considers potentials V of the form $V(\underline{x}) = V_1(x_1) + \cdots + V_d(x_d)$ where each V_i has polynomial growth (to what extend is polynomial growth necessary?) and dimension-free L^1 and L^∞ bounds for $R_V^a, 0 \leq a \leq 1$, are examined (what about dimension-free L^p bounds?). Given the form of the potential, matters are reduced to one dimensional estimates and then a careful analysis gives the desired dimension-free bounds.

3. Conclusion

All-in-all, Maciej Kucharski's phd thesis is outstanding. It rates among the best phd theses written by the 16 phd students I have supervised over the years and some of my students have gone on to be award-winning research mathematicians (for example, Neal Bez has recently won the Spring Prize from the Japanese Mathematical Society and Jonathan Hickman has won one of the Frontiers of Science Awards 2018-22 in Beijing). The thesis was a pleasure to read.

References

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Sincerely yours,

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