Report on the Ph.D. thesis of Adrian Portillo Fernandez, "Canonical quotients in model theory".

Anand Pillay, University of Notre Dame

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This is an excellent thesis by Portillo Fernandez and I strongly recommend that he be awarded a Ph.D.

Although the title of the thesis is "canonical quotients", the work can also be described as being about continuous logic and its variants.

After the introductory and background chapters, the thesis presents four more chapters. Chapters 3 and 4 on (hyperdefinable) stable quotients of groups and invariant types in NIP theories are closely related. The work presented there is joint with the advisor Krzysztof Krupinski and has already been published in two papers, [KP22] and [KP23b]. Chapter 5 on n-dependence in continuous logic, is due to Portillo alone. Chapter 6 on "nice" quotients and topological dynamics is also joint with the advisor Krupinski.

I will describe below both the mathematical background and original contributions of Portillo's thesis. I will also write afterwards a selective list of minor technical comments and questions.

Model theory studies first order theories T, often complete. Given such a theory T we study definability and also definable groups in models of T. In the background is a rich and sophisticated theory of definability for theories which are "close" to having the property of *stability*. When this general theory is applied to specific first order theories (such as differentially closed fields) it can and has resulted in some spectacular results in other areas of mathematics (algebraic and arithmetic geometry, combinatorics, functional transcendence,...). It is natural to take quotients of definable sets with respect to definable equivalence relations. But around the mid 1990's a systematic study of quotients of definable sets by "type-definable" equivalence relations

E (i.e. where E is given by a possibly infinite intersection of definable sets) was intiated. These are called "hyperdefinable sets". They played a big role in the solution (2000's) to some conjectures I raised about relations between groups definable in o-minimal theories and compact Lie groups. They also played a big role in Hrushovski's results on approximate subgroups. Trying to understand the "logic of hyperimaginaries" was also partly behind the development of "continuous logic" in the 2000's and subsequently, where formulas are now allowed to be real valued, rather than only Boolean-valued.

So Portillo's thesis is about hyperdefinable sets and groups, generalizing certain results from first order theories to continuous theories, as well as using topological dynamical methods to study quotients of type spaces. It belongs to "pure" model theory, although one could foresee applications.

Now to more details. I will use the references from the bibliography of the thesis. In 2009 I gave a seminar in Lyon about "maximal stable quotients" in first order theories with NIP (not the independence property). The motivation came partly from the role of "stable domination" in the theory of algebraically closed valued fields (Haskell, Hrushovski, Macpherson) which had applications (Hrushovski-Loeser) to rigid geometry. Anyway my seminar required a proper definition of when a hyperdefinable set is stable. I claimed several results with rather sketchy "proofs". This included (i) in a NIP theory, a definable group has a unique maximal stable hyperdefinable quotient G/G^{st}, and (ii) a version of for invariant types; for a global invariant type p, there is a "smallest" quotient of p by type-definable equivalence relation with stable quotient (in a suitable sense). With my student Mike Haskel, we subsequently worked out details of (i) and published a paper [HP18]. We asked several questions, including (a) whether in suitable (distal) theories, G/G^{st} is *compact*, and (b) is there an example in a NIP theory of G with $G = G^{00}$ but G^{st} is NOT an intersection of definable subgroups of G. In Chapter 3, Portillo first gives a very nice account of the stability of hyperdefinable sets (in first order theories) in terms of real valued continuous functions on type spaces (what we called CL-formulas in a paper of Hruhovski, Krupinski, and me). Then he answers the questions (a) and (b) above. In the process of answering (a), he shows that distality is preserved when passing to hyperimaginaries. And a completely comprehensive answer is given to (b).

Chapter 4 is based on giving a rigorous account of (ii) from my Lyon talk. In fact the solution is quite different from my sketchy notes and makes use of the notion of (relatively) definable subsets of automorphism groups. Even the actual statement of the Theorem seems different from what I originally envisaged, and maybe Portillo has the optimal result. I will ask some related questions below in my technical comments. In any case this Chapter 4 is a very strong piece of work, both technically and with nice ideas.

Chapter 5 is devoted to generalizing some nice earlier results on generalized indiscernibles and n-dependent theories from classical first order theories to continuous first order theoiries. For continuous logic theories, Portillo uses the formalism of [BYU] (what I call "official" continuous logic). The first piece of work being generalized is by Lynn Scow ([Sco12], [Sco15], [Sco21]). Scow considers various "indexing structures" $\mathcal{I}$, and "$\mathcal{I}$-indiscernible sequences" in models of a given complete theory T. The classical case being where $\mathcal{I}$ is a total ordering, and we have usual indiscernible sequences. Scow introduced the "modelling property" and proved (recently) a nice result relating structural Ramsey theory to the modelling property. In Section 5.1, Portillo proves the same results but where T is now a continuous logic theory, (and where certain definitions of Scow have to be modified accordingly). This is a very nice piece of work.

It is well-known that stability (of T) can be characterized by the property that indiscernible sequences are totally indiscernible. In [Sco12] (which I guess comes out of her Ph.D. thesis) Scow adapts this to a characterization of NIP theories in terms of "graph" indiscernibles. In [CPT] from 2014 the authors study Shelah's notion of an "n-dependent theory T", where the $n = 1$ case is precisely NIP. The paper [CPT] also characterized n-dependence in terms of " hypergraph indiscernibles". Section 5.2 extends the notion of n-dependence to continuous theories and again gives a characterization in terms of hypergraph indiscerinibles. Section 5.3 does a similar thing but for hyperdefinable sets in (classical) first order theories.

This Chapter 5 of the thesis should make an attractive solo paper, publishable in a very good logic journal.

Finally I will turn to Chapter 6. The environment is again a classical first order theory, together with a study of hyperdefinability and the related quotients of type spaces by closed equivalence relations. This chapter presents a generalization and extension of [KNS17] (by Krupinski, Newelski, and Simon). The latter paper is about topological dynamical invariants, specifically the "Ellis group" or "ideal group" attached to the dynamical system $(S_X(M), Aut(M))$ where M is a saturated model, and X a $\emptyset$-definable set (or even type-definable over $\emptyset$). The main result of [KNS] is that the Ellis group does not depend on the choice of the saturated model, and hence is an invariant of the theory T. In Section 6.1 the methods from [KNS] are summarized and expressed in the language of "infinitary definability patters" from [Hru22]. Anyway the aim of Chapter 6 is to study certain canonical *factors* of $(S_X(M), Aut(M))$, namely systems $(\mathcal{Y}, Aut(M))$ given by a continuous surjective map from $S_X(M)$ to $\mathcal{Y}$ commuting with the action of $Aut(M)$. Using slightly different notation from the thesis, I will call these $\mathcal{Y}_{WAP}$ and $\mathcal{Y}_{tame}$. These are connected with classical notions from topological dynamics, namely WAP flows and tame flows. So for example $(\mathcal{Y}_{WAP}, Aut(M))$ is the greatest $Aut(M)$-invariant factor of $(S_X(M), Aut(M))$ which is WAP. Substantially summarizing, there are two kinds of new results in the thesis. The first relates the stable (and NIP) hyperdefinable sets from Chapter 3 to WAP and tame factors. For example, given a type-definable over $\emptyset$ equivalence relation E on X, and taking E' to be the induced closed $Aut(M)$-invariant equivalence relation on $S_X(M)$, X/E is stable iff $(S_X(M)/E', Aut(M))$ is WAP. The second new result is extending [KNS] to the the flows $(\mathcal{Y}_{WAP}, Aut(M))$ and $(\mathcal{Y}_{tame}, Aut(M))$, by showing that the Ellis groups do not depend on the choice of the saturated model M. To prove these results, some new machinery is introduced, in addition to what is presented in 6.1. Again this is a very nice piece of work, which should appear in a top logic journal or very good generalist mathematics journal.

To reiterate, I strongly recommend that Adrian Portillo Frenandez be awarded a Ph. D. for this great thesis.

Some detailed comments.

Here are some relatively minor comments and a few questions.

Abstract, top of p. 4 "bullet"

p. 3, line 17, "unit"

p. 3, line 27 "unstable" in place of "non-stable"

p. 8, line -8 "endow" not "endorse"

p. 9. Definition 2.18 is not so well-expressed. In the first bullet point, of course any definable set is defined over a small (finite set). The conventions in the first two bullet points and the last one, are a bit contradictory. Namely in second bullet point, type-definable means over some small set of parameters, and in last bullet point, invariant means over $\emptyset$.

p. 10, After Definition 2.1.12, "indiscernible", and next line "the" compactness ...

p. 11, After Definition 2.1.17. "The", "Shelah".

p. 11-12. Section 2.2. You do not describe the semantics of continuous logic,

namely given a structure M, and formula $\phi(\bar{x})$ with free variables $\bar{x}$ and tuple $\bar{a}$, $\phi(\bar{a})(M)$ is a real number. $\phi(\bar{x}) = 0$ is not a formula. ETC., Also the connective $-$ with . above is not defined.
p. 13, After Fact 2.2.8 erase "an"
p. 14, At the beginning of Section 2.3, refer to the original paper, Coordinatization and canonical bases in simple theories by Hart, Kim, Pillay (JSL 2000) where hyperimaginaries were first introduced and all these notions of types of hperimaginaries over hyperimaginaries etc. were defined.
p. 14, Remark 2.3.2, last line, agrees with E "on" .... Next line, erase "the"
p. 15. Fact 2.39, second line. Insert "is" before "a".
p. 15. Section 2.3.1. Too much hand-waving. You refer to Hanson's thesis, but it does not appear to be on arXiv. One should refer to something published. In Fact 2.3.11, you talk about a "continuous formula". What does this mean as T is a first order theory?
p. 16. First line. Is $\mathcal{L}'$ supposed to be a relational language?
p. 16. Fact 2.4.3 needs a reference.
p. 16, Fact 2.5.2, second line, erase "the".
p. 16, Fact 2.5.3, second bullet point, "idempotents". Third bullet point, maybe "identity" in place of "neutral". Also say that the group is called the Elllis group of the original flow, by model theorists, although the dynamics people mean something else by "Ellis group".
p. 17, Definition 2.5.6, third line "$S_y(A)$".
p. 17. Fact 2.5.7. Say π is a partial type, amd say somewhere what $S_\pi(\mathfrak{C})$ means.
p. 18. Definition 2.5.9. Say what the weak topology on $C(X)$ is.
p. 18, Fact 2.5.11. Recall also what you mean by a unital subalgebra. And closed in what topology on $C(X)$?
p. 20. Third paragraph. I would not call this folklore. I suggest to either give precise details (about what the pseudometric is), or just drop this paragraph, and say that you will make use of a formalism from [HKP22] and [HKP21]. And will adapt results from [BYU10] to this formalism (rather than say you will use results from [BYU10].
p. 21, line 5. You mention the space $S_\phi(M)$. What is its topology?
p. 21. It would be more natural to state Proposition 3.1 in terms of the contrapositive (i.e. equality of types).
p. 23, line 8, "the" Stone-Weierstrass ...
p. 23, Remark 3.1.5, maybe "homeomorphism" in place of "bijection".
p. 24, Definition 3.1.8. Give a reference for the original definition of generi-

cally stable.
p. 24, line -6. "the" compactness ...
p. 25. Definition 3.10 as stated is a bit strange. Would be better to say: X/E is weakly stable if for every set A such that X/E is over A, for every A-indiscernible sequence $(a_i, b_i)_{i<\omega}$ with $a_i, b_i \in X/E$ we have $tp(a_i, b_j) = tp(a_j, b_i)$ for all $i \neq j$ (or equiv. for all $i < j$).
p. 32, line -9. Give a reference for non-constant totally indiscernible sequences being non distal.
p. 36, 2 lines above Proposition 3.4.2, "a" in place of "the".
p. 40 + . Chapter 4. I know that this chapter is close to or identical to a paper appearing (or about to appear) in JSL. But the notation is rather unclear and ambiguous, especially regarding what sets of parameters are where and are small with respect to what models. For example in Lemma 4.1.2, the assumption is that $q \in S(M)$ is A-invariant for SOME $A \subseteq M$. Is M supposed to be ω-saturated over A? Maybe the assumption should not be that q is A-invariant but that q does not split over A (if $a, b \in M$ have same type over A then $\phi(x, a) \in q$ iff $\phi(x, b) \in q$).

In the paragraph on p. 41 beginning "Given a partial type ...." are X and π supposed to be type-definable over arbitrary (maybe non small) subsets of $\mathfrak{C}$? So why not say so? In Lemma 4.1.3, I guess p is supposed to be the A-invariant complete type over $\mathfrak{C}$ from the first paragraph of the chapter. Also $\mathcal{M}$ should contain A. Anyway p is an example of the X from the previous paragraph.

In Proposition 4.1.4 is B small (wrt $\mathfrak{C}$)? Is B assumed to contain A? Where does A appear in the proof of 4.1.4?

p. 43. Somewhere it should be said that if for example M is small in $\mathfrak{C}$ and contains A, and π is defined over M and defines an equiv. relation on $p|M(\mathcal{C}')$ then the quotient of $p|M(\mathcal{C}')$ by π is stable iff the quotient of $p|M(\mathcal{C})$ by $\pi(\mathcal{C})$ is stable.
p. 43, line -3. What do you mean E' is stable?
General question about the main Theorem 4.2.7 of this chapter. Even assuming the saturation, does it imply my claim (Proposition 2) from the Lyon lecture notes, that given say p over monster $\mathfrak{C}$ which is say $\emptyset$ invariant, then there is small A in $\mathfrak{C}$ and type-definable over A equiv. relation E_A on $p|A$ (i.e. on realizations in $\mathcal{C}$), such for no small B containing A is there similar E_B on $p|B$ which properly refines $E|A$ on $p|B$? (So working completely in $\mathfrak{C}$.)

p. 63. Definition 5.1.3. Erase "$\epsilon > 0$", on line 3. Otherwise Chapter 5 seems well-written.
p. 81. Second line of section 6.1. Why write $S(\mathfrak{C}) = S_X(\mathfrak{C})$?
General comment on the various quotients. If E is a type-definable over $\emptyset$ equivalence relation on X, then E induces a closed $Aut(\mathfrak{C})$-invariant equivalence relation on $S_X(\mathfrak{C})$, which you call $\tilde{E}$. On the other hand, if I start with an $Aut(\mathfrak{C})$-invariant closed equivalence relation F on $S_X(\mathfrak{C})$, it need not come from a type-definable over $\emptyset$ equivalence relation E on X, namely it need not be the case that $F = \tilde{E}$ for some such E. (Is this is the case for the pEq if $p_= = q_=$ as discussed in the proof of 6.4.5.?) So 6.4.5. is not so surprizing and maybe this could be mentioned.

Anand Pillay
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