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REPORT ON THE PH.D DISSERTATION "BOUNDARY REPRESENTATIONS OF HYPERBOLIC GROUPS" BY LUKASZ GARNCAREK

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Representation theory is one of the most fundamental and applicable tools in group theory. The study of topological groups (in particular, locally compact groups) leads quite naturally to the study of *unitary representations*, i.e the representations of group elements as unitary operators on a Hilbert space. Notably, for abelian groups unitary representation theory is a synonym for Fourier analysis. Outside the world of abelain groups, unitary representation theory had an enormous success in the study of semisimple Lie groups (and algebraic groups over local fields), and this subject, though in great parts already well developed and understood, is still a major line of research in mathematics, with many applications in diverged fields, e.g Number Theory, Geometry and Analysis.

However, the systematic study of unitary representations of non-abelain *countable* groups is not yet as developed. The task of initiating such a study is undertaken in Garncarek's dissertation for a major class of groups - the class of Gromov hyperbolic groups. This is a wide class (in many senses, a generic group is Gromov hyperbolic) consisting of groups for which the word metric roughly has a negative curvature. This class received a lot of attention in the last three decays, and its study is in the heart of Geometric Group Theory. In many ways, hyperbolic groups behave similarly to (lattices in rank one) semisimple groups. It is an essential result in the representation theory of semisimple groups that every unitary representation could be fully modeled on function spaces on the group boundary. Garncarek takes this result as the starting point of his discussion, and focuses on those unitary representations arising in such a geometric way, considering the Gromov ideal boundary associated with any hyperbolic group and endowing it with its Patterson-Sullivan measure.

Garncarek's explicit setting is as follows. Given any hyperbolic group Γ and a group norm ℓ on Γ which is in the word class (that is, for any finite generating set S there exists a constant $K > 0$ such that for every $g \in \Gamma$, $\ell(g) \geq K|g|_S$, where $|g|_S$ is the word length of g w.r.t the generating set S), one constructs a measure on the Gromov boundary of Γ, $\partial\Gamma$, known as the Patterson-Sullivan measure, which we will denote by μ. Garncarek considers the Hilbert space $H = L^2(\partial\Gamma, \mu)$ and the

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quasi-regular representation of Γ on H given by

$$\rho : \Gamma \to U(H), \quad \rho(g)f(x) = \sqrt{\frac{dg\mu}{d\mu}(x)} f(g^{-1}x).$$

The main result of the dissertation is the proof of irreducibility of every representation in this class. Moreover, it is proven that for a given group Γ and two lengths ℓ_1 and ℓ_2 on Γ, the associated quasi-regular representations are unitary equivalent if and only if the two lengths are equivalent in a very strong sense: up to a homothety their difference is uniformly bounded.

Garncarek's results are the first of their kind available for such a wide class of groups as the class of Gromov hyperbolic groups. It is important to note at this point that for a general countable group there exists no explicit construction of an irreducible unitary representation (though every group has an abundant of such representations by the general theory). In fact, for groups which are not virtually nilpotent, only a handful of examples were known prior to Garncarek's achievement. One class of examples for which similar theorems were known before is the class of fundamental groups of negatively curved compact manifolds, considered in [BM11]. Such groups are necessarily Gromov hyperbolic, but the converse is far from being true. That is, the class considered by Garncarek is much wider then any previously known class for which such theorem applies.

The general method of proof taken by Garncarek is the same as the one taken in [BM11]. However, there is a major difference - at a crucial point Garncarek takes a novel and ingenious approach. This novel approach makes an important progress, and enables him to treat the problem in a higher generality. Since this is Garncarek's central contribution, I will describe the essence of this new approach here, in spite of the fact that it requires diving into some technicalities. In the heart of the proof lies the approximation of some explicit operator $T \in B(H)$ by weighted sums of unitaries $\rho(g)$, $g \in \Gamma$. The challenge is to come up with weights which guarantee the right asymptotics. In the previous approach this was achieved by a comparison to the asymptotics growth of the number of closed geodesic loops on the manifold, a problem that was previously solved in Margulis' dissertation (for which an English translation appeared recently, [Mar04]). In the lack of an analogue for Margulis' thesis in his more general setting, Garncarek had to come up with a new approach to the problem. His solution is as follows: he observed that the operator T is an integration operator against a bounded kernel, that is a function $k \in L^\infty(\partial\Gamma \times \partial\Gamma)$, and then explained how can one get explicitly the weight of any $g \in \Gamma$ as a result of integrating the function k against a corresponding domain $V_g \subset \partial\Gamma \times \partial\Gamma$ which he defined geometrically. This solution is so natural and elegant!

Apart of their great aesthetic value, Garncarek's results are of fundamental importance. For example, I suspect that one can use Garncarek's weights in order to generalize Margulis' thesis to the class of Gromov hyperbolic groups, as well as to

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give a generalization of Selberg's trace formula [Sel56]. I have no doubt that further important applications will be found in the ergodic theory of Gromov hyperbolic groups. Moreover, the robustness of his methods gives hope that his results could be generalized to other groups with hyperbolic properties, e.g the mapping class groups, and could be related to current development in the ergodic theory of such groups, see for example [EM11].

As a whole, the dissertation is very well written. The style is concise and elegant and, although it is quite technical and computational in nature, all the ideas are expressed clearly. As mentioned above, the results are of fundamental importance and the methods are novel and ingenious. I strongly recommend that Lukasz Garncarek will be awarded a doctoral degree for his achievements appearing in his dissertation. In fact, I find the dissertation outstanding and I recommend the dissertation to be nominated for the prestigious doctoral dissertations awards (e.g. Award of the Minister).

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