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Oszacowania norm transformat Riesza

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Norm estimates for Riesz transforms

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Abstract

The goal of the dissertation is to investigate various kinds of Riesz transforms on $\mathbb{R}^d$ with focus on obtaining dimension-free estimates of their L^p norms.

In the first part we handle classical Riesz transforms and maximal operators associated with them. We begin by using Fourier transform techniques to obtain a dimension-free estimate of the L^2 norm of the maximal Riesz transform in terms of the corresponding Riesz transform with an explicit constant. In order to accomplish this we factorize the maximal Riesz transform, following Mateu and Verdera, into the 'maximal part' and the 'Riesz part', namely

$$R_j^* = M^* R_j,$$

and estimate the Fourier multiplier associated with M^* in a dimension-free way.

Next, we use the real method of rotations and the complex method of rotations of Iwaniec and Martin to generalize this result to Riesz transforms of higher orders and to L^p norms for $1 < p < \infty$. We express the operator M^* as an integral of the Hilbert transform, thus obtaining a dimension-free estimate which is additionally explicit in terms of dependency on p.

In the second part we turn our attention to Riesz transforms related to Schrödinger operators, i.e. operators of the form

$$R_V^a = V^a L^{-a}, \qquad L = -\tfrac{1}{2}\Delta + V,$$

where Δ is the Laplacian, V is a non-negative potential, and L is called the Schrödinger operator. First we use complex interpolation to prove some general results on L^p-boundedness ($1 < p < \infty$) of the operators R_V^a for locally integrable potentials. Then, using the Feynman–Kac formula and probabilistic methods we give conditions for the potential under which the operators R_V^a are bounded on L^1 and L^∞. In particular our results apply to potentials with power or exponential growth.

Finally, using similar methods, we show that if the potential V is of the form

$$V(x) = V_1(x) + \cdots + V_d(x),$$

where each V_i acts only on the i-th coordinate of the argument x and has polynomial growth with the exponent not greater than 2, then the L^1 and L^∞ norms of R_V^a can be estimated independently of the dimension. We achieve this by factorizing the semigroup associated with L into one-dimensional factors, estimating them separately and then combining the results.

Chapters 2 and 4 are based on joint works with Błażej Wróbel and Chapter 3 is based on joint work with Błażej Wróbel and Jacek Zienkiewicz.

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Contents

Chapter 1

Introduction

1.1 Riesz transforms — overview

Harmonic analysis is a branch of mathematics concerned with investing functions via their decomposition into some kind of simpler 'basic' parts, in particular via their decomposition using the Fourier transform defined for a function $f : \mathbb{R}^d \to \mathbb{C}$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \xi} \, dx.$$

Problems posed in the field of harmonic analysis often involve operators, i.e. functions taking functions as arguments and returning other functions as values, among which the most investigated are singular integral operators. These are operators of the form

$$Tf(x) = \int_{\mathbb{R}^d} K(x, y) f(y) \, dy,$$

where the function $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, called the kernel, behaves as $|x - y|^{-d}$ as $|x - y|$ tends to 0. Usually we also assume that the kernel K satisfies some regularity conditions, e.g.

$$\nabla_x K(x, y) + \nabla_y K(x, y) \leqslant C|x - y|^{-d-1}$$

for some constant $C > 0$.

The simplest multivariate singular integral operators are the Riesz transforms R_j defined by

$$R_j f(x) = \lim_{t \to 0^+} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}} \int_{|x-y|>t} \frac{x_j - y_j}{|x - y|^{d+1}} f(y) \, dy \tag{1.1.1}$$

for $j = 1, \ldots, d$. Equivalently, they can be defined via the Fourier transform as

$$\widehat{R_j f}(\xi) = -i \frac{\xi_j}{|\xi|} \widehat{f}(\xi). \tag{1.1.2}$$

The Riesz transforms have been studied for several dozens years now. It is well known that they are bounded on $L^p(\mathbb{R}^d)$ spaces for $1 < p < \infty$ due to the theory of Calderón and Zygmund initiated in [8]. Moreover, Stein showed in [47] that the vector of Riesz transforms has its $L^p(\mathbb{R}^d)$ norm bounded independently of the dimension d. More precisely, he proved the following theorem.

Theorem (Stein, [47]). *For $1 < p < \infty$ there is a constant C_p independent of the dimension d such that*

$$\left\| \left(\sum_{j=1}^{d} |R_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \leqslant C_p \|f\|_{L^p(\mathbb{R}^d)}.$$

In the course of investigating the Riesz transforms, several questions arise. One of them is: what will happen if we replace the limit in the definition (1.1.1) with the supremum? This leads to the definition of the maximal Riesz transform R_j^* and an auxiliary operator R_j^t called the truncated Riesz transform

$$R_j^* f(x) = \sup_{t>0} \left| R_j^t f(x) \right| \qquad \text{and} \qquad R_j^t f(x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}} \int_{|x-y|>t} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\, dy.$$

Probably one of the best known theorems providing estimates of maximal operators is the Cotlar's inequality. It says that if T is a singular integral operator with kernel satisfying some regularity conditions, then the following estimate holds

$$T^* f(x) \leqslant C_d \left(M(T(f)) + M(f) \right),$$

where C_d is a positive constant depending on the dimension, M is the Hardy–Littlewood maximal operator, and T^* is the maximal operator associated with T defined analogously to the maximal Riesz transform.

In regards to our work, another important results concerning the maximal Riesz transforms due to Mateu and Verdera [37] states that for $1 < p < \infty$ the $L^p(\mathbb{R}^d)$ norm of $R_j^* f$ can be controlled by the $L^p(\mathbb{R}^d)$ norm of $R_j f$, namely

Theorem (Mateu, Verdera, [37]). *For $1 < p < \infty$ there is a constant $C_{p,d}$ depending on p and d such that*

$$\left\| R_j^* f \right\|_{L^p(\mathbb{R}^d)} \leqslant C_{p,d} \|R_j f\|_{L^p(\mathbb{R}^d)}.$$

Chapter 2 is devoted to improving the above result, albeit only in the case $p = 2$, to one with the constant $C_{p,d}$ independent of the dimension d.

Another question one may ask is whether there are any natural generalizations of the classical Riesz transforms. Let P be a homogeneous, harmonic polynomial of degree k. The k-th order Riesz transform R_P associated with the polynomial P is then defined as

$$R_P f(x) = \lim_{t \to 0^+} R_P^t f(x), \qquad \text{where} \qquad R_P^t f(x) = \frac{\Gamma\left(\frac{k+d}{2}\right)}{\pi^{d/2} \Gamma\left(\frac{k}{2}\right)} \int_{|x-y|>t} \frac{P(x-y)}{|x-y|^{d+k}} f(y)\, dy, \tag{1.1.3}$$

or, equivalently, via the Fourier transform as

$$\widehat{R_P f}(\xi) = (-i)^k \frac{P(\xi)}{|\xi|^k} \widehat{f}(\xi). \tag{1.1.4}$$

Higher order Riesz transforms were studied by Duoandikoetxea and Rubio de Francia, who proved in [17] a result analogous to the aforementioned theorem of Stein, namely that the vector of higher order Riesz transforms of a fixed degree has its $L^p(\mathbb{R}^d)$ norm bounded independently of the dimension.

Having defined the higher order Riesz transforms, wa may also define the maximal Riesz transform of order k as

$$R_P^* f(x) = \sup_{t>0} \left| R_P^t f(x) \right|.$$

Similarly to the first-order case, Mateu, Orobitg, Pérez and Verdera proved in [35, 36] that for $1 < p < \infty$ the $L^p(\mathbb{R}^d)$ norm of $R_P^* f$ can be controlled by the $L^p(\mathbb{R}^d)$ norm of $R_P f$, namely

Theorem (Mateu, Orobitg, Pérez, Verdera, [35, 36]). *For $1 < p < \infty$ there is a constant $C_{p,k,d}$ depending on p, k and d such that*

$$\|R_P^* f\|_{L^p(\mathbb{R}^d)} \leqslant C_{p,k,d} \|R_P f\|_{L^p(\mathbb{R}^d)}.$$

Chapter 3 is devoted to improving the above result to one with the constant $C_{p,k,d}$ independent of the dimension d.

1.2 Riesz transforms associated with Schrödinger operators

In the second part of the dissertation we slightly change the object of investigation. In order to define it, we first need to introduce the Schrödinger operator

$$L = -\frac{1}{2}\Delta + V,$$

where Δ is the Laplacian and V is a non-negative function in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ called the potential. For $a > 0$ we define the Riesz transform associated with the Schrödinger operator L by the formula

$$R_V^a f(x) = V^a(x) \cdot L^{-a} f(x) = \frac{V^a(x)}{\Gamma(a)} \cdot \int_0^\infty e^{-tL} f(x)\, t^{a-1}\, dt,$$

where e^{-tL} is the semigroup generated by L. Rigorous definitions of the Schrödinger operator L, the associated semigroup e^{-tL} and the Riesz transform R_V^a are more complicated than in the case of the classical Riesz transforms; the relevant details can be found in Section 4.1.

Unlike in the case of the classical Riesz transforms, here it is not straightforward to give one 'canonical' result regarding the $L^p(\mathbb{R}^d)$ boundedness of the operators R_V^a. Nonetheless, there exist numerous partial results with varying assumptions on V and a, which we present in Section 1.3.2. In Chapter 4 we provide a general result on $L^p(\mathbb{R}^d)$ boundedness of R_V^a for $1 < p \leqslant 2$ and locally integrable potentials V and another result on $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$ boundedness for a certain class of potentials, including potentials with power and exponential growth. As for the dimension-free estimates of the $L^p(\mathbb{R}^d)$ norms of the operators R_V^a, not much is known. Chapter 5 is devoted to proving dimension-free results for a certain class of potentials.

1.3 Summary of known results

1.3.1 Classical Riesz transforms

Classical Riesz transforms have been studied by numerous authors. First, it follows from the theory of Calderón and Zygmund, see [9, 8], that they are bounded on the $L^p(\mathbb{R}^d)$ space for $1 < p < \infty$. Then, Iwaniec and Martin [26] calculated the $L^p(\mathbb{R}^d)$ norm of the first-order Riesz transform to be the same as the $L^p(\mathbb{R})$ norm of the Hilbert transform, i.e.

Theorem (Iwaniec, Martin, [26, Theorem 1.1])**.** *For each* $1 < p < \infty$ *and* $j = 1, \dots, d$ *we have*

$$\|R_j\|_{L^p(\mathbb{R}^d)} = \begin{cases} \tan\left(\frac{\pi}{2p}\right) & \text{if } 1 < p \leqslant 2 \\ \cot\left(\frac{\pi}{2p}\right) & \text{if } 2 \leqslant p < \infty \end{cases}$$

Their result is based on [41, Theorem 4.1], where Pichorides calculated the norm of the Hilbert transform.

As for the vector-valued estimates, Stein proved in [47] that the vector of Riesz transforms has $L^p(\mathbb{R}^d)$ bounds which are independent of the dimension. More precisely,

Theorem (Stein, [47])**.** *For* $1 < p < \infty$ *there is a constant* C_p *independent of the dimension* d *such that*

$$\left\| \left(\sum_{j=1}^{d} |R_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \leqslant C_p \|f\|_{L^p(\mathbb{R}^d)}. \tag{1.3.1}$$

Unlike in the case of a single Riesz transform, the optimal constant in (1.3.1) is not known. The best results to date are $C(p-1)^{-1}$ for small values of p given by Bañuelos and Wang in [3] (see also [15]) and $C\cot\left(\frac{\pi}{2p}\right)$ for large values of p, which follows from [26].

It is also worth noting that Duoandikoetxea and Rubio de Francia proved in [17] a counterpart of the above Stein's theorem for higher order Riesz transforms, namely

Theorem (Duoandikoetxea, Rubio de Francia, [17, Théorème 2])**.** *For* $1 < p < \infty$ *there is a constant* $C_{p,k}$ *independent of the dimension* d *such that*

$$\left\| \left(\sum_{P \in \mathcal{P}_k} |R_P f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \leqslant C_{p,k} \|f\|_{L^p(\mathbb{R}^d)},$$

where $\mathcal{P}_k$ *is the orthogonal basis of the space of spherical harmonics of degree* k*. Moreover, for fixed odd* k *we have*

$$C_{p,k} = O((p-1)^{-1-k/2}) \quad \text{as} \quad p \to 1 \qquad \text{and} \qquad C_{p,k} = O(p) \quad \text{as} \quad p \to \infty$$

and for even k *we have*

$$C_{p,k} = O((p-1)^{-2-k/2}) \quad \text{as} \quad p \to 1 \qquad \text{and} \qquad C_{p,k} = O(p^2) \quad \text{as} \quad p \to \infty.$$

Moving on to the maximal Riesz transforms, the main estimates regarding them are due to Mateu, Orobitg, Pérez and Verdera in a series of three papers: [37] (first order Riesz transforms), [36, Section 2] (even order higher Riesz transforms) and [35, Section 4] (odd order higher Riesz transforms). The estimates for the Riesz transforms following from this series of papers are summarized in the following theorem.

Theorem (Mateu, Orobitg, Pérez, Verdera, [35, 36])**.** *For $1 < p < \infty$ there is a constant $C_{p,k,d}$ depending on p, k and d such that*

$$\|R_P^* f\|_{L^p(\mathbb{R}^d)} \leqslant C_{p,k,d} \|R_P f\|_{L^p(\mathbb{R}^d)}.$$

Recently Liu, Melentijević and Zhu in [33] partially improved the results of Mateu, Orobitg, Pérez and Verdera to a dimension-free estimate in the case of first-order Riesz transforms and $p \in [2, \infty)$. Their theorem reads

Theorem 1.3.1 (Liu, Melentijević, Zhu, [33])**.** *For every $f \in L^p(\mathbb{R}^d)$ with $p \geqslant 2$ we have*

$$\left\|R_j^* f\right\|_{L^p(\mathbb{R}^d)} \leqslant \left(2 + \frac{1}{\sqrt{2}}\right)^{2/p} \|R_j f\|_{L^p(\mathbb{R}^d)}.$$

1.3.2 Riesz transforms associated with Schrödinger operators

As mentioned before, since the class of Riesz transforms associated with Schrödinger operators is more diverse than the in the case of classical Riesz transforms, it is harder to provide a general result on their $L^p(\mathbb{R}^d)$ boundedness. The topic has been investigated by many authors, see for example [1, 2, 4, 12, 13, 18, 19, 44, 54]. However, we present a wide array of partial results with varying assumptions on V and a. The cases of $a = \frac{1}{2}$ and $a = 1$ attracted the most attention.

The first, well known, result dates back to 1970s and concerns the case $a = 1$. It states that for a locally integrable non-negative potential V the operator R_V^1 is bounded on $L^1(\mathbb{R}^d)$ and, in fact, that it is a contraction, see for example [21], [27, Lemma 6] and [2, Theorem 4.3]. Then Shen proved two related theorems for the potentials belonging to the reverse Hölder class B_q for $q \geqslant \frac{d}{2}$: [43, Theorem 3.1] asserts the $L^p(\mathbb{R}^d)$ boundedness of R_V^1 if $1 \leqslant p \leqslant q$ and in [43, Theorem 5.10] it is shown that the $L^p(\mathbb{R}^d)$ norm of $R_V^{1/2}$ is bounded whenever $1 \leqslant p \leqslant 2q$. Both results were later improved by Auscher and Ben Ali to $1 < q \leqslant \infty$, see [2, Theorem 1.1 and Theorem 1.2].

The next two results address polynomial potentials. In [18, Theorem 4.5] Dziubański proved that for such potentials the operator R_V^a with any $a \geqslant 0$ is bounded on the $L^p(\mathbb{R}^d)$ space for $1 \leqslant p \leqslant \infty$. Then Urban and Zienkiewicz showed in [54, Theorem 1.1] that if the potential is a polynomial satisfying a certain C. Fefferman condition, then R_V^1 is bounded on the $L^\infty(\mathbb{R}^d)$ space and, by interpolation with the first presented result, on all $L^p(\mathbb{R}^d)$ spaces for $1 \leqslant p \leqslant \infty$. Moreover, its norm is estimated independently of the dimension d.

The last two results concern the harmonic oscillator, i.e. the case when $V(x) = |x|^2$. The first result [5, Lemma 3] gives $L^p(\mathbb{R}^d)$ boundedness, $1 \leqslant p \leqslant \infty$, of R_V^a for all values of $a > 0$. On the other hand, the second one addresses only the case of $a = \frac{1}{2}$, however the achieved bound on the $L^p(\mathbb{R}^d)$ norm of the operator does not depend on the dimension d, see [24], [34] and [28, Theorem 8]. This, together with [54, Theorem 1.1], are the only dimension-free norm estimates for the Riesz transforms associated with Schrödinger operators that we are aware of.

1.4 Outline of the thesis and overview of the methods

The dissertation consists of two parts: the first part, containing Chapters 2 and 3, is devoted to estimates of first order and higher order classical maximal Riesz transforms. In the second part, consisting of Chapters 4 and 5, we handle Riesz transforms associated with Schrödinger operators.

In Chapter 2 we investigate first order Riesz transforms. The main result is a dimension-free estimate for the $L^2(\mathbb{R}^d)$ norm of the maximal truncated Riesz transform in terms of the $L^2(\mathbb{R}^d)$ norm of the Riesz transform, specifically we prove

Theorem 1.4.1. *For every $f \in L^2(\mathbb{R}^d)$ we have*

$$\left\|R_j^* f\right\|_{L^2(\mathbb{R}^d)} \leqslant 2 \cdot 10^8 \|R_j f\|_{L^2(\mathbb{R}^d)}.$$

As a consequence, we also derive

Corollary 1.4.2. *For every $f \in L^2(\mathbb{R}^d)$ we have*

$$\left\| \left(\sum_{j=1}^d \left|R_j^* f\right|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^d)} \leqslant 2 \cdot 10^8 \|f\|_{L^2(\mathbb{R}^d)}.$$

The first ingredient in the proof of Theorem 1.4.1 is a factorization of the truncated Riesz transform $R_j^t = M^t(R_j)$, where M^t, $t > 0$, is a family of multiplier operators, see Section 2.1. This reduces the task of proving Theorem 1.4.1 to estimating the norm of the operator $M^* f(x) = \sup_{t>0} \left|M^t f(x)\right|$.

The second ingredient is based on the technique initiated by Bourgain in [7]. It consists of estimating the Fourier multiplier associated with the family M^t in a dimension-free way and then applying this estimate to the square function inequality

$$M^* f = \sup_{t>0} \left|M^t f\right| \leqslant \sup_{n\in\mathbb{Z}} \left|M^{2^n} f\right| + \left(\sum_{n\in\mathbb{Z}} \sup_{t\in[2^n,2^{n+1}]} \left|M^t f - M^{2^n} f\right|^2 \right)^{1/2} \tag{1.4.1}$$

in order to deduce the desired estimate of the maximal operator M^*. The Fourier multiplier estimates may be found in Section 2.2, while their application to (1.4.1) are contained in Section 2.3.

Chapter 2 is based on [30].

In Chapter 3 we generalize the results of Chapter 2 to higher order Riesz transforms and $L^p(\mathbb{R}^d)$ spaces for $1 < p < \infty$. The main results are the following two theorems. By $\mathcal{H}_k$ we denote the space of spherical harmonics of degree k.

Theorem 1.4.3. *Take $p \in (1, \infty)$ and let k be a non-negative integer. Let $\mathcal{P}_k$ be a subset of $\mathcal{H}_k$. Then there is a constant $A(p,k)$ independent of the dimension d and such that*

$$\left\| \left(\sum_{P\in\mathcal{P}_k} |R_P^* f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \leqslant A(p,k) \left\| \left(\sum_{P\in\mathcal{P}_k} |R_P f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)},$$

where $f \in L^p(\mathbb{R}^d)$. Moreover, for fixed k we have

$$A(p,k) = O(p^{5/2+k/2}) \quad \text{as} \quad p \to \infty \qquad \text{and} \qquad A(p,k) = O((p-1)^{-5/2-k/2}) \quad \text{as} \quad p \to 1.$$

In particular, if $\mathcal{P}_k$ contains one element P, then Theorem 1.4.3 immediately gives

$$\|R_P^* f\|_{L^p(\mathbb{R}^d)} \leqslant A(p,k)\|R_P f\|_{L^p(\mathbb{R}^d)}.$$

In this case however, we can slightly improve the constant $A(p,k)$.

Theorem 1.4.4. *Take $p \in (1,\infty)$ and let k be a non-negative integer. Let P be a spherical harmonic of degree k. Then there is a constant $B(p,k)$ independent of the dimension d and such that*

$$\|R_P^* f\|_{L^p(\mathbb{R}^d)} \leqslant B(p,k)\|R_P f\|_{L^p(\mathbb{R}^d)},$$

where $f \in L^p(\mathbb{R}^d)$. Moreover, for fixed k we have

$$B(p,k) = O(p^{2+k/2}) \quad \text{as} \quad p \to \infty \qquad \text{and} \qquad B(p,k) = O((p-1)^{-2-k/2}) \quad \text{as} \quad p \to 1.$$

Combination of Theorem 1.4.3 and a result of Duoandikoetxea and Rubio de Francia [17, Théorème 2] yields a generalization of Corollary 1.4.2. Denote by $a(d,k)$ the dimension of $\mathcal{H}_k$ and let $\{Y_j\}_{j=1,\dots,a(d,k)}$ be an orthogonal basis of $\mathcal{H}_k$ normalized by the condition

$$\int_{S^{d-1}} |Y_j(\omega)|^2 \, d\omega = \frac{1}{a(d,k)}; \tag{1.4.2}$$

here $d\omega$ denotes the probabilistic spherical measure. Then we have

Corollary 1.4.5. *Take $p \in (1,\infty)$ and let k be a non-negative integer. Then there is a constant $G(p,k)$ independent of the dimension d and such that*

$$\left\| \left(\sum_{j=1}^{a(d,k)} \left| R_{Y_j}^* f \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \leqslant G(p,k)\|f\|_{L^p(\mathbb{R}^d)},$$

where $f \in L^p(\mathbb{R}^d)$. Moreover, for fixed and odd k we have

$$G(p,k) = O(p^{7/2+k/2}) \quad \text{as} \quad p \to \infty \qquad \text{and} \qquad G(p,k) = O((p-1)^{-7/2-k}) \quad \text{as} \quad p \to 1$$

and for even k we have

$$G(p,k) = O(p^{9/2+k/2}) \quad \text{as} \quad p \to \infty \qquad \text{and} \qquad G(p,k) = O((p-1)^{-9/2-k}) \quad \text{as} \quad p \to 1.$$

Theorem 1.4.3 and Theorem 1.4.4 are first proved only in the case of odd k which is simpler as it employs the real method of rotations. Then we use the complex method of rotations to generalize the argument to all natural k. In both proofs the first step is to use the same kind of factorization as in Chapter 2, i.e. $R_P^t = M_k^t(R_P)$, where M_k^t, $t > 0$, is a family of multiplier operators. This step is described in detail in Section 3.1.

Then we need to find a useful expression for the operator M_k^t and this is the place where the proofs of odd case and general case split. In the odd case we express M_k^t in terms of the

Riesz transforms associated with the orthogonal basis $\{Y_j\}_{j=1,\dots,a(d,k)}$ of $\mathcal{H}_k$ normalized by the condition (1.4.2), which yields

$$M_k^t = (-1)^k \sum_{j=1}^{a(d,k)} R_{Y_j}^t R_{Y_j}.$$

Finally we use the real method of rotations to express the operators $R_{Y_j}^t$ in terms of the Hilbert transform, which gives us a dimension-free estimate of their norm. The application of the real method of rotations is found in Section 3.2.

The real method of rotations works only for odd kernels, so in the general case we have to use the complex method of rotations of Iwaniec and Martin [26] and this requires extending the operators from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{C}^d)$. Because of that we need to obtain a different expression for the operator M_k^t better suited to extension to $\mathbb{C}^d$ and restriction back to $\mathbb{R}^d$. Instead of taking the orthogonal basis $\{Y_j\}_{j=1,\dots,a(d,k)}$ of $\mathcal{H}_k$, for each multi-index $j = (j_1, \dots, j_k)$ with pairwise distinct elements we take the monomial $P_j(x) := x_{j_1} \cdots x_{j_k}$. The reason behind this change is that the basis $\{Y_j\}_{j=1,\dots,a(d,k)}$ may not be orthogonal after extension to $\mathbb{C}^d$, while the monomials P_j are orthogonal both on $\mathbb{R}^d$ and on $\mathbb{C}^d$. We also average the resulting sum over the special orthogonal group $SO(d)$. Then we obtain

$$M_k^t f(x) = C(d,k) \int_{SO(d)} \sum_{j\in I} (R_{P_j}^t R_{P_j} f)_U(x)\, d\mu(U),$$

where T_U is the conjugation of an operator T by $U \in SO(d)$ and I denotes the set of multi-indices $j = (j_1, \dots, j_k)$ with pairwise distinct elements. The details of the averaging procedure can be found in Section 3.3.

The third step is similar to the odd case: we use the complex method of rotations, preceded by extension of the operator R^t on $\mathbb{R}^d$ to the operator $\widetilde{R}^t$ on $\mathbb{C}^d$, to express the operator

$$R^t = \sum_{j\in I} R_{P_j}^t R_{P_j}$$

in terms of the Hilbert transform. The application of the method of rotations is described in Section 3.4.

Lastly, we need to deduce the estimates for R^t from the estimates for $\widetilde{R}^t$. The complex method of rotations of Iwaniec and Martin includes a restriction procedure, see [26, Section 4], however the resulting restricted operator is not the same as the initial operator R^t, hence we need to estimate the difference between the two of them, which is done in Section 3.5.

Chapter 3 is based on [31].

In Chapter 4 we turn our attention to Schrödinger operators, i.e. operators of the form

$$L = -\frac{1}{2}\Delta + V,$$

where Δ is the Laplacian and V is a non-negative function in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ called the potential. Specifically, we investigate Riesz transforms associated with them, which for $a > 0$ are given by

$$R_V^a f(x) = V^a(x) \cdot L^{-a} f(x) = \frac{V^a(x)}{\Gamma(a)} \cdot \int_0^\infty e^{-tL} f(x)\, t^{a-1}\, dt, \tag{1.4.3}$$

where e^{-tL} is the semigroup generated by L.

There are two main results in this chapter. Firstly, we prove $L^p(\mathbb{R}^d)$, $1 < p \leqslant 2$, boundedness for a wide class of potentials.

Theorem 1.4.6. *Let $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and take $p \in (1,2]$. Then for all $0 \leqslant a \leqslant 1/p$ the Riesz transform R_V^a is bounded on $L^p(\mathbb{R}^d)$.*

Theorem 1.4.6 is derived as a consequence of the endpoint bounds for $R_V^{1/2}$ on $L^2(\mathbb{R}^d)$, see Proposition 4.1.3, and for R_V^1 on $L^1(\mathbb{R}^d)$ ([2, Theorem 4.3], see also [21, 27]) together with the interpolation result given below.

Theorem 1.4.7. *Let $0 < a_0 < a_1$. Assume that $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is such that $R_V^{a_1}$ is bounded on $L^{p_1}(\mathbb{R}^d)$ for some $p_1 \in [1,\infty)$ and $R_V^{a_0}$ is bounded on $L^1(\mathbb{R}^d)$. Then, R_V^a is bounded on $L^p(\mathbb{R}^d)$ for every p and a such that $\frac{1}{p} = \theta + \frac{1-\theta}{p_1}$ and $a = \theta a_0 + (1-\theta)a_1$ with some $\theta \in (0,1)$.*

The above theorem is proved via Stein's complex interpolation theorem.

The other main result concerns $L^\infty(\mathbb{R}^d)$ and $L^1(\mathbb{R}^d)$ boundedness of R_V^a for specific classes of non-negative potentials V, for which we assume a certain condition relating the value $V(x)$ and the speed at which $V(y)$ decreases for y in a ball around x. The main classes of potentials to which our results may be applied are given in the following theorem. We will say that some property holds *globally* if there is a compact set $F \subseteq \mathbb{R}^d$ such that the property holds for almost all $x \in \mathbb{R}^d \setminus F$.

Theorem 1.4.8. *Let $V\colon \mathbb{R}^d \to [0,\infty)$ be a function in $L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$. Then in all the three cases*

1. *$V(x) \approx 1$ globally*
2. *For some $\alpha > 0$ globally*
3. *For some $\beta > 1$ globally*

each of the Riesz transforms R_V^a, $a > 0$, is bounded on $L^\infty(\mathbb{R}^d)$ and on $L^1(\mathbb{R}^d)$.

To prove the theorem, we first notice that the semigroup e^{-tL}, and in consequence the operator R_V^a, are positivity preserving. Hence in order to obtain a bound on the $L^\infty(\mathbb{R}^d)$ norm of R_V^a it suffices to bound the quantity $R_V^a(\mathbb{1})(x)$ by a constant that does not depend on x. Thus we only need to handle the following integral

$$V^a(x) \cdot \int_0^\infty e^{-tL}(\mathbb{1})(x)\, t^{a-1}\, dt,$$

where $\mathbb{1}$ is the constant 1 function. This is where the main ingredient of the proof comes in, which is the Feynman–Kac formula

$$e^{-tL} f(x) = \mathbb{E}_x \left[e^{-\int_0^t V(X_s)\, ds} f(X_t) \right]. \tag{1.4.4}$$

Here $X_t = (X_t^1, \ldots, X_t^d)$ is the standard d-dimensional Brownian motion starting at x. In our case we need this formula only for $f = \mathbb{1}$, i.e.

$$e^{-tL}(\mathbb{1})(x) = \mathbb{E}_x \left[e^{-\int_0^t V(X_s)\, ds} \right]. \tag{1.4.5}$$

Then, for $k = 0, 1, 2, \ldots$ and some fixed $x \in \mathbb{R}^d$ we divide the underlying probability space into events 'up to time t the value of the potential $V(X_s)$ was always at least $\frac{V(x)}{2^k}$'. More formally for fixed $x \in \mathbb{R}^d$ and $k = 0, 1, 2, \ldots$ we introduce the sets

$$A_k = \left\{ y \in \mathbb{R}^d : \frac{V(x)}{2^k} \leqslant V(y) \right\}$$

and

$$\Omega_k = \{\omega \in \Omega : X_s(\omega) \in A_k \text{ for almost all } s \in [0, t]\} .$$

Using them we split the expected value in (1.4.5) in the following manner

$$\begin{aligned} &e^{-tL}(\mathbb{1})(x) = \\ &\quad \mathbb{E}_x \left[e^{-\int_0^t V(X_s)\, ds} \mathbb{1}_{\Omega_0} \right] + \sum_{k=1}^{K} \mathbb{E}_x \left[e^{-\int_0^t V(X_s)\, ds} \mathbb{1}_{\Omega_k \cap \Omega_{k-1}^c} \right] + \mathbb{E}_x \left[e^{-\int_0^t V(X_s)\, ds} \mathbb{1}_{\Omega_K^c} \right] \\ &\quad \leqslant e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} \mathbb{P}\left(\Omega_k \cap \Omega_{k-1}^c\right) + \mathbb{P}\left(\Omega_K^c\right) \end{aligned}$$

with $K = \lfloor \log_2 V(x) \rfloor$. Lastly, we estimate the probabilities appearing in the expression above using bounds for the normal distribution and the complementary error function. A detailed description is found in Section 4.3.

The case of $L^1(\mathbb{R}^d)$ estimates is similar, but more complex. First we use duality between the spaces $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$ in order to reduce the task of estimating the $L^1(\mathbb{R}^d)$ norm of the operator $R_V^a = V^a L^{-a}$ to estimating the $L^\infty(\mathbb{R}^d)$ norm of the operator $L^{-a}V^a$. Similarly to the previous case, we use the positivity-preserving property of L^{-a} and we remain with the goal of bounding the quantity

$$L^{-a}(V^a)(x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL}(V^a)(x)\, t^{a-1}\, dt$$

by a constant independent of x. The main part of the proof is again the Feynman–Kac formula used to estimate the semigroup e^{-tL}, however now the semigroup is applied to the function V^a instead of $\mathbb{1}$. For this reason the calculations, although similar to the $L^\infty(\mathbb{R}^d)$ case, are more complex, see Section 4.4 for more details.

Chapter 4 is based on [29].

In Chapter 5 we aim at improving the result of the previous chapter to a bound that does not depend on the dimension d. However, this comes at a price of narrowing down the class of permissible potentials to ones of the form

$$V(x) = V_1(x) + \cdots + V_d(x), \tag{1.4.6}$$

where each V_i acts only on the i-th coordinate of the argument x and has polynomial growth with the exponent not greater than 2, i.e. there are absolute constants m and M such that

$$m|x_i|^\alpha \leqslant V_i(x) \leqslant M|x_i|^\alpha \tag{1.4.7}$$

for some $0 < \alpha \leqslant 2$. The main theorem of the chapter is

Theorem 1.4.9. *Fix* $\alpha > 0$ *and let* V *given by* (1.4.6) *satisfy* (1.4.7). *Then there is a constant* $C > 0$ *depending on* m, M, α *and independent of the dimension* d *such that*

$$\|R_V^a f\|_{L^\infty(\mathbb{R}^d)} \leqslant C\|f\|_{L^\infty(\mathbb{R}^d)}, \quad f \in L^\infty(\mathbb{R}^d).$$

As a by-product of our considerations we also obtain $L^1(\mathbb{R}^d)$ estimates for R_V^a, but only for a limited range of a. The reason for this is that we need to use concavity of the function x^a.

Theorem 1.4.10. *Fix* $\alpha > 0$ *and let* V *given by* (1.4.6) *satisfy* (1.4.7). *For* $a \leqslant 1$ *there is a constant* $C > 0$ *depending on* m, M, α *and independent of the dimension* d *such that*

$$\|R_V^a f\|_{L^1(\mathbb{R}^d)} \leqslant C\|f\|_{L^1(\mathbb{R}^d)}, \quad f \in L^1(\mathbb{R}^d).$$

The methods used to prove the above results are similar to the ones in Chapter 4, although we have to be more careful as we want to obtain a dimension-free estimate.

It is interesting to note that we do not need any explicit formulas for the potential V or for the semigroup e^{-tL}. This is in contrast to previous dimension-free result, which addressed only the case of $V(x) = |x|^2$ and of polynomial V.

The particular structure of V (1.4.6) lets us write

$$L = \sum_{i=1}^{d} L_i, \quad \text{where } L_i = -\frac{1}{2}\frac{\partial^2}{\partial x_i^2} + V_i, \tag{1.4.8}$$

and, as a consequence, factorize the semigroup e^{-tL} in the following way

$$e^{-tL} = \prod_{i=1}^{d} e^{-tL_i} \quad \text{and hence} \quad e^{-tL}(\mathbb{1}) = \prod_{i=1}^{d} e^{-tL_i}(\mathbb{1}). \tag{1.4.9}$$

This is the key property allowing us reduce the problem to one-dimensional estimates of the semigroups e^{-tL_i} and as a result to get estimates that does not depend on the dimension d. In Section 5.2 we prove that the one-dimensional semigroups e^{-tL_i} decay exponentially in t and $V(x)$ for small values of t, i.e. we have

$$e^{-tL_i}(\mathbb{1})(x) \leqslant e^{-c_N t V_i(x)} \quad \text{for } t \leqslant N,\ x \in \mathbb{R}^d.$$

It is noteworthy that the constant in front of the exponential in the above estimate is 1, which means that we can multiply one-dimensional bounds to estimate the full semigroup e^{-tL} without constants growing with the dimension. The proof is divided into three cases depending on the value of $|x_i|$ and $tV_i(x)$ but in all of them the main ingredient is the Feynman–Kac formula (1.4.4).

In Section 5.3 we use results from Section 5.2 and a similar result [29, Lemma 3.1] giving an exponential decay of the semigroup for large values of t, namely

$$e^{-tL_i}(\mathbb{1})(x) \leqslant e^{-ct} \quad \text{for } t \geqslant N,\ x \in \mathbb{R}^d,$$

to estimate the L^∞ norm of R_V^a.

Finally in Section 5.4 we estimate the $L^1(\mathbb{R}^d)$ norm of the Riesz transform R_V^a. We use duality between $L^\infty(\mathbb{R}^d)$ and $L^1(\mathbb{R}^d)$ spaces which reduces estimating the $L^1(\mathbb{R}^d)$ norm of the operator $R_V^a = V^a L^{-a}$ to estimating the $L^\infty(\mathbb{R}^d)$ norm of the adjoint operator

$$(L^{-a}V^a)f(x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL}(V^a f)(x)\, t^{a-1}\, dt.$$

Again, using the positivity-preserving property of the semigroup e^{-tL} reduces the task to estimating $e^{-tL}(V^a)$. In this case, although the factorization (1.4.9) of the semigroup as an operator still applies, it does not behave well when the semigroup is applied to V^a instead of the constant function $\mathbb{1}$, hence we use the following formula

$$e^{-tL}(V) = \sum_{i=1}^d e^{-tL}(V_i) = \sum_{i=1}^d e^{-tL^i}(\mathbb{1})\, e^{-tL_i}(V_i), \quad \text{where } L^i = L - L_i,$$

which again allow us to use the one-dimensional estimates to obtain the desired dimension-free result.

1.5 Notation

We finish the introduction with a description of the notation and conventions used in the dissertation.

1. We abbreviate $L^p(\mathbb{R}^d)$ to L^p and $\|\cdot\|_{L^p}$ to $\|\cdot\|_p$. For a sublinear operator T acting on L^p we denote its operator norm by $\|T\|_{p\to p}$. We let $\mathcal{S}$ be the space of Schwartz functions on $\mathbb{R}^d$. Slightly abusing the notation we say that a sublinear operator T is bounded on L^p if it is bounded on $\mathcal{S}$ in the L^p norm.

2. For a Banach space E the symbol $L^p(\mathbb{R}^d; E)$ stands for the space of weakly measurable functions $f\colon \mathbb{R}^d \to E$ with the norm $\|f\|_{L^p(\mathbb{R}^d;E)} = (\int_{\mathbb{R}^d} \|f(x)\|_E^p\, dx)^{1/p}$. Similarly, for a finite set F by $\ell^p(F;E)$ we denote the Banach space of E-valued sequences $\{f_s\}_{s\in F}$ with the norm $\|f\|_{\ell^p(F;E)} = (\sum_{s\in F} \|f_s\|_E^p)^{1/p}$.

3. For an exponent $p \in [1,\infty]$ we let q be its conjugate exponent satisfying
$$\frac{1}{p} + \frac{1}{q} = 1.$$
For $1 < p < \infty$ we also set
$$p^* = \max(p, (p-1)^{-1}).$$

4. The Fourier transform is defined for $f \in L^1$ and $\xi \in \mathbb{R}^d$ by the formula
$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi}\, dx.$$

5. The gamma function is defined for $s > 0$ by the formula
$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt.$$

We will use Stirling's formula [40, 5.6.1]

$$\sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x} \leqslant \Gamma(x) \leqslant \sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x+\frac{1}{12x}}, \qquad x > 0. \tag{1.5.1}$$

and its asymptotic form

$$\Gamma(s) \sim \sqrt{2\pi}s^{s-\frac{1}{2}}e^{-s}, \qquad s \to \infty. \tag{1.5.2}$$

We will also need estimates for the ratio of two gamma functions:

$$\Gamma(s+\alpha) \sim s^{\alpha}\Gamma(s), \qquad s \to \infty, \tag{1.5.3}$$

see [40, 5.11.12], and Gautschi's inequality [40, 5.6.4]

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}, \qquad x > 0,\ s \in (0,1). \tag{1.5.4}$$

6. The symbol S^{d-1} stands for the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$. We also write

$$S_{d-1} = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \tag{1.5.5}$$

to denote the unnormalized surface area of S^{d-1}.

7. For a set A by $\mathbb{1}_A$ we denote its characteristic function. The symbol $\mathbb{1}$ stands for the constant function 1.

Part I

Classical Riesz transforms

Chapter 2

L^2 estimates for the maximal Riesz transform

In this chapter we investigate the Riesz transforms, the maximal Riesz transforms, and their relations, in particular we ask whether it is possible to control the L^p norm of the maximal Riesz transform by the norm of the Riesz transform in a dimension-free manner.

To be more specific, for a Schwartz function f on $\mathbb{R}^d$ let us define the Riesz transform R_j, $j = 1, \dots, d$, by

$$R_j f(x) = \lim_{t \to 0^+} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}} \int_{|x-y|>t} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\, dy. \tag{2.0.1}$$

It is well known that R_j may be defined equivalently by the Fourier transform as

$$\widehat{R_j f}(\xi) = -i \frac{\xi_j}{|\xi|} \widehat{f}(\xi), \quad \xi \in \mathbb{R}^d; \tag{2.0.2}$$

as it will turn out, both definitions are useful in our case. We define also the maximal Riesz transform

$$R_j^* f(x) = \sup_{t>0} \left| R_j^t f(x) \right|,$$

where R_j^t, called the truncated Riesz transform, is given by

$$R_j^t f(x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}} \int_{|x-y|>t} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\, dy.$$

In [37] Mateu and Verdera proved that for $1 < p < \infty$ the L^p norm of $R_j^* f$ can be controlled by the L^p norm of $R_j f$, namely

Theorem (Mateu, Verdera, [37]). *For $1 < p < \infty$ there is a constant $C_{p,d}$ depending on p and d such that*

$$\left\| R_j^* f \right\|_p \leqslant C_{p,d} \| R_j f \|_p. \tag{2.0.3}$$

The main purpose of this chapter is to improve this estimate in the case of $p = 2$ to a dimension-free bound with an explicit constant.

Theorem 2.0.1. *For every $f \in L^2$ we have*

$$\left\| R_j^* f \right\|_2 \leqslant 2 \cdot 10^8 \| R_j f \|_2.$$

Remark. There is nothing special in $2 \cdot 10^8$ and clearly, optimizing our method, one may get a better constant instead. We wrote down an explicit constant in order to get an impression of its magnitude.

Remark. The theorem is true for all dimensions d, however we restrict our proof to the case $d \geqslant 4$ due to technical reasons and from now on we assume that $d \geqslant 4$. The case $1 \leqslant d \leqslant 3$ follows from [37, Theorem 1].

Note that Theorem 2.0.1 combined with Plancherel's theorem and (2.0.2) easily implies a dimension-free bound for the norm of the vector of maximal Riesz transforms on L^2.

Corollary 2.0.2. *For every $f \in L^2$ we have*

$$\left\| \left(\sum_{j=1}^{d} \left| R_j^* f \right|^2 \right)^{1/2} \right\|_2 \leqslant 2 \cdot 10^8 \|f\|_2.$$

As described in Section 1.4, the proof of Theorem 2.0.1 consist of factorization of the truncated Riesz transform $R_j^t = M^t(R_j)$ and then estimating the Fourier multiplier associated with the operator M^t.

Before we move on to the proof, we establish some notation and facts used in this chapter.

1. The symbol K_j^t stands for the kernel of the operator R_j^t which is

$$K_j^t(x) = \gamma \cdot \mathbb{1}_{|x|>t}(x) \frac{x_j}{|x|^{d+1}}, \qquad x \in \mathbb{R}^d,$$

   and γ denotes the constant

$$\gamma = \gamma_d = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}}.$$

2. In the proof of Theorem 2.1.4 we shall need a numerical inequality (see e.g. [38, Lemma 2.5]) which says that for any $n \in \mathbb{Z}$ and continuous function $g\colon [2^n, 2^{n+1}] \to \mathbb{C}$ we have

$$\begin{aligned} &\sup_{t \in [2^n, 2^{n+1}]} |g(t) - g(2^n)| \\ &\leqslant \sqrt{2} \sum_{l=0}^{\infty} \left(\sum_{m=0}^{2^l-1} \left| g(2^n + 2^{n-l}(m+1)) - g(2^n + 2^{n-l}m) \right|^2 \right)^{1/2}. \end{aligned} \tag{2.0.4}$$

3. Lastly, we will need the Poisson semigroup defined for $f \in L^2$ by

$$\widehat{P_t f}(\xi) = p_t(\xi) \widehat{f}(\xi) \qquad \text{with} \qquad p_t(\xi) = e^{-t\frac{|\xi|}{\sqrt{d}}}. \tag{2.0.5}$$

   We denote by $P_*(f)$ and $g(f)$ the maximal function and the square function associated with this semigroup, i.e.

$$P_* f(x) = \sup_{t>0} |P_t f(x)| \quad \text{and} \quad g(f)(x) = \left(\int_0^\infty t \left| \frac{d}{dt} P_t f(x) \right|^2 dt \right)^{1/2}.$$

From [45, pp. 47–51] and [16, Theorem VIII.7.7] we know that for $f \in L^2$ we have

$$\|P_* f\|_2 \leqslant 4\|f\|_2 \quad \text{and} \quad \|g(f)\|_2 \leqslant \frac{1}{\sqrt{2}}\|f\|_2. \tag{2.0.6}$$

We will also need the so-called Poisson projections given by

$$S_n = P_{2^{n-1}} - P_{2^n}, \quad n \in \mathbb{Z}.$$

The sequence $(S_n)_{n\in\mathbb{Z}}$ is then a resolution of the identity on L^2 which means that

$$f = \sum_{n\in\mathbb{Z}} S_n f, \quad f \in L^2. \tag{2.0.7}$$

Moreover, S_n satisfies

$$S_n f(x) = -\int_{2^{n-1}}^{2^n} \frac{d}{dt} P_t f(x)\, dt,$$

so by the Cauchy–Schwartz inequality we have

$$|S_n f(x)|^2 \leqslant 2^{n-1} \int_{2^{n-1}}^{2^n} \left|\frac{d}{dt} P_t f(x)\right|^2 dt \leqslant \int_{2^{n-1}}^{2^n} t \left|\frac{d}{dt} P_t f(x)\right|^2 dt.$$

Now summing over $n \in \mathbb{Z}$ and using (2.0.6) leads us to the conclusion that

$$\left\| \left(\sum_{n\in\mathbb{Z}} |S_n f|^2 \right)^{1/2} \right\|_2 \leqslant \frac{1}{\sqrt{2}}\|f\|_2. \tag{2.0.8}$$

2.1 Factorization

The goal of this section is to prove the factorization $R_j^t f = M^t(R_j f)$ of the truncated Riesz transform. Hence, we begin with the definition of the function which will be a part of the Fourier multiplier associated with the operator M^t. Let $m : [0, +\infty) \to \mathbb{C}$ be the function

$$m(x) = \frac{2^{\frac{d}{2}} \Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}} \int_{2\pi x}^{\infty} r^{-\frac{d}{2}} J_{\frac{d}{2}}(r)\, dr. \tag{2.1.1}$$

For $\operatorname{Re} \nu > -\frac{1}{2}$ the symbol J_ν denotes the Bessel function of the first kind defined by

$$J_\nu(t) = \frac{t^\nu}{2^\nu \Gamma\left(\nu + \frac{1}{2}\right)\sqrt{\pi}} \int_{-1}^{1} e^{its} \left(1 - s^2\right)^{\nu - \frac{1}{2}} ds, \qquad t \geqslant 0, \tag{2.1.2}$$

see e.g. [22, B.1]. It is known that for $\nu \geqslant 0$ the Bessel function satisfies $|J_\nu(t)| \leqslant 1$ (see [40, 10.14.1]) and $|J_\nu(t)| \leqslant C(\nu) t^\nu$ (see [40, 10.14.4]). Using the assumption that $d \geqslant 4$ we thus see that (2.1.1) defines a bounded continuous function on $[0, \infty)$.

Lemma 2.1.1. *For each $t > 0$ the multiplier associated with the j-th truncated Riesz transform defined in* (2.0.1) *equals*

$$\widehat{K_j^t}(\xi) = -i \frac{\xi_j}{|\xi|} m(t|\xi|),$$

where $\xi \in \mathbb{R}^d$, $\xi \neq 0$.

Proof. First observe that $K_j^t(x) = K_j^1\left(\frac{x}{t}\right)t^{-d}$ which means that $\widehat{K_j^t}(\xi) = \widehat{K_j^1}(t\xi)$ and we can focus on $K_j := K_j^1$. Then we write

$$K_j(x) = \gamma x_j \chi_{|x|>1}(x)\frac{1}{|x|^{d+1}} = x_j K(x) \qquad \text{with} \qquad K(x) := \gamma \chi_{|x|>1}(x)\frac{1}{|x|^{d+1}}$$

so that $\widehat{K_j} = -\frac{1}{2\pi i}\partial_j \widehat{K}$. Since K is radial, its Fourier transform $\widehat{K}$ is also radial and has the form $\widehat{K}(\xi) = h(|\xi|)$, where

$$h(x) = 2\pi\gamma\, x^{-\frac{d}{2}+1}\int_1^\infty r^{-\frac{d}{2}-1}J_{\frac{d}{2}-1}(2\pi r x)\,dr, \tag{2.1.3}$$

see e.g. [22, B.5]. Recalling the estimate $\left|J_{d/2-1}(x)\right| \leqslant 1$ we see that for $x > 0$ the integral in (2.1.3) is convergent and the function h is well defined. Since by [40, 10.6.6] the Bessel function satisfies

$$\frac{1}{x}\frac{d}{dx}\frac{J_\alpha(x)}{x^\alpha} = -\frac{J_{\alpha+1}}{x^{\alpha+1}}, \tag{2.1.4}$$

differentiating (2.1.3) for $x > 0$ we obtain

$$h'(x) = -\gamma(2\pi)^{\frac{d}{2}+1}\int_{2\pi x}^\infty r^{-\frac{d}{2}}J_{\frac{d}{2}}(r)\,dr.$$

Passing with the derivative under the integral sign in (2.1.3) can be easily justified with the aid of the Leibniz integral rule. In summary we have proved that

$$\widehat{K_j}(\xi) = -\frac{1}{2\pi i}\frac{\xi_j}{|\xi|}h'(|\xi|) = -i\frac{\xi_j}{|\xi|}\left(-\frac{1}{2\pi}h'(|\xi|)\right)$$

and noticing that $-h'(|\xi|) = 2\pi m(|\xi|)$ completes the reasoning. □

Let M^t, $t > 0$, be defined by

$$\widehat{M^t f}(\xi) = m(t|\xi|)\widehat{f}(\xi), \qquad f \in L^2, \tag{2.1.5}$$

and set

$$M^* f(x) = \sup_{t>0}\left|M^t f(x)\right|. \tag{2.1.6}$$

Since m is a bounded function, Plancherel's theorem implies that M^t defines a bounded operator on L^2. Moreover, since m is continuous, we see that if $f \in \mathcal{S}$, then for each $x \in \mathbb{R}^d$ the mapping $t \mapsto M^t f(x)$ is continuous. In particular for such f the supremum in the definition of $M^* f(x)$ may be restricted to rational numbers, which shows that the function $M^* f(x)$ is Borel measurable.

As a corollary of Lemma 2.1.1 we shall obtain a factorization of R_j^t in terms of M^t which is crucial for our purposes, see Corollary 2.1.3. For its proof we need a lemma on the density of $R_j(L^p)$ in L^p. For $p = 2$ this is an easy consequence of Plancherel's theorem and (2.0.2).

Lemma 2.1.2. *Let $1 < p < \infty$ and $j = 1, \ldots, d$. Then the space $R_j(L^p) \cap \mathcal{S}$ is dense in L^p. In particular $R_j(L^p)$ is dense in L^p.*

Proof. Throughout the proof we fix $1 < p < \infty$ and $j = 1, \dots, d$. It is sufficient to prove that $(R_j)^2(L^p) \cap \mathcal{S}$ is dense in L^p and this is our goal. Here the symbol $(R_j)^2$ denotes the two-fold composition $(R_j)^2 = R_j \circ R_j$.

For $t > 0$ and $f \in L^1 + L^\infty$ denote

$$T_t^j f(x) = (4\pi t)^{-1/2} \int_{\mathbb{R}} \exp\Big(-|x_j - y_j|^2/4t\Big) f(x_1, \dots, x_{j-1}, y_j, x_{j+1}, \dots, x_d)\, dy_j.$$

Then $\{T_t^j\}_{t>0}$ is the heat semigroup on $\mathbb{R}$ applied to the j-th coordinate of $\mathbb{R}^d$. It is a symmetric diffusion semigroup in the sense of Stein [45, Chapter 3]. Applying the Fourier transform for $f \in \mathcal{S}$ we obtain

$$\widehat{T_t^j f}(\xi) = \exp\big(-4\pi^2 t \xi_j^2\big) \widehat{f}(\xi), \qquad \xi \in \mathbb{R}^d. \tag{2.1.7}$$

In particular if $f \in \mathcal{S}$, then also $T_t^j f \in \mathcal{S}$.

Take $f \in \mathcal{S}$. It is easy to show (see [22, Proposition 5.1.17]) that

$$(R_j)^2(\Delta f) = -\partial_j^2 f, \tag{2.1.8}$$

where Δ denotes the Laplacian on $\mathbb{R}^d$. Using the Fourier inversion formula together with (2.1.7) and (2.1.8) we obtain for each $t > 0$

$$T_t^j f - f = \int_0^t T_s^j(\partial_j^2 f)\, ds = -\int_0^t T_s^j((R_j)^2(\Delta f))\, ds = -(R_j)^2 \left(\int_0^t T_s^j(\Delta f)\, ds \right). \tag{2.1.9}$$

The integrals in (2.1.9) are Bochner integrals on L^2. Since $f \in \mathcal{S}$, we see that $\lim_{t\to\infty} T_t^j f = 0$ both a.e. and in the L^2 norm. Now invoking the L^p boundedness of the maximal operator $f \mapsto \sup_{t>0} \left|T_t^j f\right|$ (see [45, Chapter III, Section 3]) and the dominated convergence theorem we deduce that also $\lim_{t\to\infty} \left\|T_t^j f\right\|_p = 0$. Thus, denoting $g_t = \int_0^t T_s^j(\Delta f)\, ds$ and coming back to (2.1.9) we see that $(R_j)^2(g_t) \in \mathcal{S}$ and

$$\lim_{t\to\infty} \left\|(R_j)^2(g_t) - f\right\|_p = 0.$$

Noticing that $g_t \in \mathcal{S}$ we conclude that any $f \in \mathcal{S}$ may be approximated arbitrarily close in the L^p norm by an element of $(R_j)^2(\mathcal{S}) \cap \mathcal{S}$. At this point the density of $\mathcal{S}$ in L^p completes the proof. □

Having proved Lemma 2.1.2 we now have all the ingredients for justifying the factorization. Recall that the operators M^t and M^* are defined by (2.1.5) and (2.1.6), respectively.

Corollary 2.1.3. *Let $j = 1, \dots, d$. Then for each $t > 0$ the truncated Riesz transform factorizes as*

$$R_j^t f = M^t(R_j f), \qquad f \in L^2. \tag{2.1.10}$$

Moreover the maximal operator M^* is bounded on all L^p spaces, $1 < p < \infty$, and the optimal constant C_p in the inequality $\left\|R_j^* f\right\|_p \leqslant C_p \|R_j f\|_p$ equals $\|M^*\|_{p\to p}$.

Proof. Recalling (2.0.2) the decomposition (2.1.10) follows immediately from Lemma 2.1.1.

When studying the L^p boundedness of M^* by Lemma 2.1.2 it suffices to consider M^*g with $g \in R_j(L^p) \cap \mathcal{S}$. Note that for such g the function M^*g is measurable. Clearly, (2.1.10) implies $C_p \leqslant \|M^*\|_{p\to p}$. Applying (2.0.3) we see that M^* is bounded on $R_j(L^p) \cap \mathcal{S}$, which is a dense subset of L^p by Lemma 2.1.2. Thus, using again (2.1.10) we see that $\|M^*\|_{p\to p} \leqslant C_p$. This completes the proof of the corollary. □

By Corollary 2.1.3, Theorem 2.0.1 is equivalent to the following result.

Theorem 2.1.4. *Let M^* be defined as in* (2.1.6)*. Then for every $f \in L^2$ we have*

$$\|M^* f\|_2 \leqslant 2 \cdot 10^8 \|f\|_2.$$

2.2 Multiplier estimates

Now we focus on proving Theorem 2.1.4 and on the operator M^*.

First we prove estimates for the multiplier m. We start with small arguments.

Lemma 2.2.1. *For $0 \leqslant x \leqslant \sqrt{d}$ we have*

$$|m(x) - 1| \leqslant 20 \frac{x}{\sqrt{d}}.$$

Proof. By [40, 10.22.43] we know that $m(0) = 1$, so

$$m(x) - 1 = m(x) - m(0) = -\frac{2^{\frac{d}{2}}\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}} \int_0^{2\pi x} r^{-\frac{d}{2}} J_{\frac{d}{2}}(r)\, dr. \tag{2.2.1}$$

Now [22, B.6] gives

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} + S_\nu(x)$$

with S_ν satisfying

$$|S_\nu(x)| \leqslant \frac{2^{-\nu} x^{\nu+1}}{(\nu+1)\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}.$$

Hence, using (1.5.4) we estimate (2.2.1) as follows (recall that $\frac{x}{\sqrt{d}} \leqslant 1$)

$$\begin{aligned} |m(x) - 1| &\leqslant \frac{2\pi x \Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}+1\right)\sqrt{\pi}} + \frac{1}{\left(\frac{d}{2}+1\right)\pi} \int_0^{2\pi x} r\, dr \\ &\leqslant \frac{2\sqrt{2\pi}x}{\sqrt{d}} + \frac{4\pi x^2}{d} \leqslant 20\frac{x}{\sqrt{d}}. \end{aligned}$$

□

Our estimate for $m(x)$ when x is large will be based on an inequality for the Bessel function J_ν. This is essentially a restatement of [39, Lemma 4.1]. We present the proof in order to keep track of the constants.

Lemma 2.2.2. *For each $t \geqslant 0$ and $\nu \geqslant 0$ we have*

$$|J_\nu(t)| \leqslant \frac{2100\, t^\nu}{2^\nu \Gamma\left(\nu+\frac{1}{2}\right)\sqrt{\nu\pi}} \left(e^{-\frac{t}{\sqrt{\nu}}} + e^{-\frac{\nu}{5}}\right).$$

Proof. Define for $t \geqslant 0$ and $\nu \geqslant 0$

$$M(t) := \sqrt{\nu} \int_{-1}^{1} e^{its\sqrt{\nu}} \left(1 - s^2\right)^{\nu - \frac{1}{2}} \, ds = \int_{-\sqrt{\nu}}^{\sqrt{\nu}} e^{its} \left(1 - \frac{s^2}{\nu}\right)^{\nu - \frac{1}{2}} ds. \tag{2.2.2}$$

Then, using the definition of the Bessel function (2.1.2) we see that

$$J_\nu(t) = \frac{t^\nu}{2^\nu \Gamma\left(\nu + \frac{1}{2}\right) \sqrt{\nu\pi}} M\left(\frac{t}{\sqrt{\nu}}\right).$$

Therefore in order to prove the lemma it suffices to show that

$$|M(t)| \leqslant 2100 \left(e^{-t} + e^{-\frac{\nu}{5}}\right), \quad t \geqslant 0, \tag{2.2.3}$$

and till the end of the proof we focus on justifying (2.2.3).

We begin by splitting the second integral in (2.2.2) into two parts

$$|M(t)| \leqslant \left| \int_{\frac{\sqrt{\nu}}{2} \leqslant |s| \leqslant \sqrt{\nu}} e^{its} \left(1 - \frac{s^2}{\nu}\right)^{\nu - \frac{1}{2}} ds \right| + \left| \int_{|s| \leqslant \frac{\sqrt{\nu}}{2}} e^{its} \left(1 - \frac{s^2}{\nu}\right)^{\nu - \frac{1}{2}} ds \right|. \tag{2.2.4}$$

Then we observe that

$$\left| \int_{\frac{\sqrt{\nu}}{2} \leqslant |s| \leqslant \sqrt{\nu}} e^{its} \left(1 - \frac{s^2}{\nu}\right)^{\nu - \frac{1}{2}} ds \right| \leqslant 2\sqrt{\nu} \left(\frac{3}{4}\right)^{\nu - \frac{1}{2}} \leqslant 6e^{-\frac{\nu}{4}}$$

since $1 - \frac{s^2}{\nu} \leqslant \frac{3}{4}$ for $|s| \geqslant \frac{\sqrt{\nu}}{2}$. This means that we can move on to estimating the second integral in (2.2.4). To do this we will change the contour of integration. However this will work only if $\nu > \frac{4}{3}$, so we take care of $\nu \leqslant \frac{4}{3}$ first. In this case we use (1.5.4) to write

$$\left| \int_{|s| \leqslant \frac{\sqrt{\nu}}{2}} e^{its} \left(1 - \frac{s^2}{\nu}\right)^{\nu - \frac{1}{2}} ds \right| \leqslant \sqrt{\nu} \int_{-1}^{1} \left(1 - s^2\right)^{\nu - \frac{1}{2}} ds = \sqrt{\nu} \frac{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma\left(\nu + 1\right)} \leqslant \sqrt{\pi} \leqslant 3e^{-\frac{\nu}{5}}.$$

Now assume that $\nu > \frac{4}{3}$ and let $C = C_0 \cup C_1 \cup C_2 \cup C_3$ be the rectangle with the parametrization

$$\begin{aligned}
C_0(s) &:= s && \text{for} \quad s \in \left[-\tfrac{\sqrt{\nu}}{2}, \tfrac{\sqrt{\nu}}{2}\right], \\
C_1(s) &:= is + \frac{\sqrt{\nu}}{2} && \text{for} \quad s \in [0, 1], \\
C_2(s) &:= -s + i && \text{for} \quad s \in \left[-\tfrac{\sqrt{\nu}}{2}, \tfrac{\sqrt{\nu}}{2}\right], \\
C_3(s) &:= i(1 - s) - \frac{\sqrt{\nu}}{2} && \text{for} \quad s \in [0, 1].
\end{aligned}$$

The function $z \mapsto e^{itz} \left(1 - \frac{z^2}{\nu}\right)^{\nu - \frac{1}{2}}$ is holomorphic in the disk $\{z \in \mathbb{C} : |z| < \sqrt{\nu}\}$, which for $\nu > \frac{4}{3}$ contains the rectangle C, hence the Cauchy integral theorem gives

$$\begin{aligned}
\left| \int_{|s| \leqslant \frac{\sqrt{\nu}}{2}} e^{its} \left(1 - \frac{s^2}{\nu}\right)^{\nu - \frac{1}{2}} ds \right| &\leqslant \sum_{j \in \{1,3\}} \left| \int_0^1 e^{itC_j(s)} \left(1 - \frac{C_j(s)^2}{\nu}\right)^{\nu - \frac{1}{2}} ds \right| \\
&\quad + \left| \int_{|s| \leqslant \frac{\sqrt{\nu}}{2}} e^{it(i-s)} \left(1 - \frac{(s-i)^2}{\nu}\right)^{\nu - \frac{1}{2}} ds \right|.
\end{aligned}$$

The first term can be estimated as

$$\sum_{j\in\{1,3\}}\left|\int_0^1 e^{itC_j(s)}\left(1-\frac{C_j(s)^2}{\nu}\right)^{\nu-\frac{1}{2}} ds\right| \leqslant 2\left(\frac{3}{4}+\frac{1}{\nu}+\frac{1}{\sqrt{\nu}}\right)^{\nu-\frac{1}{2}} \leqslant 600e^{-\frac{\nu}{5}}.$$

Since $e^{it(i-s)}=e^{-t}e^{-its}$, now it suffices to show that

$$\left|\int_{|s|\leqslant\frac{\sqrt{\nu}}{2}} e^{-its}\left(1-\frac{(s-i)^2}{\nu}\right)^{\nu-\frac{1}{2}} ds\right| \leqslant 2100.$$

Recall that $\nu>\frac{4}{3}$ and observe that

$$\left|1-\frac{(s-i)^2}{\nu}\right| \leqslant 1-\frac{s^2-1}{\nu}+\frac{2|s|}{\nu} \leqslant \begin{cases} 1+\frac{6}{\nu}, & \text{if } |s|\leqslant\frac{5}{2} \\ 1-\frac{s^2}{25\nu} & \text{if } \frac{5}{2}\leqslant|s|\leqslant\frac{\sqrt{\nu}}{2} \end{cases}$$

and thus

$$\begin{aligned}\left|\int_{|s|\leqslant\frac{\sqrt{\nu}}{2}} e^{-its}\left(1-\frac{(s-i)^2}{\nu}\right)^{\nu-\frac{1}{2}} ds\right| &\leqslant 5\left(1+\frac{6}{\nu}\right)^{\nu-\frac{1}{2}}+\int_{|s|\leqslant\frac{\sqrt{\nu}}{2}}\left(1-\frac{s^2}{25\nu}\right)^{\nu-\frac{1}{2}} ds \\ &\leqslant 5e^6+5\sqrt{\nu}\int_{-1}^{1}\left(1-s^2\right)^{\nu-\frac{1}{2}} ds \\ &= 5e^6+5\sqrt{\nu}\frac{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma\left(\nu+1\right)} \\ &\leqslant 5e^6+5\sqrt{\pi}\leqslant 2100.\end{aligned}$$

This completes the proof of (2.2.3) and thus also the proof of Lemma 2.2.2.

□

Applying Lemma 2.2.2 we now justify an estimate of m for large arguments.

Lemma 2.2.3. *For $x\geqslant\sqrt{d}$ we have*

$$|m(x)|\leqslant 6\cdot 10^4\frac{\sqrt{d}}{x}.$$

Proof. We consider two cases. First we take $x\geqslant d$. Recalling that $|J_\nu(x)|\leqslant 1$ and $d\geqslant 4$ we can estimate the integral in (2.1.1) by

$$\left|\int_{2\pi x}^{\infty} r^{-\frac{d}{2}}J_{\frac{d}{2}}(r)\,dr\right| \leqslant \int_{2\pi x}^{\infty} r^{-\frac{d}{2}}\,dr \leqslant 2\frac{(2\pi x)^{1-\frac{d}{2}}}{d-2} \leqslant 4\frac{(2\pi d)^{1-\frac{d}{2}}}{x}.$$

Including the constant in (2.1.1) and using (1.5.1) for $d\geqslant 4$ gives

$$|m(x)|\leqslant 8\frac{(\pi d)^{1-\frac{d}{2}}}{x}\Gamma\left(\frac{d+1}{2}\right)\leqslant\frac{8}{x}\sqrt{2\pi}\pi d(2\pi e)^{-\frac{d}{2}}e^{\frac{1}{6(d+1)}}\leqslant\frac{1}{x}\leqslant\frac{\sqrt{d}}{x}. \tag{2.2.5}$$

The second case is when $\sqrt{d}\leqslant x\leqslant d$. Then the integral in (2.1.1) can be split into two parts: from $2\pi x$ to $2\pi d$ and from $2\pi d$ to infinity; namely

$$m(x)=\frac{2^{\frac{d}{2}}\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}}\left(\int_{2\pi x}^{2\pi d} r^{-\frac{d}{2}}J_{\frac{d}{2}}(r)\,dr+\int_{2\pi d}^{\infty} r^{-\frac{d}{2}}J_{\frac{d}{2}}(r)\,dr\right)=I_1(x)+I_2.$$

The second integral can be estimated as in (2.2.5)

$$|I_2| = |m(d)| \leqslant \frac{\sqrt{d}}{d} \leqslant \frac{\sqrt{d}}{x}.$$

To handle I_1, we use Lemma 2.2.2 which gives

$$\begin{aligned} |I_1(x)| &\leqslant \frac{2100\sqrt{2}}{\pi\sqrt{d}} \left(\int_{2\pi x}^{2\pi d} e^{-\frac{r\sqrt{2}}{\sqrt{d}}}\, dr + \int_{2\pi x}^{2\pi d} e^{-\frac{d}{10}}\, dr \right) \\ &\leqslant \frac{2100}{\pi} e^{-\frac{2\sqrt{2}\pi x}{\sqrt{d}}} + 4200\sqrt{2} d e^{-\frac{d}{10}} \leqslant 6 \cdot 10^4 \frac{\sqrt{d}}{x}. \end{aligned}$$

In the last inequality we used the fact that $e^{-x} \leqslant \frac{1}{x}$ for $x \geqslant 0$. □

We will also need an estimate of the derivative of m.

Lemma 2.2.4. *For all $x \geqslant 0$ we have*

$$\left| x m'(x) \right| \leqslant 10^4.$$

Proof. Differentiating (2.1.1) gives

$$m'(x) = -\frac{2\sqrt{\pi}\Gamma\left(\frac{d+1}{2}\right)}{(\pi x)^{\frac{d}{2}}} J_{\frac{d}{2}}(2\pi x).$$

If $x \geqslant d$, then we can estimate $J_{\frac{d}{2}}$ by 1 and use (1.5.1) to get

$$\left| x m'(x) \right| \leqslant \frac{2\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d-1}{2}} d^{\frac{d}{2}-1}} \leqslant 2\sqrt{2}\pi d (2\pi e)^{-\frac{d}{2}} e^{\frac{1}{6(d+1)}} \leqslant 3.$$

Otherwise, when $x < d$, we use Lemma 2.2.2 which yields

$$\begin{aligned} \left| x m'(x) \right| &\leqslant x \frac{2\sqrt{\pi}\Gamma\left(\frac{d+1}{2}\right)}{(\pi x)^{\frac{d}{2}}} \frac{2100\sqrt{2}\,(2\pi x)^{\frac{d}{2}}}{2^{\frac{d}{2}}\Gamma\left(\frac{d+1}{2}\right)\sqrt{d\pi}} \left(e^{-\frac{2\sqrt{2}\pi x}{\sqrt{d}}} + e^{-\frac{d}{10}} \right) \\ &= 4200\sqrt{2} \frac{x}{\sqrt{d}} \left(e^{-\frac{2\sqrt{2}\pi x}{\sqrt{d}}} + e^{-\frac{d}{10}} \right) \leqslant \frac{2100}{\pi} + \frac{4200\sqrt{10}}{\sqrt{e}} \leqslant 10^4. \end{aligned}$$

□

2.3 Square function estimates

Having established the technical results regarding the multiplier m, we move on to the proof of Theorem 2.1.4. We estimate M^* as follows

$$M^* f = \sup_{t>0} \left| M^t f \right| \leqslant \sup_{n \in \mathbb{Z}} \left| M^{2^n} f \right| + \left(\sum_{n \in \mathbb{Z}} \sup_{t \in [2^n, 2^{n+1}]} \left| M^t f - M^{2^n} f \right|^2 \right)^{1/2}. \tag{2.3.1}$$

To bound the first part, we compare it with the maximal function of the Poisson semigroup

$$\sup_{n \in \mathbb{Z}} \left| M^{2^n} f \right| \leqslant \sup_{n \in \mathbb{Z}} \left| M^{2^n} f - P_{2^n} f \right| + |P_* f|. \tag{2.3.2}$$

Since by (2.0.6) the norm of P_* is bounded on L^2 by 4, to estimate the first term in (2.3.1) it is enough to take care of the first term in (2.3.2).

Theorem 2.3.1. *For every $f \in L^2$ we have*

$$\left\| \sup_{n\in\mathbb{Z}} \left| M^{2^n} f \right| \right\|_2 \leqslant 1.3 \cdot 10^5 \|f\|_2.$$

Proof. As noted above in order to prove the theorem it is enough to show that

$$\left\| \sup_{n\in\mathbb{Z}} \left| M^{2^n} f - P_{2^n} f \right| \right\|_2 \leqslant 1.2 \cdot 10^5 \|f\|_2.$$

Estimating the supremum by the sum and using Plancherel's theorem we arrive at

$$\left\| \sup_{n\in\mathbb{Z}} \left| M^{2^n} f - P_{2^n} f \right| \right\|_2^2 \leqslant \sum_{n\in\mathbb{Z}} \left\| M^{2^n} f - P_{2^n} f \right\|_2^2 = \sum_{n\in\mathbb{Z}} \left\| \widehat{M^{2^n} f} - \widehat{P_{2^n} f} \right\|_2^2.$$

Recall that the multiplier symbol associated with M^t is $m(t|\xi|)$ by (2.1.10) and the one of the Poisson semigroup P_t is $e^{-t\frac{|\xi|}{\sqrt{d}}}$ by its definition (2.0.5). Combining these facts leads to

$$\sum_{n\in\mathbb{Z}} \left\| \widehat{M^{2^n} f} - \widehat{P_{2^n} f} \right\|_2^2 = \sum_{n\in\mathbb{Z}} \left\| \left(m(2^n|\xi|) - e^{-2^n \frac{|\xi|}{\sqrt{d}}} \right) \widehat{f} \right\|_2^2. \tag{2.3.3}$$

Now we need to estimate the expression inside the norm. We split the analysis into two cases in order to use Lemma 2.2.1 and Lemma 2.2.3. First assume that $2^n|\xi| \leqslant \sqrt{d}$. Then by Lemma 2.2.1 and the fact that $1 - e^{-x} \leqslant x$ we have

$$\left| m(2^n|\xi|) - e^{-2^n \frac{|\xi|}{\sqrt{d}}} \right| \leqslant |m(2^n|\xi|) - 1| + \left| e^{-2^n \frac{|\xi|}{\sqrt{d}}} - 1 \right| \leqslant 21 \frac{2^n|\xi|}{\sqrt{d}}. \tag{2.3.4}$$

If, on the other hand, $2^n|\xi| \geqslant \sqrt{d}$, then we use Lemma 2.2.3 and the fact that $e^{-x} \leqslant \frac{1}{x}$ for $x > 0$ to get

$$\left| m(2^n|\xi|) - e^{-2^n \frac{|\xi|}{\sqrt{d}}} \right| \leqslant 6 \cdot 10^4 \frac{\sqrt{d}}{2^n|\xi|}. \tag{2.3.5}$$

Combining (2.3.4) and (2.3.5) gives

$$\left| m(2^n|\xi|) - e^{-2^n \frac{|\xi|}{\sqrt{d}}} \right| \leqslant 6 \cdot 10^4 \min \left(\frac{2^n|\xi|}{\sqrt{d}}, \left(\frac{2^n|\xi|}{\sqrt{d}} \right)^{-1} \right).$$

Squaring, summing over $n \in \mathbb{Z}$, and using the fact that for any $x > 0$ we have

$$\sum_{n\in\mathbb{Z}} \min \left(4^n x, (4^n x)^{-1} \right) \leqslant 4$$

leads to

$$\sum_{n\in\mathbb{Z}} \left| m(2^n|\xi|) - e^{-2^n \frac{|\xi|}{\sqrt{d}}} \right|^2 \leqslant 4 \cdot (6 \cdot 10^4)^2, \qquad \xi \in \mathbb{R}^d.$$

Plugging the inequality above into (2.3.3) finally gives

$$\sum_{n\in\mathbb{Z}} \left\| \left(m(2^n|\xi|) - e^{-2^n \frac{|\xi|}{\sqrt{d}}} \right) \widehat{f} \right\|_2^2 \leqslant (1.2 \cdot 10^5)^2 \left\| \widehat{f} \right\|_2^2 = (1.2 \cdot 10^5)^2 \|f\|_2^2.$$

This completes the proof of Theorem 2.3.1. ∎

Now we estimate the norm of the second term in (2.3.1).

Theorem 2.3.2. *For every $f \in \mathcal{S}$ we have*

$$\left\| \left(\sum_{n\in\mathbb{Z}} \sup_{t\in[2^n,2^{n+1}]} \left| M^t f - M^{2^n} f \right|^2 \right)^{1/2} \right\|_2 \leqslant 1.7 \cdot 10^8 \|f\|_2.$$

Proof. Since $f \in \mathcal{S}$, we see that for each $x \in \mathbb{R}^d$ the function $t \mapsto M^t f(x)$ is continuous. Hence, an application of the numerical inequality (2.0.4) is legitimate. Using this inequality, the resolution of identity (2.0.7) given by S_n, and the triangle inequality on the space $L^2(\ell^2)$ we obtain

$$\left\| \left(\sum_{n\in\mathbb{Z}} \sup_{t\in[2^n,2^{n+1}]} \left| M^t f - M^{2^n} f \right|^2 \right)^{1/2} \right\|_2$$
$$\leqslant \sqrt{2} \sum_{l=0}^{\infty} \sum_{k\in\mathbb{Z}} \left\| \left(\sum_{n\in\mathbb{Z}} \sum_{m=0}^{2^l-1} \left| \left(M^{2^n+2^{n-l}(m+1)} - M^{2^n+2^{n-l}m} \right) S_{k+n} f \right|^2 \right)^{1/2} \right\|_2. \tag{2.3.6}$$

Then we estimate the norm in the above expression in two ways.

Similarly to the previous proof, by Plancherel's theorem we have

$$\left\| \left(\sum_{n\in\mathbb{Z}} \sum_{m=0}^{2^l-1} \left| \left(M^{2^n+2^{n-l}(m+1)} - M^{2^n+2^{n-l}m} \right) S_{k+n} f \right|^2 \right)^{1/2} \right\|_2^2$$
$$= \sum_{n\in\mathbb{Z}} \sum_{m=0}^{2^l-1} \left\| \left(m\left((2^n + 2^{n-l}(m+1))|\xi| \right) - m\left((2^n + 2^{n-l}m)|\xi| \right) \right) \cdot \left(e^{-2^{n+k}\frac{|\xi|}{\sqrt{d}}} - e^{-2^{n+k-1}\frac{|\xi|}{\sqrt{d}}} \right) \widehat{f} \right\|_2^2$$

We estimate the first factor in the norm using Lemmas 2.2.1 and 2.2.3

$$\left| m\left((2^n + 2^{n-l}(m+1))|\xi| \right) - m\left((2^n + 2^{n-l}m)|\xi| \right) \right| \leqslant 3 \cdot 10^5 \min\left(\frac{2^n|\xi|}{\sqrt{d}}, \left(\frac{2^n|\xi|}{\sqrt{d}} \right)^{-1} \right)$$

and the second one by

$$\left| e^{-2^{n+k}\frac{|\xi|}{\sqrt{d}}} - e^{-2^{n+k-1}\frac{|\xi|}{\sqrt{d}}} \right| \leqslant 3 \min\left(\frac{2^{n+k}|\xi|}{\sqrt{d}}, \left(\frac{2^{n+k}|\xi|}{\sqrt{d}} \right)^{-1} \right).$$

The product of the right-hand sides of the two inequalities above can be further estimated by $10^6 \cdot 2^{-|k|}$, which gives

$$\left\| \left(\sum_{n\in\mathbb{Z}} \sum_{m=0}^{2^l-1} \left| \left(M^{2^n+2^{n-l}(m+1)} - M^{2^n+2^{n-l}m} \right) S_{k+n} f \right|^2 \right)^{1/2} \right\|_2^2$$
$$\leqslant 10^{12} \cdot 2^{-|k|} 2^l \int_{\mathbb{R}^d} \sum_{n\in\mathbb{Z}} \min\left(\frac{2^n|\xi|}{\sqrt{d}}, \left(\frac{2^n|\xi|}{\sqrt{d}} \right)^{-1} \right) \left| \widehat{f}(\xi) \right|^2 d\xi \leqslant (2\cdot 10^6)^2 2^{-|k|} 2^l \|f\|_2^2. \tag{2.3.7}$$

For the second way of estimating (2.3.6) note that Lemma 2.2.4 implies

$$\begin{aligned}&\left| m\left((2^n+2^{n-l}(m+1))|\xi|\right) - m\left((2^n+2^{n-l}m)|\xi|\right)\right| \\ &\leqslant \int_{2^n+2^{n-l}m}^{2^n+2^{n-l}(m+1)} \left|t|\xi|m'(t|\xi|)\right| \frac{dt}{t} \leqslant 10^4\cdot 2^{-l}.\end{aligned}$$

We use the above inequality, Plancherel's theorem and (2.0.8) to continue (2.3.6) as follows

$$\begin{aligned}&\left\| \left(\sum_{n\in\mathbb{Z}} \sum_{m=0}^{2^l-1} \left| \left(M^{2^n+2^{n-l}(m+1)} - M^{2^n+2^{n-l}m}\right) S_{k+n} f\right|^2 \right)^{1/2} \right\|_2^2 \\ &\leqslant 10^8\cdot 2^{-l} \sum_{n\in\mathbb{Z}} \|S_{k+n}f\|_2^2 \leqslant 10^8\cdot 2^{-l}\|f\|_2^2. \end{aligned} \tag{2.3.8}$$

Putting (2.3.7) and (2.3.8) together we reach

$$\begin{aligned}\left\| \left(\sum_{n\in\mathbb{Z}} \sup_{t\in[2^n,2^{n+1}]} \left| M^t f - M^{2^n} f\right|^2 \right)^{1/2} \right\|_2 &\leqslant 2\cdot 10^6\sqrt{2}\sum_{l=0}^{\infty}\sum_{k\in\mathbb{Z}} 2^{-\frac{l}{2}} \min\left(1, 2^{l-\frac{|k|}{2}}\right)\|f\|_2 \\ &\leqslant 1.7\cdot 10^8 \|f\|_2.\end{aligned}$$

The proof of Theorem 2.3.2 is completed. □

In the light of (2.3.1), Theorem 2.3.1 and Theorem 2.3.2, the proof of Theorem 2.0.1 is concluded.

Chapter 3

L^p estimates for the higher order maximal Riesz transform

In this chapter we employ the method of rotations in order to generalize the results of Chapter 2 to higher order Riesz transforms and to L^p spaces for $1 < p < \infty$. First, we use the real method of rotations to establish the estimates for odd order Riesz transforms and then we use complex method of rotations, which is more involved and requires some additional steps in the proof, to include also the case of even order Riesz transforms.

Fix a positive integer k and denote by $\mathcal{H}_k = \mathcal{H}_k^d$ the space of spherical harmonics of degree k on the Euclidean unit sphere S^{d-1}. Throughout the chapter we identify $P \in \mathcal{H}_k$ with the corresponding solid spherical harmonic. Via this identification $P \in \mathcal{H}_k$ is a harmonic polynomial on $\mathbb{R}^d$ which is homogeneous of degree k, i.e. satisfies $P(x) = |x|^k P(x/|x|)$, $x \in \mathbb{R}^d$.

For $P \in \mathcal{H}_k$ the Riesz transform $R = R_P$ is defined by the kernel

$$K_P(x) = K(x) = \gamma_k \frac{P(x)}{|x|^{d+k}} \qquad \text{with} \qquad \gamma_k = \frac{\Gamma\left(\frac{k+d}{2}\right)}{\pi^{d/2}\Gamma\left(\frac{k}{2}\right)}, \tag{3.0.1}$$

more precisely,

$$R_P f(x) = \lim_{t \to 0^+} R_P^t f(x), \qquad \text{where} \qquad R_P^t f(x) = \gamma_k \int_{|x-y|>t} \frac{P(x-y)}{|x-y|^{d+k}} f(y)\, dy. \tag{3.0.2}$$

The operator R_P^t is called the truncated Riesz transform. In the particular case of $k = 1$ and $P_j(x) = x_j$ the operators R_{P_j}, $j = 1, \ldots, d$, coincide with the classical first order Riesz transforms studied in Chapter 2. It is well known, see [46, p. 73], that the Fourier multiplier associated with the Riesz transform R_P equals

$$m_P(\xi) = (-i)^k \frac{P(\xi)}{|\xi|^k}, \qquad \xi \in \mathbb{R}^d. \tag{3.0.3}$$

By the above formula m_P is bounded and Plancherel's theorem implies the L^2 boundedness of R_P. The L^p boundedness of the single Riesz transforms R_P for $1 < p < \infty$ follows from the Calderón–Zygmund method of rotations [8].

Similarly to the previous chapter, the main object of investigation is the maximal Riesz transform defined by

$$R_P^* f(x) = \sup_{t>0} \left| R_P^t f(x) \right|.$$

The research was inspired by results of Mateu, Orobitg, Pérez and Verdera [35], [36], who proved the following result.

Theorem (Mateu, Orobitg, Pérez, Verdera, [35, 36]). *For $1 < p < \infty$ there is a constant $C_{p,k,d}$ depending on p, k and d such that*

$$\|R_P^* f\|_p \leqslant C_{p,k,d} \|R_P f\|_p.$$

In this chapter we improve the above theorem by estimating the constant $C_{p,k,d}$ independently of the dimension d. Our results are summarized in the following two theorems.

Theorem 3.0.1. *Take $p \in (1, \infty)$ and let k be a non-negative integer. Let $\mathcal{P}_k$ be a subset of $\mathcal{H}_k$. Then there is a constant $A(p,k)$ independent of the dimension d and such that*

$$\left\| \left(\sum_{P \in \mathcal{P}_k} |R_P^* f|^2 \right)^{1/2} \right\|_p \leqslant A(p,k) \left\| \left(\sum_{P \in \mathcal{P}_k} |R_P f|^2 \right)^{1/2} \right\|_p ,$$

where $f \in L^p$. Moreover, for fixed k we have

$$A(p,k) = O(p^{5/2+k/2}) \quad \text{as} \quad p \to \infty \qquad \text{and} \qquad A(p,k) = O((p-1)^{-5/2-k/2}) \quad \text{as} \quad p \to 1.$$

This theorem is a generalization of Theorem 2.0.1 from Chapter 2 to higher order Riesz transforms and to L^p spaces for $1 < p < \infty$. In particular, if $\mathcal{P}_k$ contains one element P, then Theorem 3.0.1 immediately gives

$$\|R_P^* f\|_p \leqslant A(p,k) \|R_P f\|_p.$$

In this case however, we can slightly improve the constant $A(p,k)$.

Theorem 3.0.2. *Take $p \in (1, \infty)$ and let k be a non-negative integer. Let P be a spherical harmonic of degree k. Then there is a constant $B(p,k)$ independent of the dimension d and such that*

$$\|R_P^* f\|_p \leqslant B(p,k) \|R_P f\|_p,$$

where $f \in L^p$. Moreover, for fixed k we have

$$B(p,k) = O(p^{2+k/2}) \quad \text{as} \quad p \to \infty \qquad \text{and} \qquad B(p,k) = O((p-1)^{-2-k/2}) \quad \text{as} \quad p \to 1.$$

Combination of Theorem 3.0.1 and a result of Duoandikoetxea and Rubio de Francia [17, Théorème 2] yields a generalization of Corollary 2.0.2. Denote by $a(d,k)$ the dimension of $\mathcal{H}_k$ and let $\{Y_j\}_{j=1,\dots,a(d,k)}$ be an orthogonal basis of $\mathcal{H}_k$ normalized by the condition

$$\int_{S^{d-1}} |Y_j(\omega)|^2 \, d\omega = \frac{1}{a(d,k)},$$

where $d\omega$ is the uniform probabilistic measure on S^{d-1}. Then we have

Corollary 3.0.3. *Take $p \in (1,\infty)$ and let k be a non-negative integer. Then there is a constant $G(p,k)$ independent of the dimension d and such that*

$$\left\| \left(\sum_{j=1}^{a(d,k)} \left| R^*_{Y_j} f \right|^2 \right)^{1/2} \right\|_p \leqslant G(p,k) \|f\|_p,$$

where $f \in L^p$. Moreover, for fixed and odd k we have

$$G(p,k) = O(p^{7/2+k/2}) \quad \text{as} \quad p \to \infty \qquad \text{and} \qquad G(p,k) = O((p-1)^{-7/2-k}) \quad \text{as} \quad p \to 1$$

and for even k we have

$$G(p,k) = O(p^{9/2+k/2}) \quad \text{as} \quad p \to \infty \qquad \text{and} \qquad G(p,k) = O((p-1)^{-9/2-k}) \quad \text{as} \quad p \to 1.$$

Interestingly, we are not aware of any way of proving Corollary 3.0.3 which does not use Theorem 3.0.1.

In the case of odd k the proof is simpler and consists of three steps

1. We factorize the operator R^t_P into $R^t_P = M^t_k(R_P)$.
2. We express the operator M^t_k in terms of Riesz transforms as
$$M^t_k = (-1)^k \sum_{j=1}^{a(d,k)} R^t_{Y_j} R_{Y_j}.$$
3. We use the real method of rotations to express M^t_k in terms of the Hilbert transform and we estimate the $L^p(\mathbb{R}^d)$ norm of M^t_k.

In the case of general integer k we use the complex method of rotations, which requires an additional step of extending the operators from $\mathbb{R}^d$ to $\mathbb{C}^d$ and then restricting them back to $\mathbb{R}^d$. The steps of the proof are as follows.

1. We factorize the operator R^t_P into $R^t_P = M^t_k(R_P)$.
2. We express the operator M^t_k in terms of Riesz transforms as
$$M^t_k f(x) = C(d,k) \int_{SO(d)} \sum_{j \in I} (R^t_{P_j} R_{P_j} f)_U(x)\, d\mu(U). \tag{3.0.4}$$
Note that in the even case we use P_j instead of Y_j, since it is not clear to us whether the functions Y_j remain orthogonal after extension to $\mathbb{C}^d$.
3. We extend the operator $R^t = \sum_{j \in I} R^t_{P_j} R_{P_j}$ on $\mathbb{R}^d$ to the operator $\widetilde{R}^t$ on $\mathbb{C}^d$ and apply the complex method of rotations of Iwaniec and Martin [26] in order to express $\widetilde{R}^t$ in terms of the complex Hilbert transform. Then we estimate the $L^p(\mathbb{C}^d)$ norm of operator R^t.
4. We deduce the estimates for R^t from the estimates for $\widetilde{R}^t$.

Before we move on to the proof, we establish some notation specific to this chapter.

1. The letters d and k stand for the dimension and for the order of the Riesz transforms, respectively.

2. For $k \in \mathbb{N}$ we let $\mathcal{D}(k)$ be the linear span of $\{R_P(f) : P \in \mathcal{H}_k, f \in \mathcal{S}\}$. Since R_P is bounded on L^p for $1 < p < \infty$, the space $\mathcal{D}(k)$ is then a subspace of each of the L^p spaces.

3. The symbol C_Δ stands for a constant that possibly depends on $\Delta > 0$. We write C without a subscript when the constant is universal in the sense that it may depend only on k but not on the dimension d nor on any other quantity.

4. For two quantities X and Y we write $X \lesssim_\Delta Y$ if $X \leqslant C_\Delta Y$ for some constant $C_\Delta > 0$ that depends only on Δ. We abbreviate $X \lesssim Y$ when C is a universal constant. We also write $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold simultaneously. By $X \lesssim^\Delta Y$ we mean that $X \leqslant C^\Delta Y$ with a universal constant C. Note that in this case $X^{1/\Delta} \lesssim Y^{1/\Delta}$.

5. By ω we denote the uniform measure on S^{d-1} normalized by the condition $\omega(S^{d-1}) = 1$. By σ we denote the uniform measure on S^{d-1} normalized by the condition $\sigma(S^{d-1}) = S_{d-1}$.
   We write ζ for the uniform measure on S^{2d-1} normalized by the condition $\zeta(S^{2d-1}) = 1$.
   We write θ for the uniform measure on S^{2d-1} normalized by the condition $\theta(S^{2d-1}) = S_{2d-1}$.

6. We let
$$\gamma_k = \gamma_{k,d} := \frac{\Gamma\left(\frac{k+d}{2}\right)}{\pi^{d/2}\Gamma\left(\frac{k}{2}\right)} \qquad \text{and} \qquad \widetilde{\gamma}_k = \gamma_{k,2d} = \frac{\Gamma\left(d+\frac{k}{2}\right)}{\pi^{d}\Gamma\left(\frac{k}{2}\right)}. \tag{3.0.5}$$

7. We will also need the following formula
$$2\int_0^\infty \frac{r^{d-1}}{(1+r^2)^{d+\alpha}}\,dr = B\left(\frac{d}{2}, \frac{d}{2}+\alpha\right) = \frac{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}+\alpha)}{\Gamma(d+\alpha)}, \tag{3.0.6}$$
   valid for $\alpha \geqslant 0$. This follows from change of variables $r^2 \to r$ followed by formulas for Euler's Beta function $B(a,b)$ from [40, 5.12.1, 5.12.3].

3.1 Factorization

The goal of this section is to show that a factorization formula similar to the one used in Chapter 2 exists also for higher order Riesz transforms R_P^t. Proposition below is implicit in [35, Section 4] and [36, pp. 1435–1436].

Proposition 3.1.1. *Let $k \in \mathbb{N}$. Then there exists a family of operators M_k^t, $t > 0$, which are bounded on L^p, $1 < p < \infty$, and such that for all $P \in \mathcal{H}_k$ we have*
$$R_P^t f = M_k^t(R_P f), \tag{3.1.1}$$
where $f \in L^p$. Each M_k^t is a convolution operator with a radial convolution kernel b_k^t. Moreover, when $P \in \mathcal{H}_k$ and $f \in \mathcal{S}$, then for a.e. $x \in \mathbb{R}^d$ the function $t \mapsto M_k^t(R_P f)(x)$ is continuous on $(0, \infty)$.

Proof. We consider separately the cases of k odd or even starting with k odd.

Let $N = \frac{k-1}{2}$ and denote by B the open Euclidean ball of radius 1 in $\mathbb{R}^d$. Similarly to the proof in [35, pp. 3674–3675], we want to show that the function

$$b(x) = b_{k,d}(x) := \sum_{j=1}^{d} R_j\left(y_j \cdot h(y)\right)(x), \tag{3.1.2}$$

where

$$h(y) = c_d \frac{1}{|y|^{d+1}} \mathbb{1}_{B^c}(y) + (\beta_1 + \beta_2|y|^2 + \cdots + \beta_N|y|^{2N-2})\mathbb{1}_B(y)$$

satisfies the formula

$$R_P(b)(x) = K_P(x)\mathbb{1}_{B^c}(x). \tag{3.1.3}$$

Here $\beta_1, \ldots, \beta_N$ and c_d are constants which depend only on k and d and whose exact values are irrelevant for our considerations, and K_P, R_P are defined in (3.0.1), (3.0.2), respectively, and R_j is the j-th first-order Riesz transform. The important point is that (3.1.3) remains true for any $P \in \mathcal{H}_k$. We provide a sketch of the proof of (3.1.3) contained in [35, pp. 3674–3675] for the reader's convenience.

Throughout the proof C stands for any constant depending only on k and d. Consider the fundamental solution of $(-\Delta)^{1/2}\Delta^N$, that is, a function E such that

$$(-\Delta)^{1/2}\Delta^N E = \delta,$$

where δ is the Dirac delta at the origin. One can take E as a solution of

$$\Delta^N E = \frac{C}{|x|^{d-1}},$$

where the constant C is chosen so that $\widehat{\frac{C}{|x|^{d-1}}}(\xi) = \frac{1}{|\xi|}$. Consider the function

$$\varphi(x) = E(x)\mathbb{1}_{B^c}(x) + \left(A_0 + A_1|x|^2 + \cdots + A_{2N}|x|^{4N}\right)\mathbb{1}_B(x),$$

where the constants $A_0, A_1, \ldots, A_{2N}$ are chosen so that the derivatives of φ up to order $2N$ extend continuously to the boundary of B. Then, in computing the distributional derivatives of φ, one can apply $2N+1$ times the Green–Stokes' theorem and the boundary terms will vanish. This yields

$$\begin{aligned}(-\Delta)^{1/2}\Delta^N\varphi(x) &= (-\Delta)^{1/2}\left(\frac{C}{|x|^{d-1}}\mathbb{1}_{B^c}(x) + \left(\alpha_0 + \alpha_1|x|^2 + \cdots + \alpha_N|x|^{2N}\right)\mathbb{1}_B(x)\right)\\ &= \sum_{j=1}^{d} R_j\left(C\frac{x_j}{|x|^{d+1}}\mathbb{1}_{B^c}(x) + x_j\left(\beta_1 + \beta_2|x|^2 + \cdots + \beta_N|x|^{2N-2}\right)\mathbb{1}_B(x)\right) =: b(x),\end{aligned}$$

where the last identity is the definition of b. Since

$$\varphi = E * (-\Delta)^{1/2}\Delta^N\varphi,$$

taking derivatives of both sides we obtain

$$P(\partial)\varphi = P(\partial)E * (-\Delta)^{1/2}\Delta^N\varphi.$$

To compute $P(\partial)E$ we take the Fourier transform

$$\widehat{P(\partial)E}(\xi) = P(2\pi i\xi)\widehat{E}(\xi) = C\frac{P(\xi)}{|\xi|^k}.$$

On the other hand, it is well known, see [46, p. 73], that

$$\widehat{\frac{P(x)}{|x|^{d+k}}}(\xi) = C\frac{P(\xi)}{|\xi|^k}.$$

We conclude that we have

$$P(\partial)E(x) = C\frac{P(x)}{|x|^{d+k}}.$$

Thus

$$P(\partial)\varphi = C\frac{P(x)}{|x|^{d+k}} * (-\Delta)^{1/2}\Delta^N\varphi = CR_P(b).$$

The only thing left is the computation of $P(\partial)\varphi$. We have, by [36, Corollary 2], that

$$P(\partial)\varphi = CK_P(x)\mathbb{1}_{B^c}(x) + P(\partial)\left(A_0 + A_1|x|^2 + \cdots + A_{k-1}|x|^{2k-2}\right)\mathbb{1}_B(x)$$

To finish the proof, we need to show that

$$P(\partial)(|x|^{2j}) = 0, \quad \text{for } 1 \leqslant j \leqslant k-1, \tag{3.1.4}$$

which will let us write

$$CR_P(b) = P(\partial)\varphi = CK_P\mathbb{1}_{B^c}.$$

Taking the Fourier transform of both sides of (3.1.4) gives

$$\widehat{P(\partial)(|x|^{2j})} = c_jP(\xi)\Delta^j\delta,$$

where c_j is a constant depending on j and d. Let ψ be a test function. Then, since P is harmonic, we get

$$\left\langle P\Delta^j\delta, \psi\right\rangle = \left\langle \Delta^j\delta, P\psi\right\rangle = \left\langle \Delta^{j-1}\delta, 2\nabla P\cdot\nabla\psi + P\Delta\psi\right\rangle.$$

Iterating this computation we obtain

$$\left\langle P\Delta^j\delta, \psi\right\rangle = \langle\delta, D\rangle = D(0),$$

where D is a linear combination of products of the form $\partial^\alpha\psi\cdot\partial^\beta P$ with multi-indices β of length $|\beta| \leqslant j \leqslant d-1$. Therefore $\partial^\beta P$ is a homogeneous polynomial of degree at least $d-j \geqslant 1$, and so $\partial^\beta P(0) = 0$. This yields $D(0) = 0$ and completes the proof of (3.1.4) and (3.1.3).

Denote by H the radial profile of the Fourier transform of h, i.e. $H(|\xi|) = \widehat{h}(\xi)$ for $\xi \in \mathbb{R}^d$. By taking the Fourier transform of (3.1.2) it is straightforward to see that b is a radial function. This follows since the multiplier symbol of R_j is $-i\frac{\xi_j}{|\xi|}$ and

$$(\widehat{y_jh(y)})(\xi) = \frac{\xi_j}{-2\pi i|\xi|}H'(|\xi|),$$

so that

$$\mathcal{F}b(\xi) = \sum_{j=1}^{d} \frac{\xi_j^2}{2\pi|\xi|^2} \cdot H'(|\xi|) = \frac{1}{2\pi} H'(|\xi|)$$

is indeed radial and so is b.

Let $b^t(x) = b_k^t(x) := t^{-d} b(\frac{x}{t})$ be the L^1 dilation of b; clearly b^t is still radial. The dilation invariance of R_P together with (3.1.3) leads us to the expression

$$K_P(x)\mathbb{1}_{B^c}(\tfrac{x}{t}) = R_P(b^t)(x). \tag{3.1.5}$$

Let M_k^t be the convolution operator

$$M_k^t f(x) = b^t * f(x).$$

It follows from [35, Section 4] that M_k^t is bounded on L^p spaces whenever $1 < p < \infty$. Moreover, in view of (3.1.5) we see that

$$R_P^t f = R_P(b^t) * f = b^t * R_P(f) = M_k^t(R_P f).$$

Finally, for $f \in \mathcal{S}$, $P \in \mathcal{H}_k$, and $x \in \mathbb{R}^d$ the mapping $t \mapsto R_P^t f(x)$ is continuous on $(0, \infty)$. Thus, also $M_k^t(R_P f)(x)$ is a continuous function of $t > 0$ for a.e. x. This completes the proof of the proposition in the case when k is odd.

It remains to consider k even. Denote $N = \frac{k}{2}$. Then, as in the odd case, we show that that the function

$$b(x) = b_{k,d}(x) := (\alpha_0 + \alpha_1|x|^2 + \cdots + \alpha_{N-1}|x|^{2(N-1)})\mathbb{1}_B(x)$$

satisfies the formula

$$R_P(b)(x) = K_P(x)\mathbb{1}_{B^c}(x). \tag{3.1.6}$$

Here $\alpha_1, \ldots, \alpha_{N-1}$ are constants which depend only on k and d and whose exact value is irrelevant for our considerations. As in the case of odd k, the important point is that (3.1.6) remains true for any $P \in \mathcal{H}_k$.

The proof of (3.1.6) is similar to the proof of (3.1.3) except that we start with the fundamental solution of Δ^N instead of $(-\Delta)^{1/2}\Delta^N$. The result is that the function b does not feature Riesz transforms R_j. Details can be found in [36, pp. 1435–1436].

Using (3.1.6) we proceed as in the proof in the case when k is odd. Let $b^t(x) = b_k^t(x) := t^{-d} b(\frac{x}{t})$ be the L^1 dilation of b. Since b is clearly radial the same is true of b^t. Let M_k^t be the convolution operator

$$M_k^t f(x) = b^t * f(x).$$

It follows from [36, Section 2] that M_k^t is bounded on L^p spaces whenever $1 < p < \infty$. Moreover, in view of (3.1.6) we see that

$$R_P^t f = R_P(b^t) * f = b^t * R_P(f) = M_k^t(R_P f).$$

Moreover, for $f \in \mathcal{S}$, $P \in \mathcal{H}_k$, and $x \in \mathbb{R}^d$ the mapping $t \mapsto R_P^t f(x)$ is continuous on $(0, \infty)$ and therefore so is $t \mapsto M_k^t(R_P f)(x)$. This completes the proof of the proposition. □

As a corollary of Proposition 3.1.1 we see that in order to justify Theorems 3.0.1 and 3.0.2 it suffices to control vector- and scalar-valued maximal functions corresponding to the operators M_k^t. Note that by Proposition 3.1.1 for $f \in \mathcal{D}(k)$ we have

$$\sup_{t>0} \left|M_k^t f(x)\right| = \sup_{t\in\mathbb{Q}_+} \left|M_k^t f(x)\right|.$$

In particular $\sup_{t>0} \left|M_k^t f(x)\right|$ is measurable for such f, although possibly being infinite for some x. Define

$$M^* f(x) = \sup_{t\in\mathbb{Q}_+} \left|M_k^t f(x)\right|. \tag{3.1.7}$$

Proposition 3.1.1 reduces our task to proving the following two theorems.

Theorem 3.1.2. *Fix $k \in \mathbb{N}$. For each $p \in (1,\infty)$ there is a constant $A(p,k)$ independent of the dimension d and such that for any $S \in \mathbb{N}$ we have*

$$\left\|\left(\sum_{s=1}^{S} |M^* f_s|^2\right)^{1/2}\right\|_p \leqslant A(p,k)\left\|\left(\sum_{s=1}^{S} |f_s|^2\right)^{1/2}\right\|_p,$$

where $f_1,\ldots,f_S \in L^p$. Furthermore $A(p,k)$ satisfies $A(p,k) \lesssim_k (p^*)^{5/2+k/2}$.

Theorem 3.1.3. *Fix $k \in \mathbb{N}$. For each $p \in (1,\infty)$ there is a constant $B(p,k)$ independent of the dimension d and such that*

$$\|M^* f\|_p \leqslant B(p,k)\|f\|_p,$$

whenever $f \in L^p$. Moreover $B(p,k)$ satisfies $B(p,k) \lesssim_k (p^*)^{2+k/2}$.

At this point the proof splits into the odd k case and the general case. We first deal with odd k, however the proofs are independent and the reader may wish to skip directly to Section 3.3 where the general case begins.

Until the end of Section 3.2 k is a fixed odd integer. In proving the above theorems we shall need a useful expression for M_k^t. The next two propositions let us express M_k^t in terms of the Riesz transforms and truncated Riesz transforms.

In what follows we denote by $a(d,k)$ the dimension of $\mathcal{H}_k$. For further reference we recall the formula

$$\dim \mathcal{H}_k = a(d,k) = \binom{d+k-1}{k} - \binom{d+k-3}{k-2} \approx d^k.$$

We will also need an orthogonal basis $\{Y_j\}_{j=1,\ldots,a(d,k)}$ of the space $\mathcal{H}_k$ normalized by the condition

$$\int_{S^{d-1}} |Y_j(\omega)|^2\, d\omega = \frac{1}{a(d,k)} \approx d^{-k}. \tag{3.1.8}$$

Lemma 3.1.4. *Let $\{Y_j\}_{j=1,\ldots,a(d,k)}$ be as above. Then we have*

$$f = (-1)^k \sum_{j=1}^{a(d,k)} (R_{Y_j})^2 f, \qquad f \in L^2. \tag{3.1.9}$$

Proof. Let $\widetilde{Y}_j = \left(\frac{a(d,k)}{S_{d-1}}\right)^{1/2} Y_j$ so that $\int_{S^{d-1}} |\widetilde{Y}_j(\sigma)|^2\, d\sigma = 1$, where $d\sigma$ denotes the uniform measure on S^{d-1} normalized by the condition $\sigma(S^{d-1}) = S_{d-1}$. Using [48, Corollary 2.9 b), p. 144] we see that for all $\theta \in S^{d-1}$ it holds

$$\sum_{j=1}^{a(d,k)} \widetilde{Y}_j(\theta)^2 = \frac{a(d,k)}{S_{d-1}}, \quad \text{so that} \quad \sum_{j=1}^{a(d,k)} Y_j(\theta)^2 = 1. \tag{3.1.10}$$

Taking the Fourier transform of both sides of (3.1.9) and using the formula (3.0.3) we are left with showing that

$$\sum_{j=1}^{a(d,k)} Y_j(\xi)^2 = |\xi|^{2k}, \qquad \xi \in \mathbb{R}^d.$$

The above equality follows from (3.1.10) and the homogeneity $Y_j(\xi) = |\xi|^k \cdot Y_j(\xi/|\xi|)$. This completes the proof of the lemma. □

Proposition 3.1.5. *Let $k \in \mathbb{N}_{\text{odd}}$ and $t > 0$ and take the basis $\{Y_j\}_{j=1}^{a(d,k)}$ normalized as in* (3.1.8)*. Then, for all $f \in L^p$, $1 < p < \infty$, we have*

$$M_k^t f = (-1)^k \sum_{j=1}^{a(d,k)} R_{Y_j}^t R_{Y_j} f. \tag{3.1.11}$$

Proof. We apply Proposition 3.1.1 to the functions $R_{Y_j} f$, $f \in L^p$, and then sum over $j = 1, \ldots, a(d,k)$. This gives

$$M_k^t \sum_{j=1}^{a(d,k)} (R_{Y_j})^2 f = \sum_{j=1}^{a(d,k)} R_{Y_j}^t R_{Y_j} f,$$

and together with Lemma 3.1.4 completes the proof of the proposition. □

3.2 The real method of rotations

In this section we apply the method of rotations to the Riesz transforms $R_{Y_j}^t$ in order to express them in terms of the Hilbert transform, which is a one-dimensional operator, hence giving us a dimension-free estimate for the Riesz transforms.

We apply the method of rotations, specifically [22, 5.2.20], to the operator $R_{Y_j}^t$ with the kernel $K_{Y_j} \mathbb{1}_{|x|\geqslant t}$. Since the kernel is odd, the application of the method of rotations is legitimate. It yields

$$R_{Y_j}^t f(x) = \frac{\pi\gamma_k}{2} \int_{S^{d-1}} Y_j(\sigma) H_\sigma^t f(x)\, d\sigma, \tag{3.2.1}$$

where H_σ^t is the truncated directional Hilbert transform given by

$$H_\sigma^t f(x) = \frac{1}{\pi} \int_{|y|>t} \frac{f(x - y\sigma)}{y}\, dy.$$

In terms of the normalized surface measure $d\omega$ equality (3.2.1) becomes

$$R_{Y_j}^t f(x) = \frac{\pi\Gamma\left(\frac{k+d}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{d}{2}\right)} \int_{S^{d-1}} Y_j(\omega) H_\omega^t f(x)\, d\omega \tag{3.2.2}$$

and the limiting case of (3.2.2) is then

$$R_{Y_j} f(x) = \frac{\pi\Gamma\left(\frac{k+d}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{d}{2}\right)} \int_{S^{d-1}} Y_j(\omega) H_\omega f(x)\, d\omega, \tag{3.2.3}$$

where H_ω is the directional Hilbert transform given by

$$H_\omega f(x) = \lim_{t\to 0^+} H_\omega^t f(x).$$

For further reference we note that when k is fixed then

$$\frac{\pi\Gamma\left(\frac{k+d}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{d}{2}\right)} \approx d^{k/2}. \tag{3.2.4}$$

Proposition 3.1.5 and identity (3.2.2) let us express the operator M_k^t in terms of the directional Hilbert transform and the Riesz transform in the following way

$$M_k^t f(x) = (-1)^k \sum_{j=1}^{a(d,k)} R_{Y_j}^t R_{Y_j} f(x) = (-1)^k \frac{\pi\Gamma\left(\frac{k+d}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{d}{2}\right)} \int_{S^{d-1}} H_\omega^t \left[\sum_{j=1}^{a(d,k)} Y_j(\omega) R_{Y_j} f\right](x)\, d\omega \tag{3.2.5}$$

From (3.2.5) we can see that in order to estimate the operator M_k^* we need to estimate the maximal directional Hilbert transform H_ω^* and the Riesz transforms R_{Y_j}. The estimates are summarized in the two following propositions.

Proposition 3.2.1. *For each $1 < p < \infty$ and $f_1, \ldots, f_S \in L^p$ we have*

$$\left\| \left(\sum_{s=1}^S |H_\omega^* f_s|^2 \right)^{1/2} \right\|_p \lesssim p^* \left\| \left(\sum_{s=1}^S |f_s|^2 \right)^{1/2} \right\|_p \tag{3.2.6}$$

uniformly in $\omega \in S^{d-1}$ and the dimension d.

Proposition 3.2.2. *Fix $k \in \mathbb{N}_{\text{odd}}$. Then for each $1 < p < \infty$ and $f, f_1, \ldots, f_S \in L^p$ we have*

$$\left\| \left(\sum_{s=1}^S \sum_{j=1}^{a(d,k)} \left|R_{Y_j} f_s\right|^2 \right)^{1/2} \right\|_p \lesssim_k p^* p^{1/2} q^{\frac{k+1}{2}} \left\| \left(\sum_{s=1}^S |f_s|^2 \right)^{1/2} \right\|_p, \tag{3.2.7}$$

$$\left\| \left(\sum_{j=1}^{a(d,k)} \left|R_{Y_j} f\right|^2 \right)^{1/2} \right\|_p \lesssim_k p^* q^{k/2} \|f\|_p, \tag{3.2.8}$$

uniformly in the dimension d.

We begin with the proof of Proposition 3.2.1.

Proof of Proposition 3.2.1. First we will reduce the inequality to its one-dimensional case. Assume that we have proved the following inequality

$$\left\| \left(\sum_{s=1}^S |H^* f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})} \lesssim p^* \left\| \left(\sum_{s=1}^S |f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})}, \tag{3.2.9}$$

where H^* is the one-dimensional maximal Hilbert transform, i.e.

$$H^*f(x) = \sup_{t>0}\left|\frac{1}{\pi}\int_{|y|>t}\frac{f(x-y)}{y}\,dy\right|, \qquad f\in L^p(\mathbb{R}), \tag{3.2.10}$$

and $f_1,\dots,f_S\in L^p(\mathbb{R})$. We want to show that (3.2.9) implies (3.2.6).

Observe that for any $A\in SO(d)$ and $F\in L^p(\mathbb{R}^d)$ we have

$$\begin{aligned} H^*_{Ae_1}F(x) &= \sup_{t>0}\left|\frac{1}{\pi}\int_{|y|>t}\frac{F(x-y\cdot Ae_1)}{y}\,dy\right| \\ &= \sup_{t>0}\left|\frac{1}{\pi}\int_{|y|>t}\frac{(F\circ A)(A^{-1}x-ye_1)}{y}\,dy\right| = H^*_{e_1}(F\circ A)(A^{-1}x), \end{aligned}$$

where $e_1=(1,0,\dots,0)\in\mathbb{R}^d$. This means that in order to prove (3.2.6) we only need to show the following inequality

$$\left\|\left(\sum_{s=1}^S\left|H^*_{e_1}F_s\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)} \lesssim p^*\left\|\left(\sum_{s=1}^S|F_s|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)}, \tag{3.2.11}$$

where $F_1,\dots,F_s\in L^p(\mathbb{R}^d)$. Indeed, assume that (3.2.11) holds and take any $\omega\in S^{d-1}$ and $A\in SO(d)$ such that $\omega=Ae_1$. Then

$$\begin{aligned} \left\|\left(\sum_{s=1}^S|H^*_\omega F_s|^2\right)^{1/2}\right\|^p_{L^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d}\left(\sum_{s=1}^S\left|H^*_{Ae_1}F_s(x)\right|^2\right)^{p/2}dx \\ &= \int_{\mathbb{R}^d}\left(\sum_{s=1}^S\left|H^*_{e_1}(F_s\circ A)(A^{-1}x)\right|^2\right)^{p/2}dx \\ &= \int_{\mathbb{R}^d}\left(\sum_{s=1}^S\left|H^*_{e_1}(F_s\circ A)(x)\right|^2\right)^{p/2}dx \\ &\lesssim p^*\left\|\left(\sum_{s=1}^S|F_s\circ A|^2\right)^{1/2}\right\|^p_{L^p(\mathbb{R}^d)} = p^*\left\|\left(\sum_{s=1}^S|F_s|^2\right)^{1/2}\right\|^p_{L^p(\mathbb{R}^d)}, \end{aligned}$$

which is exactly (3.2.6). Now we need to prove (3.2.11). For $x=(x_1,x_2,\dots,x_d)\in\mathbb{R}^d$ denote $x_1'=(x_2,\dots,x_d)\in\mathbb{R}^{d-1}$. Then

$$\begin{aligned} &\left\|\left(\sum_{s=1}^S\left|H^*_{e_1}F_s\right|^2\right)^{1/2}\right\|^p_{L^p(\mathbb{R}^d)} = \int_{\mathbb{R}^d}\left(\sum_{s=1}^S\left|H^*_{e_1}F_s(x)\right|^2\right)^{p/2}dx \\ &= \int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}\left(\sum_{s=1}^S\sup_{t>0}\left|\frac{1}{\pi}\int_{|y|>t}\frac{F_s(x_1-y,x_2,\dots,x_d)}{y}\,dy\right|^2\right)^{p/2}dx_1\,dx_1' \\ &\lesssim p^*\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}\left(\sum_{s=1}^S|F_s(x_1,\dots,x_d)|^2\right)^{p/2}dx_1\,dx_1' = p^*\left\|\left(\sum_{s=1}^S|F_s|^2\right)^{1/2}\right\|^p_{L^p(\mathbb{R}^d)}. \end{aligned}$$

In the inequality we used (3.2.9) with $f_s(z) = F_s(z, x_2, \dots, x_d)$. Thus, we have shown that (3.2.9) implies (3.2.6) and hence we can focus on proving (3.2.6).

We split the operator H^* defined in (3.2.10) into two parts. To this end let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a smooth even function satisfying $\varphi(x) = 1$ for $|x| < 2$, $\varphi(x) = 0$ for $|x| > 4$. Define $\varphi_t(x) = \varphi(x/t)$ and let

$$\chi_t(x) = \frac{1}{\pi x} \mathbb{1}_{|x|>t}(x)$$

be the kernel of H^t. Then

$$\begin{aligned} H^* f(x) &\leqslant \sup_{t>0} |(\varphi_t \chi_t * f)(x)| + \sup_{t>0} |((1-\varphi_t)\chi_t * f)(x)| \\ &=: H^*_\varphi f(x) + H^*_{1-\varphi} f(x) \\ &\lesssim \mathcal{M} f(x) + H^*_{1-\varphi} f(x), \end{aligned}$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator on $\mathbb{R}$. Since [22, Theorem 5.6.6] gives us vector-valued estimates for $\mathcal{M}$ we get

$$\left\| \left(\sum_{s=1}^{S} \left| H^*_\varphi f_s \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})} \lesssim p^* \left\| \left(\sum_{s=1}^{S} |f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})}.$$

The remaining ingredient is to prove

$$\left\| \left(\sum_{s=1}^{S} \left| H^*_{1-\varphi} f_s \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})} \lesssim p^* \left\| \left(\sum_{s=1}^{S} |f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})}. \tag{3.2.12}$$

We will apply [22, Theorem 5.6.1] with

$$\mathcal{B}_1 = \ell^2\left(\{1,\dots,S\}\right) \qquad \text{and} \qquad \mathcal{B}_2 = \ell^2\left(\{1,\dots,S\}; L^\infty(\mathbb{Q}_+)\right)$$

and

$$\vec{K}(x)(u) = ((1-\varphi_t)\chi_t(x)\cdot u_1, \dots, (1-\varphi_t)\chi_t(x)\cdot u_S) \in \mathcal{B}_2 \tag{3.2.13}$$

for any sequence $u = (u_s)_{s=1}^S \in \mathcal{B}_1$. Then, taking $e_s = (0,\dots,1,\dots,0)$, with 1 on the s-th coordinate, we see that the operator $\vec{T}$ defined in [22, 5.6.4] satisfies

$$\vec{T}\left(\sum_{s=1}^{S} f_s e_s \right)(z) = \left(H^t_{1-\varphi} f_1(x), \dots, H^t_{1-\varphi} f_S(x) \right) \tag{3.2.14}$$

and

$$\left\| \vec{T}\left(\sum_{s=1}^{S} f_s e_s \right)(x) \right\|_{\mathcal{B}_2} = \left(\sum_{s=1}^{S} \left| H^*_{1-\varphi} f_s(x) \right|^2 \right)^{1/2}$$

for any sequence $(f_s)_{s=1}^S$ of smooth functions that vanish at infinity. In order to use [22, Theorem 5.6.1] we need to verify conditions (5.6.1), (5.6.2) and (5.6.3) from [22] and check that $\vec{T}$ is bounded from $L^2(\mathbb{R}, \mathcal{B}_1)$ to $L^2(\mathbb{R}, \mathcal{B}_2)$.

Condition (5.6.1) is a straightforward consequence of (3.2.13). It is also not hard to verify that $\int_{\varepsilon \leqslant |x| \leqslant 1} \vec{K}(x)\,dx = 0$, so that condition (5.6.3) is satisfied with $\vec{K}_0 = 0$.

We shall now justify (5.6.2). Denote $\widetilde{\varphi}_t := 1 - \varphi_t$ and $g_t = \widetilde{\varphi}_t \chi_t$ so that

$$g_t(x) = \frac{\widetilde{\varphi}_t(x)}{\pi x}.$$

Since

$$\left\|\vec{K}(x-y) - \vec{K}(x)\right\|_{\mathcal{B}_1 \to \mathcal{B}_2} = \sup_{t>0} |g_t(x-y) - g_t(x)|,$$

we have

$$\begin{aligned} \left\|\vec{K}(x-y) - \vec{K}(x)\right\|_{\mathcal{B}_1 \to \mathcal{B}_2} &= \frac{1}{\pi} \sup_{t>0} \left| \frac{\widetilde{\varphi}_t(x-y)}{x-y} - \frac{\widetilde{\varphi}_t(x)}{x} \right| \\ \leqslant \frac{1}{\pi} \sup_{t>0} \left| \frac{\widetilde{\varphi}_t(x-y) - \widetilde{\varphi}_t(x)}{x-y} \right| &+ \frac{1}{\pi} \sup_{t>0} \left| \widetilde{\varphi}_t(x) \left(\frac{1}{x-y} - \frac{1}{x} \right) \right|. \end{aligned} \tag{3.2.15}$$

Hence, the proof of (5.6.2) boils down to estimating the two terms in (3.2.15) under the assumption $|x| \geqslant 2|y|$. We begin with the first term. Since $|x| \geqslant 2|y|$ we have $|x| \approx |x-y|$. Hence, in order for the expression inside the absolute value to be nonzero, t has to be comparable to $|x|$ and $|x-y|$. In that case, using the smoothness of φ we obtain

$$\left| \frac{\widetilde{\varphi}_t(x-y) - \widetilde{\varphi}_t(x)}{x-y} \right| \lesssim \frac{|y|}{t} \frac{1}{|x-y|} \approx \frac{|y|}{|x||x-y|} \approx \frac{|y|}{|x|^2}$$

In the second term of (3.2.15) we omit $\widetilde{\varphi}_t$ and get

$$\left| \frac{1}{x-y} - \frac{1}{x} \right| = \frac{|y|}{|x||x-y|} \approx \frac{|y|}{|x|^2}.$$

This means that we have proved that

$$\left\|\vec{K}(x-y) - \vec{K}(x)\right\|_{\mathcal{B}_1 \to \mathcal{B}_2} \lesssim \frac{|y|}{|x|^2}$$

for $|x| \geqslant 2|y|$. Integrating this yields

$$\int_{|x| \geqslant 2|y|} \left\|\vec{K}(x-y) - \vec{K}(x)\right\|_{\mathcal{B}_1 \to \mathcal{B}_2} dx \lesssim |y| \int_{|x| \geqslant 2|y|} \frac{1}{|x|^2} dx \approx 1$$

so that condition (5.6.2) is satisfied.

It remains to justify the boundedness of $\vec{T}$ from $L^2(\mathbb{R}, \mathcal{B}_1)$ to $L^2(\mathbb{R}, \mathcal{B}_2)$. We have the pointwise bound

$$H^*_{1-\varphi} f(x) \lesssim \mathcal{M} f(x) + H^* f(x).$$

Therefore the desired L^2 boundedness of $\vec{T}$ is a consequence of (3.2.14) and the $L^2(\mathbb{R})$ boundedness of H^*. This allows us to use [22, Theorem 5.6.1] and completes the proof of (3.2.12) and hence also the proof of Proposition 3.2.1. □

Before we move on to the estimates for the Riesz transforms R_{Y_j}, it will be convenient to state explicitly [17, Lemme, p. 195] for later reference.

Lemma 3.2.3. *For $P \in \mathcal{H}_k$ and $q \in [1, \infty)$ we have*

$$\left(\int_{S^{d-1}} |P(\omega)|^q \, d\omega \right)^{1/q} \leqslant q^{k/2} \left(\int_{S^{d-1}} |P(\omega)|^2 \, d\omega \right)^{1/2}.$$

Recall the the functions Y_j form an orthogonal basis of the space $\mathcal{H}_k$ normalized by the condition

$$\int_{S^{d-1}} |Y_j(\omega)|^2 \, d\omega = \frac{1}{a(d,k)} \approx d^{-k}.$$

We justify (3.2.7) and (3.2.8) separately, starting with the latter.

Proof of (3.2.8). Take numbers $\lambda_j(x,f) = \lambda_j(x)$, $j = 1, \ldots, a(d,k)$, such that

$$\left(\sum_{j=1}^{a(d,k)} \left| R_{Y_j} f(x) \right|^2 \right)^{1/2} = \sum_{j=1}^{a(d,k)} \lambda_j(x) R_{Y_j} f(x), \qquad \sum_{j=1}^{a(d,k)} \lambda_j^2(x) = 1.$$

Using (3.2.3) and (3.2.4) followed by Hölder's inequality we obtain

$$\begin{aligned} \left\| \left(\sum_{j=1}^{a(d,k)} \left| R_{Y_j} f \right|^2 \right)^{1/2} \right\|_p^p &= \int_{\mathbb{R}^d} \left| \sum_{j=1}^{a(d,k)} \lambda_j(x) R_{Y_j} f(x) \right|^p dx \\ &\lesssim^p d^{kp/2} \int_{\mathbb{R}^d} \left| \int_{S^{d-1}} \sum_{j=1}^{a(d,k)} \lambda_j(x) Y_j(\omega) H_\omega f(x) \, d\omega \right|^p dx \\ &\leqslant d^{kp/2} \int_{\mathbb{R}^d} \left(\int_{S^{d-1}} \left| \sum_{j=1}^{a(d,k)} \lambda_j(x) Y_j(\omega) \right|^q d\omega \right)^{p/q} \int_{S^{d-1}} |H_\omega f(x)|^p \, d\omega \, dx. \end{aligned} \tag{3.2.16}$$

Now we deal with the first inner integral in (3.2.16). Since $Y_j \in \mathcal{H}_k^d$ for $j = 1, \ldots, a(d,k)$, for fixed x the function $\omega \mapsto \sum_{j=1}^{a(d,k)} Y_j(\omega)\lambda_j(x)$ also belongs to $\mathcal{H}_k^d$. Using Lemma 3.2.3, orthogonality of the functions Y_j, $j = 1, \ldots, a(d,k)$, condition (3.1.8), and the formula $\sum_{j=1}^{a(d,k)} \lambda_j^2(x) = 1$ we get

$$\begin{aligned} &\left(\int_{S^{d-1}} \left| \sum_{j=1}^{a(d,k)} \lambda_j(x) Y_j(\omega) \right|^q d\omega \right)^{1/q} \lesssim q^{k/2} \left(\int_{S^{d-1}} \left| \sum_{j=1}^{a(d,k)} \lambda_j(x) Y_j(\omega) \right|^2 d\omega \right)^{1/2} \\ &= q^{k/2} \left(\int_{S^{d-1}} \sum_{j=1}^{a(d,k)} \lambda_j^2(x) Y_j(\omega)^2 \, d\omega \right)^{1/2} \lesssim q^{k/2} \left(d^{-k} \sum_{j=1}^{a(d,k)} \lambda_j^2(x) \right)^{1/2} \leqslant q^{k/2} d^{-k/2}. \end{aligned} \tag{3.2.17}$$

Applying (3.2.17) and coming back to (3.2.16) we obtain

$$\left\| \left(\sum_{j=1}^{a(d,k)} \left| R_{Y_j} f \right|^2 \right)^{1/2} \right\|_p \lesssim q^{k/2} \left(\int_{S^{d-1}} \|H_\omega f\|_p^p \, d\omega \right)^{1/p}.$$

Now Proposition 3.2.1 completes the proof of (3.2.8). □

We are now ready to prove (3.2.7). This is similar to the proof of (3.2.8) with an addition of Khintchine's inequalities. For $s = 1, 2, \ldots$ we let $\{r_s\}$ be the Rademacher functions, see [22, Appendix C]. These form an orthonormal set in $L^2([0,1])$. Moreover we have Khintchine's inequalities ([22, Appendix C.2])

$$\left\| \sum_{j=1}^{\infty} a_j r_j \right\|_{L^p([0,1])} \lesssim p^{\frac{1}{2}} \left(\sum_{j=1}^{\infty} |a_j|^2 \right)^{1/2} \tag{3.2.18}$$

and

$$\left(\sum_{j=1}^{\infty}|a_j|^2\right)^{1/2} \lesssim \left\|\sum_{j=1}^{\infty} a_j r_j\right\|_{L^p([0,1])} \tag{3.2.19}$$

for any complex sequence $(a_s)_{s=1}^{\infty}$ and $1 \leqslant p < \infty$. The explicit bounds on constants in (3.2.18) and (3.2.19) follow from explicit values of the optimal constants established by Haagerup [23] together with Stirling's formula (1.5.2).

Proof of (3.2.7). Take numbers $\lambda_{j,s}(x,\{f_s\}) = \lambda_{j,s}(x)$, $j = 1,\ldots,a(d,k)$, $s = 1,\ldots,S$, such that

$$\left(\sum_{j=1}^{a(d,k)}\sum_{s=1}^{S}\left|R_{Y_j}f_s(x)\right|^2\right)^{1/2} = \sum_{s=1}^{S}\sum_{j=1}^{a(d,k)}\lambda_{j,s}(x)R_{Y_j}f_s(x), \qquad \sum_{s=1}^{S}\sum_{j=1}^{a(d,k)}\lambda_{j,s}^2(x) = 1. \tag{3.2.20}$$

Using (3.2.20), (3.2.3), and (3.2.4) we obtain

$$\begin{aligned}\left\|\left(\sum_{j=1}^{a(d,k)}\sum_{s=1}^{S}\left|R_{Y_j}f_s\right|^2\right)^{1/2}\right\|_p^p &= \int_{\mathbb{R}^d}\left|\sum_{s=1}^{S}\sum_{j=1}^{a(d,k)}\lambda_{j,s}(x)R_{Y_j}f_s(x)\right|^p dx\\ &\lesssim^p d^{kp/2}\int_{\mathbb{R}^d}\left|\int_{S^{d-1}}\sum_{s=1}^{S}\sum_{j=1}^{a(d,k)}\lambda_{j,s}(x)Y_j(\omega)H_\omega f_s(x)\,d\omega\right|^p dx.\end{aligned} \tag{3.2.21}$$

Orthogonality of the Rademacher functions $\{r_s\}$ and Hölder's inequality imply

$$\begin{aligned}&d^{kp/2}\int_{\mathbb{R}^d}\left|\int_{S^{d-1}}\sum_{s=1}^{S}\sum_{j=1}^{a(d,k)}\lambda_{j,s}(x)Y_j(\omega)H_\omega f_s(x)\,d\omega\right|^p dx\\ &= d^{kp/2}\int_{\mathbb{R}^d}\left|\int_{S^{d-1}}\int_0^1\left(\sum_{s=1}^{S}\sum_{j=1}^{a(d,k)}r_s(\xi)\lambda_{j,s}(x)Y_j(\omega)\right)\left(\sum_{s=1}^{S}r_s(\xi)H_\omega f_s(x)\right)d\xi\,d\omega\right|^p dx\\ &\leqslant d^{kp/2}\int_{\mathbb{R}^d}\left(\int_{S^{d-1}}\int_0^1\left|\sum_{s=1}^{S}\sum_{j=1}^{a(d,k)}r_s(\xi)\lambda_{j,s}(x)Y_j(\omega)\right|^q d\xi\,d\omega\right)^{p/q}\\ &\qquad\times\int_{S^{d-1}}\int_0^1\left|\sum_{s=1}^{S}r_s(\xi)H_\omega f_s(x)\right|^p d\xi\,d\omega\,dx.\end{aligned} \tag{3.2.22}$$

Denote

$$Q_{S,q}(x) := \left(\int_{S^{d-1}}\int_0^1\left|\sum_{s=1}^{S}\sum_{j=1}^{a(d,k)}r_s(\xi)\lambda_{j,s}(x)Y_j(\omega)\right|^q d\xi\,d\omega\right)^{1/q}$$

Then, coming back to (3.2.21) and using Khintchine's inequality (3.2.18) to the second factor in the last inequality in (3.2.22) we reach

$$\left\|\left(\sum_{j=1}^{a(d,k)}\sum_{s=1}^{S}\left|R_{Y_j}f_s\right|^2\right)^{1/2}\right\|_p^p \lesssim^p p^{p/2}d^{kp/2}\|Q_{S,q}\|_\infty^p\int_{S^{d-1}}\int_{\mathbb{R}^d}\left(\sum_{s=1}^{S}|H_\omega f_s(x)|^2\right)^{p/2} dx\,d\omega.$$

Thus, Proposition 3.2.1 implies

$$\left\| \left(\sum_{j=1}^{a(d,k)} \sum_{s=1}^{S} \left| R_{Y_j} f_s \right|^2 \right)^{1/2} \right\|_p \lesssim p^* p^{1/2} d^{k/2} \|Q_{S,q}\|_\infty \left\| \left(\sum_{s=1}^{S} |f_s|^2 \right)^{1/2} \right\|_p .$$

Therefore, the proof of (3.2.7) will be completed if we justify that

$$\|Q_{S,q}\|_\infty \lesssim q^{\frac{k+1}{2}} d^{-k/2}. \tag{3.2.23}$$

The proof of (3.2.23) splits into two cases.

If $q \geqslant 2$, we apply Khintchine's inequality (3.2.18), Minkowski's inequality and Lemma 3.2.3, obtaining

$$\begin{aligned}(Q_{S,q}(x))^q &\lesssim^q q^{q/2} \int_{S^{d-1}} \left(\sum_{s=1}^{S} \left| \sum_{j=1}^{a(d,k)} \lambda_{j,s}(x) Y_j(\omega) \right|^2 \right)^{q/2} d\omega \\ &\leqslant q^{q/2} \left(\sum_{s=1}^{S} \left(\int_{S^{d-1}} \left| \sum_{j=1}^{a(d,k)} \lambda_{j,s}(x) Y_j(\omega) \right|^q d\omega \right)^{2/q} \right)^{q/2} \\ &\lesssim^q q^{q/2} q^{kq/2} \left(\sum_{s=1}^{S} \int_{S^{d-1}} \left| \sum_{j=1}^{a(d,k)} \lambda_{j,s}(x) Y_j(\omega) \right|^2 d\omega \right)^{q/2},\end{aligned}$$

uniformly in $x \in \mathbb{R}^d$. Here an application of Lemma 3.2.3 is justified since $Y_j \in \mathcal{H}_k^d$ for $j = 1, \ldots, a(d,k)$ and thus also the sum $\sum_{j=1}^{a(d,k)} \lambda_{j,s}(x) Y_j$ belongs to $\mathcal{H}_k^d$ for each fixed $x \in \mathbb{R}^d$. Now, using the orthogonality of Y_j, $j = 1, \ldots, a(d,k)$, condition (3.1.8) and the formula $\sum_{s=1}^{S} \sum_{j=1}^{a(d,k)} \lambda_{j,s}^2(x) = 1$ we see that

$$\begin{aligned}(Q_{S,q}(x))^q &\lesssim^q q^{q/2} q^{kq/2} \left(\sum_{s=1}^{S} \int_{S^{d-1}} \sum_{j=1}^{a(d,k)} \lambda_{j,s}^2(x)\, Y_j(\omega)^2 \, d\omega \right)^{q/2} \\ &= q^{q/2} q^{kq/2} \left(d^{-k} \sum_{s=1}^{S} \sum_{j=1}^{a(d,k)} \lambda_{j,s}^2(x) \right)^{q/2} \lesssim q^{q/2} q^{kq/2} d^{-kq/2}.\end{aligned}$$

Therefore, (3.2.23) is justified in the case $q \geqslant 2$.

If on the other hand $1 < q < 2$, an application of Hölder's inequality together with (3.2.23) in the case $q = 2$ shows that

$$Q_{S,q}(x) \leqslant Q_{S,2}(x) \lesssim d^{-k/2}.$$

This completes the proof of (3.2.23) and thus also the proof of (3.2.7) from Proposition 3.2.2.

□

We are now ready to prove Theorem 3.1.2 and Theorem 3.1.3 for odd k and we start with the proof of Theorem 3.1.3.

Proof of Theorem 3.1.3. Using (3.2.5) and (3.2.4) we see that

$$|M^* f(x)| \lesssim d^{k/2} \int_{S^{d-1}} H^*_\omega \left[\sum_{j=1}^{a(d,k)} Y_j(\omega) R_{Y_j} f \right] (x)\, d\omega, \qquad x \in \mathbb{R}^d.$$

Hence, Minkowski's integral inequality followed by Proposition 3.2.1 show that

$$\|M^* f\|_p \lesssim p^* d^{k/2} \int_{S^{d-1}} \left\| \sum_{j=1}^{a(d,k)} Y_j(\omega) R_{Y_j} f \right\|_p d\omega.$$

Using Hölder's inequality and Fubini's theorem we obtain

$$\|M^* f\|_p \lesssim p^* d^{k/2} \left(\int_{\mathbb{R}^d} \int_{S^{d-1}} \left| \sum_{j=1}^{a(d,k)} Y_j(\omega) R_{Y_j} f(x) \right|^p d\omega\, dx \right)^{1/p}. \tag{3.2.24}$$

Since for fixed x the function $\omega \mapsto \sum_{j=1}^{a(d,k)} Y_j(\omega) R_{Y_j} f(x)$ belongs to $\mathcal{H}^d_k$, applying Lemma 3.2.3 we obtain

$$\left(\int_{S^{d-1}} \left| \sum_{j=1}^{a(d,k)} Y_j(\omega) R_{Y_j} f(x) \right|^p d\omega \right)^{1/p} \lesssim p^{k/2} \left(\int_{S^{d-1}} \left| \sum_{j=1}^{a(d,k)} Y_j(\omega) R_{Y_j} f(x) \right|^2 d\omega \right)^{1/2}.$$

Using orthogonality and (3.1.8) we thus see that

$$\left(\int_{S^{d-1}} \left| \sum_{j=1}^{a(d,k)} Y_j(\omega) R_{Y_j} f(x) \right|^p d\omega \right)^{1/p} \lesssim d^{-k/2}\, p^{k/2} \left(\sum_{j=1}^{a(d,k)} \left| R_{Y_j} f(x) \right|^2 \right)^{1/2}, \tag{3.2.25}$$

which, together with (3.2.24) leads to

$$\|M^* f\|_p \lesssim p^* p^{k/2} \left\| \left(\sum_{j=1}^{a(d,k)} \left| R_{Y_j} f \right|^2 \right)^{1/2} \right\|_p.$$

Thus, (3.2.8) from Proposition 3.2.2 completes the proof of Theorem 3.1.3 if k is odd.

□

We finish this section with the proof of Theorem 3.1.2.

Proof of Theorem 3.1.2. Using (3.2.5), (3.2.4), and Minkowski's integral inequality on the space $\ell^2(\{1, \dots, S\}; L^\infty(\mathbb{Q}_+))$ we see that

$$\left(\sum_{s=1}^{S} |M^* f_s(x)|^2 \right)^{1/2} \lesssim d^{k/2} \int_{S^{d-1}} \left(\sum_{s=1}^{S} \left(H^*_\omega \Big[\sum_{j=1}^{a(d,k)} Y_j(\omega) R_{Y_j} f_s \Big](x) \right)^2 \right)^{1/2} d\omega, \qquad x \in \mathbb{R}^d.$$

Thus, another application of Minkowski's integral inequality followed by Proposition 3.2.1 gives

$$\left\| \left(\sum_{s=1}^{S} |M^* f_s|^2 \right)^{1/2} \right\|_p \lesssim p^* d^{k/2} \int_{S^{d-1}} \left\| \left(\sum_{s=1}^{S} \left| \sum_{j=1}^{a(d,k)} Y_j(\omega) R_{Y_j} f_s \right|^2 \right)^{1/2} \right\|_p d\omega.$$

Using Khintchine's inequality (3.2.19) followed by Hölder's inequality on S^{d-1} we see that

$$\begin{aligned}
&\left\| \left(\sum_{s=1}^{S} |M^* f_s|^2 \right)^{1/2} \right\|_p \\
&\lesssim p^* d^{k/2} \int_{S^{d-1}} \left(\int_{\mathbb{R}^d} \int_0^1 \left| \sum_{s=1}^{S} r_s(\xi) \sum_{j=1}^{a(d,k)} Y_j(\omega) R_{Y_j} f_s(x) \right|^p d\xi\, dx \right)^{1/p} d\omega \\
&\lesssim p^* d^{k/2} \left(\int_{\mathbb{R}^d} \int_0^1 \int_{S^{d-1}} \left| \sum_{j=1}^{a(d,k)} Y_j(\omega) R_{Y_j} \left[\sum_{s=1}^{S} r_s(\xi) f_s(x) \right] \right|^p d\omega\, d\xi\, dx \right)^{1/p} .
\end{aligned}$$

Finally, (3.2.25) followed by (3.2.8) from Proposition 3.2.2 and Khintchine's inequality (3.2.18) give

$$\begin{aligned}
&\left\| \left(\sum_{s=1}^{S} |M^* f_s|^2 \right)^{1/2} \right\|_p \lesssim p^* p^{k/2} \left(\int_{\mathbb{R}^d} \int_0^1 \left(\sum_{j=1}^{a(d,k)} \left| R_{Y_j} \left[\sum_{s=1}^{S} r_s(\xi) f_s(x) \right] \right|^2 \right)^{p/2} d\xi\, dx \right)^{1/p} \\
&\lesssim (p^*)^{2+k/2} \left(\int_{\mathbb{R}^d} \int_0^1 \left| \sum_{s=1}^{S} r_s(\xi) f_s(x) \right|^p d\xi\, dx \right)^{1/p} \lesssim (p^*)^{5/2+k/2} \left(\int_{\mathbb{R}^d} \left(\sum_{s=1}^{S} |f_s|^2 \right)^{p/2} dx \right)^{1/p} .
\end{aligned}$$

The proof of Theorem 3.1.2 in the odd k case is thus completed. □

3.3 Averaging

From now until the end of the chapter k is a fixed positive integer. In this section we describe the averaging procedure. The averaging procedure will allow us to pass from M^* to another maximal operator that is better suited for applications in Sections 3.4 and 3.5. Before moving on, we establish some notation. For a multi-index

$$j = (j_1, \ldots, j_k) \in \{1, \ldots, d\}^k \quad \text{we write} \quad P_j(x) = x_j := x_{j_1} \cdots x_{j_k}$$

and denote by R_j the Riesz transform R_{P_j} associated with the monomial P_j. The truncated transform R_j^t and the maximal transform R_j^* are defined analogously. We also abbreviate $K_j(x) = K_{P_j}(x)$ and $K_j^t(x) = K_{P_j}^t(x)$. As we will be mainly interested in multi-indices with different components, we define

$$I = \{j \in \{1, \ldots, d\}^k : j_m \neq j_l \text{ for } m \neq l\}.$$

Note that the set I is non-empty only when $d \geqslant k$. Thus in the rest of the proof we assume that this is the case. The result for $d < k$ follows from [35, 36].

The averaging procedure will provide an expression for M_k^t in terms of the Riesz transforms R_j and R_j^t postulated in (3.0.4). For $f \in L^p$, $1 < p < \infty$, denote

$$R^t f := \sum_{j \in I} R_j^t R_j f \qquad \text{and let} \qquad R^* f := \sup_{t \in \mathbb{Q}_+} \left| R^t f \right|.$$

Note that both R^t and R^* are well defined on all L^p spaces for $1 < p < \infty$. Indeed, R_j^t and R_j are bounded on L^p and the supremum in the definition of R^* runs over a countable set thus defining a measurable function.

Let $SO(d)$ be the special orthogonal group in dimension d. Since it is compact, it has a bi-invariant Haar measure μ such that $\mu(SO(d)) = 1$. For $U \in SO(d)$ and a sublinear operator T on L^2 we denote by T_U the conjugation by U, i.e. the operator acting via

$$T_U f(x) = T(f \circ U^{-1}))(Ux). \tag{3.3.1}$$

Proposition 3.3.1. *Fix $k \in \mathbb{N}$. Then there is a constant $C(d,k) \in \mathbb{R}$ such that*

$$M_k^t f(x) = C(d,k) \int_{SO(d)} [(R^t)_U f](x)\, d\mu(U) \tag{3.3.2}$$

for all $t > 0$ and $f \in L^p$. Moreover, $|C(d,k)|$ has an estimate from above by a constant that depends only on k but not on the dimension d, so that

$$\left(\sum_{s=1}^{S} |M^* f_s(x)|^2 \right)^{1/2} \lesssim \int_{SO(d)} \left(\sum_{s=1}^{S} |[(R^*)_U f_s](x)|^2 \right)^{1/2} d\mu(U), \tag{3.3.3}$$

for $S \in \mathbb{N}$ and $f_1, \ldots, f_S \in L^p$.

Proof. Let A be the operator

$$A = \sum_{j \in I} R_j^2, \tag{3.3.4}$$

which by (3.0.3) means that its multiplier symbol equals

$$a(\xi) = (-i)^{2k} \sum_{j \in I} \frac{\xi_j^2}{|\xi|^{2k}} = (-1)^k \sum_{j \in I} \frac{\xi_j^2}{|\xi|^{2k}}.$$

Let $\widetilde{A}$ be the operator with the multiplier symbol

$$\widetilde{a}(\xi) := \int_{SO(d)} a(U\xi)\, d\mu(U) = (-1)^k \sum_{j \in I} \int_{SO(d)} \frac{((U\xi)_j)^2}{|\xi|^{2k}}\, d\mu(U). \tag{3.3.5}$$

Then $\widetilde{a}$ being radial and homogeneous of order 0 is constant.

The first step in the proof of the proposition is to show that

$$|\widetilde{a}| \approx 1 \tag{3.3.6}$$

uniformly in the dimension d. Note that each of the integrals on the right hand side of (3.3.5) has the same value independently of $j \in I$, so that

$$\widetilde{a}(\xi) = (-1)^k |I| \int_{SO(d)} \frac{((U\xi)_{(1,\ldots,k)})^2}{|\xi|^{2k}}\, d\mu(U);$$

here $|I|$ stands for the number of elements in I. Since $\widetilde{a}$ is constant, we can integrate it over S^{d-1} with respect to the probabilistic measure $d\omega$ and change the order of integration to get

$$\begin{aligned}\widetilde{a} &= (-1)^k |I| \int_{S^{d-1}} \int_{SO(d)} (U\omega)^2_{(1,\ldots,k)}\, d\mu(U)\, d\omega \\ &= (-1)^k |I| \int_{SO(d)} \int_{S^{d-1}} (U\omega)^2_{(1,\ldots,k)}\, d\omega\, d\mu(U).\end{aligned}$$

Now notice that the inner integral does not depend on U, which means that

$$\widetilde{a} = (-1)^k |I| \int_{S^{d-1}} \omega_1^2 \cdots \omega_k^2 \, d\omega. \tag{3.3.7}$$

Since k is fixed, by an elementary argument we get $|I| = \frac{d!}{(d-k)!} \approx d^k$. Thus it remains to show that

$$\int_{S^{d-1}} \omega_1^2 \cdots \omega_k^2 \, d\omega \approx d^{-k}. \tag{3.3.8}$$

Formula (3.3.8) is given in [51, (10)]. It can be also easily computed by the method from [25, Chapter 3.4]; for the sake of completeness we provide a brief argument.

Consider the integral $J = \int_{\mathbb{R}^d} x_1^2 \cdots x_k^2 e^{-|x|^2} \, dx$. Since J is a product of one-dimensional integrals we see that $J = \Gamma\left(\frac{3}{2}\right)^k \Gamma\left(\frac{1}{2}\right)^{d-k}$, while using polar coordinates gives

$$J = S_{d-1} \int_{S^{d-1}} \omega_1^2 \cdots \omega_k^2 \, d\omega \int_0^\infty r^{2k+d-1} e^{-r^2} \, dr,$$

where S_{d-1} is defined by (1.5.5). Altogether we have justified that

$$\int_{S^{d-1}} \omega_1^2 \cdots \omega_k^2 \, d\omega \approx \frac{\Gamma\left(\frac{1}{2}\right)^{d-k}}{S_{d-1} \Gamma\left(k + \frac{d}{2}\right)}.$$

Since k is fixed and d is arbitrarily large, using (1.5.5), Stirling's formula for the Γ function (1.5.2) and the known identity $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ we obtain

$$\begin{aligned}
\int_{S^{d-1}} \omega_1^2 \cdots \omega_k^2 \, d\omega &\approx \frac{\sqrt{k + \frac{d}{2}} \left(\frac{d}{2e}\right)^{d/2}}{\sqrt{\frac{d}{2}} \left(\frac{k+\frac{d}{2}}{e}\right)^{k+d/2}} \\
&\approx \frac{e^{-d/2}}{e^{-k-d/2}} \left(\frac{k + \frac{d}{2}}{d/2}\right)^{-d/2} \left(k + \frac{d}{2}\right)^{-k} \\
&\approx d^{-k}
\end{aligned}$$

This gives (3.3.8) and concludes the proof of (3.3.6).

Let now m^t be the multiplier symbol of M_k^t. Then, from Proposition 3.1.1 we see that $m^t = \widehat{b^t}$ is radial, so that

$$\begin{aligned}
m^t(\xi) = \tilde{a}^{-1} \tilde{a} \, m^t(\xi) &= \tilde{a}^{-1} \int_{SO(d)} m^t(\xi) \, a(U\xi) \, d\mu(U) \\
&= \tilde{a}^{-1} \int_{SO(d)} m^t(U\xi) \, a(U\xi) \, d\mu(U).
\end{aligned}$$

Using properties of the Fourier transform the above equality implies that

$$M_k^t f(x) = \tilde{a}^{-1} \int_{SO(d)} [(M_k^t A)_U](f)(x) \, d\mu(U).$$

Recalling (3.3.4) we apply (3.1.1) from Proposition 3.1.1 and obtain

$$M_k^t A = \sum_{j \in I} M_k^t R_j R_j = \sum_{j \in I} R_j^t R_j = R^t;$$

here an application of (3.1.1) is allowed since each R_j corresponds to the monomial x_j which is in $\mathcal{H}_k$ when $j \in I$. In summary, we justified that

$$M_k^t f(x) = \tilde{a}^{-1} \int_{SO(d)} [(R^t)_U](f)(x)\, d\mu(U), \qquad f \in \mathcal{D}(k), \tag{3.3.9}$$

which is (3.3.2) with $C(d,k) = \tilde{a}^{-1}$.

It remains to justify (3.3.3). This follows from (3.1.7), (3.3.9), and (3.3.6), together with the norm inequality

$$\left\| \int_{SO(d)} F_{s,t}(U)\, d\mu(U) \right\|_X \leqslant \int_{SO(d)} \|F_{s,t}(U)\|_X \, d\mu(U);$$

on the Banach space $X = \ell^2(\{1,\ldots,S\}; \ell^\infty(\mathbb{Q}_+))$, with $F_{s,t}(U) = (R^t)_U(f_s)(x)$ and x being fixed.

The proof of Proposition 3.3.1 is thus completed. □

Since conjugation by $U \in SO(d)$ is an isometry on all L^p spaces, in view of $\mu(SO(d)) = 1$ and Minkowski's integral inequality Proposition 3.3.1 eq. (3.3.3) allows us to deduce Theorems 3.1.2 and 3.1.3 from the two theorems below.

Theorem 3.3.2. *Fix $k \in \mathbb{N}$. For each $p \in (1,\infty)$ there is a constant $A(p,k)$ independent of the dimension d and such that for any $S \in \mathbb{N}$ we have*

$$\left\| \left(\sum_{s=1}^S |R^* f_s|^2 \right)^{1/2} \right\|_p \lesssim A(p,k) \left\| \left(\sum_{s=1}^S |f_s|^2 \right)^{1/2} \right\|_p,$$

where $f_1, \ldots, f_S \in L^p$. Moreover, $A(p,k)$ satisfies $A(p,k) \lesssim_k (p^*)^{5/2+k/2}$.

Theorem 3.3.3. *Fix $k \in \mathbb{N}$. For each $p \in (1,\infty)$ there is a constant $B(p,k)$ independent of the dimension d and such that*

$$\|R^* f\|_p \lesssim B(p,k) \|f\|_p.$$

whenever $f \in L^p$. Moreover, $B(p,k)$ satisfies $B(p,k) \lesssim_k (p^*)^{2+k/2}$.

3.4 Extension to $\mathbb{C}^d$ and the complex method of rotations

Here we extend the operators R^t acting on $L^p(\mathbb{R}^d)$ to the operators $\widetilde{R}^t$ acting on $L^p(\mathbb{C}^d)$ Then we apply the complex method of rotations of Iwaniec and Martin [26] to $\widetilde{R}^t$.

Let $P \in \mathcal{H}_k$. For $z = (x_1 + iy_1, \ldots, x_d + iy_d)$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ we denote

$$\widetilde{K}_P(z) = \widetilde{\gamma}_k \frac{P(z)}{|z|^{2d+k}} \qquad \text{with} \qquad \widetilde{\gamma}_k = \frac{\Gamma\left(d + \frac{k}{2}\right)}{\pi^d \Gamma\left(\frac{k}{2}\right)}, \tag{3.4.1}$$

and define, for $f \in \mathcal{S}(\mathbb{C}^d)$,

$$\widetilde{R}_P f(z) = \lim_{t \to 0} \widetilde{R}_P^t f(z), \qquad \text{where} \qquad \widetilde{R}_P^t f(z) = \widetilde{\gamma}_k \int_{w \in \mathbb{C}^d : |z-w| > t} \frac{P(z-w)}{|z-w|^{2d+k}} f(w)\, dw. \tag{3.4.2}$$

In [26] the authors considered the extension on the multiplier level whereas we need to write it on the kernel level. This makes no difference for the operator $\widetilde{R}_P$. However, the multiplier symbol corresponding to $\widetilde{R}_P^t$ does not have a simple formula, thus writing the extension on a kernel level seems the only reasonable option here.

Formulas (3.4.1) and (3.4.2) lead us to define the extension of R^t by

$$\widetilde{R}^t = \widetilde{R}_k^t := \sum_{j\in I} \widetilde{R}_j^t \widetilde{R}_j. \tag{3.4.3}$$

Using the complex method of rotations [26, Section 6] we will prove $L^p(\mathbb{C}^d)$ estimates for

$$\widetilde{R}^* f(z) = \sup_{t\in\mathbb{Q}_+} \left|\widetilde{R}^t f(z)\right|.$$

Theorem 3.4.1. *Fix $k \in \mathbb{N}$. For each $p \in (1,\infty)$ there is a constant $A(p,k)$ independent of the dimension d and such that for any $S \in \mathbb{N}$ we have*

$$\left\| \left(\sum_{s=1}^S \left|\widetilde{R}^* f_s\right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{C}^d)} \leqslant A(p,k) \left\| \left(\sum_{s=1}^S |f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{C}^d)},$$

whenever $f_1,\dots,f_S \in L^p(\mathbb{C}^d)$. Moreover, $A(p,k)$ satisfies $A(p,k) \lesssim_k (p^*)^{5/2+k/2}$.

Theorem 3.4.2. *Fix $k \in \mathbb{N}$. For each $p \in (1,\infty)$ there is a constant $B(p,k)$ independent of the dimension d and such that*

$$\left\|\widetilde{R}^* f\right\|_{L^p(\mathbb{C}^d)} \leqslant B(p,k) \|f\|_{L^p(\mathbb{C}^d)},$$

whenever $f \in L^p(\mathbb{C}^d)$. Moreover, $B(p,k)$ satisfies $B(p,k) \lesssim_k (p^*)^{2+k/2}$.

The reminder of this section will be devoted to the proofs of Theorem 3.4.1 and Theorem 3.4.2. From these results we shall obtain Theorem 3.3.2 and Theorem 3.3.3 provided we develop a restriction procedure from $\mathbb{C}^d$ to $\mathbb{R}^d$. As we already remarked this is not straightforward, since the restriction of the complex truncated Riesz transform is not the real truncated Riesz transform. Details of the restriction and estimates for the resulting operators are given in Section 3.5.

We now focus on the proofs of Theorem 3.4.1 and Theorem 3.4.2. Let $P \in \mathcal{H}_k$. We will show that for $F \in \mathcal{S}(\mathbb{C}^d)$ we have

$$2\pi \int_{\mathbb{C}^d} F(w)\, dw = \int_{S^{2d-1}} \int_{\mathbb{C}} F(\lambda\theta)|\lambda|^{2d-2}\, d\lambda\, d\theta,$$

where $d\theta$ stands for the spherical measure on S^{2d-1} normalized by the condition $\theta(S^{2d-1}) = S_{2d-1}$.

We begin by identifying $\mathbb{C}^d$ with $\mathbb{R}^{2d}$ and using spherical coordinates to get

$$\int_{\mathbb{R}^{2d}} F(x)\, dx = \int_0^\infty \int_{S^{2d-1}} F(r\theta) r^{2d-1}\, d\theta\, dr.$$

Then we integrate both sides from 0 to 2π with respect to a new variable φ which gives

$$2\pi \int_{\mathbb{R}^{2d}} F(x)\, dx = \int_0^{2\pi} \int_0^\infty \int_{S^{2d-1}} F(r\theta) r^{2d-1}\, d\theta\, dr\, d\varphi.$$

At this point we interpret θ as en element of $\mathbb{C}^d$ so that we can introduce a new variable $\theta' = e^{-i\varphi}\theta$ and use integration by substitution to obtain

$$2\pi \int_{\mathbb{R}^{2d}} F(x)\,dx = \int_0^{2\pi}\int_0^\infty \int_{S^{2d-1}} F(re^{i\varphi}\theta) r^{2d-1}\,d\theta\,dr\,d\varphi.$$

We recognize $\lambda = re^{i\varphi}$ as an arbitrary element of $\mathbb{C}$ which lets us write

$$2\pi \int_{\mathbb{C}^d} F(w)\,dw = \int_{S^{2d-1}}\int_{\mathbb{C}} F(\lambda\theta)|\lambda|^{2d-2}\,d\lambda\,d\theta$$

thus proving the claim.

Take $f \in \mathcal{S}(\mathbb{C}^d)$. Applying the above identity with $F(w) = \widetilde{\gamma}_k \frac{P(w)}{|w|^{2d+k}}\mathbb{1}_{|w|>t}(w)\,f(z-w)$ gives

$$\begin{aligned}
\widetilde{R}_P^t f(z) &= \widetilde{\gamma}_k \int_{\mathbb{C}^d} \frac{P(w)}{|w|^{2d+k}}\mathbb{1}_{|w|>t}(w)\,f(z-w)\,dw \\
&= \frac{\widetilde{\gamma}_k}{2\pi}\int_{S^{2d-1}}\int_{\mathbb{C}} \frac{P(\lambda\theta)}{|\lambda|^{2d+k}}\mathbb{1}_{|\lambda|>t}(\lambda)\,f(z-\lambda\theta)|\lambda|^{2d-2}\,d\lambda\,d\theta \\
&= \frac{\widetilde{\gamma}_k}{2\pi}\int_{S^{2d-1}} P(\theta)\int_{\mathbb{C}} \left(\frac{\lambda}{|\lambda|}\right)^k \frac{f(z-\lambda\theta)}{|\lambda|^2}\mathbb{1}_{|\lambda|>t}(\lambda)\,d\lambda\,d\theta,
\end{aligned}$$

where in the last equality above we used the k-homogeneity of P. This means that we got

$$\widetilde{R}_P^t f(z) = \frac{\widetilde{\gamma}_k}{2\pi}\int_{S^{2d-1}} P(\theta)\widetilde{H}_{\theta,k}^t f(z)\,d\theta, \tag{3.4.4}$$

where

$$\widetilde{H}_{\theta,k}^t f(z) = \widetilde{H}_\theta^t f(z) := \int_{\mathbb{C}} \left(\frac{\lambda}{|\lambda|}\right)^k \frac{f(z-\lambda\theta)}{|\lambda|^2}\mathbb{1}_{|\lambda|>t}(\lambda)\,d\lambda$$

is the truncated directional k-th power of the complex Hilbert transform. Identity (3.4.4) can be written in terms of the probabilistic spherical measure $d\zeta$ on S^{2d-1} in the following way

$$\widetilde{R}_P^t f(z) = \frac{\Gamma\left(d+\frac{k}{2}\right)}{\pi\Gamma\left(d\right)\Gamma\left(\frac{k}{2}\right)}\int_{S^{2d-1}} P(\zeta)\widetilde{H}_\zeta^t f(z)\,d\zeta. \tag{3.4.5}$$

The limiting case of (3.4.5) is then

$$\widetilde{R}_P f(z) = \frac{\Gamma\left(d+\frac{k}{2}\right)}{\pi\Gamma\left(d\right)\Gamma\left(\frac{k}{2}\right)}\int_{S^{2d-1}} P(\zeta)\widetilde{H}_\zeta f(z)\,d\zeta, \tag{3.4.6}$$

where

$$\widetilde{H}_\zeta f(z) = \widetilde{H}_{\zeta,k} f(z) = \text{p.v.}\int_{\mathbb{C}} \left(\frac{\lambda}{|\lambda|}\right)^k \frac{f(z-\lambda\zeta)}{|\lambda|^2}\,d\lambda$$

is the directional k-th power of the complex Hilbert transform. Identities (3.4.5) and (3.4.6) were initially established for $f \in \mathcal{S}(\mathbb{C}^d)$. However, a density argument based on the $L^p(\mathbb{C}^d)$ boundedness of $\widetilde{H}_\zeta^t$ and $\widetilde{H}_\zeta$ allows us to write these identities for all $f \in L^p(\mathbb{C}^d)$. For further reference we note that when k is fixed then

$$\frac{\Gamma\left(d+\frac{k}{2}\right)}{\pi\Gamma\left(d\right)\Gamma\left(\frac{k}{2}\right)} \approx d^{k/2}. \tag{3.4.7}$$

In the proofs of Theorem 3.4.1 and Theorem 3.4.2, similarly to the odd case, we shall need boundedness properties of the maximal operator

$$\widetilde{H}^*_\zeta f(z) = \widetilde{H}^*_{\zeta,k} f(z) := \sup_{t\in\mathbb{Q}_+} \left|\widetilde{H}^t_\zeta f(z)\right|$$

associated with $\widetilde{H}^t_\zeta$ and of the Riesz transforms $\widetilde{R}_j$.

Proposition 3.4.3. *For each $1 < p < \infty$ we have*

$$\left\|\left(\sum_{s=1}^S \left|\widetilde{H}^*_\zeta f_s\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{C}^d)} \lesssim p^* \left\|\left(\sum_{s=1}^S |f_s|^2\right)^{1/2}\right\|_{L^p(\mathbb{C}^d)}$$

uniformly in $\zeta \in S^{2d-1}$ and the dimension d.

Proposition 3.4.4. *Fix $k \in \mathbb{N}$. Then for each $1 < p < \infty$ we have*

$$\left\|\left(\sum_{s=1}^S \sum_{j\in I} \left|\widetilde{R}_j f_s\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{C}^d)} \lesssim_k p^* p^{1/2} q^{\frac{k+1}{2}} \left\|\left(\sum_{s=1}^S |f_s|^2\right)^{1/2}\right\|_{L^p(\mathbb{C}^d)}, \tag{3.4.8}$$

$$\left\|\left(\sum_{j\in I} \left|\widetilde{R}_j f\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{C}^d)} \lesssim_k p^* q^{k/2} \|f\|_{L^p(\mathbb{C}^d)}, \tag{3.4.9}$$

uniformly in the dimension d.

The proofs of Propositions 3.4.3 and 3.4.4 are analogous to the proofs of Propositions 3.2.1 and 3.2.2 for the most part, however we include them for completeness. We begin with Proposition 3.4.3.

Proof of Proposition 3.4.3. A (complex) rotational invariance argument analogous to the one used in the proof of Proposition 3.2.1 reduces the inequality to its two-dimensional case

$$\left\|\left(\sum_{s=1}^S \left|\widetilde{H}^*_k f_s\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{C})} \lesssim p^* \left\|\left(\sum_{s=1}^S |f_s|^2\right)^{1/2}\right\|_{L^p(\mathbb{C})},$$

where $\widetilde{H}^*_k$ is k-th power of the two-dimensional maximal complex Hilbert transform, i.e.

$$\widetilde{H}^*_k f(z) = \sup_{t>0} \left| \int_{|w|>t} \left(\frac{w}{|w|}\right)^k \frac{f(z-w)}{|w|^2}\, dw \right|, \qquad f \in L^p(\mathbb{C}),$$

and $f_1, \ldots, f_S \in L^p(\mathbb{C})$.

We split the operator $\widetilde{H}^*_k$ into two parts. To this end let $\varphi\colon \mathbb{C} \to \mathbb{R}$ be a smooth radial function satisfying $\varphi(z) = 1$ for $|z| < 2$, $\varphi(z) = 0$ for $|z| > 4$. Define $\varphi_t(z) = \varphi(z/t)$ and let

$$\chi_t(z) = \left(\frac{z}{|z|}\right)^k \frac{1}{|z|^2} \mathbb{1}_{|z|>t}(z)$$

be the kernel of $\widetilde{H}_k^t$. Then

$$\begin{aligned}\widetilde{H}_k^* f(z) &\leqslant \sup_{t>0} |(\varphi_t \chi_t * f)(z)| + \sup_{t>0} |((1-\varphi_t)\chi_t * f)(z)| \\ &=: \widetilde{H}_\varphi^* f(z) + \widetilde{H}_{1-\varphi}^* f(z) \\ &\lesssim \mathcal{M} f(z) + \widetilde{H}_{1-\varphi}^* f(z),\end{aligned}$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator on $\mathbb{R}^2$. Since [22, Theorem 5.6.6] gives us vector-valued estimates for $\mathcal{M}$ we get

$$\left\| \left(\sum_{s=1}^{S} \left| \widetilde{H}_\varphi^* f_s \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{C})} \lesssim p^* \left\| \left(\sum_{s=1}^{S} |f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{C})}.$$

The remaining ingredient is to prove

$$\left\| \left(\sum_{s=1}^{S} \left| \widetilde{H}_{1-\varphi}^* f_s \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{C})} \lesssim p^* \left\| \left(\sum_{s=1}^{S} |f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{C})}. \tag{3.4.10}$$

We will apply [22, Theorem 5.6.1] with

$$\mathcal{B}_1 = \ell^2(\{1, \dots, S\}) \qquad \text{and} \qquad \mathcal{B}_2 = \ell^2(\{1, \dots, S\}; L^\infty(\mathbb{Q}_+))$$

and

$$\vec{K}(z)(u) = ((1-\varphi_t)\chi_t(z) \cdot u_1, \dots, (1-\varphi_t)\chi_t(z) \cdot u_S) \in \mathcal{B}_2 \tag{3.4.11}$$

for any sequence $u = (u_s)_{s=1}^S \in \mathcal{B}_1$. Then, taking $e_s = (0, \dots, 1, \dots, 0)$, with 1 on the s-th coordinate, we see that the operator $\vec{T}$ defined in [22, 5.6.4] satisfies

$$\vec{T}\left(\sum_{s=1}^{S} f_s e_s \right)(z) = \left(\widetilde{H}_{1-\varphi}^t f_1(z), \dots, \widetilde{H}_{1-\varphi}^t f_S(z) \right) \tag{3.4.12}$$

and

$$\left\| \vec{T}\left(\sum_{s=1}^{S} f_s e_s \right)(z) \right\|_{\mathcal{B}_2} = \left(\sum_{s=1}^{S} \left| \widetilde{H}_{1-\varphi}^* f_s(z) \right|^2 \right)^{1/2}$$

for any sequence $(f_s)_{s=1}^S$ of smooth functions that vanish at infinity. In order to use [22, Theorem 5.6.1] we need to verify conditions (5.6.1), (5.6.2) and (5.6.3) from [22] and check that $\vec{T}$ is bounded from $L^2(\mathbb{C}, \mathcal{B}_1)$ to $L^2(\mathbb{C}, \mathcal{B}_2)$.

Condition (5.6.1) is a straightforward consequence of (3.4.11). It is also not hard to verify that $\int_{\varepsilon \leqslant |z| \leqslant 1} \vec{K}(z)\, dz = 0$, so that condition (5.6.3) is satisfied with $\vec{K}_0 = 0$.

We shall now justify (5.6.2). Denote $\widetilde{\varphi}_t := 1 - \varphi_t$ and $g_t = \widetilde{\varphi}_t \chi_t$ so that

$$g_t(z) = \widetilde{\varphi}_t(z) \frac{z^k}{|z|^{k+2}}.$$

Since

$$\left\| \vec{K}(z-w) - \vec{K}(z) \right\|_{\mathcal{B}_1 \to \mathcal{B}_2} = \sup_{t>0} |g_t(z-w) - g_t(z)|,$$

we have

$$\left\|\vec{K}(z-w)-\vec{K}(z)\right\|_{\mathcal{B}_1\to\mathcal{B}_2}=\sup_{t>0}\left|\widetilde{\varphi}_t(z-w)\frac{(z-w)^k}{|z-w|^{k+2}}-\widetilde{\varphi}_t(z)\frac{z^k}{|z|^{k+2}}\right|$$
$$\leqslant \sup_{t>0}\left|(\widetilde{\varphi}_t(z-w)-\widetilde{\varphi}_t(z))\,\frac{(z-w)^k}{|z-w|^{k+2}}\right|+\sup_{t>0}\left|\widetilde{\varphi}_t(z)\left(\frac{(z-w)^k}{|z-w|^{k+2}}-\frac{z^k}{|z|^{k+2}}\right)\right|. \tag{3.4.13}$$

Hence, the proof of (5.6.2) reduces to estimating the two terms in (3.4.13) under the assumption $|z|\geqslant 2|w|$. We begin with the first term. Since $|z|\geqslant 2|w|$ we have $|z|\approx|z-w|$. Hence, in order for the expression inside the absolute value to be nonzero, t has to be comparable to $|z|$ and $|z-w|$. In that case, using the smoothness of φ we obtain

$$\left|(\widetilde{\varphi}_t(z-w)-\widetilde{\varphi}_t(z))\,\frac{(z-w)^k}{|z-w|^{k+2}}\right|\lesssim\frac{|w|}{t}\frac{1}{|z-w|^2}\approx\frac{|w|}{|z||z-w|^2}\approx\frac{|w|}{|z|^3}.$$

In the second term of (3.4.13) we omit $\widetilde{\varphi}_t$ and get

$$\left|\frac{(z-w)^k}{|z-w|^{k+2}}-\frac{z^k}{|z|^{k+2}}\right|\leqslant\left|\frac{(z-w)^k}{|z-w|^{k+2}}-\frac{(z-w)^k}{|z|^{k+2}}\right|+\left|\frac{(z-w)^k}{|z|^{k+2}}-\frac{z^k}{|z|^{k+2}}\right|$$
$$=|z-w|^k\frac{\left||z|^{k+2}-|z-w|^{k+2}\right|}{|z-w|^{k+2}|z|^{k+2}}+\frac{1}{|z|^{k+2}}\left|(z-w)^k-z^k\right|\approx\frac{|w|}{|z|^3}.$$

This means that we have proved that

$$\left\|\vec{K}(z-w)-\vec{K}(z)\right\|_{\mathcal{B}_1\to\mathcal{B}_2}\lesssim\frac{|w|}{|z|^3}$$

for $|z|\geqslant 2|w|$. Integrating this yields

$$\int_{|z|\geqslant 2|w|}\left\|\vec{K}(z-w)-\vec{K}(z)\right\|_{\mathcal{B}_1\to\mathcal{B}_2}dz\lesssim|w|\int_{|z|\geqslant 2|w|}\frac{1}{|z|^3}dz\approx 1$$

so that condition (5.6.2) is satisfied.

It remains to justify the boundedness of $\vec{T}$ from $L^2(\mathbb{C},\mathcal{B}_1)$ to $L^2(\mathbb{C},\mathcal{B}_2)$. We have the pointwise bound

$$\widetilde{H}^*_{1-\varphi}f(z)\lesssim\mathcal{M}f(z)+\widetilde{H}^*_kf(z).$$

Therefore the desired L^2 boundedness of $\vec{T}$ is a consequence of (3.4.12) and the $L^2(\mathbb{C})$ boundedness of $\widetilde{H}^*_k$. This allows us to use [22, Theorem 5.6.1] and completes the proof of (3.4.10) hence also the proof of Proposition 3.4.3. □

Proposition 3.4.4 can be proved by an iterative application of its $k=1$ case together with Khintchine's inequalities. However, such an approach produces worse constants than those in (3.4.8) and (3.4.9). An important ingredient in the proof are properties of the functions

$$\zeta_j=(x_{j_1}+iy_{j_1})\cdots(x_{j_k}+iy_{j_k}).$$

Recall that in the proof of the odd case we used functions Y_j which form an orthogonal basis of $\mathcal{H}^d_k$. However it is not clear whether their extensions to $\mathbb{C}^d$ are still orthogonal on S^{2d-1}, which is why we need to introduce new functions ζ_j orthogonal both on S^{d-1} and on S^{2d-1}. Moreover, we have

$$\int_{S^{2d-1}} |\zeta_j|^2 \, d\zeta \lesssim d^{-k}. \tag{3.4.14}$$

Indeed, all the integrals on the left hand side of (3.4.14) are equal for $j \in I$ and thus

$$\begin{aligned}\int_{S^{2d-1}} |\zeta_j|^2 \, d\zeta &= \frac{1}{|I|} \int_{S^{2d-1}} \sum_{j \in I} |\zeta_j|^2 \, d\zeta \leqslant \frac{1}{|I|} \int_{S^{2d-1}} \sum_{j \in \{1,\ldots,d\}^k} |\zeta_j|^2 \, d\zeta \\ &= \frac{1}{|I|} \int_{S^{2d-1}} |\zeta|^{2k} \, d\zeta \lesssim d^{-k},\end{aligned}$$

since $|I| \approx d^k$.

We justify (3.4.8) and (3.4.9) separately, starting with the latter.

Proof of (3.4.9). Take numbers $\lambda_j(z,f) = \lambda_j(z)$, $j \in I$, such that

$$\left(\sum_{j \in I} \left| \widetilde{R}_j f(z) \right|^2 \right)^{1/2} = \sum_{j \in I} \lambda_j(z) \widetilde{R}_j f(z), \qquad \sum_{j \in I} \lambda_j^2(z) = 1.$$

Using (3.4.6) and (3.4.7) followed by Hölder's inequality we obtain

$$\begin{aligned}\left\| \left(\sum_{j \in I} \left| \widetilde{R}_j f \right|^2 \right)^{1/2} \right\|_p^p &= \int_{\mathbb{C}^d} \left| \sum_{j \in I} \lambda_j(z) \widetilde{R}_j f(z) \right|^p dz \\ \lesssim^p d^{kp/2} &\int_{\mathbb{C}^d} \left| \int_{S^{2d-1}} \sum_{j \in I} \lambda_j(z) \zeta_j \widetilde{H}_\zeta f(z) \, d\zeta \right|^p dz \\ \leqslant d^{kp/2} &\int_{\mathbb{C}^d} \left(\int_{S^{2d-1}} \left| \sum_{j \in I} \lambda_j(z) \zeta_j \right|^q d\zeta \right)^{p/q} \int_{S^{2d-1}} \left| \widetilde{H}_\zeta f(z) \right|^p d\zeta \, dz.\end{aligned} \tag{3.4.15}$$

Now we deal with the first inner integral in (3.4.15). Since $\zeta_j \in \mathcal{H}_k^{2d}$ for $j \in I$, for fixed z the function $\zeta \mapsto \sum_{j \in I} \zeta_j \lambda_j(z)$ also belongs to $\mathcal{H}_k^{2d}$. Using Lemma 3.2.3, orthogonality of the functions ζ_j, $j \in I$, inequality (3.4.14), and the formula $\sum_{j \in I} \lambda_j(z)^2 = 1$ we get

$$\begin{aligned}&\left(\int_{S^{2d-1}} \left| \sum_{j \in I} \lambda_j(z) \zeta_j \right|^q d\zeta \right)^{1/q} \lesssim q^{k/2} \left(\int_{S^{2d-1}} \left| \sum_{j \in I} \lambda_j(z) \zeta_j \right|^2 d\zeta \right)^{1/2} \\ &= q^{k/2} \left(\int_{S^{2d-1}} \sum_{j \in I} \lambda_j(z)^2 |\zeta_j|^2 \, d\zeta \right)^{1/2} \lesssim q^{k/2} \left(d^{-k} \sum_{j \in I} \lambda_j(z)^2 \right)^{1/2} \leqslant q^{k/2} d^{-k/2}.\end{aligned} \tag{3.4.16}$$

Applying (3.4.16) and coming back to (3.4.15) we obtain

$$\left\| \left(\sum_{j \in I} \left| \widetilde{R}_j f \right|^2 \right)^{1/2} \right\|_p \lesssim q^{k/2} \left(\int_{S^{2d-1}} \left\| \widetilde{H}_\zeta f \right\|_{L^p(\mathbb{C}^d)}^p d\zeta \right)^{1/p}.$$

Now Proposition 3.4.3 completes the proof of (3.4.9). □

We are now ready to prove (3.4.8). This is similar to the proof of (3.4.9) with an addition of Khintchine's inequalities (3.2.18) and (3.2.19).

Proof of (3.4.8). Take numbers $\lambda_{j,s}(z,\{f_s\}) = \lambda_{j,s}(z)$, $j \in I$, $s = 1,\ldots,S$, such that

$$\left(\sum_{j\in I}\sum_{s=1}^{S}\left|\widetilde{R}_j f_s(z)\right|^2\right)^{1/2} = \sum_{s=1}^{S}\sum_{j\in I}\lambda_{j,s}(z)\widetilde{R}_j f_s(z), \qquad \sum_{s=1}^{S}\sum_{j\in I}\lambda_{j,s}^2(z) = 1. \tag{3.4.17}$$

Using (3.4.17), (3.4.6), and (3.4.7) we obtain

$$\begin{aligned}\left\|\left(\sum_{j\in I}\sum_{s=1}^{S}\left|\widetilde{R}_j f_s\right|^2\right)^{1/2}\right\|_p^p &= \int_{\mathbb{C}^d}\left|\sum_{s=1}^{S}\sum_{j\in I}\lambda_{j,s}(z)\widetilde{R}_j f_s(z)\right|^p dz \\ &\lesssim^p d^{kp/2}\int_{\mathbb{C}^d}\left|\int_{S^{2d-1}}\sum_{s=1}^{S}\sum_{j\in I}\lambda_{j,s}(z)\zeta_j\widetilde{H}_\zeta f_s(z)\,d\zeta\right|^p dz. \end{aligned} \tag{3.4.18}$$

Orthogonality of the Rademacher functions $\{r_s\}$ and Hölder's inequality imply

$$\begin{aligned} &d^{kp/2}\int_{\mathbb{C}^d}\left|\int_{S^{2d-1}}\sum_{s=1}^{S}\sum_{j\in I}\lambda_{j,s}(z)\zeta_j\widetilde{H}_\zeta f_s(z)\,d\zeta\right|^p dz \\ &= d^{kp/2}\int_{\mathbb{C}^d}\left|\int_{S^{2d-1}}\int_0^1\left(\sum_{s=1}^{S}\sum_{j\in I}r_s(\xi)\lambda_{j,s}(z)\zeta_j\right)\left(\sum_{s=1}^{S}r_s(\xi)\widetilde{H}_\zeta f_s(z)\right)d\xi\,d\zeta\right|^p dz \\ &\leqslant d^{kp/2}\int_{\mathbb{C}^d}\left(\int_{S^{2d-1}}\int_0^1\left|\sum_{s=1}^{S}\sum_{j\in I}r_s(\xi)\lambda_{j,s}(z)\zeta_j\right|^q d\xi\,d\zeta\right)^{p/q} \\ &\qquad\times\int_{S^{2d-1}}\int_0^1\left|\sum_{s=1}^{S}r_s(\xi)\widetilde{H}_\zeta f_s(z)\right|^p d\xi\,d\zeta\,dz. \end{aligned} \tag{3.4.19}$$

Denote

$$Q_{S,q}(z) := \left(\int_{S^{2d-1}}\int_0^1\left|\sum_{s=1}^{S}\sum_{j\in I}r_s(\xi)\lambda_{j,s}(z)\zeta_j\right|^q d\xi\,d\zeta\right)^{1/q}$$

Then, coming back to (3.4.18) and using Khintchine's inequality (3.2.18) to the second factor in the last inequality in (3.4.19) we reach

$$\left\|\left(\sum_{j\in I}\sum_{s=1}^{S}\left|\widetilde{R}_j f_s\right|^2\right)^{1/2}\right\|_p^p \lesssim^p p^{p/2}d^{kp/2}\|Q_{S,q}\|_{L^\infty(\mathbb{C}^d)}^p\int_{S^{2d-1}}\int_{\mathbb{C}^d}\left(\sum_{s=1}^{S}\left|\widetilde{H}_\zeta f_s(z)\right|^2\right)^{p/2} dz\,d\zeta.$$

Thus, Proposition 3.4.3 implies

$$\left\|\left(\sum_{j\in I}\sum_{s=1}^{S}\left|\widetilde{R}_j f_s\right|^2\right)^{1/2}\right\|_p \lesssim p^* p^{1/2}d^{k/2}\|Q_{S,q}\|_{L^\infty(\mathbb{C}^d)}\left\|\left(\sum_{s=1}^{S}|f_s|^2\right)^{1/2}\right\|_{L^p(\mathbb{C}^d)}.$$

Therefore, the proof of (3.4.8) will be completed if we justify that

$$\|Q_{S,q}\|_{L^\infty(\mathbb{C}^d)} \lesssim q^{\frac{k+1}{2}} d^{-k/2}. \tag{3.4.20}$$

The proof of (3.4.20) splits into two cases.

If $q \geqslant 2$, we apply Khintchine's inequality (3.2.18), Minkowski's inequality and Lemma 3.2.3, obtaining

$$\begin{aligned}
(Q_{S,q}(z))^q &\lesssim^q q^{q/2} \int_{S^{2d-1}} \left(\sum_{s=1}^{S} \left| \sum_{j\in I} \lambda_{j,s}(z)\zeta_j \right|^2 \right)^{q/2} d\zeta \\
&\leqslant q^{q/2} \left(\sum_{s=1}^{S} \left(\int_{S^{2d-1}} \left| \sum_{j\in I} \lambda_{j,s}(z)\zeta_j \right|^q d\zeta \right)^{2/q} \right)^{q/2} \\
&\lesssim^q q^{q/2} q^{kq/2} \left(\sum_{s=1}^{S} \int_{S^{2d-1}} \left| \sum_{j\in I} \lambda_{j,s}(z)\zeta_j \right|^2 d\zeta \right)^{q/2},
\end{aligned}$$

uniformly in $z \in \mathbb{C}^d$. Here an application of Lemma 3.2.3 is justified since $\zeta_j \in \mathcal{H}_k^{2d}$ for $j \in I$ and thus also the sum $\sum_{j\in I} \lambda_{j,s}(z)\zeta_j$ belongs to $\mathcal{H}_k^{2d}$ for each fixed $z \in \mathbb{C}^d$. Now, using the orthogonality of ζ_j, $j \in I$, inequality (3.4.14) and the formula $\sum_{s=1}^{S} \sum_{j\in I} \lambda_{j,s}^2(z) = 1$ we see that

$$\begin{aligned}
(Q_{S,q}(z))^q &\lesssim^q q^{q/2} q^{kq/2} \left(\sum_{s=1}^{S} \int_{S^{2d-1}} \sum_{j\in I} \lambda_{j,s}(z)^2 |\zeta_j|^2 \, d\zeta \right)^{q/2} \\
&= q^{q/2} q^{kq/2} \left(d^{-k} \sum_{s=1}^{S} \sum_{j\in I} \lambda_{j,s}(z)^2 \right)^{q/2} \lesssim q^{q/2} q^{kq/2} d^{-kq/2}.
\end{aligned}$$

Therefore, (3.4.20) is justified in the case $q \geqslant 2$.

If on the other hand $1 < q < 2$, an application of Hölder's inequality together with (3.4.20) in the case $q = 2$ shows that

$$Q_{S,q}(z) \leqslant Q_{S,2}(z) \lesssim d^{-k/2}.$$

This completes the proof of (3.4.20) and thus also the proof of (3.4.8) from Proposition 3.4.4. □

We are now ready to prove Theorem 3.4.1 and Theorem 3.4.2. In both the proofs we shall need the formula

$$\widetilde{R}^t f(z) = \frac{\Gamma\left(d + \frac{k}{2}\right)}{\pi \Gamma\left(d\right) \Gamma\left(\frac{k}{2}\right)} \int_{S^{2d-1}} \widetilde{H}_\zeta^t \left[\sum_{j\in I} \zeta_j \widetilde{R}_j f \right] (z) \, d\zeta, \tag{3.4.21}$$

which follows from (3.4.3) and (3.4.5). We start with the proof of Theorem 3.4.2.

Proof of Theorem 3.4.2. Using (3.4.21) and (3.4.7) we see that

$$\left|\widetilde{R}^* f(z)\right| \lesssim d^{k/2} \int_{S^{2d-1}} \widetilde{H}^*_\zeta \left[\sum_{j\in I} \zeta_j \widetilde{R}_j f\right](z)\, d\zeta, \qquad z \in \mathbb{C}^d.$$

Hence, Minkowski's integral inequality followed by Proposition 3.4.3 show that

$$\left\|\widetilde{R}^* f\right\|_{L^p(\mathbb{C}^d)} \lesssim p^* d^{k/2} \int_{S^{2d-1}} \left\|\sum_{j\in I} \zeta_j \widetilde{R}_j f\right\|_{L^p(\mathbb{C}^d)} d\zeta.$$

Using Hölder's inequality and Fubini's theorem we obtain

$$\left\|\widetilde{R}^* f\right\|_{L^p(\mathbb{C}^d)} \lesssim p^* d^{k/2} \left(\int_{\mathbb{C}^d} \int_{S^{2d-1}} \left|\sum_{j\in I} \zeta_j \widetilde{R}_j f(z)\right|^p d\zeta\, dz\right)^{1/p}. \tag{3.4.22}$$

Since for fixed z the function $\zeta \mapsto \sum_{j\in I} \zeta_j \widetilde{R}_j f(z)$ belongs to $\mathcal{H}^{2d}_k$, applying Lemma 3.2.3 we obtain

$$\left(\int_{S^{2d-1}} \left|\sum_{j\in I} \zeta_j \widetilde{R}_j f(z)\right|^p d\zeta\right)^{1/p} \lesssim p^{k/2} \left(\int_{S^{2d-1}} \left|\sum_{j\in I} \zeta_j \widetilde{R}_j f(z)\right|^2 d\zeta\right)^{1/2}.$$

Using orthogonality and (3.4.14) we thus see that

$$\left(\int_{S^{2d-1}} \left|\sum_{j\in I} \zeta_j \widetilde{R}_j f(z)\right|^p d\zeta\right)^{1/p} \lesssim d^{-k/2} p^{k/2} \left(\sum_{j\in I} \left|\widetilde{R}_j f(z)\right|^2\right)^{1/2}, \tag{3.4.23}$$

which, together with (3.4.22) leads to

$$\left\|\widetilde{R}^* f\right\|_{L^p(\mathbb{C}^d)} \lesssim p^* p^{k/2} \left\|\left(\sum_{j\in I} \left|\widetilde{R}_j f\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{C}^d)}.$$

Thus, (3.4.9) from Proposition 3.4.4 completes the proof of Theorem 3.4.2. □

We finish this section with the proof of Theorem 3.4.1.

Proof of Theorem 3.4.1. Using (3.4.21), (3.4.7), and Minkowski's integral inequality on the space $\ell^2(\{1,\ldots,S\}; L^\infty(\mathbb{Q}_+))$ we see that

$$\left(\sum_{s=1}^S \left|\widetilde{R}^* f_s(z)\right|^2\right)^{1/2} \lesssim d^{k/2} \int_{S^{2d-1}} \left(\sum_{s=1}^S \left(\widetilde{H}^*_\zeta \left[\sum_{j\in I} \zeta_j \widetilde{R}_j f_s\right](z)\right)^2\right)^{1/2} d\zeta, \qquad z \in \mathbb{C}^d.$$

Thus, another application of Minkowski's integral inequality followed by Proposition 3.4.3 gives

$$\left\|\left(\sum_{s=1}^S \left|\widetilde{R}^* f_s\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{C}^d)} \lesssim p^* d^{k/2} \int_{S^{2d-1}} \left\|\left(\sum_{s=1}^S \left|\sum_{j\in I} \zeta_j \widetilde{R}_j f_s\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{C}^d)} d\zeta.$$

Using Khintchine's inequality (3.2.19) followed by Hölder's inequality on S^{2d-1} we see that

$$\begin{aligned}&\left\|\left(\sum_{s=1}^{S}\left|\widetilde{R}^*f_s\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{C}^d)}\\&\lesssim p^*d^{k/2}\int_{S^{2d-1}}\left(\int_{\mathbb{C}^d}\int_0^1\left|\sum_{s=1}^{S}r_s(\xi)\sum_{j\in I}\zeta_j\widetilde{R}_jf_s(z)\right|^p d\xi\,dz\right)^{1/p}d\zeta\\&\lesssim p^*d^{k/2}\left(\int_{\mathbb{C}^d}\int_0^1\int_{S^{2d-1}}\left|\sum_{j\in I}\zeta_j\widetilde{R}_j\left[\sum_{s=1}^{S}r_s(\xi)f_s(z)\right]\right|^p d\zeta\,d\xi\,dz\right)^{1/p}.\end{aligned}$$

Finally, (3.4.23) followed by (3.4.9) from Proposition 3.4.4 and Khintchine's inequality (3.2.18) give

$$\begin{aligned}&\left\|\left(\sum_{s=1}^{S}\left|\widetilde{R}^*f_s\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{C}^d)}\lesssim p^*p^{k/2}\left(\int_{\mathbb{C}^d}\int_0^1\left(\sum_{j\in I}\left|\widetilde{R}_j\left[\sum_{s=1}^{S}r_s(\xi)f_s(z)\right]\right|^2\right)^{p/2}d\xi\,dz\right)^{1/p}\\&\lesssim (p^*)^{2+k/2}\left(\int_{\mathbb{C}^d}\int_0^1\left|\sum_{s=1}^{S}r_s(\xi)f_s(z)\right|^p d\xi\,dz\right)^{1/p}\lesssim (p^*)^{5/2+k/2}\left(\int_{\mathbb{C}^d}\left(\sum_{s=1}^{S}|f_s|^2\right)^{p/2}dz\right)^{1/p}.\end{aligned}$$

The proof of Theorem 3.4.1 is thus completed. □

3.5 Restriction to the initial Riesz transforms

The purpose of this section is twofold. Firstly, we restrict the maximal operator $\widetilde{R}^*$ acting on $L^p(\mathbb{C}^d)$ to a maximal operator $\mathcal{R}^*$ acting on $L^p(\mathbb{R}^d)$. This is done in a way which preserves estimates for the norms. However, the restricted maximal operator $\mathcal{R}^*$ is not the same as R^*. Therefore, we need to estimate their difference, which is done in the second part of Section 3.5.

3.5.1 Bounding the restriction $\mathcal{R}^*$ of $\widetilde{R}^*$.

In the previous section in Theorems 3.4.1 and 3.4.2 we proved dimension-free estimates for the operator $\widetilde{R}^*$ acting on $L^p(\mathbb{C}^d)$. An approach similar to [26, Chapter 4] leads to dimension-free estimates for the restriction of this operator to $L^p(\mathbb{R}^d)$ which we now describe.

To elaborate, for $x\in\mathbb{R}^d$ and $t>0$ we define the restricted kernel $\mathcal{K}_j^t(x)$ by

$$\begin{aligned}\mathcal{K}_j^t(x)&=\widetilde{\gamma}_kS_{d-1}\frac{x_j}{|x|^{d+k}}\int_{\sqrt{\frac{t^2}{|x|^2}-1}}^{\infty}\frac{r^{d-1}}{(1+r^2)^{d+k/2}}\,dr,\qquad \text{for } |x|\leqslant t,\\\mathcal{K}_j^t(x)&=K_j^t(x),\qquad \text{for } |x|>t.\end{aligned}\tag{3.5.1}$$

Recall that K_j^t is the kernel given by (3.0.1) when $P_j(x)=x_{j_1}\cdots x_{j_k}$, $j\in I$. A short computation based on (1.5.5), (3.0.5), and (3.0.6) gives, for $x \neq 0$,

$$\begin{aligned}&\lim_{t\to 0^+} \widetilde{\gamma}_k S_{d-1} \frac{x_j}{|x|^{d+k}} \int_{\sqrt{\frac{t^2}{|x|^2}-1}}^{\infty} \frac{r^{d-1}}{(1+r^2)^{d+k/2}}\,dr \\ &= \frac{\Gamma(d+\frac{k}{2})}{\pi^{d/2}\Gamma(\frac{k}{2})\Gamma(\frac{d}{2})} \int_0^\infty \frac{2r^{d-1}}{(1+r^2)^{d+k/2}}\,dr \cdot \frac{x_j}{|x|^{d+k}} = \gamma_k \frac{x_j}{|x|^{d+k}} = K_j(x).\end{aligned} \tag{3.5.2}$$

For $f \in L^p(\mathbb{R}^d)$ we let $\mathcal{R}_j^t f = f * \mathcal{K}_j^t$ and we define

$$\mathcal{R}^t f = \sum_{j\in I} \mathcal{R}_j^t R_j f$$

and

$$\mathcal{R}^* f = \sup_{t\in\mathbb{Q}_+} \left|\mathcal{R}^t f\right|.$$

A transference argument leads to the two results below. The proofs of Theorems 3.5.1 and 3.5.2 are based on ideas from [26, Section 4]. However, compared to [26, Section 4] extra difficulties arise. These complications stem from the fact that we need to restrict compositions of singular integral operators instead of just one singular integral operator. Furthermore, useful formulas for the multiplier symbols of $\widetilde{R}_j^t$ or $\mathcal{R}_j^t$ are not available.

Theorem 3.5.1. *Fix $k \in \mathbb{N}$. For each $p \in (1,\infty)$ there is a constant $A(p,k)$ independent of the dimension d and such that for any $S \in \mathbb{N}$ we have*

$$\left\| \left(\sum_{s=1}^S |\mathcal{R}^* f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \leqslant A(p,k) \left\| \left(\sum_{s=1}^S |f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)},$$

whenever $f_1,\dots,f_S \in L^p(\mathbb{R}^d)$. Moreover, $A(p,k)$ satisfies $A(p,k) \lesssim_k (p^*)^{5/2+k/2}$.

Theorem 3.5.2. *Fix $k \in \mathbb{N}$. For each $p \in (1,\infty)$ there is a constant $B(p,k)$ independent of the dimension d and such that*

$$\|\mathcal{R}^* f\|_{L^p(\mathbb{R}^d)} \leqslant B(p,k) \|f\|_{L^p(\mathbb{R}^d)},$$

whenever $f \in L^p(\mathbb{R}^d)$. Moreover, $B(p,k)$ satisfies $B(p,k) \lesssim_k (p^*)^{2+k/2}$.

The restriction procedure from Theorems 3.4.1 and 3.4.2 to Theorems 3.5.1 and 3.5.2 will result in the kernels $\widetilde{K}_j$ and $\widetilde{K}_j^t$ defined in (3.4.1) being integrated over their imaginary component iy in $\mathbb{R}^d$. This is the origin of the kernel $\mathcal{K}_j^t$ as the next lemma justifies.

Lemma 3.5.3. *For each $t > 0$ and $x \in \mathbb{R}^d$ it holds*

$$\int_{\mathbb{R}^d} \widetilde{K}_j^t(x+iy)\,dy = \mathcal{K}_j^t(x). \tag{3.5.3}$$

Proof. To justify (3.5.3) consider two cases: $|x| > t$ and $|x| \leqslant t$. In the first case, integrating in polar coordinates in $\mathbb{R}^d$ and noting that $\int_{S^{d-1}} P_j(x+ir\omega)\,d\omega = P_j(x)$ gives

$$\begin{aligned}\int_{\mathbb{R}^d} \widetilde{K}_j^t(x+iy)\,dy &= \int_{y\in\mathbb{R}^d\colon |x+iy|>t} \widetilde{\gamma}_k \frac{P_j(x+iy)}{|x+iy|^{2d+k}}\,dy = \int_{\mathbb{R}^d} \widetilde{\gamma}_k \frac{P_j(x+iy)}{|x+iy|^{2d+k}}\,dy \\ &= \widetilde{\gamma}_k S_{d-1} P_j(x) \int_0^\infty \frac{r^{d-1}}{\left(|x|^2+r^2\right)^{d+k/2}}\,dr = \widetilde{\gamma}_k S_{d-1} \frac{P_j(x)}{|x|^{d+k}} \int_0^\infty \frac{r^{d-1}}{(1+r^2)^{d+k/2}}\,dr \\ &= K_j(x) = K_j^t(x) = \mathcal{K}_j^t(x).\end{aligned}$$

In the fourth equality above we used change of the variables $r \to r|x|$ and then we used (3.5.2). Similarly, in the second case $|x| \leqslant t$ we obtain

$$\int_{y\in\mathbb{R}^d\colon |x+iy|>t} \widetilde{\gamma}_k \frac{P_j(x+iy)}{|x+iy|^{2d+k}}\,dy = \widetilde{\gamma}_k S_{d-1} P_j(x) \int_{\sqrt{t^2-|x|^2}}^{\infty} \frac{r^{d-1}}{\Big(|x|^2+r^2\Big)^{d+k/2}}\,dr$$
$$= \mathcal{K}_j^t(x),$$

where in the second equality we used the change of variable $r \to r|x|$. Thus (3.5.3) is justified. $\square$

We first present the proof of Theorem 3.5.2. The proof of Theorem 3.5.1 is similar. We merely need a technically more involved duality argument instead of (3.5.4) below and an application of Theorem 3.4.1 instead of Theorem 3.4.2.

Proof of Theorem 3.5.2. By Lebesgue's monotone convergence theorem we may restrict the supremum to a finite set of positive numbers $\{t_1, \ldots, t_N\}$, as long as our final estimate is independent of N. Further, a density argument shows that it suffices to consider $f \in \mathcal{S}(\mathbb{R}^d)$.

For $F\colon \mathbb{C}^d \to \mathbb{C}$ and $u > 0$ we let $(\delta_u F)(x+iy) = F(x+iuy)$ and define

$$\widetilde{R}^{t,u}(F)(x+iy) := (\delta_{u^{-1}} \circ \widetilde{R}^t \circ \delta_u)(F)(x+iy) = \widetilde{R}^t(\delta_u F)(x+iu^{-1}y).$$

Using Theorem 3.4.2 it is straightforward to see that

$$\Big\|\sup_{n=1,\ldots,N} \Big|\widetilde{R}^{t_n,u} F\Big|\Big\|_{L^p(\mathbb{C}^d)} \leqslant B(p,k)\|F\|_{L^p(\mathbb{C}^d)}.$$

Note that by duality between the spaces $L^p(\mathbb{C}^d; \ell^\infty(\{t_1,\ldots,t_N\}))$ and $L^q(\mathbb{C}^d; \ell^1(\{t_1,\ldots,t_N\}))$ the above inequality can be rewritten in the following equivalent form

$$\Big|\sum_{n=1}^N \langle \widetilde{R}^{t_n,u} F,\, G_n\rangle_{L^2(\mathbb{C}^d)}\Big| \leqslant B(p,k)\|F\|_{L^p(\mathbb{C}^d)} \Big\|\sum_{n=1}^N |G_n|\Big\|_{L^q(\mathbb{C}^d)}, \tag{3.5.4}$$

where $G_n \in L^q(\mathbb{C}^d)$, $n = 1, \ldots, N$.

Let $\eta \in \mathcal{S}(\mathbb{R}^d)$ be a fixed function such that $\|\eta\|_{L^p(\mathbb{R}^d)} = 1$ and take $f \in \mathcal{S}(\mathbb{R}^d)$. Denoting

$$F(x+iy) := (f \otimes \eta)(x,y) = f(x)\cdot\eta(y), \qquad x, y \in \mathbb{R}^d$$

we claim that

$$\lim_{u\to 0^+} \langle \widetilde{R}^{t,u} F,\, G\rangle_{L^2(\mathbb{C}^d)} = \langle \mathcal{R}^t(f) \otimes \eta,\, G\rangle_{L^2(\mathbb{C}^d)} \tag{3.5.5}$$

for any function $G \in \mathcal{S}(\mathbb{C}^d)$ and all $t > 0$.

Assume for a moment that the claim holds. Fix $\varepsilon \in (0,1)$ and let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be a function of $L^q(\mathbb{R}^d)$ norm 1 and such that $|\langle \eta, \psi\rangle_{L^2(\mathbb{R}^d)}| \geqslant (1-\varepsilon)$. Take $g_n \in \mathcal{S}(\mathbb{R}^d)$, $n = 1, \ldots, N$. Then, substituting $F = f \otimes \eta$ and $G_n = g_n \otimes \psi$ in (3.5.4) we have

$$\Big|\sum_{n=1}^N \langle \widetilde{R}^{t_n,u}(f\otimes\eta),\, g_n \otimes \psi\rangle_{L^2(\mathbb{C}^d)}\Big| \leqslant B(p,k)\|f\otimes\eta\|_{L^p(\mathbb{C}^d)} \Big\|\sum_{n=1}^N |g_n \otimes \psi|\Big\|_{L^q(\mathbb{C}^d)}.$$

At this point claim (3.5.5) implies

$$\left|\sum_{n=1}^{N}\langle \mathcal{R}^{t_n} f,\, g_n\rangle_{L^2(\mathbb{R}^d)}\right|\left|\langle \eta,\psi\rangle_{L^2(\mathbb{R}^d)}\right| \leqslant B(p,k)\|f\|_{L^p(\mathbb{R}^d)}\left\|\sum_{n=1}^{N}|g_n|\right\|_{L^q(\mathbb{R}^d)}.$$

Now, using duality between the spaces $L^p(\mathbb{R}^d;\ell^\infty(\{t_1,\dots,t_N\}))$ and $L^q(\mathbb{R}^d;\ell^1(\{t_1,\dots,t_N\}))$ together with the density of Schwartz functions in $L^q(\mathbb{R}^d)$ we conclude that

$$(1-\varepsilon)\left\|\sup_{n=1,\dots,N}\left|\mathcal{R}^{t_n} f\right|\right\|_{L^p(\mathbb{R}^d)} \leqslant B(p,k)\|f\|_{L^p(\mathbb{R}^d)}.$$

Since $\varepsilon\in(0,1)$ was arbitrary this completes the proof of Theorem 3.5.2.

It remains to verify claim (3.5.5). Since $\widetilde{R}^t=\sum_{j\in I}\widetilde{R}^t_j\widetilde{R}_j$ it is easy to see that

$$\widetilde{R}^{t,u}F=\sum_{j\in I}\widetilde{R}^{t,u}_j\widetilde{R}^u_j F,$$

where, for $F=f\otimes\eta$, we denote

$$\widetilde{R}^{t,u}_j(F)(x+iy)=\widetilde{R}^t_j(\delta_u F)(x+iu^{-1}y),\qquad \widetilde{R}^u_j(F)(x+iy)=\widetilde{R}_j(\delta_u F)(x+iu^{-1}y).$$

Thus, it is enough to justify that

$$\lim_{u\to 0^+}\langle \widetilde{R}^{t,u}_j\widetilde{R}^u_j F,\, G\rangle_{L^2(\mathbb{C}^d)}=\langle (\mathcal{R}^t_j R_j f)\otimes\eta,\, G\rangle_{L^2(\mathbb{C}^d)} \tag{3.5.6}$$

for $j\in I$, $t>0$, and $G\in\mathcal{S}(\mathbb{C}^d)$.

Fix $j\in I$ and $t>0$ and denote by m^t and m the multiplier symbols on $\mathbb{C}^d$ corresponding to the operators $\widetilde{R}^t_j$ and $\widetilde{R}_j$, respectively. Then $\delta_u(m^t)$ and $\delta_u(m)$ are the multiplier symbols corresponding to the operators $\widetilde{R}^{t,u}_j$ and $\widetilde{R}^u_j$, respectively. Thus, identifying $\mathbb{C}^d$ with $\mathbb{R}^{2d}$, taking the Fourier transform on $\mathbb{R}^{2d}$, and using Plancherel's theorem we see that

$$\langle \widetilde{R}^{t,u}_j\widetilde{R}^u_j F,\, G\rangle_{L^2(\mathbb{C}^d)}=\langle \delta_u(m)\delta_u(m^t)\mathcal{F}[F],\mathcal{F}[G]\rangle_{L^2(\mathbb{C}^d)}. \tag{3.5.7}$$

By formula (3.0.3) (applied on $\mathbb{R}^{2d}$) and definitions (3.4.1), (3.4.2) for $P_j(z):=z_j=z_{j_1}\cdots z_{j_k}$ we have

$$\delta_u(m)(\xi,\tau)=(-i)^k\frac{P_j(\xi+iu\tau)}{|\xi+iu\tau|^k},$$

for $\xi,\tau\in\mathbb{R}^d$. Hence, for $\xi\neq 0$ and $\tau\in\mathbb{R}^d$ it holds that $\lim_{u\to 0^+}m(\xi,u\tau)=m(\xi,0)=(-i)^k\frac{P_j(\xi)}{|\xi|^k}$. Another application of (3.0.3) (this time on $\mathbb{R}^d$) shows that the function $m_0(\xi):=m(\xi,0)$ is the multiplier symbol of the operator R_j acting on $L^2(\mathbb{R}^d)$.

Since the operators $\widetilde{R}^t_j$ and $\widetilde{R}_j$ are both bounded on $L^2(\mathbb{C}^d)$ the functions $\delta_u(m)$ and $\delta_u(m^t)$ are in $L^\infty(\mathbb{C}^d)$ uniformly in $u>0$. Thus, coming back to (3.5.7) and using Lebesgue's dominated convergence theorem we see that

$$\lim_{u\to 0^+}\langle \widetilde{R}^{t,u}_j\widetilde{R}^u_j F,\, G\rangle_{L^2(\mathbb{C}^d)}=\lim_{u\to 0^+}\langle \delta_u(m^t)\,\mathcal{F}[F],\,\overline{m_0}\,\mathcal{F}[G]\rangle_{L^2(\mathbb{C}^d)},$$

provided the limit on the right hand side exists. By definition of m_0 applying again Plancherel's theorem we obtain

$$\lim_{u\to 0^+}\langle \widetilde{R}^{t,u}_j\widetilde{R}^u_j F,\, G\rangle_{L^2(\mathbb{C}^d)}=\lim_{u\to 0^+}\langle \widetilde{R}^{t,u}_j F,\,(R_j\otimes I)^*G\rangle_{L^2(\mathbb{C}^d)}, \tag{3.5.8}$$

provided the limit on the right hand side exists. In the above formula $R_j \otimes I$ denotes the operator R_j acting only on the $\mathbb{R}^d$ coordinates of a function defined on $\mathbb{C}^d$ and the adjoint is taken with respect to the inner product on $L^2(\mathbb{C}^d)$. Now, if we justify that

$$\lim_{u\to 0^+} \langle \widetilde{R}_j^{t,u} F,\, (R_j \otimes I)^* G\rangle_{L^2(\mathbb{C}^d)} = \langle \mathcal{R}_j^t(f)\otimes\eta,\, (R_j\otimes I)^* G\rangle_{L^2(\mathbb{C}^d)} \tag{3.5.9}$$

and use the formula

$$\langle \mathcal{R}_j^t(f)\otimes\eta,\, (R_j\otimes I)^* G\rangle_{L^2(\mathbb{C}^d)} = \langle (\mathcal{R}_j^t R_j f)\otimes\eta,\, G\rangle_{L^2(\mathbb{C}^d)}$$

together with (3.5.8), then we will complete the proof of claim (3.5.6).

Since the operators $\widetilde{R}_j^{t,u}$ are bounded on $L^2(\mathbb{C}^d)$ uniformly with respect to $u > 0$, to prove (3.5.9) it suffices to show that

$$\lim_{u\to 0^+} \langle \widetilde{R}_j^{t,u} F, \widetilde{G}\,\rangle_{L^2(\mathbb{C}^d)} = \langle \mathcal{R}_j^t(f)\otimes\eta,\, \widetilde{G}\rangle_{L^2(\mathbb{C}^d)}, \tag{3.5.10}$$

where $\widetilde{G} \in \mathcal{S}(\mathbb{C}^d)$, and use the density of Schwartz functions in $L^2(\mathbb{C}^d)$. For $z = x + iy$, $x, y \in \mathbb{R}^d$, we have

$$\begin{aligned}
&\widetilde{R}_j^{t,u}(F)(z) = \widetilde{R}_j^{t,u}(f\otimes\eta)(z) = u^{-d}\delta_{u^{-1}}(\widetilde{K}_j^t) * (f\otimes\eta)(z)\\
&= \int_{\mathbb{R}^d} f(x-x') \int_{y'\in\mathbb{R}^d\colon |x'+iu^{-1}y'|>t} \widetilde{\gamma}_k u^{-d} \frac{P_j(x'+iu^{-1}y')}{|x'+iu^{-1}y'|^{2d+k}} \eta(y-y')\, dy'\, dx'\\
&= \int_{\mathbb{R}^d}\int_{y'\in\mathbb{R}^d\colon |x'+iy'|>t} f(x-x')\, \widetilde{\gamma}_k \frac{P_j(x'+iy')}{|x'+iy'|^{2d+k}} \eta(y-uy')\, dy'\, dx'.
\end{aligned} \tag{3.5.11}$$

Moreover, we will show that for fixed $t > 0$ it holds

$$f(x-x')\widetilde{\gamma}_k \frac{P_j(x'+iy')}{|x'+iy'|^{2d+k}} \mathbb{1}_{|x'+iy'|>t} \in L^1(\mathbb{C}^d). \tag{3.5.12}$$

uniformly in $x \in \mathbb{R}^d$. Recall that P_j is a homogeneous polynomial of degree k and that f is a Schwartz function. Hence

$$\begin{aligned}
\int_{\mathbb{R}^d}\int_{|x'+iy'|>t} \left| f(x-x') \frac{P_j(x'+iy')}{|x'+iy'|^{2d+k}} \right| dy'\, dx' &\lesssim \int_{\mathbb{R}^d}\int_{|x'+iy'|>t} \frac{|f(x-x')|}{|x'+iy'|^{2d}}\, dy'\, dx'\\
&\leqslant \int_{\mathbb{R}^d} |f(x-x')| \left(\int_{|y'|\leqslant t} \frac{dy'}{t^{2d}} + \int_{|y'|>t} \frac{dy'}{|y'|^{2d}} \right) dx' \lesssim_{t,d} \|f\|_1.
\end{aligned}$$

Hence, taking the limit as $u \to 0^+$ in (3.5.11) and using Lebesgue's dominated convergence theorem followed by Lemma 3.5.3 we obtain

$$\begin{aligned}
&\lim_{u\to 0^+} \widetilde{R}_j^{t,u}(F)(z) = \eta(y) \int_{\mathbb{R}^d} f(x-x') \int_{y'\in\mathbb{R}^d\colon |x'+iy'|>t} \widetilde{\gamma}_k \frac{P_j(x'+iy')}{|x'+iy'|^{2d+k}}\, dy'\, dx'\\
&= \eta(y)\int_{\mathbb{R}^d} f(x-x')\mathcal{K}_j^t(x')\, dx' = \eta(y)\mathcal{R}_j^t f(x) = (\mathcal{R}_j^t(f)\otimes\eta)(x,y),
\end{aligned} \tag{3.5.13}$$

for $x, y \in \mathbb{R}^d$. Moreover, another application of (3.5.12) shows that $\widetilde{R}_j^{t,u}(F) \in L^\infty(\mathbb{C}^d)$ uniformly in $u > 0$. Now, since $\widetilde{G} \in \mathcal{S}(\mathbb{C}^d)$ using again Lebesgue's dominated convergence theorem followed by (3.5.13) we reach

$$\lim_{u\to 0^+} \langle \widetilde{R}_j^{t,u} F, \widetilde{G}\,\rangle_{L^2(\mathbb{C}^d)} = \langle \lim_{u\to 0^+} \widetilde{R}_j^{t,u} F, \widetilde{G}\,\rangle_{L^2(\mathbb{C}^d)} = \langle \mathcal{R}_j^t(f)\otimes\eta,\, \widetilde{G}\rangle_{L^2(\mathbb{C}^d)}.$$

This justifies (3.5.10), hence, also claim (3.5.6). The proof of Theorem 3.5.2 is thus completed. □

Now we prove Theorem 3.5.1. Note that we will use claim (3.5.5) justified in the proof of Theorem 3.5.2.

Proof of Theorem 3.5.1. By Lebesgue's monotone convergence theorem we may restrict the supremum in the definition of $\mathcal{R}^*$ to a finite set of positive numbers $\{t_1, \dots, t_N\}$, as long as our final estimate is independent of N. Further, a density argument shows that it suffices to consider $f_1, \dots, f_S \in \mathcal{S}(\mathbb{R}^d)$.

For $F\colon \mathbb{C}^d \to \mathbb{C}$ and $u > 0$ we let $(\delta_u F)(x+iy) = F(x+iuy)$ and define

$$\widetilde{R}^{t,u}(F)(x+iy) := (\delta_{u^{-1}} \circ \widetilde{R}^t \circ \delta_u)(F)(x+iy) = \widetilde{R}^t(\delta_u F)(x+iu^{-1}y).$$

Using Theorem 3.4.1 it is straightforward to see that

$$\left\| \left(\sum_{s=1}^{S} \sup_{n=1,\dots,N} \left| \widetilde{R}^{t_n,u} F_s \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{C}^d)} \leqslant A(p,k) \left\| \left(\sum_{s=1}^{S} |F_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{C}^d)}.$$

Note that by duality between the spaces $L^p(\mathbb{C}^d; E_\infty)$ and $L^q(\mathbb{C}^d; E_1)$, where

$$E_\infty = \ell^2\left(\{1,\dots,S\}; \ell^\infty(\{t_1,\dots,t_N\})\right) \quad \text{and} \quad E_1 = \ell^2\left(\{1,\dots,S\}; \ell^1(\{t_1,\dots,t_N\})\right),$$

the above inequality can be rewritten in the following equivalent form

$$\begin{aligned} &\left| \sum_{s=1}^{S} \sum_{n=1}^{N} \langle \widetilde{R}^{t_n,u} F_s,\, G_{n,s} \rangle_{L^2(\mathbb{C}^d)} \right| \\ &\qquad \leqslant A(p,k) \left\| \left(\sum_{s=1}^{S} |F_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{C}^d)} \left\| \left(\sum_{s=1}^{S} \left(\sum_{n=1}^{N} |G_{n,s}| \right)^2 \right)^{1/2} \right\|_{L^q(\mathbb{C}^d)}, \end{aligned} \tag{3.5.14}$$

where $G_{n,s} \in L^q(\mathbb{C}^d, E_1)$.

Let $\eta \in \mathcal{S}(\mathbb{R}^d)$ be a fixed function such that $\|\eta\|_{L^p(\mathbb{R}^d)} = 1$, take $f_1, \dots, f_S \in \mathcal{S}(\mathbb{R}^d)$ and denote

$$F_s(x+iy) := (f_s \otimes \eta)(x,y) = f_s(x) \cdot \eta(y), \qquad x, y \in \mathbb{R}^d, \quad s = 1, \dots, S.$$

Fix $\varepsilon \in (0,1)$ and let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be a function of $L^q(\mathbb{R}^d)$ norm 1 and such that $|\langle \eta, \psi \rangle_{L^2(\mathbb{R}^d)}| \geqslant (1-\varepsilon)$. Take $g_{n,s} \in \mathcal{S}(\mathbb{R}^d)$, $n = 1, \dots, N$, $s = 1, \dots, S$. Then, substituting $F_s = f_s \otimes \eta$ and $G_{n,s} = g_{n,s} \otimes \psi$ in (3.5.14) we have

$$\begin{aligned} &\left| \sum_{s=1}^{S} \sum_{n=1}^{N} \langle \widetilde{R}^{t_n,u}(f_s \otimes \eta),\, g_{n,s} \otimes \psi \rangle_{L^2(\mathbb{C}^d)} \right| \\ &\qquad \leqslant A(p,k) \left\| \left(\sum_{s=1}^{S} |f_s \otimes \eta|^2 \right)^{1/2} \right\|_{L^p(\mathbb{C}^d)} \left\| \left(\sum_{s=1}^{S} \left(\sum_{n=1}^{N} |g_{n,s} \otimes \psi| \right)^2 \right)^{1/2} \right\|_{L^q(\mathbb{C}^d)}. \end{aligned}$$

At this point claim (3.5.5) from the previous proof implies

$$\left|\sum_{s=1}^{S}\sum_{n=1}^{N}\langle \mathcal{R}^{t_n} f_s,\, g_{n,s}\rangle_{L^2(\mathbb{R}^d)}\right|\left|\langle \eta,\psi\rangle_{L^2(\mathbb{R}^d)}\right| \leqslant A(p,k)\left\|\left(\sum_{s=1}^{S}|f_s|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)}\left\|\left(\sum_{s=1}^{S}\left(\sum_{n=1}^{N}|g_{n,s}|\right)^2\right)^{1/2}\right\|_{L^q(\mathbb{R}^d)}.$$

Now, using duality between the spaces $L^p(\mathbb{R}^d;E_\infty)$ and $L^q(\mathbb{R}^d;E_1)$ together with the density of Schwartz function in $L^q(\mathbb{R}^d)$ we conclude that

$$(1-\varepsilon)\left\|\left(\sum_{s=1}^{S}\sup_{n=1,\ldots,N}\left|\mathcal{R}^{t_n} f_s\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)} \leqslant A(p,k)\left\|\left(\sum_{s=1}^{S}|f_s|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)}.$$

Since $\varepsilon\in(0,1)$ was arbitrary this completes the proof of Theorem 3.5.1. □

3.5.2 Bounding the difference between R^t and $\mathcal{R}^t$

Define the difference kernels on $\mathbb{R}^d$ by

$$E_j^t(x) := K_j^t(x) - \mathcal{K}_j^t(x). \tag{3.5.15}$$

Recall that by definitions (3.0.1) of K_j^t and (3.5.1) of $\mathcal{K}_j^t$ we have $E_j^t(x) = -\mathcal{K}_j^t(x)$ if $|x|\leqslant t$ and $E_j^t(x)=0$ if $|x|>t$. We let D_j be the operator on $L^p(\mathbb{R})$ given by $D_j^t f = E_j^t * f$ and define

$$D^t f = \sum_{j\in I} D_j^t R_j f, \qquad D^* f = \sup_{t\in\mathbb{Q}_+}\left|D^t f\right|. \tag{3.5.16}$$

Clearly,

$$R^t = \mathcal{R}^t + D^t,$$

so using Theorems 3.5.1 and 3.5.2 we reduce Theorems 3.3.2 and 3.3.3 to the following two statements.

Theorem 3.5.4. *Fix $k\in\mathbb{N}$. For each $p\in(1,\infty)$ there is a constant $A(p,k)$ independent of the dimension d and such that for any $S\in\mathbb{N}$ we have*

$$\left\|\left(\sum_{s=1}^{S}|D^* f_s|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)} \leqslant A(p,k)\left\|\left(\sum_{s=1}^{S}|f_s|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)},$$

whenever $f_1,\ldots,f_S\in L^p(\mathbb{R}^d)$. Moreover, $A(p,k)$ satisfies $A(p,k)\lesssim_k (p^*)^{5/2+k/2}$.

Theorem 3.5.5. *Fix $k\in\mathbb{N}$. For each $p\in(1,\infty)$ there is a constant $B(p,k)$ independent of the dimension d and such that*

$$\|D^* f\|_{L^p(\mathbb{R}^d)} \leqslant B(p,k)\|f\|_{L^p(\mathbb{R}^d)},$$

whenever $f\in L^p(\mathbb{R}^d)$. Moreover, $B(p,k)$ satisfies $B(p,k)\lesssim_k (p^*)^{2+k/2}$.

The proofs of the above two theorems will follow the scheme of the proofs of Theorems 3.4.1 and 3.4.2. The main difference lies in the application of the method of rotations. It has to be appropriate for the operator D^t. For $t>0$ we let I^t be the function on $(0,\infty)$ given by

$$I^t(r) = \mathbb{1}_{(0,t)}(r)\int_{\sqrt{\frac{t^2}{r^2}-1}}^{\infty} \frac{s^{d-1}}{(1+s^2)^{d+k/2}}\,ds, \qquad r>0. \tag{3.5.17}$$

Using the definitions (3.5.1) and (3.5.15) and integrating in polar coordinates in $\mathbb{R}^d$ we obtain

$$\begin{aligned} -D_j^t f(x) &= \int_{\mathbb{R}^d} \widetilde{\gamma}_k S_{d-1} \frac{y_j}{|y|^{d+k}} I^t(|y|) f(x-y)\,dy \\ &= \widetilde{\gamma}_k S_{d-1}^2 \int_0^t \int_{S^{d-1}} \frac{\omega_j}{r} I^t(r) f(x-r\omega)\,d\omega\,dr \\ &= \gamma_k S_{d-1} \int_{S^{d-1}} \omega_j \mathcal{H}_\omega^t f(x)\,d\omega = \frac{2\Gamma\left(\frac{k+d}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{d}{2}\right)} \int_{S^{d-1}} \omega_j \mathcal{H}_\omega^t f(x)\,d\omega, \end{aligned} \tag{3.5.18}$$

where

$$\mathcal{H}_\omega^t f(x) = \frac{\widetilde{\gamma}_k}{\gamma_k} S_{d-1} \int_0^t I^t(r) \frac{f(x-r\omega)}{r}\,dr. \tag{3.5.19}$$

Let now $\mathcal{H}_\omega^* f(x) = \sup_{t\in\mathbb{Q}_+} \left|\mathcal{H}_\omega^t f(x)\right|$. The next proposition serves as a replacement for Proposition 3.4.3.

Proposition 3.5.6. *For each $1<p<\infty$ we have*

$$\left\| \left(\sum_{s=1}^S |\mathcal{H}_\omega^* f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \lesssim p^* \left\| \left(\sum_{s=1}^S |f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \tag{3.5.20}$$

uniformly in $\omega\in S^{d-1}$ and the dimension d.

Proof. For $f\colon \mathbb{R}^d\to\mathbb{C}$, $\omega\in S^{d-1}$ and $t>0$ we let

$$\mathcal{M}_\omega^t f(x) = \frac{1}{t}\int_{-t}^t |f(x-r\omega)|\,dr, \qquad \mathcal{M}_\omega^* f(x) = \sup_{t>0} \left|\mathcal{M}_\omega^t f(x)\right|,$$

be the directional Hardy–Littlewood averaging operator and the directional Hardy–Littlewood maximal function. Using Fubini's theorem and one-dimensional estimates for the Hardy–Littlewood maximal function, see e.g. [22, Theorem 5.6.6], we obtain

$$\left\| \left(\sum_{s=1}^S |\mathcal{M}_\omega^* f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \lesssim p^* \left\| \left(\sum_{s=1}^S |f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)},$$

uniformly in $\omega\in S^{d-1}$. Thus, to prove (3.5.20) it suffices to show the pointwise estimate

$$\mathcal{H}_\omega^t f(x) \lesssim \mathcal{M}_\omega^t f(x)$$

uniformly in $x\in\mathbb{R}^d$, $\omega\in S^{d-1}$, with implicit constants independent of the dimension.

This bound will follow if we justify that

$$\frac{\widetilde{\gamma}_k}{\gamma_k} S_{d-1} \frac{I^t(r)}{r} \lesssim \frac{1}{t}, \tag{3.5.21}$$

with the implicit constant being uniform in $t>0$, $0\leqslant r\leqslant t$, and the dimension d. Note that for $s\geqslant(\frac{t^2}{r^2}-1)^{1/2}$ we have $\frac{1}{r}\leqslant\frac{\sqrt{s^2+1}}{t}$. Hence, recalling (3.5.17) and using (3.0.6) we obtain

$$\begin{aligned}\frac{\widetilde{\gamma}_k}{\gamma_k}S_{d-1}\frac{I^t(r)}{r}&\leqslant\frac{\widetilde{\gamma}_k}{\gamma_k}S_{d-1}\frac{1}{t}\int_{\sqrt{\frac{t^2}{r^2}-1}}^{\infty}\frac{s^{d-1}}{(1+s^2)^{d+(k-1)/2}}\,ds\\&\leqslant S_{d-1}\frac{\widetilde{\gamma}_k}{\gamma_k}\frac{1}{t}\int_0^{\infty}\frac{s^{d-1}}{(1+s^2)^{d+(k-1)/2}}\,ds=S_{d-1}\frac{\widetilde{\gamma}_k}{\gamma_k}\frac{\Gamma\left(\frac{d+k-1}{2}\right)\Gamma\left(\frac{d}{2}\right)}{2\Gamma\left(d+\frac{k-1}{2}\right)}\cdot\frac{1}{t}.\end{aligned}$$

Applying (1.5.5) and (3.0.5) we reach

$$\begin{aligned}S_{d-1}\frac{\widetilde{\gamma}_k}{\gamma_k}\frac{I^t(r)}{r}&\leqslant\frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\frac{\Gamma(d+\frac{k}{2})}{\pi^{d/2}\Gamma\left(\frac{d+k}{2}\right)}\frac{\Gamma\left(\frac{d+k-1}{2}\right)\Gamma\left(\frac{d}{2}\right)}{2\Gamma\left(d+\frac{k-1}{2}\right)}\cdot\frac{1}{t}\\&=\frac{\Gamma(d+\frac{k}{2})}{\Gamma\left(d+\frac{k-1}{2}\right)}\cdot\frac{\Gamma\left(\frac{d+k-1}{2}\right)}{\Gamma\left(\frac{d+k}{2}\right)}\cdot\frac{1}{t}.\end{aligned}$$

Since k is fixed, using (1.5.3) we conclude that

$$S_{d-1}\frac{\widetilde{\gamma}_k}{\gamma_k}\frac{I^t(r)}{r}\lesssim\frac{\left(d+\frac{k-1}{2}\right)^{1/2}}{\left(\frac{d}{2}+\frac{k-1}{2}\right)^{1/2}}\cdot\frac{1}{t}\lesssim\frac{1}{t}.$$

Thus, we completed the proof of (3.5.21) and hence also the proof of Proposition 3.5.6.

□

We will also need vector-valued estimates for $\{R_j(f_s)\}$, $j\in I$, $s=1,\ldots,d$. The following proposition can be deduced from Proposition 3.4.4 if we proceed along the lines of [26, Section 4].

Proposition 3.5.7. *For each $1<p<\infty$ we have*

$$\left\|\left(\sum_{s=1}^{S}\sum_{j\in I}|R_jf_s|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)}\lesssim p^*p^{1/2}q^{\frac{k+1}{2}}\left\|\left(\sum_{s=1}^{S}|f_s|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)},\tag{3.5.22}$$

$$\left\|\left(\sum_{j\in I}|R_jf|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)}\lesssim p^*q^{k/2}\|f\|_{L^p(\mathbb{R}^d)},\tag{3.5.23}$$

uniformly in the dimension d.

Proof. In contrast to the proofs of Theorem 3.5.1 and Theorem 3.5.2 here we apply the methods from [26, Section 4] in a direct way. Therefore we shall be brief. Let $n=k=d$ and identify $\mathbb{C}^d$ with $\mathbb{R}^{2d}$.

For the proof (3.5.22) we take $E=\ell^2(\{1,\ldots,S\})$ and $F=\ell^2(\{1,\ldots,S\}\times I)$. The operator $\mathbf{T}$ is defined by

$$\mathbf{T}(\{f_s\}_{s=1,\ldots,S})=\{\widetilde{R}_j(f_s)\}_{(s,j)\in\{1,\ldots,S\}\times I}.$$

Using (3.0.3) for $P(z)=z_{j_1}\cdots z_{j_k}$ one can check that the restricted operator $\mathbf{T}_0$ is then

$$\mathbf{T}_0(\{f_s\}_{s=1,\ldots,S})=\{R_j(f_s)\}_{(s,j)\in\{1,\ldots,S\}\times I}.$$

Hence, [26, eq. (45)] together with (3.4.8) lead to (3.5.22).

The proof of (3.5.23) is similar. We take $E = \mathbb{C}$ and $F = \ell^2(I)$. The operators $\mathbf{T}$ and $\mathbf{T_0}$ are defined as above. The desired inequality follows from [26, eq. (45)] together with (3.4.9). □

We are finally ready to justify Theorems 3.5.4 and 3.5.5. At this point the proofs mimic the corresponding proofs of Theorems 3.4.1 and 3.4.2.

Proof of Theorem 3.5.4. We proceed analogously to the proof of Theorem 3.4.1 on p. 56, nevertheless we present the proof for completeness.

Observe that it follows from (3.5.16) and (3.5.18) that

$$D^t f(x) = -\frac{2\Gamma\left(\frac{k+d}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{d}{2}\right)} \int_{S^{d-1}} \mathcal{H}^t_\omega \left[\sum_{j\in I} \omega_j R_j f\right](x)\, d\omega. \tag{3.5.24}$$

Using this identity, estimate (3.4.7) with $\frac{d}{2}$ in place of d, and Minkowski's integral inequality on the space $\ell^2(\{1,\dots,S\}; L^\infty(\mathbb{Q}_+))$ we see that

$$\left(\sum_{s=1}^S |D^* f_s(x)|^2\right)^{1/2} \lesssim d^{k/2} \int_{S^{d-1}} \left(\sum_{s=1}^S \left(\mathcal{H}^*_\omega\left[\sum_{j\in I}\omega_j R_j f_s\right](x)\right)^2\right)^{1/2} d\omega, \qquad x\in\mathbb{R}^d.$$

Thus, another application of Minkowski's integral inequality followed by Proposition 3.5.6 gives

$$\left\|\left(\sum_{s=1}^S |D^* f_s|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)} \lesssim p^* d^{k/2} \int_{S^{d-1}} \left\|\left(\sum_{s=1}^S \left|\sum_{j\in I}\omega_j R_j f_s\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)} d\omega.$$

Using Khintchine's inequality (3.2.19) followed by Hölder's inequality on S^{d-1} we see that

$$\begin{aligned}
&\left\|\left(\sum_{s=1}^S |D^* f_s|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)} \\
&\lesssim p^* d^{k/2} \int_{S^{d-1}} \left(\int_{\mathbb{R}^d}\int_0^1 \left|\sum_{s=1}^S r_s(\xi)\sum_{j\in I}\omega_j R_j f_s(x)\right|^p d\xi\, dx\right)^{1/p} d\omega \\
&\lesssim p^* d^{k/2} \left(\int_{\mathbb{R}^d}\int_0^1\int_{S^{d-1}} \left|\sum_{j\in I}\omega_j R_j\left[\sum_{s=1}^S r_s(\xi) f_s(x)\right]\right|^p d\omega\, d\xi\, dx\right)^{1/p}.
\end{aligned}$$

Since for fixed x and ξ the function $\omega \mapsto \sum_{j\in I}\omega_j R_j\left[\sum_{s=1}^S r_s(\xi) f_s(x)\right]$ belongs to $\mathcal{H}^d_k$, applying Lemma 3.2.3 we obtain

$$\left(\int_{S^{d-1}} \left|\sum_{j\in I}\omega_j R_j\left[\sum_{s=1}^S r_s(\xi) f_s(x)\right]\right|^p d\omega\right)^{1/p} \lesssim p^{k/2} \left(\int_{S^{d-1}} \left|\sum_{j\in I}\omega_j R_j\left[\sum_{s=1}^S r_s(\xi) f_s(x)\right]\right|^2 d\omega\right)^{1/2}.$$

Then we use orthogonality of the functions ω_j with respect to the inner product on S^{d-1} and an estimate for their L^2 norms similar to (3.4.14), followed by (3.5.23) from Proposition 3.5.7 and Khintchine's inequality (3.2.18) to get

$$\left\| \left(\sum_{s=1}^{S} |D^* f_s|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \lesssim p^* p^{k/2} d^{k/2} \left(\int_{\mathbb{R}^d} \int_0^1 \left(\sum_{j\in I} \left| R_j \left[\sum_{s=1}^{S} r_s(\xi) f_s(x) \right] \right|^2 \right)^{p/2} d\xi\, dx \right)^{1/p}$$
$$\lesssim (p^*)^{2+k/2} \left(\int_{\mathbb{R}^d} \int_0^1 \left| \sum_{s=1}^{S} r_s(\xi) f_s(x) \right|^p d\xi\, dx \right)^{1/p} \lesssim (p^*)^{5/2+k/2} \left(\int_{\mathbb{R}^d} \left(\sum_{s=1}^{S} |f_s|^2 \right)^{p/2} dx \right)^{1/p}.$$

The proof of Theorem 3.5.4 is thus completed. □

Proof of Theorem 3.5.5. We proceed analogously to the proof of Theorem 3.4.2 on p. 55.

Using (3.5.24) and (3.4.7) with $\frac{d}{2}$ in place of d we see that

$$|D^* f(x)| \lesssim d^{k/2} \int_{S^{d-1}} \mathcal{H}_\omega^* \left[\sum_{j\in I} \omega_j R_j f \right](x)\, d\omega, \qquad x \in \mathbb{R}^d.$$

Hence, Minkowski's integral inequality followed by Proposition 3.5.6 show that

$$\|D^* f\|_{L^p(\mathbb{R}^d)} \lesssim p^* d^{k/2} \int_{S^{d-1}} \left\| \sum_{j\in I} \omega_j R_j f \right\|_{L^p(\mathbb{R}^d)} d\omega.$$

Using Hölder's inequality and Fubini's theorem we obtain

$$\|D^* f\|_{L^p(\mathbb{R}^d)} \lesssim p^* d^{k/2} \left(\int_{\mathbb{R}^d} \int_{S^{d-1}} \left| \sum_{j\in I} \omega_j R_j f(x) \right|^p d\omega\, dx \right)^{1/p}. \tag{3.5.25}$$

Since for fixed x the function $\omega \mapsto \sum_{j\in I} \omega_j R_j f(x)$ belongs to $\mathcal{H}_k^d$, applying Lemma 3.2.3 we obtain

$$\left(\int_{S^{d-1}} \left| \sum_{j\in I} \omega_j R_j f(x) \right|^p d\omega \right)^{1/p} \lesssim p^{k/2} \left(\int_{S^{d-1}} \left| \sum_{j\in I} \omega_j R_j f(x) \right|^2 d\omega \right)^{1/2}.$$

Using orthogonality of ω_j and a version of (3.4.14) for ω_j we thus see that

$$\left(\int_{S^{d-1}} \left| \sum_{j\in I} \omega_j R_j f(x) \right|^p d\omega \right)^{1/p} \lesssim d^{-k/2} p^{k/2} \left(\sum_{j\in I} |R_j f(x)|^2 \right)^{1/2},$$

which, together with (3.5.25) leads to

$$\|D^* f\|_{L^p(\mathbb{R}^d)} \lesssim p^* p^{k/2} \left\| \left(\sum_{j\in I} |R_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)}.$$

Thus, (3.5.23) from Proposition 3.5.7 completes the proof of Theorem 3.5.5. □

Part II

Riesz transforms associated with Schrödinger operators

Chapter 4

L^p estimates for Riesz transforms associated with Schrödinger operators

In the second part of the dissertation we consider a class of Riesz transforms related to the Schrödinger operator

$$L = -\frac{1}{2}\Delta + V,$$

where Δ is the Laplacian on $\mathbb{R}^d$ and V is a non-negative locally integrable function called the potential. The operator L is rigorously defined via quadratic forms, see Section 4.1. The Riesz transforms are formally given for $a > 0$ by

$$R_V^a f(x) = V^a(x) \cdot \left(-\tfrac{1}{2}\Delta + V\right)^{-a} f(x) = \frac{V^a(x)}{\Gamma(a)} \cdot \int_0^\infty e^{-tL} f(x)\, t^{a-1}\, dt, \tag{4.0.1}$$

where e^{-tL} is the semigroup generated by L.

In this chapter we consider a wide range of potentials and prove results on L^p boundedness of the operators R_V^a with norm estimates which depend on the dimension d, unlike in other chapters of the dissertation. In Chapter 5 we focus on a specific class of potentials which lets us obtain dimension-free estimates of the L^p norm of R_V^a.

There are two main results in this chapter. First we consider a general locally integrable potential and prove L^p boundedness of the Riesz transform associated with it, namely

Theorem 4.0.1. *Let $V \in L^1_{\mathrm{loc}}$ and take $p \in (1, 2]$. Then for all $0 \leqslant a \leqslant \frac{1}{p}$ the Riesz transform R_V^a is bounded on L^p.*

The theorem generalizes several earlier results described in Section 1.3.2. It is derived as a consequence of the endpoint bounds for $R_V^{1/2}$ on L^2, see Proposition 4.1.3 and for R_V^1 on L^1 ([2, Theorem 4.3], see also [21, 27]) together with the interpolation result given below.

Theorem 4.0.2. *Let $0 < a_0 < a_1$. Assume that $V \in L^1_{\mathrm{loc}}$ is such that $R_V^{a_1}$ is bounded on L^{p_1} for some $p_1 \in [1, \infty)$ and $R_V^{a_0}$ is bounded on L^1. Then, R_V^a is bounded on L^p for every p and a such that $\frac{1}{p} = \theta + \frac{1-\theta}{p_1}$ and $a = \theta a_0 + (1-\theta) a_1$ with some $\theta \in (0, 1)$.*

The other results concern L^∞ and L^1 boundedness of R_V^a for specific classes of non-negative potentials V, for which we assume a certain condition relating the value $V(x)$ and the speed at which $V(y)$ decreases for y in a ball around x. The main classes of potentials to which our results apply are given in the following theorem. In order to make the presentation clearer, we will say that some property holds *globally* if there is a compact set $F \subseteq \mathbb{R}^d$ such that the property holds for almost all $x \in \mathbb{R}^d \setminus F$.

Theorem 4.0.3. *Let $V\colon \mathbb{R}^d \to [0,\infty)$ be a function in $L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$. Then in all the three cases*

1. *$V(x) \approx 1$ globally*
2. *For some $\alpha > 0$ we have $V(x) \approx |x|^\alpha$ globally*
3. *For some $\beta > 1$ we have $V(x) \approx \beta^{|x|}$ globally*

each of the Riesz transforms R^a_V, $a > 0$, is bounded on L^∞ and on L^1.

In the proof of the above theorem we first use the positivity-preserving property of the semigroup e^{-tL} so that we only need to bound the quantity

$$R^a_V(\mathbb{1})(x) = V^a(x) \cdot \int_0^\infty e^{-tL}(\mathbb{1})(x)\, t^{a-1}\, dt.$$

We estimate it using the Feynman–Kac formula

$$e^{-tL}(\mathbb{1})(x) = \mathbb{E}_x \left[e^{-\int_0^t V(X_s)\, ds} \right]$$

by splitting the underlying probability space into events relating the value of $V(X_s)$ and $V(x)$ in a way which facilitates the estimates and which was described in Section 1.4.

The case of L^1 estimates is similar, but more complex. First we use duality between the spaces L^1 and L^∞ in order to reduce the task of estimating the L^1 norm of the operator $R^a_V = V^a L^{-a}$ to estimating the L^∞ norm of the operator $L^{-a}V^a$. Similarly to the previous case, we use the positivity-preserving property of L^{-a} and we remain with the goal of bounding the quantity

$$L^{-a}(V^a)(x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL}(V^a)(x)\, t^{a-1}\, dt$$

by a constant independent of x.

Before we move on to the proof we establish notations used in this chapter.

1. We say that f is a finitely simple function if it is a simple function supported in a compact subset of $\mathbb{R}^d$. Such functions are clearly dense in L^p, $1 \leqslant p < \infty$.

   The space of smooth compactly supported functions on $\mathbb{R}^d$ is denoted by C^∞_c.

2. For $x \in \mathbb{R}^d$ and $r > 0$ we denote by $B(x,r) := \{y \in \mathbb{R}^d : |x-y| \leqslant r\}$ the closed Euclidean ball of radius r.

   For a Lebesgue-measurable subset $A \subseteq \mathbb{R}^d$ we denote by $|A|$ its Lebesgue measure.

3. For a random variable X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $A \subseteq \mathbb{R}$ we denote $\mathbb{P}(X \in A) := \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\})$. We abbreviate *almost everywhere* and *almost every* to *a.e.*

4. The symbol C_Δ stands for a constant that possibly depends on $\Delta > 0$. We write C without a subscript when the constant is universal in the sense that it may depend only on the dimension d and the parameter a of the Riesz transform.

5. For two quantities A and B we write $A \lesssim B$ if $A \leqslant CB$ for some constant $C > 0$ which may depend on d and a. If both $A \lesssim B$ and $B \lesssim A$ hold, then we write $A \approx B$.

   If A and B are functions on $\mathbb{R}^d$, then $A \lesssim B$ means that $A \leqslant CB$ for almost all $x \in \mathbb{R}^d$.

   For two functions A and B on $\mathbb{R}^d$ we write $A \leqslant_g B$ if $A(x) \leqslant B(x)$ for almost all $x \notin F$ for some compact set F. The same convention applies to the symbols $\lesssim$ and $\approx$.

4.1 Definitions and general results on L^p for $1 \leqslant p < \infty$

The main goal of this section is to define the Riesz transforms R_V^a, $a > 0$, on L^p and to prove L^p boundedness results for these operators valid for general classes of non-negative potentials V. Throughout this section we take $1 \leqslant p < \infty$. The case of $p = \infty$ is addressed in the next section.

Our general definition on L^p will be based on semigroups related to $-\frac{1}{2}\Delta + V$ that are given by the spectral theorem. Let $V \in L^1_{\text{loc}}$ be an a.e. non-negative potential. This assumption is in force throughout the chapter even if this is not stated explicitly. Whenever we write $V(x)$ we mean the value at x of a particular representative of the equivalence class of V in the space L^1_{loc}. The same is true for any expression in which similar ambiguity may arise. We follow closely the approach in [2, Section 3] (see also [11]) and define the Schrödinger operator L via quadratic forms. Consider the sesquilinear form

$$Q(u, v) = \int_{\mathbb{R}^d} \tfrac{1}{2} \langle \nabla u, \nabla v \rangle + V u \overline{v} \tag{4.1.1}$$

on the domain

$$\operatorname{Dom}(Q) = \{ f \in L^2 : \nabla f \in L^2 \text{ and } V^{1/2} f \in L^2 \},$$

where ∇f denotes the distributional gradient of f. We equip the domain with the norm

$$\|f\|_V = \left(\|f\|_2^2 + \tfrac{1}{2} \|\nabla f\|_2^2 + \left\| V^{1/2} f \right\|_2^2 \right)^{1/2},$$

which turns it into a Hilbert space with $C_c^\infty(\mathbb{R}^d)$ as a dense subspace. Since Q is bounded below and non-negative, there is a unique positive self-adjoint operator L such that

$$\langle Lu, v \rangle = Q(u, v), \quad u \in \operatorname{Dom}(L),\ v \in \operatorname{Dom}(Q)$$

and its square root $L^{1/2}$, defined on $\operatorname{Dom}(L^{1/2}) = \operatorname{Dom}(Q)$, satisfies

$$\left\| L^{1/2} f \right\|_2^2 = \tfrac{1}{2} \|\nabla f\|_2^2 + \left\| V^{1/2} f \right\|_2^2, \quad f \in C_c^\infty(\mathbb{R}^d). \tag{4.1.2}$$

By [2, Section 3] the semigroup e^{-tL} is positivity-preserving and pointwise dominated by the heat semigroup, hence it is a contraction on L^p for $1 \leqslant p \leqslant \infty$.

Let $a > 0$. For $f \in L^p$, $1 \leqslant p < \infty$, and $\varepsilon > 0$ we define

$$(L + \varepsilon I)^{-a} f := \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL} f \; t^{a-1} e^{-\varepsilon t} \, dt, \tag{4.1.3}$$

Since the semigroup e^{-tL} is a strongly continuous semigroup of contractions on L^p, the integral in (4.1.3) is well defined as a Bochner integral on L^p. It is also not hard to see that for $f \in L^2$ the operator defined by (4.1.3) coincides with $(L + \varepsilon I)^{-a}$ given by the spectral theorem. Moreover, if f is an a.e. non-negative function in L^p then

$$L^{-a} f(x) := \lim_{\varepsilon \to 0^+} \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL} f(x) \, t^{a-1} e^{-\varepsilon t} \, dt, \tag{4.1.4}$$

exists x-a.e. as a monotone pointwise limit, although it may be infinite. In either case

$$L^{-a} f(x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL} f(x) \, t^{a-1} \, dt \tag{4.1.5}$$

by the monotone convergence theorem. For $a > 0$ and a non-negative function $f \in L^p$ we let

$$R_V^a f(x) := V^a(x) L^{-a} f(x), \qquad x \in \mathbb{R}^d. \tag{4.1.6}$$

This is well defined x-a.e. though possibly equal to infinity. Additionally, for $a = 0$ we set R_V^0 to be the identity operator.

Definition 4.1.1. Let $1 \leqslant p < \infty$ and $a > 0$. We say that the Riesz transform R_V^a is bounded on L^p if there is a constant $C > 0$ such that

$$\|R_V^a f\|_p \leqslant C \|f\|_p, \tag{4.1.7}$$

for all non-negative finitely simple functions $f \in L^p$.

Note that if R_V^a is bounded on L^p, then for each finitely simple function f the quantity $R_V^a |f|$ given by (4.1.6) is finite for a.e. $x \in \mathbb{R}^d$. Since $\left|e^{-tL} f\right| \leqslant e^{-tL}|f|$ we see that in this case

$$V^a(x) \int_0^\infty e^{-tL} f(x) \, t^{a-1} \, dt$$

is finite x-a.e.. Thus, whenever R_V^a is bounded on L^p the integral above is a natural definition of $R_V^a f$, first for finitely simple functions and then, by density, for arbitrary functions in L^p.

Using Stein's complex interpolation theorem and functional calculus for symmetric contraction semigroups [10] we now prove an interpolation result for the operators R_V^a. Similar method was applied in [2, Section 6], where the authors proved the L^p boundedness of $R_V^{1/2}$ for $1 < p < 2(q + \varepsilon)$ by using Stein's complex interpolation theorem together with the L^p boundedness of R_V^1. They considered non-negative potentials belonging to a reverse Hölder class B_q.

Theorem 4.1.1. *Let $0 \leqslant a_0 < a_1$. Assume that $V \in L^1_{\mathrm{loc}}$ is an a.e. non-negative potential such that $R_V^{a_0}$ is bounded on L^{p_0} and $R_V^{a_1}$ is bounded on L^{p_1} for some $p_0, p_1 \in (1, \infty)$. Then, R_V^a is bounded on L^p for every p and a such that $\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$ and $a = \theta a_0 + (1 - \theta) a_1$ with some $\theta \in (0, 1)$.*

Proof. Let $\varepsilon > 0$ and denote $F(\varepsilon) \coloneqq \{x \in \mathbb{R}^d : \varepsilon < V(x) < \varepsilon^{-1}\}$. It is enough to justify that

$$R^{a,\varepsilon} f(x) \coloneqq (\mathbb{1}_{F(\varepsilon)} V^a)(x) \cdot \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL} f(x)\, t^{a-1} e^{-\varepsilon t}\, dt,$$

satisfies for all simple functions f the bound

$$\|R^{a,\varepsilon} f\|_p \leqslant C \|f\|_p, \tag{4.1.8}$$

uniformly in $\varepsilon > 0$ and with $C > 0$ being a constant. Indeed, if (4.1.8) holds, then taking $\varepsilon \to 0^+$ we obtain the L^p boundedness of R_V^a, first (with the aid of monotone convergence theorem) for non-negative simple functions and then for all functions in L^p.

Thus, in the remainder of the proof we fix $\varepsilon > 0$ and focus on justifying (4.1.8). Denote $S = \{z \in \mathbb{C} : a_0 < \operatorname{Re} z < a_1\}$. Then, for $z \in \overline{S}$ and $\varepsilon > 0$ the function $m_z^\varepsilon(\lambda) = (\lambda + \varepsilon)^{-z}$ is a bounded function on $[0, \infty)$, hence, by the spectral theorem $(L + \varepsilon I)^{-z}$ is well defined as a bounded operator on L^2. We let

$$T_z f \coloneqq (\mathbb{1}_{F(\varepsilon)} V^z) \cdot (L + \varepsilon I)^{-z} f, \qquad f \in L^2. \tag{4.1.9}$$

Since $(L + \varepsilon I)^{-b}$ given by the spectral theorem coincides with

$$\frac{1}{\Gamma(b)} \int_0^\infty e^{-tL} f\, t^{b-1} e^{-\varepsilon t}\, dt,$$

for every $b > 0$ we have

$$R^{b,\varepsilon} f = T_b f, \qquad f \in L^2.$$

Thus, in order to justify (4.1.8) it suffices to prove a uniform in $\varepsilon > 0$ bound for the L^p norm of T_a.

This will be achieved by Stein's complex interpolation theorem. Note first that for f, g being finitely simple functions the pairing

$$h(z) = \langle T_z f, g \rangle, \qquad z \in \overline{S},$$

gives a function which is holomorphic in S. To see this observe that (4.1.3) still holds with complex $a \in S$. Combining this observation with the definition (4.1.9) of T_z it is easy to see that h is indeed holomorphic. Additionally, the spectral theorem gives the bound

$$|h(z)| \leqslant C(\varepsilon, f, g), \tag{4.1.10}$$

valid for $z \in \overline{S}$. Altogether $\{T_z\}_{z \in \overline{S}}$ is an analytic family of operators of admissible growth.

It remains to bound the operator T_z for $\operatorname{Re} z = a_0$ and $\operatorname{Re} z = a_1$; this is the place where we use the assumptions on $R_V^{a_j}$. We let $z = a_j + i\tau$ for $\tau \in \mathbb{R}$, $j = 0, 1$ and write

$$T_z = (\mathbb{1}_{F(\varepsilon)} V^z) \cdot (L + \varepsilon I)^{-z} = (\mathbb{1}_{F(\varepsilon)} V^{i\tau}) T_{a_j} (L + \varepsilon I)^{-i\tau},$$

from which we see that

$$\|T_z\|_{p_j} \leqslant \left\|T_{a_j}\right\|_{p_j} \left\|(L + \varepsilon I)^{-i\tau}\right\|_{p_j}. \tag{4.1.11}$$

Since $(L+\varepsilon I)$ generates a symmetric contraction semigroup and $p_j \in (1,\infty)$, by e.g. [10] the imaginary powers $(L+\varepsilon I)^{-i\tau}$ satisfy

$$\left\|(L+\varepsilon I)^{-i\tau}\right\|_{p_j} \lesssim e^{\frac{\pi|\tau|}{2}}, \tag{4.1.12}$$

uniformly in $\varepsilon > 0$ Moreover, we have

$$\left|T_{a_j}(f)(x)\right| = |R^{a_j,\varepsilon} f(x)| \leqslant R_V^{a_j}|f|(x), \qquad x \in \mathbb{R}^d.$$

Thus, coming back to (4.1.11) and using our assumptions on the L^{p_j} boundedness of $R_V^{a_j}$ we obtain, for $z = a_j + i\tau$, $j = 0,1$,

$$\|T_z\|_{p_j} \lesssim e^{\frac{\pi|\tau|}{2}}, \qquad \tau \in \mathbb{R}.$$

Finally, applying Stein's complex interpolation theorem, see e.g. [22, Theorem 1.3.7], we obtain the L^p boundedness of R_V^a. □

Theorem 4.1.1 immediately leads to the following corollary.

Corollary 4.1.2. *Let $0 \leqslant a_0 < a_1$ and assume that both $R_V^{a_0}$ and $R_V^{a_1}$ are bounded on L^p for some $1 < p < \infty$. Then R_V^a is bounded on L^p for every $a_0 \leqslant a \leqslant a_1$. In particular if $R_V^{a_1}$ is bounded on L^p, then R_V^a is bounded on L^p for every $0 \leqslant a \leqslant a_1$.*

Proof. We apply Theorem 4.1.1 with $p_0 = p_1 = p$. For the second part recall that R_V^0 is the identity operator. □

It is straightforward to see that the Riesz transform $R_V^{1/2}$ is bounded on L^2. Using Corollary 4.1.2 we now extend the L^2 boundedness to the operators R_V^a with $0 \leqslant a \leqslant \frac{1}{2}$.

Proposition 4.1.3. *Let $V \in L^1_{\mathrm{loc}}$ be an a.e. non-negative potential. If $0 \leqslant a \leqslant \frac{1}{2}$, then R_V^a extends to a contraction on L^2.*

Proof. By formula (4.1.2) we have

$$\left\|V^{1/2} f\right\|_2 \leqslant \left\|L^{1/2} f\right\|_2, \qquad f \in C_c^\infty; \tag{4.1.13}$$

here $L^{1/2}$ is the self-adjoint operator with domain $\mathrm{Dom}(L^{1/2}) = \mathrm{Dom}(Q)$, while Q is the sesquilinear form given by (4.1.1). Using the fact that self-adjoint operators are closed we get $\mathrm{Dom}(L^{1/2}) \subseteq \mathrm{Dom}(V^{1/2})$ and

$$\left\|V^{1/2} f\right\|_2 \leqslant \left\|L^{1/2} f\right\|_2, \qquad f \in \mathrm{Dom}(L^{1/2}). \tag{4.1.14}$$

For each fixed $\varepsilon > 0$ the operator $(L+\varepsilon I)^{-1/2}$ is bounded on L^2 by the spectral theorem. Taking $f = (L+\varepsilon I)^{-1/2} g$ with $g \in L^2$ in (4.1.14) we get

$$\left\|V^{1/2}(L+\varepsilon I)^{-1/2} g\right\|_2 \leqslant \left\|L^{1/2}(L+\varepsilon I)^{-1/2} g\right\|_2, \qquad g \in L^2. \tag{4.1.15}$$

If g is a non-negative function on L^2 then by definitions (4.1.3), (4.1.6) and the monotone convergence theorem we have $\lim_{\varepsilon\to 0^+} \left\|V^{1/2}(L+\varepsilon I)^{-1/2} g\right\|_2 = \left\|R_V^{1/2} g\right\|_2$. The right-hand side of (4.1.15) converges to $\|g\|_2$ as $\varepsilon \to 0^+$ by the spectral theorem. Therefore we justified that $\left\|R_V^{1/2} g\right\|_2 \leqslant \|g\|_2$ for non-negative $g \in L^2$. This implies that $R_V^{1/2}$ is a contraction on L^2.

At this stage an application of Corollary 4.1.2 shows that R_V^a is bounded on L^2 whenever $0 \leqslant a \leqslant \frac{1}{2}$. The contractivity of R_V^a is not a direct consequence of the corollary. However, it is easy to justify once we follow the proof of Theorem 4.1.1 and use the spectral theorem to enhance inequality (4.1.12) to

$$\left\|(L + \varepsilon I)^{-i\tau}\right\|_2 \leqslant 1, \qquad \tau \in \mathbb{R}.$$

□

When $p_0 = 1$ we have a slightly weaker variant of Theorem 4.1.1 with the restriction $a_0, a_1 > 0$. This is caused by the unboundedness of the imaginary powers $L^{i\tau}$, $\tau \in \mathbb{R}$, on L^1.

Theorem 4.1.4. *Let $0 < a_0 < a_1$. Assume that $V \in L^1_{\mathrm{loc}}$ is such that $R_V^{a_1}$ is bounded on L^{p_1} for some $p_1 \in [1, \infty)$ and $R_V^{a_0}$ is bounded on L^1. Then, R_V^a is bounded on L^p for every p and a such that $\frac{1}{p} = \theta + \frac{1-\theta}{p_1}$ and $a = \theta a_0 + (1-\theta) a_1$ with some $\theta \in (0,1)$.*

Proof. The proof is similar to that of Theorem 4.1.1. For $\varepsilon > 0$ we define the sets $F(\varepsilon)$ and the operators $R^{a,\varepsilon}$ as in that proof. Once again it suffices to justify (4.1.8).

Let $S = \{z \in \mathbb{C} : a_0 < \operatorname{Re} z < a_1\}$ and define the family of operators $\{T_z\}_{z \in \overline{S}}$ as in (4.1.9). Since this time $a_0 > 0$ the formula

$$T_z f = (\mathbb{1}_{F(\varepsilon)} V^z) \cdot \frac{1}{\Gamma(z)} \int_0^\infty e^{-tL} f \, t^{z-1} e^{-\varepsilon t} \, dt, \qquad f \in L^2, \tag{4.1.16}$$

holds for $z \in \overline{S}$. Moreover, $\{T_z\}_{z \in S}$ is a family of analytic operators of admissible growth; this can be justified as in the proof of Theorem 4.1.1. Hence, in order to apply Stein's complex interpolation theorem it remains to bound $\|T_z\|_{p_j}$ for $z = a_j + i\tau$, $j = 0, 1$. Using (4.1.16) and the asymptotics for the gamma function $|\Gamma(a_j + i\tau)| \approx |\tau|^{a_j - 1/2} e^{-\frac{\pi|\tau|}{2}}$, see [40, 5.11.9], we obtain the pointwise bound

$$|T_z f(x)| \lesssim e^{\pi|\tau|} (\mathbb{1}_{F(\varepsilon)} V^{a_j})(x) \cdot \int_0^\infty e^{-tL} |f|(x) \, t^{a_j - 1} e^{-\varepsilon t} \, dt \lesssim e^{\pi|\tau|} R_V^{a_j} |f|(x)$$

valid for $z = a_j + i\tau$, $j = 0, 1$. Hence, the L^1 boundedness of $R_V^{a_0}$ together with the L^{p_1} boundedness of $R_V^{a_1}$ give

$$\|T_z\|_1 \lesssim e^{\pi|\tau|}, \qquad z = a_0 + i\tau, \quad \tau \in \mathbb{R},$$

and

$$\|T_z\|_{p_1} \lesssim e^{\pi|\tau|}, \qquad z = a_1 + i\tau, \quad \tau \in \mathbb{R}.$$

Thus, using Stein's complex interpolation theorem we complete the proof. □

Analogously to the L^2 case one particular Riesz transform R_V^1 is always bounded on L^1, see [2, Theorem 4.3] and [21, 27]. Interpolating this result with Proposition 4.1.3 we obtain the following theorem.

Theorem 4.1.5. *Let $V \in L^1_{\rm loc}$ and take $p \in (1,2]$. Then for all $0 \leqslant a \leqslant \frac{1}{p}$ the Riesz transform R_V^a is bounded on L^p.*

Proof. The L^2 boundedness of $R_V^{1/2}$ is guaranteed by Proposition 4.1.3. The L^1 boundedness of R_V^1 is justified in [2, Theorem 4.3]. Hence, Theorem 4.1.4 gives the L^p boundedness of R_V^a whenever $a = \theta + \frac{1-\theta}{2} = \frac{1}{p}$. Finally, Corollary 4.1.2 extends the boundedness on L^p to $0 \leqslant a \leqslant \frac{1}{p}$. □

4.2 Definitions and a counterexample on L^∞

Here the approach from the previous section is invalid since e^{-tL} does not necessarily extend to a strongly continuous semigroup on L^∞. However, for certain classes of potentials the operator e^{-tL}, $t > 0$, can be also expressed by the celebrated Feynman–Kac formula

$$e^{-tL}f(x) = \mathbb{E}_x\left[e^{-\int_0^t V(X_s)\,ds} f(X_t)\right], \qquad f \in L^p, \tag{4.2.1}$$

where $1 \leqslant p < \infty$. The expectation $\mathbb{E}_x$ is taken with regards to the Wiener measure of the standard d-dimensional Brownian motion $\{X_s\}_{s>0}$ starting at $x \in \mathbb{R}^d$; here $X_s = (X_s^1, \ldots, X_s^d)$. Since the potential V is a.e. non-negative, identity (4.2.1) is true whenever $V \in L^2_{\rm loc}$ belongs to the local Kato class $K_d^{\rm loc}$. This follows for example from [50, Remark 4.14] once we recall that for $V \in L^2_{\rm loc}$ the operator $-\frac{\Delta}{2} + V$ is essentially self-adjoint on C_c^∞, hence its Friedrichs extension is its unique self-adjoint extension. We will not need the definition of the local Kato class in the dissertation; for our purpose it is important to note that $L^q_{\rm loc} \subseteq K_d^{\rm loc}$ whenever $q \geqslant 1$ satisfies $q > \frac{d}{2}$, see [32, Lemma 4.105]. Therefore (4.2.1) is true for $V \in L^q_{\rm loc}$ whenever $q > \frac{d}{2}$ and $q \geqslant 2$, in particular for $V \in L^\infty_{\rm loc}$. The right-hand side of (4.2.1) makes sense also for $f \in L^\infty$, see [32, Section 4.2.4], which leads us to the definition

$$e^{-tL}f(x) \coloneqq \mathbb{E}_x\left[e^{-\int_0^t V(X_s)\,ds} f(X_t)\right], \qquad f \in L^\infty, \quad t > 0. \tag{4.2.2}$$

To deal with measurability questions we need a technical lemma on the continuity of $e^{-tL}f$.

Lemma 4.2.1. *Assume that $q > \frac{d}{2}$ and $q \geqslant 2$ and let $V \in L^q_{\rm loc}$ be an a.e. non-negative potential. Then for all $f \in L^\infty$ the function $e^{-tL}f(x)$ given by (4.2.2) is jointly continuous in $(t,x) \in (0,\infty) \times \mathbb{R}^d$. In particular $e^{-tL}(\mathbb{1})(x)$ is jointly continuous in t and x.*

Proof. Since $L^q_{\rm loc} \subseteq K_d^{\rm loc}$ it follows from [50, Proposition 3.5] that e^{-tL} is an integral operator with its kernel $K_t(x,y)$ being a jointly continuous functions of (t,x,y). Since $V \geqslant 0$ we also have $K_t(x,y) \leqslant (2\pi t)^{-d/2} \exp\left(\frac{|x-y|^2}{2t}\right)$ and therefore for each $N > 0$ it holds

$$\int_{|x-y|>N} K_t(x,y)|f(y)|\,dy \leqslant \pi^{-d/2}\|f\|_\infty \int_{|w|\geqslant \frac{N}{\sqrt{2t}}} e^{-|w|^2}\,dw. \tag{4.2.3}$$

Consider now $(t,x) \to (t_0,x_0)$ and let $\varepsilon > 0$ be arbitrarily small. Splitting

$$e^{-tL}f(x) = \int_{|x-y|\leqslant N} K_t(x,y)f(y)\,dy + \int_{|x-y|>N} K_t(x,y)f(y)\,dy$$

and using (4.2.3) we see that for $N = N(\varepsilon)$ large enough holds

$$\left| e^{-tL} f(x) - \int_{|x-y|\leqslant N} K_t(x,y) f(y)\, dy \right| \leqslant \varepsilon$$

uniformly in $\frac{t_0}{2} < t < 2t_0$ and $|x - x_0| < 1$. Moreover, for such (t,x) we see that $C\|f\|_\infty \mathbb{1}_{|y|\leqslant N+|x_0|+1}$ is an integrable majorant of $\mathbb{1}_{|x-y|\leqslant N} K_t(x,y) f(y)$. Thus, using Lebesgue's dominated convergence theorem we obtain

$$\limsup_{(t,x)\to(t_0,x_0)} \left| e^{-tL} f(x) - e^{-t_0 L} f(x_0) \right| \leqslant 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary this completes the proof. □

Now, take $a > 0$ and let $V \in L^\infty_{\mathrm{loc}}$ be an a.e. non-negative potential. For a non-negative function $f \in L^\infty$ we define the Riesz transform R_V^a by

$$R_V^a f(x) = V^a(x) \cdot \frac{1}{\Gamma(a)} \int_0^\infty \mathbb{E}_x \left[e^{-\int_0^t V(X_s)\, ds} f(X_t) \right] t^{a-1}\, dt, \qquad f \in L^\infty. \tag{4.2.4}$$

Note that by Lemma 4.2.1 the function $R_V^a f(x)$ is then a measurable function on $\mathbb{R}^d$ possibly being infinite for some x. Moreover, by (4.2.1) the L^∞ definition (4.2.4) coincides with the L^p definition (4.1.6) whenever f is a finitely simple function.

Since the semigroup is positivity preserving we have

$$\left| e^{-tL} f(x) \right| \leqslant e^{-tL}(\|f\|_\infty \mathbb{1})(x) = \|f\|_\infty e^{-tL}(\mathbb{1})(x), \qquad f \in L^\infty, \tag{4.2.5}$$

which leads to the following definition of the L^∞ boundedness of R_V^a.

Definition 4.2.1. We say that the Riesz transform R_V^a is bounded on L^∞ if

$$\|R_V^a(\mathbb{1})\|_\infty < \infty. \tag{4.2.6}$$

Note that if (4.2.6) holds, then by (4.2.5) for every $f \in L^\infty$ we have $|R_V^a(f)(x)| \leqslant \|f\|_\infty R_V^a(\mathbb{1})(x)$ so that

$$\|R_V^a(f)\|_\infty \leqslant C\|f\|_\infty, \qquad f \in L^\infty, \tag{4.2.7}$$

with $C = \|R_V^a(\mathbb{1})\|_\infty$.

Since

$$R_V^a(\mathbb{1})(x) = V^a(x) \cdot \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL}(\mathbb{1})(x)\, t^{a-1}\, dt \tag{4.2.8}$$

it is apparent that in order for R_V^a to be finite a.e. on $\operatorname{supp} V$ the monotone function $t \mapsto e^{-tL}(\mathbb{1})(x)$ must converge to 0 as $t \to \infty$. This however is not always the case.

Proposition 4.2.2. *Let $d \geqslant 3$ and let V be a non-negative potential on $\mathbb{R}^d$ which is compactly supported and not identically equal to zero. Assume that $V \in L^q$ with $q > \frac{d}{2}$ and $q \geqslant 2$. Then, for any $a > 0$ we have $R_V^a(\mathbb{1})(x) = \infty$ for all x such that $V(x) \neq 0$. In particular R_V^a is unbounded on L^∞.*

Proof. Fix $a>0$. For $x\in\mathbb{R}^d$ we let $w(x)=\lim_{s\to\infty} e^{-sL}(\mathbb{1})(x)$. From [20, Lemma 2.4] there exists a constant $\delta>0$ such that $\delta<w(x)\leqslant 1$ uniformly in $x\in\mathbb{R}^d$. Since by the semigroup property $w(x)=e^{-tL}(w)(x)$ for any $t>0$, we see that $e^{-tL}(\mathbb{1})(x)\geqslant e^{-tL}(w)(x)\geqslant\delta$ uniformly in $x\in\mathbb{R}^d$. Consequently, the integral $\int_0^\infty e^{-tL}(\mathbb{1})(x)\,t^{a-1}\,dt$ is infinite for a.e. x and so is $R_V^a(\mathbb{1})(x)$ as long as $V(x)\neq 0$. □

The definition below is meant to guarantee the x-a.e. finiteness of $R_V^a f(x)$.

Definition 4.2.2. Let $V\in L^\infty_{\mathrm{loc}}$ be an a.e. non-negative potential and let $\delta>0$. We say that the semigroup e^{-tL} has an exponential decay of order δ (ED(δ) for short) if there exists a constant $C>0$ such that

$$\left\|e^{-tL}(\mathbb{1})\right\|_\infty\leqslant Ce^{-\delta t},\qquad t>0. \tag{ED(δ)}$$

The assumption ($ED(\delta)$) implies $|R_V^a f(x)|\leqslant C\delta^{-a}V^a(x)\|f\|_\infty$ x-a.e.. Note, however, that this may not be enough to conclude that $\|R_V^a(\mathbb{1})\|_\infty<\infty$.

4.3 L^∞ boundedness for classes of potentials

Throughout this section we assume that $V\in L^\infty_{\mathrm{loc}}$. Here our goal is to estimate the L^∞ norm of R_V^a for classes of potentials V. As mentioned in Definition 4.2.1 this is the same as estimating $\|R_V^a(\mathbb{1})\|_\infty$ with $R_V^a(\mathbb{1})$ defined by (4.2.8).

Before we dive into details, we prove a general result concerning the L^∞ decay of the semigroup e^{-tL} defined in (4.2.2). We will use Lemma 4.3.1 below to prove the L^∞ and L^1 boundedness of R_V^a for concrete examples of potentials V in Theorem 4.0.3. Here π denotes a $(d-1)$-dimensional hyperplane in $\mathbb{R}^d$. For $N>0$ we let P be the strip

$$P=P_N:=\{x\in\mathbb{R}^d:\operatorname{dist}(x,\pi)\leqslant N\}\qquad\text{and set}\qquad \chi=\mathbb{1}_P.$$

Lemma 4.3.1. *Let $N>0$ and assume that the potential $V\in L^\infty_{\mathrm{loc}}$ is uniformly positive outside the strip P_N. More precisely we assume that V is non-negative a.e. and that there is $c>0$ such that $V(x)\geqslant c$ for a.e. x satisfying* $\operatorname{dist}(x,\pi)>N$. *Then the semigroup e^{-tL} has $ED(\delta)$ with $\delta=\frac12\min\left(c,\frac{1}{8N^2}\right)$. More precisely, there is a universal constant $C>0$ such that for $t>0$ and $x\in\mathbb{R}^d$ it holds*

$$e^{-tL}(\mathbb{1})(x)\leqslant C\,e^{-\delta t}.$$

To prove the above lemma we will need an auxiliary fact. Lemma 4.3.2 below can be deduced from [32, Lemma 4.105]. For the sake of completeness we give a more direct proof below.

Lemma 4.3.2. *For all $k>0$, $t>0$, and $x\in\mathbb{R}^d$ we have*

$$\mathbb{E}_x\left[e^{2\int_0^t k\chi(X_s)\,ds}\right]\leqslant C\,e^{8N^2k^2t}, \tag{4.3.1}$$

where $C>0$ is a universal constant.

Proof. We prove this fact in the case $\pi = \{0\} \times \mathbb{R}^{d-1}$ and $P = [-N, N] \times \mathbb{R}^{d-1}$. The general result follows from the invariance of Brownian motion under orthogonal transformations (see [42, p. 5]) and the fact that the bound is independent of x. Since in this case $\chi(X_s) = \mathbb{1}_{[-N,N]}(X_s^1)$ it suffices to prove the lemma in the dimension $d = 1$. In particular in the proof we take $x \in \mathbb{R}$.

The main tool of our proof is the local time of Brownian motion defined for $y \in \mathbb{R}$ in the one-dimensional case as

$$L_t(y) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{[y-\varepsilon,\, y+\varepsilon]}(Y_s)\, ds,$$

where $\{Y_s\}_{s>0}$ is the standard one-dimensional Brownian motion starting at 0. It is well known that

$$\int_0^t f(Y_s)\, ds = \int_{\mathbb{R}} f(y) L_t(y)\, dy$$

for any locally integrable function f, see [6, (5.4)]. In particular, we have

$$\int_0^t \mathbb{1}_{[-N-x,\, N-x]}(Y_s)\, ds = \int_{-N-x}^{N-x} L_t(y)\, dy. \tag{4.3.2}$$

The law of $L_t(y)$ was computed by Takács [52]. From a paper of Doney and Yor [14], see the last identity in Section 3 on p. 277 (with $\mu = 0$ and $x = y$) and [14, eq. (1.4)], it follows that the distribution of $L_t(y)$ is given by

$$c_{y,t}\delta_0 + f_{y,t}(z)\, dz$$

on $[0, +\infty)$, where δ_0 denotes the Dirac measure at 0,

$$f_{y,t}(z) = \frac{\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{(|y|+z)^2}{2t}}, \qquad y \in \mathbb{R}, \quad z > 0, \tag{4.3.3}$$

and $c_{y,t} < 1$ is a normalizing constant whose value is irrelevant for us.

Using (4.3.2) and Jensen's inequality for $x \in \mathbb{R}$ we obtain

$$\begin{aligned}
\mathbb{E}_x\left[e^{2\int_0^t k\chi(X_s)\, ds}\right] &= \mathbb{E}_0\left[e^{2\int_{-N-x}^{N-x} kL_t(y)\, dy}\right] \leqslant \frac{1}{2N}\mathbb{E}_0\left[\int_{-N-x}^{N-x} e^{4NkL_t(y)}\, dy\right] \\
&\leqslant \frac{1}{2N}\int_{-N-x}^{N-x}\left(1 + \int_0^\infty e^{4Nkz} f_{y,t}(z)\, dz\right) dy \\
&= 1 + \frac{1}{2N}\int_0^\infty e^{4Nkz}\int_{-N-x}^{N-x} f_{y,t}(z)\, dy\, dz
\end{aligned}$$

The 1+ term in the second line comes from the atom of the distribution of $L_t(y)$ at $z = 0$. Since the function $y \mapsto f_{y,t}(z)$ is radially decreasing, we can change the limits of the inner integral to $[-N, N]$, possibly increasing its value. Thus, using (4.3.3) gives

$$\begin{aligned}
&1 + \frac{1}{2N}\int_0^\infty e^{4Nkz}\int_{-N-x}^{N-x} f_{y,t}(z)\, dy\, dz \leqslant 1 + \frac{1}{2N}\int_0^\infty e^{4Nkz}\int_{-N}^{N} f_{y,t}(z)\, dy\, dz \\
&= 1 + \frac{\sqrt{2}}{N\sqrt{\pi t}}\int_0^\infty e^{4Nkz}\int_0^N e^{-\frac{(y+z)^2}{2t}}\, dy\, dz.
\end{aligned} \tag{4.3.4}$$

First we deal with the inner integral. Since y and z are positive, we estimate it by

$$\int_0^N e^{-\frac{(y+z)^2}{2t}}\,dy \leqslant \int_0^N e^{-\frac{z^2}{2t}}\,dy = Ne^{-\frac{z^2}{2t}}.$$

Plugging the above estimate into (4.3.4), we obtain

$$\begin{aligned}\mathbb{E}_x\left[e^{2\int_0^t k\chi(X_s)\,ds}\right] &\lesssim 1+\sqrt{\frac{2}{\pi t}}\int_0^\infty e^{4Nkz-\frac{z^2}{2t}}\,dz\\ &\lesssim e^{8N^2k^2t},\end{aligned}$$

which completes the proof of Lemma 4.3.2. □

Now we prove Lemma 4.3.1. It is noteworthy that the quadratic dependence on k on the right-hand side of (4.3.1) is crucial in the proof.

Proof of Lemma 4.3.1. We want to make use of the assumption that the potential V is uniformly positive outside the set P together with the previous lemma. We achieve this by an appropriate application of the Cauchy–Schwarz inequality.

Recall that $\chi = \mathbb{1}_P$ and take $k \in (0, c]$. Since the potential $2(V + k\chi)$ is bounded below by $2k$, using Cauchy–Schwarz inequality we estimate

$$\begin{aligned}e^{-tL}(\mathbb{1})(x) &= \mathbb{E}_x\left[e^{-\int_0^t V(X_s)\,ds}\right] = \mathbb{E}_x\left[e^{-\int_0^t V(X_s)+k\chi(X_s)\,ds}\,e^{\int_0^t k\chi(X_s)ds}\right]\\ &\leqslant \left(\mathbb{E}_x\left[e^{-2\int_0^t V(X_s)+k\chi(X_s)\,ds}\right]\right)^{1/2}\left(\mathbb{E}_x\left[e^{2\int_0^t k\chi(X_s)\,ds}\right]\right)^{1/2}\\ &\leqslant e^{-kt}\mathbb{E}_x\left[e^{2\int_0^t k\chi(X_s)\,ds}\right]^{1/2}.\end{aligned} \tag{4.3.5}$$

Applying Lemma 4.3.2 for k satisfying $4N^2k^2 \leqslant \frac{k}{2}$ we get

$$e^{-tL}(\mathbb{1})(x) \lesssim e^{-kt+4N^2k^2t} \leqslant e^{-\frac{kt}{2}}, \qquad x \in \mathbb{R}^d.$$

In particular, the above estimate holds for $k = \min(c, (8N^2)^{-1})$ and the proof is completed. □

Now we focus on our goal, which is estimating the quantity

$$\Gamma(a)\,R_V^a(\mathbb{1})(x) = V^a(x)\int_0^\infty e^{-tL}(\mathbb{1})(x)\,t^{a-1}\,dt \tag{4.3.6}$$

independently of $x \in \mathbb{R}^d$. We will do this by splitting the integral in (4.3.6) into two parts and estimating them separately.

Before stating the result we need to introduce a quantity ρ which plays a crucial role in our assumptions. For $u \geqslant 1$ and $x \in \mathbb{R}^d$ we define

$$\rho = \rho_x(u) = \sup\left\{r \geqslant 0 : \tfrac{V(x)}{u} \leqslant V(y) \;\text{ for a.e. } y \in B(x,r)\right\}; \tag{4.3.7}$$

recall that $B(x, r)$ denotes the closed Euclidean ball of radius r in $\mathbb{R}^d$. Consequently, $\rho_x(u)$ is the radius of the largest closed ball around x in which the potential V is at least $\frac{V(x)}{u}$ a.e. We note that $\rho_x(u)$ is a non-decreasing function of u with values in $[0, \infty]$. We also set

$$r_k = r_k(x) = \rho_x(2^k) \quad \text{for } k = 0, 1, \ldots. \tag{4.3.8}$$

Our main assumption will be phrased in terms of

$$I^a(V)(x) \coloneqq \int_1^{\max(1,V(x))} s^{a-1} e^{-\frac{\rho_x^2(s)}{4d}}\, ds \qquad \text{for a.e. } x \in \mathbb{R}^d. \tag{4.3.9}$$

If $\rho_x(s) = \infty$, then we define $e^{-\frac{\rho_x^2(s)}{4d}} = 0$.

First we estimate the integral in (4.3.6) from 0 to 1. Recall that implicit constants in $\lesssim$ and $\approx$ do not depend on $x \in \mathbb{R}^d$ but may depend on $a > 0$ and d.

Lemma 4.3.3. *Let V be an a.e. non-negative potential and let $a > 0$. Then we have*

$$V(x)^a \int_0^1 e^{-tL}(\mathbb{1})(x)\, t^{a-1}\, dt \lesssim I^a(V)(x) + 1 \qquad \textit{for a.e. } x \in \mathbb{R}^d.$$

Proof. First if $V(x) \leqslant 2$, then

$$V(x)^a \int_0^1 e^{-tL}(\mathbb{1})(x)\, t^{a-1}\, dt \lesssim 1.$$

From now on we focus on the other case $V(x) > 2$. Define $K = K(x) = \lfloor \log_2 V(x) \rfloor$. For fixed $x \in \mathbb{R}^d$ and $k = 0, 1, 2, \dots$ we introduce the sets

$$A_k = \left\{ y \in \mathbb{R}^d : \tfrac{V(x)}{2^k} \leqslant V(y) \right\} \tag{4.3.10}$$

and

$$\Omega_k = \{\omega \in \Omega : X_s(\omega) \in A_k \text{ for almost all } s \in [0,t]\}, \tag{4.3.11}$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying probability space for the d-dimensional Brownian motion $\{X_s\}_{s>0}$ started at 0.

Note that both the families $\{A_k\}$ and $\{\Omega_k\}$ are increasing in k. Using the Feynman–Kac formula (4.2.2) we write

$$\begin{aligned} &e^{-tL}(\mathbb{1})(x) = \\ &\quad \mathbb{E}_x\left[e^{-\int_0^t V(X_s)\, ds}\, \mathbb{1}_{\Omega_0}\right] + \sum_{k=1}^{K} \mathbb{E}_x\left[e^{-\int_0^t V(X_s)\, ds}\, \mathbb{1}_{\Omega_k \cap \Omega_{k-1}^c}\right] + \mathbb{E}_x\left[e^{-\int_0^t V(X_s)\, ds}\, \mathbb{1}_{\Omega_K^c}\right] \\ &\quad \leqslant e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}}\, \mathbb{P}\left(\Omega_k \cap \Omega_{k-1}^c\right) + \mathbb{P}\left(\Omega_K^c\right). \end{aligned} \tag{4.3.12}$$

We need to estimate the probabilities in the above formula. This will be achieved with the aid of inequality

$$\mathbb{P}\left(\Omega_k^c\right) \leqslant \mathbb{P}\left(\sup_{0 \leqslant s \leqslant t} |X_s - x| \geqslant r_k\right). \tag{4.3.13}$$

Before moving further we justify (4.3.13). To prove this inequality we will show that

$$\left\{\omega \in \Omega : \sup_{0 \leqslant s \leqslant t} |X_s(\omega) - x| < r_k \right\} \subseteq \Omega_k$$

up to a set of $\mathbb{P}$ measure 0. More precisely, we will demonstrate that for $\mathbb{P}$ almost all $\omega \in \Omega$ we have the implication

$$\text{if } \sup_{0 \leqslant s \leqslant t} |X_s(\omega) - x| < r_k \quad \text{then also } X_s(\omega) \in A_k \text{ for almost all } s \in [0,t]. \tag{4.3.14}$$

To this end take $\omega \in \Omega$ such that $\sup_{0\leqslant s\leqslant t} |X_s(\omega) - x| < r_k$. Using the definitions (4.3.7) and (4.3.8) of ρ and r_k we see that there is a set $N \subseteq \mathbb{R}^d$ of measure 0 such that

$$\text{if } X_s(\omega) \notin N \text{ then } \frac{V(x)}{2^k} \leqslant V(X_s(\omega)),$$

By the definition (4.3.10) of A_k this statement is the same as the implication

$$\text{if } X_s(\omega) \notin N \text{ then } X_s(\omega) \in A_k.$$

Define $f_\omega(s) := X_s(\omega)$, $s \in [0,t]$, and let $\widetilde{N}(\omega) = f_\omega^{-1}[N] \subseteq [0,t]$. Then $s \notin \widetilde{N}(\omega)$ if and only if $X_s(\omega) \notin N$. We shall now demonstrate that $\left|\widetilde{N}(\omega)\right| = 0$ for $\mathbb{P}$ almost all $\omega \in \Omega$. Observe that

$$\left|\widetilde{N}(\omega)\right| = |\{s \in [0,t] : X_s(\omega) \in N\}| = \int_0^t \mathbb{1}_{\{X_s(\omega)\in N\}}(s,\omega)\, ds.$$

Calculating the expected value of the above expression and using Fubini's theorem give

$$\begin{aligned}\mathbb{E}\left[\left|\widetilde{N}\right|\right] &= \mathbb{E}\left[\int_0^t \mathbb{1}_{\{X_s(\omega)\in N\}}(s,\omega)\, ds\right] = \int_0^t \mathbb{E}\left[\mathbb{1}_{\{X_s(\omega)\in N\}}(s,\omega)\right]\, ds \\ &= \int_0^t \mathbb{P}\left(X_s(\omega) \in N\right)\, ds = 0.\end{aligned}$$

The last equality follows from the fact that $|N| = 0$ and that each of the variables X_s has a continuous distribution. Since $\left|\widetilde{N}(\omega)\right|$ is non-negative, it has to be 0 for $\mathbb{P}$ almost all $\omega \in \Omega$.

Hence we have proved that for $\mathbb{P}$ almost all $\omega \in \Omega$ there is a set $\widetilde{N}(\omega) \subseteq [0,t]$ of Lebesgue measure 0 and such that

$$\text{if } s \notin \widetilde{N}(\omega) \text{ then } X_s(\omega) \in A_k.$$

This proves (4.3.14) and in consequence (4.3.13).

Now we come back to calculating the probabilities in (4.3.12). The right-hand side of inequality (4.3.13) is the probability that X_s exits the ball of radius r_k centered at x. We can estimate it from above by the probability that X_s exits an inscribed cube whose sides are parallel to the coordinate axes. The length of its diagonal equals $a\sqrt{d} = 2r_k$, where a is the cube's side length, so we get

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leqslant s\leqslant t} |X_s - x| \geqslant r_k\right) &\leqslant \mathbb{P}\left(\sup_{0\leqslant s\leqslant t} \max_i \left|X_s^i - x_i\right| \geqslant \frac{a}{2}\right) = \mathbb{P}\left(\max_i \sup_{0\leqslant s\leqslant t} \left|X_s^i - x_i\right| \geqslant \frac{a}{2}\right) \\ &\leqslant d\cdot\mathbb{P}\left(\sup_{0\leqslant s\leqslant t} \left|X_s^1 - x_1\right| \geqslant \frac{a}{2}\right) \\ &\leqslant d\cdot\mathbb{P}\left(\sup_{0\leqslant s\leqslant t} (X_s^1 - x_1) \geqslant \frac{a}{2}\right) + d\cdot\mathbb{P}\left(\inf_{0\leqslant s\leqslant t} (X_s^1 - x_1) \leqslant -\frac{a}{2}\right) \\ &= 2d\cdot\mathbb{P}\left(\sup_{0\leqslant s\leqslant t} (X_s^1 - x_1) \geqslant \frac{a}{2}\right) = 4d\cdot\mathbb{P}\left((X_t^1 - x_1) \geqslant \frac{a}{2}\right) \\ &\leqslant 4d\,\mathrm{erfc}\left(\frac{r_k}{\sqrt{2td}}\right) \leqslant 4de^{-\frac{r_k^2}{2td}}.\end{aligned} \tag{4.3.15}$$

The last equality in (4.3.15) follows from the reflection principle for Brownian motion, while the last inequality is a well-known bound for the complementary error function erfc, see e.g. [40, eq. (7.8.3)].

Consequently,

$$\mathbb{P}\left(\Omega_k^c\right) \leqslant 4de^{-\frac{r_k^2}{2td}} \tag{4.3.16}$$

and coming back to (4.3.12) for $0 < t < 1$ we get

$$\begin{aligned} e^{-tL}(\mathbb{1})(x) &\lesssim e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} e^{-\frac{r_{k-1}^2}{2td}} + e^{-\frac{r_K^2}{2td}} \\ &\leqslant e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} e^{-\frac{r_{k-1}^2}{2d}} + e^{-\frac{r_K^2}{2d}} \end{aligned} \tag{4.3.17}$$

Integrating and multiplying this inequality by $V(x)^a$ gives

$$V(x)^a \int_0^1 e^{-tL}(\mathbb{1})(x)\, t^{a-1}\, dt \lesssim 1 + \sum_{k=1}^{K} 2^{ka} e^{-\frac{r_{k-1}^2}{2d}} + V(x)^a e^{-\frac{r_K^2}{2d}}. \tag{4.3.18}$$

Then, for $k \geqslant 2$ we estimate each of the terms in the sum by an integral recalling that $r_k(x) = \rho_x(2^k)$ and using the fact that $\rho_x(u)$ is a non-decreasing function of u

$$2^{ka} e^{-\frac{r_{k-1}^2}{2d}} \leqslant \int_{k-2}^{k-1} 2^{(u+2)a} e^{-\frac{\rho_x^2(2^u)}{2d}}\, du. \tag{4.3.19}$$

The last term in (4.3.18) is estimated in a similar manner using additionally the fact that $V(x)^a \leqslant \int_{K-1}^{K} 2^{(u+2)a}\, du$. This yields

$$V(x)^a e^{-\frac{r_K^2}{2d}} \leqslant \int_{K-1}^{K} 2^{(u+2)a} e^{-\frac{\rho_x^2(2^u)}{2d}}\, du. \tag{4.3.20}$$

We estimate the first term of the sum in (4.3.18) by a constant and plug this, (4.3.19) and (4.3.20) into (4.3.18), which results in

$$\begin{aligned} 1 + \sum_{k=1}^{K} 2^{ka} e^{-\frac{r_{k-1}^2}{2d}} + V(x)^a e^{-\frac{r_K^2}{2d}} &\lesssim 1 + \int_0^K 2^{ua} e^{-\frac{\rho_x^2(2^u)}{2d}}\, du \\ &\leqslant 1 + \int_0^{\log_2 V(x)} 2^{ua} e^{-\frac{\rho_x^2(2^u)}{2d}}\, du. \end{aligned} \tag{4.3.21}$$

Finally we substitute $s = 2^u$ to get

$$1 + \int_0^{\log_2 V(x)} 2^{ua} e^{-\frac{\rho_x^2(2^u)}{2d}}\, du \approx 1 + \int_1^{V(x)} s^{a-1} e^{-\frac{\rho_x^2(s)}{2d}}\, ds \leqslant 1 + I^a(V)(x). \tag{4.3.22}$$

□

In the next lemma we estimate the second part of the integral from (4.3.6).

Lemma 4.3.4. *Let V be an a.e. non-negative potential and suppose that the semigroup e^{-tL} satisfies* (ED(δ)) *for some $\delta > 0$ and take $a > 0$. Then we have*

$$V(x)^a \int_1^\infty e^{-tL}(\mathbb{1})(x)\, t^{a-1}\, dt \lesssim I^a(V)(x) + 1, \qquad x \in \mathbb{R}^d. \tag{4.3.23}$$

Proof. Using the semigroup property and the positivity-preserving property of $\{e^{-tL}\}_{t>0}$ for $t \geqslant 1$ we obtain

$$\begin{aligned} e^{-tL}(\mathbb{1})(x) &= e^{-(t/2)L}[e^{-(t/2)L}(\mathbb{1})](x) \leqslant \left\| e^{-(t/2)L}(\mathbb{1}) \right\|_\infty e^{-(t/2)L}(\mathbb{1})(x) \\ &\leqslant Ce^{-\delta t/2} e^{-(1/2)L}(\mathbb{1})(x), \end{aligned} \tag{4.3.24}$$

where the last two inequalities follow from (ED(δ)) and (4.2.1). Plugging this into (4.3.23) we get

$$V(x)^a \int_1^\infty e^{-tL}(\mathbb{1})(x)\, t^{a-1}\, dt \lesssim V(x)^a e^{-L/2}(\mathbb{1})(x). \tag{4.3.25}$$

Now we are left with proving that $V^a(x)e^{-L/2}(\mathbb{1})(x) \lesssim I^a(V)(x) + 1$. If $V(x) \leqslant 2$, then this is true. Assume that $V(x) > 2$ and let $K(x) = \lfloor \log_2 V(x) \rfloor$. Recall that by (4.3.17) we have

$$e^{-L/2}(\mathbb{1})(x) \lesssim e^{-\frac{V(x)}{2}} + \sum_{k=1}^{K} e^{-\frac{V(x)}{2^{k+1}}} e^{-\frac{r_{k-1}^2}{2d}} + e^{-\frac{r_K^2}{2d}}.$$

Since $V(x)^a e^{-\frac{V(x)}{2^{k+1}}} \leqslant \left(\frac{2^{k+1}a}{e}\right)^a$, repeating calculations as in (4.3.18)–(4.3.22) we get

$$V(x)^a e^{-L/2}(\mathbb{1})(x) \lesssim 1 + \sum_{k=1}^{K} 2^{ka} e^{-\frac{r_{k-1}^2}{2d}} + V(x)^a e^{-\frac{r_K^2}{2d}} \lesssim 1 + I^a(V)(x). \tag{4.3.26}$$

In view of (4.3.25) this completes the proof of the lemma. □

Together, Lemma 4.3.3 and Lemma 4.3.4 lead to the following conclusion.

Theorem 4.3.5. *Let $V \in L^\infty_{\mathrm{loc}}$ be an a.e. non-negative potential. Suppose that the semigroup e^{-tL} has exponential decay of order $\delta > 0$ (see* (ED(δ))*). If*

$$I^a(V) \lesssim_g 1 \tag{4.3.27}$$

for some $a > 0$, then the operator R^a_V is bounded on L^∞.

Proof. We need to estimate the quantity

$$V^a(x) \int_0^\infty e^{-tL}(\mathbb{1})(x)\, t^{a-1}\, dt \tag{4.3.28}$$

independently of x. Take $N > 0$ such that $I^a(V)(x) \lesssim 1$ for almost all $|x| > N$. Then by Lemma 4.3.3 and Lemma 4.3.4 the expression (4.3.28) is uniformly bounded for a.e. $|x| > N$. If on the other hand $|x| \leqslant N$, then, since $V \in L^\infty_{\mathrm{loc}}$ and the semigroup satisfies (ED(δ)), the expression (4.3.28) is also uniformly bounded x-a.e. □

As an application of this theorem, we prove that R^a_V is bounded on L^∞ if V is of the order of a power function or an exponential function. The corollary below is a restatement of one of our main results — Theorem 4.0.3 — in the L^∞ case.

Corollary 4.3.6. *Let $V\colon \mathbb{R}^d \to [0,\infty)$ be a function in L^∞_{loc}. Then in all the three cases*

1. *$V(x) \approx 1$ globally*

2. For some $\alpha > 0$ we have $V(x) \approx |x|^\alpha$ globally

3. For some $\beta > 1$ we have $V(x) \approx \beta^{|x|}$ globally

each of the Riesz transforms R_V^a, $a > 0$, is bounded on L^∞.

Remark. More generally, the theorem also holds if in (2) and (3) we take an arbitrary norm on $\mathbb{R}^d$ instead of the Euclidean norm $|\cdot|$. The proof is the same mutatis mutandis.

Proof. In the proof implicit constants in $\lesssim$, $\gtrsim$, and $\approx$ do not depend on $x \in \mathbb{R}^d$ but may depend on $a > 0$, $\alpha > 0$ or $\beta > 1$.

Clearly in all three cases the assumptions of Lemma 4.3.1 are satisfied, so the semigroup satisfies (ED(δ)) and we only need to check that (4.3.27) holds.

In the first case $V(x)$ is bounded for almost all sufficiently large values of $|x|$ and so is $I^a(V)(x)$ for all $a > 0$.

In the second case we need to estimate from below $\rho_x(s)$ appearing in $I^a(V)$. We shall prove that $\rho_x(s) \geqslant \frac{|x|}{2}$ provided s and $|x|$ are large enough. Let N be such that for some $0 < m < M$ it holds

$$m|x|^\alpha < V(x) < M|x|^\alpha \qquad \text{for a.e.} \qquad |x| \geqslant N. \tag{4.3.29}$$

Take $|x| \geqslant 2N$ and assume that $|x-y| \leqslant \frac{|x|}{2}$. Then $2|x| \geqslant |y| \geqslant \frac{|x|}{2} \geqslant N$ so that (4.3.29) holds with y in place of x. Consequently, $V(x) \approx V(y)$ for such x and y so that for s larger than some threshold depending only on N, m and M it holds $V(y) \geqslant \frac{V(x)}{s}$. This means that for a.e. $|x| \geqslant 2N$ and uniformly large enough $s \geqslant 1$ we have $\rho_x(s) \geqslant \frac{|x|}{2}$. Thus, for any $a > 0$ we obtain

$$I^a(V)(x) \lesssim_g 1 + |x|^{a\alpha} e^{-\frac{|x|^2}{16d}} \lesssim_g 1. \tag{4.3.30}$$

as desired.

Finally we handle the third case. We shall prove that $\rho_x(s) \geqslant \frac{1}{2}\min\left(|x|, \log_\beta s\right)$ provided s and $|x|$ are large enough. Let $N > 0$ be such that for some $0 < m \leqslant 1 \leqslant M$ we have

$$m\beta^{|x|} < V(x) < M\beta^{|x|} \qquad \text{for a.e.} \qquad |x| \geqslant N. \tag{4.3.31}$$

Take $|x| \geqslant 2N$, $s > 4$, and assume that $|x-y| \leqslant \frac{1}{2}\min\left(|x|, \log_\beta s\right)$. Then, similarly to the previous paragraph, $|x| \approx |y| \geqslant N$ and (4.3.31) also holds with y in place of x. Therefore, for such x and y we have $\beta^{|y|-|x|} \approx \frac{V(y)}{V(x)}$. In particular $|y| - |x| - \gamma \leqslant \log_\beta V(y) - \log_\beta V(x)$, for some $\gamma > 0$ independent of x and y. Hence, we reach

$$-\frac{1}{2}\min\left(|x|, \log_\beta s\right) - \gamma \leqslant \log_\beta V(y) - \log_\beta V(x). \tag{4.3.32}$$

Taking s large enough we see that $-\frac{1}{2}\log_\beta s - \gamma \geqslant -\log_\beta s$ and coming back to (4.3.32) we obtain $\frac{V(x)}{s} \leqslant V(y)$. In conclusion, we proved that $\rho_x(s) \geqslant \frac{1}{2}\min\left(|x|, \log_\beta s\right)$ for a.e. $|x| \geqslant 2N$ when s is large enough (independently of x). Now, using (4.3.31) we obtain the uniform in $|x| \geqslant 2N$ bound

$$I^a(V)(x) \lesssim_g 1 + \int_1^{\beta^{|x|}} s^{a-1} e^{-\frac{(\log_\beta s)^2}{16d}}\, ds + \int_{\beta^{|x|}}^{M\beta^{|x|}} s^{a-1} e^{-\frac{|x|^2}{16d}}\, ds \lesssim_g 1, \tag{4.3.33}$$

This completes the treatment of the third case and also the proof of Corollary 4.3.6. □

4.4 L^1 boundedness for classes of potentials

In this section we estimate the L^1 norm of the operator R_V^a for $a > 0$ and various non-negative potentials $V \in L^\infty_{\mathrm{loc}}$. Recall that the assumption $V \in L^\infty_{\mathrm{loc}}$ guarantees the validity of the Feynman–Kac formula (4.2.1).

The idea is to estimate the L^∞ norm of the adjoint operator which formally is

$$(L^{-a}V^a)f = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL}(V^a f)\, t^{a-1}\, dt.$$

Using the positivity-preserving property of e^{-tL} the task naturally reduces to estimating the L^∞ norm of the function

$$\Gamma(a)L^{-a}(V^a)(x) := \int_0^\infty e^{-tL}(V^a)(x)\, t^{a-1}\, dt. \tag{4.4.1}$$

Since V may be unbounded, the expression $e^{-tL}(V^a)(x)$ may be infinite for some x in which case the x-measurability of the integral (4.4.1) is not clear. To remedy the situation we formally define

$$\Gamma(a)L^{-a}(V^a)(x) := \lim_{N\to\infty} \int_0^\infty e^{-tL}(V^a \mathbb{1}_{V<N})(x)\, t^{a-1} e^{-t/N}\, dt. \tag{4.4.2}$$

Note that each of the integrals in (4.4.2) is finite and measurable by Lemma 4.2.1, hence the limit gives a measurable function by the monotone convergence theorem. We will now show that if $L^{-a}(V^a) \in L^\infty$, then R_V^a is bounded on L^1 with norm estimate $\|R_V^a\|_1 \leqslant \|L^{-a}(V^a)\|_\infty$.

Take a finitely simple function f and assume that $L^{-a}(V^a) \in L^\infty$. The following equalities and inequalities hold provided that all the expressions are finite, which will turn out to be true. In the calculations below we use duality between the spaces L^1 and L^∞ and the fact that the semigroup e^{-tL} is symmetric and positivity preserving and that the operator L^{-a} is also positivity preserving.

$$\begin{aligned}
\|R_V^a f\|_1 &= \sup_{\substack{g\in L^\infty \\ \|g\|_\infty = 1}} \langle R_V^a f, g\rangle \leqslant \sup_{\|g\|_\infty=1} \left| \int_{\mathbb{R}^d} V(x)^a L^{-a} f(x) g(x)\, dx \right| \\
&\leqslant \frac{1}{\Gamma(a)} \sup_{\|g\|_\infty=1} \int_{\mathbb{R}^d} |g(x)| \lim_{N\to\infty} V(x)^a \mathbb{1}_{|x|<N}(x) \int_0^\infty e^{-tL}(|f|)(x)\, t^{a-1} e^{-t/N}\, dt\, dx \\
&\leqslant \frac{1}{\Gamma(a)} \lim_{N\to\infty} \int_0^\infty \int_{\mathbb{R}^d} V(x)^a \mathbb{1}_{|x|<N}(x)\, e^{-tL}(|f|)(x)\, dx\ t^{a-1} e^{-t/N}\, dt \\
&= \frac{1}{\Gamma(a)} \lim_{N\to\infty} \int_0^\infty \int_{\mathbb{R}^d} |f(x)|\, e^{-tL}(V^a \mathbb{1}_{|x|<N})(x)\, dx\ t^{a-1} e^{-t/N}\, dt \\
&\leqslant \int_{\mathbb{R}^d} L^{-a}(V^a)(x)|f(x)|\, dx \leqslant \left\| L^{-a}(V^a) \right\|_\infty \|f\|_1 < \infty.
\end{aligned}$$

Changing the order of integration and swapping limits and integrals is allowed since all the functions are non-negative and non-decreasing with respect to N. Since finitely simple functions are dense in L^1, we have shown that indeed R_V^a is bounded on L^1 and $\|R_V^a\|_1 \leqslant \|L^{-a}(V^a)\|_\infty$.

Throughout this section we estimate the L^∞ norm of $L^{-a}(V^a)$ in the form (4.4.1). This is allowed since by the assumptions which we will impose on V both $e^{-tL}(V^a)(x)$ and the integral (4.4.1) will turn out to be finite x-a.e.. This permits us to take $N=\infty$ in (4.4.2).

In what follows for $x \in \mathbb{R}^d$ and $u \geqslant 1$ we let

$$\sigma = \sigma_x(u) = \sup\{r \geqslant 0 : V(y) \leqslant uV(x) \text{ for a.e. } y \in B(x,r)\}.$$

Consequently, $\sigma_x(u)$ is the radius of the largest closed ball around x in which the potential V is at most $uV(x)$ a.e. We remark that $\sigma_x(u)$ is a non-decreasing function of u with values in $[0,\infty]$. Using the quantity $\sigma_x(u)$ we define

$$J^a(V)(x) \coloneqq \min(1, V(x)^a) \int_1^\infty s^{a-1} e^{-\frac{\sigma_x^2(s)}{8}}\, ds \qquad \text{for a.e. } x \in \mathbb{R}^d. \tag{4.4.3}$$

If $V \in L^\infty$ and $uV(x) \geqslant \|V\|_\infty$, then $V(y) \leqslant uV(x)$ for a.e. $y \in B(x,r)$ with arbitrarily large $r>0$. In this case $\sigma_x(u)=\infty$ and by convention $e^{-\frac{\sigma_x^2(s)}{8}}=0$. This is the case for instance if $V \in L^\infty$ is of constant order for large x.

We begin with estimating the integral (4.4.1) from 0 to 1. Recall that implicit constants in $\lesssim$ and $\approx$ are allowed to depend on d and $a>0$.

Proposition 4.4.1. *Let $V \in L^\infty_{\mathrm{loc}}$ be an a.e. non-negative potential and take $a>0$. Then the inequality*

$$\int_0^1 e^{-tL}(V^a)(x)\, t^{a-1}\, dt \lesssim (J^a(V)(x)+1)(I^a(V)(x)+1) \tag{4.4.4}$$

holds uniformly for a.e. $x \in \mathbb{R}^d$ that satisfies $V(x) \neq 0$.

Moreover, if V is an a.e. non-negative potential which satisfies the growth estimate $V(x) \lesssim e^{\frac{|x|^2}{4a}}$ for a.e. $x \in \mathbb{R}^d$, then

$$\int_0^1 e^{-tL}(V^a)(x)\, t^{a-1}\, dt \lesssim e^{|x|^2}, \qquad x \in \mathbb{R}^d. \tag{4.4.5}$$

Proof. **Proof of** (4.4.4). Here we consider $x \in \mathbb{R}^d$ such that $V(x) \neq 0$.

Recall that

$$A_k = \left\{y \in \mathbb{R}^d : \tfrac{V(x)}{2^k} \leqslant V(y)\right\}$$

and

$$\Omega_k = \{\omega \in \Omega : X_s(\omega) \in A_k \text{ for almost all } s \in [0,t]\}.$$

Here we shall also need

$$B_j = \left\{y \in \mathbb{R}^d :\, 2^j V(x) < V(y) \leqslant 2^{j+1} V(x)\right\}$$

and

$$\Psi_j = \Psi_j^t \coloneqq \{\omega \in \Omega : X_t(\omega) \in B_j\}.$$

Note that the sets $\{B_j\}_{j\in\mathbb{Z}}$ are pairwise disjoint and

$$\begin{aligned} e^{-tL}(V^a)(x) &= e^{-tL}\left(\sum_{j\leqslant 0} \mathbb{1}_{B_j} V^a\right)(x) + e^{-tL}\left(\sum_{j>0} \mathbb{1}_{B_j} V^a\right)(x) + e^{-tL}\left(\mathbb{1}_{V=0} V^a\right)(x) \\ &\lesssim V(x)^a e^{-tL}(\mathbb{1})(x) + \sum_{j>0} V(x)^a 2^{ja} e^{-tL}(\mathbb{1}_{B_j})(x). \end{aligned} \tag{4.4.6}$$

We shall prove that the estimates

$$\int_0^1 e^{-tL}(V^a)(x)\,t^{a-1}\,dt \lesssim (I^a(V)(x)+1)\left(\int_1^\infty s^{a-1}e^{-\frac{\sigma_x^2(s)}{8}}\,ds+1\right) \tag{4.4.7}$$

and

$$\int_0^1 e^{-tL}(V^a)(x)\,t^{a-1}\,dt \lesssim I^a(V)(x)+1+V(x)^a\left(\int_1^\infty s^{a-1}e^{-\frac{\sigma_x^2(s)}{8}}\,ds\right) \tag{4.4.8}$$

hold uniformly for x such that $V(x)\neq 0$. The inequalities (4.4.7) and (4.4.8) imply (4.4.4).

We prove (4.4.7) first. Let $K=\max(1,\lfloor\log_2 V(x)\rfloor)$ and for $k=1,\dots,K$ and $j\in\mathbb{Z}$ denote

$$r_k=\rho_x(2^k),\qquad s_j=\sigma_x(2^j).$$

Estimating the second term in (4.4.6) we use the Feynman–Kac formula (4.2.1) with $f=V^a\mathbb{1}_{B_j}$ to write

$$\sum_{j>0}e^{-tL}(V^a\mathbb{1}_{B_j})(x)\lesssim V^a(x)\sum_{j>0}2^{ja}e^{-tL}(\mathbb{1}_{B_j})(x). \tag{4.4.9}$$

Using again (4.2.1), proceeding as in the proof of Lemma 4.3.3 and applying (4.3.16) we obtain

$$\begin{aligned}
e^{-tL}(\mathbb{1}_{B_j})(x) &\leqslant e^{-tV(x)}\,\mathbb{P}(\Psi_j)+\sum_{k=1}^{K}e^{-\frac{tV(x)}{2^k}}\,\mathbb{P}\left(\Omega_{k-1}^c\cap\Psi_j\right)+\mathbb{P}\left(\Omega_K^c\cap\Psi_j\right)\\
&\leqslant \mathbb{P}(\Psi_j)^{1/2}\left(e^{-tV(x)}+\sum_{k=1}^{K}e^{-\frac{tV(x)}{2^k}}\,\mathbb{P}\left(\Omega_{k-1}^c\right)^{1/2}+\mathbb{P}\left(\Omega_K^c\right)^{1/2}\right)\\
&\lesssim \mathbb{P}(\Psi_j)^{1/2}\left(e^{-tV(x)}+\sum_{k=1}^{K}e^{-\frac{tV(x)}{2^k}}e^{-\frac{r_{k-1}^2}{4td}}+e^{-\frac{r_K^2}{4td}}\right)
\end{aligned}$$

Further, we have $\Psi_j\subseteq\{\omega\in\Omega:X_t(\omega)\notin B(x,s_j)\}$ up to a set of $\mathbb{P}$ measure 0. Indeed, a.e. $y\in B(x,s_j)$ satisfies $V(y)\leqslant 2^jV(x)$, hence it lies outside B_j. Here we also use the fact that X_t has a continuous distribution. Thus we reach

$$\begin{aligned}
\mathbb{P}(\Psi_j)\leqslant\mathbb{P}(|X_t-x|\geqslant s_j)&=\frac{1}{(2\pi t)^{d/2}}\int_{|y|\geqslant s_j}e^{-\frac{|y|^2}{2t}}\,dy\\
&\leqslant\frac{e^{-s_j^2/(4t)}}{(2\pi t)^{d/2}}\int_{|y|\geqslant s_j}e^{-\frac{|y|^2}{4t}}\,dy\lesssim e^{-\frac{s_j^2}{4t}}
\end{aligned} \tag{4.4.10}$$

so that

$$e^{-tL}(\mathbb{1}_{B_j})(x)\lesssim e^{\frac{\sigma_x^2(s)}{8t}}\left(e^{-tV(x)}+\sum_{k=1}^{K}e^{-\frac{tV(x)}{2^k}}e^{-\frac{r_{k-1}^2}{4td}}+e^{-\frac{r_K^2}{4td}}\right).$$

Putting the above bound in (4.4.6) and replacing the sum over j with an integral as in (4.3.20) and (4.3.21) we reach

$$\begin{aligned}
\sum_{j>0}V(x)^a2^{ja}e^{-tL}(\mathbb{1}_{B_j})(x)&\lesssim V(x)^a\left(e^{-tV(x)}+\sum_{k=1}^{K}e^{-\frac{tV(x)}{2^k}}e^{-\frac{r_{k-1}^2}{4td}}+e^{-\frac{r_K^2}{4td}}\right)\sum_{j>0}2^{ja}e^{-\frac{\sigma_x^2(s)}{8t}}\\
&\lesssim V(x)^a\left(e^{-tV(x)}+\sum_{k=1}^{K}e^{-\frac{tV(x)}{2^k}}e^{-\frac{r_{k-1}^2}{4td}}+e^{-\frac{r_K^2}{4td}}\right)\int_1^\infty s^{a-1}e^{-\frac{\sigma_x^2(s)}{8t}}\,ds.
\end{aligned}$$

The first term on the right-hand side of (4.4.6) was already estimated in the proof of Lemma 4.3.3 by

$$V(x)^a e^{-tL}(\mathbb{1})(x) \leqslant V(x)^a \left(e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} e^{-\frac{r_{k-1}^2}{2td}} + e^{-\frac{r_K^2}{2td}} \right),$$

see (4.3.17). Hence, coming back to (4.4.6) we reach

$$e^{-tL}(V^a)(x) \lesssim V(x)^a \left(\int_1^\infty s^{a-1} e^{-\frac{\sigma_x^2(s)}{8}}\, ds + 1 \right) \left(e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} e^{-\frac{r_{k-1}^2}{4td}} + e^{-\frac{r_K^2}{4td}} \right)$$

We use the above inequality to estimate $\int_0^1 e^{-tL}(V^a)(x)\, t^{a-1}\, dt$. From this point on the proof is a repetition of the argument in (4.3.17)–(4.3.22) that leads to (4.4.7).

Now we pass to the proof of (4.4.8). This time we merely estimate $e^{-tL}(\mathbb{1}_{B_j})(x)$ by $\mathbb{P}(\Psi_j)$. In view of (4.4.6) and (4.4.10) proceeding as in the proof of (4.4.7) we thus obtain

$$\begin{aligned} e^{-tL}(V^a)(x) &\lesssim V(x)^a \left(e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} e^{-\frac{r_{k-1}^2}{2td}} + e^{-\frac{r_K^2}{2td}} \right) + V(x)^a \sum_{j>0} 2^{ja} e^{-\frac{s_j^2}{4t}} \\ &\lesssim V(x)^a \left(e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} e^{-\frac{r_{k-1}^2}{2td}} + e^{-\frac{r_K^2}{2td}} \right) + V(x)^a \int_1^\infty s^{a-1} e^{-\frac{\sigma_x^2(s)}{8}}\, ds. \end{aligned}$$

Once again we integrate the above expression by repeating the argument in (4.3.17)–(4.3.22) and obtain (4.4.8).

Proof of (4.4.5) The growth assumption on V implies that

$$\mathbb{E}_x[V(X_t)^a] \lesssim (2\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{|y-x|^2}{2t}} e^{\frac{|y|^2}{4}}\, dy. \tag{4.4.11}$$

To estimate the above integral, we rewrite the exponent in the form

$$-\frac{|y|^2}{4} + \frac{|y-x|^2}{2t} = \frac{|y|^2(2-t) - 4\langle x, y\rangle + 2|x|^2}{4t} = \frac{\left| y\sqrt{2-t} - \frac{2x}{\sqrt{2-t}} \right|^2 - \frac{4|x|^2}{2-t} + 2|x|^2}{4t}$$

and plug it into (4.4.11) obtaining

$$\begin{aligned} \mathbb{E}_x[V(X_t)^a] &\lesssim (2\pi t)^{-d/2} \exp\left(\frac{\frac{4|x|^2}{2-t} - 2|x|^2}{4t} \right) \int_{\mathbb{R}^d} \exp\left(-\frac{\left| y\sqrt{2-t} - \frac{2x}{\sqrt{2-t}} \right|^2}{4t} \right) dy \\ &= (2\pi t)^{-d/2} \exp\left(\frac{\frac{4|x|^2}{2-t} - 2|x|^2}{4t} \right) \int_{\mathbb{R}^d} \exp\left(-\frac{\left| y\sqrt{2-t} \right|^2}{4t} \right) dy \\ &= (2\pi t)^{-d/2} \exp\left(\frac{2|x|^2 t}{4t} \right) \left(\frac{4\pi t}{2-t} \right)^{d/2} = \left(\frac{2}{2-t} \right)^{d/2} \exp\left(\frac{|x|^2}{2} \right) \end{aligned}$$

The quantity $\left(\frac{2}{2-t}\right)^{d/2}$ is bounded for $t \in [0,1]$, hence we get

$$\mathbb{E}_x[V(X_t)^a] \lesssim e^{|x|^2}, \qquad t \leqslant 1.$$

Thus, using the Feynman–Kac formula (4.2.1) we estimate

$$e^{-tL}(V^a)(x) \leqslant \mathbb{E}_x[V(X_t)^a] \lesssim e^{|x|^2},$$

so that

$$\int_0^1 e^{-tL}(V^a)(x)\, t^{a-1}\, dt \lesssim e^{|x|^2}.$$

This completes the proof of Proposition 4.4.1. □

Now we pass to the integral (4.4.1) restricted to the range $[1, \infty)$. We shall prove several results with varying assumptions on the potential V. For this reason the treatment here is significantly more complicated than in Section 4.3.

We start with a counterpart of Proposition 4.4.1. To this end we need yet another quantity

$$K_c^a(V)(x) := \min(1, V(x)^a) \int_1^\infty e^{-c\sigma_x(s)} s^{a-1}\, ds \qquad \text{for a.e. } x \in \mathbb{R}^d, \tag{4.4.12}$$

where $a, c > 0$. Note that this is essentially larger than $J^a(V)(x)$ defined by (4.4.3) and used in Proposition 4.4.1. Indeed, observe that for each $c > 0$ there is a constant M independent of x and s such that $\frac{\sigma_x^2(s)}{8} \geqslant c\sigma_x(s) - M$ for all $s \geqslant 1$ and $x \in \mathbb{R}^d$, which means that $e^{-\frac{\sigma_x^2(s)}{8}} \leqslant e^M e^{-c\sigma_x(s)}$ and in turn

$$J^a(V)(x) \lesssim K_c^a(V)(x). \tag{4.4.13}$$

Proposition 4.4.2. *Let V be an a.e. non-negative potential. Assume that the semigroup e^{-tL} satisfies* (ED(δ)) *with some $\delta > 0$. Let $a > 0$, take $b > a$ and define*

$$c = \min\left(\frac{b-a}{8b}, \frac{\delta a}{4b}\right). \tag{4.4.14}$$

Then

$$\int_1^\infty e^{-tL}(V^a)(x)\, t^{a-1}\, dt \lesssim (K_c^a(V)(x) + 1)(I^b(V)(x) + 1) \tag{4.4.15}$$

uniformly in every x such that $V(x) \neq 0$.

Moreover, if V is of exponential growth η, i.e.

$$V(x) \lesssim e^{\eta|x|}, \tag{4.4.16}$$

with $\eta < \frac{\sqrt{\delta}}{\sqrt{2d}a}$, then

$$\int_1^\infty e^{-tL}(V^a)(x)\, t^{a-1}\, dt \lesssim \exp\Big(\sqrt{d}a\eta|x|\Big), \qquad x \in \mathbb{R}^d. \tag{4.4.17}$$

Remark. The implicit constants in (4.4.15), (4.4.17) possibly depend on a, b, δ, η.

Proof. **Proof of** (4.4.15). Using the splitting into the sets B_j as in (4.4.6) and the Feynman–Kac formula (4.2.1) we obtain

$$\begin{aligned} e^{-tL}(V^a)(x) &\lesssim V(x)^a e^{-tL}(\mathbb{1})(x) + \sum_{j>0} V(x)^a 2^{ja} e^{-tL}(\mathbb{1}_{B_j})(x) \\ &\lesssim V(x)^a e^{-tL}(\mathbb{1})(x) + \sum_{j>0} V(x)^a 2^{ja} \mathbb{E}_x[e^{-\int_0^t V(X_s)\, ds} \mathbb{1}_{\Psi_j}] \end{aligned}$$

By Lemma 4.3.4 we have

$$\int_1^\infty V(x)^a e^{-tL}(\mathbb{1})(x)\, t^{a-1}\, dt \lesssim I^a(V)(x) + 1 \lesssim I^b(V)(x) + 1.$$

Hence, we only focus on the integral over the second term, namely $\int_1^\infty S_x(t)\, t^{a-1}\, dt$ with

$$S_x(t) := \sum_{j>0} V(x)^a 2^{ja} \mathbb{E}_x[e^{-\int_0^t V(X_s)\, ds} \mathbb{1}_{\Psi_j}]. \tag{4.4.18}$$

Let $p = \frac{b}{a}$ and let q be its conjugate exponent. Then Hölder's inequality gives

$$\begin{aligned} S_x(t) &\leqslant \sum_{j>0} V(x)^a 2^{ja} \left(\mathbb{E}_x[e^{-p\int_0^t V(X_s) ds}]\right)^{1/p} \left(\mathbb{E}_x[\mathbb{1}_{\Psi_j}]\right)^{1/q} \\ &\lesssim \sum_{j>0} V(x)^a 2^{ja} \left(e^{-tL}(\mathbb{1})(x)\right)^{1/p} \mathbb{P}(\Psi_j)^{1/q}. \end{aligned} \tag{4.4.19}$$

Using (4.4.19) we shall prove that

$$\int_1^\infty S_x(t)\, t^{a-1}\, dt \lesssim (I^b(V)(x) + 1) \left(\int_1^\infty e^{-c\sigma_x(s)} s^{a-1}\, ds + 1\right). \tag{4.4.20}$$

and

$$\int_1^\infty S_x(t)\, t^{a-1}\, dt \lesssim V(x)^a \left(\int_1^\infty e^{-c\sigma_x(s)} s^{a-1}\, ds\right). \tag{4.4.21}$$

These two inequalities imply that

$$\int_1^\infty S_x(t)\, t^{a-1}\, dt \lesssim (K_c^a(V)(x) + 1)(I^b(V)(x) + 1),$$

and thus are enough to complete the proof of (4.4.15).

We start with (4.4.20). Using monotonicity, the semigroup property, and (ED(δ)) we obtain that

$$e^{-tL}(\mathbb{1})(x) = e^{-tL/2}(e^{-tL/2}(\mathbb{1}))(x) \lesssim e^{-\delta t/2} e^{-L/2}(\mathbb{1})(x).$$

Hence, (4.4.19) gives

$$S_x(t) \leqslant e^{-\frac{\delta t}{2p}} \left(V(x)^{ap} e^{-L/2}(\mathbb{1})(x)\right)^{1/p} \cdot \sum_{j>0} 2^{ja} \mathbb{P}(\Psi_j)^{1/q}.$$

Since $ap = b$ a repetition of the computation in (4.3.26) shows that

$$S_x(t) \lesssim (I^b(V)(x) + 1) \cdot e^{-\frac{\delta t}{2p}} \cdot \sum_{j>0} 2^{ja} \mathbb{P}(\Psi_j)^{1/q}. \tag{4.4.22}$$

Now, using the estimate (4.4.10) for $\mathbb{P}(\Psi_j)$ we obtain

$$\sum_{j>0} 2^{ja} \mathbb{P}(\Psi_j)^{1/q} \lesssim \sum_{j>0} 2^{ja} e^{-\frac{s_j^2}{4tq}}. \tag{4.4.23}$$

Consider the integral

$$\int_1^\infty e^{-\frac{\delta t}{2p}} e^{-\frac{s_j^2}{4tq}} t^{a-1}\, dt.$$

We split it at $t = s_j$ and estimate each part separately:

$$\int_1^\infty e^{-\frac{\delta t}{2p}} e^{-\frac{s_j^2}{4tq}} t^{a-1}\, dt \leqslant \int_1^{s_j} e^{-\frac{s_j^2}{4tq}} t^{a-1}\, dt + \int_{s_j}^\infty e^{-\frac{\delta t}{2p}} t^{a-1}\, dt$$
$$\lesssim e^{-\frac{s_j}{8q}} + e^{-\frac{\delta s_j}{4p}} \lesssim e^{-cs_j}.$$

Recall that $c = \min(\frac{b-a}{8b}, \frac{\delta a}{4b})$. Formally, the splitting above only works when $s_j \geqslant 1$, however, the estimate

$$\int_1^\infty e^{-\frac{\delta t}{2p}} e^{-\frac{s_j^2}{4tq}} t^{a-1}\, dt \lesssim e^{-cs_j}$$

remains true for any $s_j \geqslant 0$. Consequently, integrating (4.4.23) we get

$$\int_1^\infty e^{-\frac{\delta t}{2p}} \cdot \sum_{j>0} 2^{ja} \mathbb{P}(\Psi_j)^{1/q} t^{a-1}\, dt \leqslant \sum_{j>0} 2^{ja} e^{-cs_j} \lesssim \int_1^\infty e^{-c\sigma_x(s)} s^{a-1}\, ds, \tag{4.4.24}$$

where in the last inequality above we used the fact that $s_j = \sigma_x(2^j)$. Combining (4.4.24) with (4.4.22) gives (4.4.20).

We pass to the proof of (4.4.21). Note that (4.4.19) and the assumption (ED(δ)) imply

$$S_x(t) \lesssim e^{-\delta t/p} \sum_{j>0} V(x)^a 2^{ja} \mathbb{P}(\Psi_j)^{1/q},$$

thus, an application of (4.4.24) produces

$$\int_1^\infty S_x(t)\, t^{a-1}\, dt \lesssim V(x)^a \int_1^\infty e^{-c\sigma_x(s)} s^{a-1}\, ds,$$

and (4.4.21) is justified.

Proof of (4.4.17). Using the Feynman–Kac formula (4.2.2) and Cauchy–Schwarz inequality we obtain

$$e^{-tL}(V^a)(x) \leqslant \mathbb{E}_x \left[V^{2a}(X_t)\right]^{1/2} \mathbb{E}_x \left[e^{-2\int_0^t V(X_s)\, ds}\right]^{1/2}$$
$$\leqslant \mathbb{E}_x \left[V^{2a}(X_t)\right]^{1/2} \left(e^{-tL}(\mathbb{1})(x)\right)^{1/2}.$$

Hence, the assumptions (ED(δ)) and (4.4.16) give

$$e^{-tL}(V^a)(x) \lesssim e^{-\delta t/2} \left(\mathbb{E}_x e^{2\eta a|X_t|}\right)^{1/2}.$$

We claim that the proof of (4.4.17) will be completed if we show that

$$\mathbb{E}_x e^{2\eta a|X_t|} \lesssim \exp\Big(2d\eta^2 a^2 t + 2\sqrt{d}\eta a|x|\Big). \tag{4.4.25}$$

Indeed, the above estimate leads to

$$\int_1^\infty e^{-tL}(V^a)(x)\, t^{a-1}\, dt \lesssim e^{\sqrt{d}\eta a|x|} \int_1^\infty \exp\big(-\tfrac{\delta t}{2} + d\eta^2 a^2 t\big)\, t^{a-1}\, dt \lesssim e^{\sqrt{d}\eta a|x|},$$

where in the last inequality we used the assumption $\eta < \frac{\sqrt{\delta}}{\sqrt{2da}}$.

It remains to justify (4.4.25). Since

$$\begin{aligned}\mathbb{E}_x\left[e^{2\eta a|X_t|}\right] &= \frac{1}{(2\pi t)^{d/2}}\int_{\mathbb{R}^d} e^{2\eta a|z|}e^{-\frac{|x-z|^2}{2t}}\,dz \leqslant \frac{1}{(2\pi t)^{d/2}}\int_{\mathbb{R}^d} e^{2\eta a\sum_{i=1}^d|z_i|}e^{-\frac{|x-z|^2}{2t}}\,dz\\ &= \prod_{i=1}^d \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{2\eta a|z_i|}e^{-\frac{|x_i-z_i|^2}{2t}}\,dz_i\end{aligned} \tag{4.4.26}$$

it suffices to focus on each of the factors in the above product separately. A simple computation shows that

$$\begin{aligned}&\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{2\eta a|z_i|}e^{-\frac{|x_i-z_i|^2}{2t}}\,dz_i \leqslant e^{2\eta a|x_i|}\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{2\eta a|z_i-x_i|}e^{-\frac{|x_i-z_i|^2}{2t}}\,dz_i\\ &= e^{2\eta a|x_i|}\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{2\eta a|y|}e^{-\frac{|y|^2}{2t}}\,dy \leqslant 2e^{2\eta a|x_i|}\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{2\eta a y}e^{-\frac{|y|^2}{2t}}\,dy\\ &= 2e^{2\eta a|x_i|}e^{\frac{(2\eta a)^2 t}{2}} = 2e^{2\eta a|x_i|}e^{2\eta^2a^2t}.\end{aligned}$$

Hence, coming back to (4.4.26) and using the inequality $\sum_{i=1}^d|x_i| \leqslant \sqrt{d}|x|$ we obtain

$$\mathbb{E}_x\left[e^{2\eta a|X_t|}\right] \leqslant 2^d e^{2d\eta^2a^2t}\prod_{i=1}^d e^{2\eta a|x_i|} \lesssim \exp\Big(2d\eta^2a^2t + 2\sqrt{d}\eta a|x|\Big),$$

thus proving the claim (4.4.25).

The proof of Proposition 4.4.2 is thus completed. □

By a comparison with the Hermite semigroup we can improve Proposition 4.4.2 in the full range $a > 0$ for potentials V which grow faster than $|x|^2$ at infinity.

Proposition 4.4.3. *Let c, b, N be positive constants. Assume that $V \in L^\infty_{\mathrm{loc}}$ is an a.e. non-negative potential that satisfies $c|x|^2 \leqslant V(x)$ for a.e. $|x| \geqslant N$ and $V(x) \lesssim e^{b|x|^2}$. Denote $\mu = \frac{d^{1/3}}{5N^2}$. Then, for each $0 < a \leqslant \frac{\mu\tanh\frac{\mu}{2}}{4b}$ we have*

$$\int_1^\infty e^{-tL}(V^a)(x)\,t^{a-1}\,dt \lesssim 1, \qquad x \in \mathbb{R}^d. \tag{4.4.27}$$

Proof. Denote by ω a C_c^∞ function which is equal to $c|x|^2$ for $|x| \leqslant N$, is bounded by $c|x|^2$, and vanishes for $|x| \geqslant 2N$. Then, for all $k \in (0, 1]$, we have

$$V(x) + k\omega(x) \geqslant ck|x|^2, \qquad \text{for a.e. } x \in \mathbb{R}^d.$$

Hence, using (4.2.2) and Cauchy–Schwarz inequality we obtain

$$\begin{aligned}e^{-tL}(V^a)(x) &= \mathbb{E}_x\left[e^{-\int_0^t V(X_s)\,ds}V^a(X_t)\right] = \mathbb{E}_x\left[e^{-\int_0^t (V+k\omega)(X_s)\,ds}V^a(X_t)\cdot e^{k\int_0^t\omega(X_s)\,ds}\right]\\ &\leqslant \left(\mathbb{E}_x\left[e^{-2\int_0^t(V+k\omega)(X_s)\,ds}V^{2a}(X_t)\right]\right)^{1/2}\cdot\left(\mathbb{E}_x\left[e^{2k\int_0^t\omega(X_s)\,ds}\right]\right)^{1/2}\\ &\leqslant \left(\mathbb{E}_x\left[e^{-2ck\int_0^t|X_s|^2\,ds}V^{2a}(X_t)\right]\right)^{1/2}\cdot\left(\mathbb{E}_x\left[e^{2k\int_0^t\omega(X_s)\,ds}\right]\right)^{1/2}\\ &= \left(e^{-t(-\frac{\Delta}{2}+2ck|x|^2)}(V^{2a})(x)\right)^{1/2}\cdot\left(\mathbb{E}_x\left[e^{2k\int_0^t\omega(X_s)\,ds}\right]\right)^{1/2}.\end{aligned} \tag{4.4.28}$$

In what follows we denote

$$\gamma = \gamma(c,k) = 2\sqrt{ck}.$$

Throughout the proof the implicit constants in $\lesssim$ depend on $k \in (0,1]$, thus also on γ. Appropriate k and γ will be fixed at a later stage. From [53, 4.1.2] or [49, 1.4] we deduce that

$$e^{-t(-\frac{\Delta}{2}+2ck|x|^2)}f(x) = e^{-t(-\frac{\Delta}{2}+\frac{\gamma^2}{2}|x|^2)}f(x) = \left(\frac{\gamma}{2\pi}\right)^{d/2}\int_{\mathbb{R}^d} K_t^\gamma(x,y)f(y)\,dy,$$

with

$$\begin{aligned} K_t^\gamma(x,y) &= \frac{1}{(\sinh\gamma t)^{d/2}}\exp\left(-\frac{\gamma}{2}\left(|x|^2+|y|^2\right)\coth\gamma t + \frac{\gamma\langle x,y\rangle}{\sinh\gamma t}\right) \\ &= \frac{1}{(\sinh\gamma t)^{d/2}}\exp\left(-\frac{\gamma|x-y|^2}{4\tanh\frac{\gamma t}{2}} - \frac{\gamma\tanh\frac{\gamma t}{2}}{4}|x+y|^2\right). \end{aligned}$$

Using the upper bound on V we estimate $e^{-t(-\frac{\Delta}{2}+\frac{\gamma^2}{2}|x|^2)}(V^{2a})$ as follows

$$\begin{aligned} &e^{-t(-\frac{\Delta}{2}+\frac{\gamma^2}{2}|x|^2)}(V^{2a})(x) \\ &\qquad\lesssim \frac{1}{(\sinh\gamma t)^{d/2}}\int_{\mathbb{R}^d} V(y)^{2a}\exp\left(-\frac{\gamma|x-y|^2}{4\tanh\frac{\gamma t}{2}} - \frac{\gamma\tanh\frac{\gamma t}{2}}{4}|x+y|^2\right)dy \\ &\qquad\lesssim e^{-\frac{d\gamma t}{2}}\int_{\mathbb{R}^d}\exp\left(2ab|y|^2 - \frac{\gamma|x-y|^2}{4\tanh\frac{\gamma t}{2}} - \frac{\gamma\tanh\frac{\gamma t}{2}}{4}|x+y|^2\right)dy \end{aligned} \tag{4.4.29}$$

Rewriting the exponents we obtain

$$\begin{aligned} &2ab|y|^2 - \frac{\gamma|x-y|^2}{4\tanh\frac{\gamma t}{2}} - \frac{\gamma\tanh\frac{\gamma t}{2}}{4}|x+y|^2 \\ &= \left(2ab - \frac{\gamma\coth\gamma t}{2}\right)\left|y + \frac{\gamma\operatorname{csch}\gamma t}{4ab-\gamma\coth\gamma t}x\right|^2 - \left(\frac{\gamma\coth\gamma t}{2} + \frac{(\gamma\operatorname{csch}\gamma t)^2}{8ab-2\gamma\coth\gamma t}\right)|x|^2. \end{aligned}$$

We see that for the integral in (4.4.29) to be finite the quantity $\varphi(t) := 2ab - \frac{\gamma\coth\gamma t}{2}$ has to be negative for all $t \geqslant 1$, which is satisfied for $a \leqslant \frac{\gamma\tanh\frac{\gamma}{2}}{4b}$ since $\frac{\gamma\tanh\frac{\gamma}{2}}{4b} < \frac{\gamma\coth\gamma t}{4b}$. For such a we have $\varphi(t) \leqslant \frac{\gamma}{2}(\tanh\frac{\gamma}{2} - \coth\gamma t)$ and

$$\begin{aligned} &\int_{\mathbb{R}^d}\exp\left(2ab|y|^2 - \frac{\gamma|x-y|^2}{4\tanh\frac{\gamma t}{2}} - \frac{\gamma\tanh\frac{\gamma t}{2}}{4}|x+y|^2\right)dy \\ &= \exp\left(-\left(\frac{\gamma\coth\gamma t}{2} + \frac{(\gamma\operatorname{csch}\gamma t)^2}{4\varphi(t)}\right)|x|^2\right)\int_{\mathbb{R}^d} e^{\varphi(t)|y|^2}\,dy \\ &\leqslant \exp\left(-\frac{\gamma}{2}\left(\coth\gamma t + \frac{\operatorname{csch}^2\gamma t}{\tanh\frac{\gamma}{2} - \coth\gamma t}\right)|x|^2\right)\left(-\frac{\pi}{\varphi(t)}\right)^{d/2}. \end{aligned}$$

Denoting $\psi(t) := \coth\gamma t + \frac{\operatorname{csch}^2\gamma t}{\tanh\frac{\gamma}{2}-\coth\gamma t}$ a calculation gives

$$\psi'(t) = -\frac{\gamma\operatorname{csch}^2\gamma t\,\cdot\,\left(-1+\tanh^2\frac{\gamma}{2}\right)}{\left(\tanh\frac{\gamma}{2} - \coth\gamma t\right)^2}.$$

Since ψ' is positive the function ψ is strictly increasing. Moreover it has a zero at $t = \frac{1}{2}$ so that for $t \geqslant 1$ we have $\psi(t) \geqslant \psi(1) = \delta > 0$ and thus we can continue the previous calculation as follows

$$\begin{aligned}&\exp\left(-\frac{\gamma}{2}\left(\coth\gamma t + \frac{\operatorname{csch}^2\gamma t}{\tanh\frac{\gamma}{2} - \coth\gamma t}\right)|x|^2\right)\left(-\frac{\pi}{\varphi(t)}\right)^{d/2}\\ &\lesssim e^{-\frac{\gamma\delta|x|^2}{2}}(-\varphi(t))^{-d/2}\end{aligned}$$

Next we need to handle the term $(-\varphi(t))^{-d/2}$. Since $a \leqslant \frac{\gamma\tanh\frac{\gamma}{2}}{4b}$ we see that

$$(-\varphi(t))^{-d/2} \lesssim \left(\gamma\left(\coth\gamma t - \tanh\frac{\gamma}{2}\right)\right)^{-d/2} \lesssim 1, \qquad t \geqslant 1.$$

Finally plugging the above estimates in (4.4.29) we get

$$e^{-t(-\frac{\Delta}{2}+\frac{\gamma^2}{2}|x|^2)}(V^{2a})(x) \lesssim e^{-\frac{d\gamma t}{2}}e^{-\frac{\gamma\delta|x|^2}{2}}, \tag{4.4.30}$$

uniformly in $x \in \mathbb{R}^d$ and $t \geqslant 1$.

Next we estimate $\left(\mathbb{E}_x\left[e^{2k\int_0^t\omega(X_s)\,ds}\right]\right)^{1/2}$. Since $\omega \leqslant 4cN^2\mathbb{1}_P$ for $P = [-2N, 2N] \times \mathbb{R}^{d-1}$, we can apply Lemma 4.3.2 with $k' = 4ckN^2$, which gives

$$\mathbb{E}_x\left[e^{2k\int_0^t\omega(X_s)\,ds}\right] \lesssim e^{512c^2k^2N^6t} = e^{32\gamma^4N^6t} \tag{4.4.31}$$

Combining (4.4.30) and (4.4.31) and coming back to (4.4.28) we reach

$$\int_1^\infty e^{-tL}(V^a)(x)\,t^{a-1}\,dt \lesssim e^{-\frac{\gamma\delta|x|^2}{4}}\int_1^\infty e^{-\frac{d\gamma t}{4}}e^{16\gamma^4N^6t}t^{a-1}\,dt \lesssim 1, \qquad x \in \mathbb{R}^d,$$

provided that $\gamma < \frac{d^{1/3}}{4N^2}$. This can be achieved by taking $k = \min(1, \frac{\mu^2}{4c})$, since for such k we have

$$\gamma = 2\sqrt{ck} \leqslant \mu < \frac{d^{1/3}}{4N^2}.$$

The proof of Proposition 4.4.3 is thus completed. □

We shall now derive L^1 boundedness of R_V^a using Proposition 4.4.1 together with one of the Propositions 4.4.2 and 4.4.3.

Combining Proposition 4.4.1 and Proposition 4.4.2 we get a theorem on the L^1 boundedness of R_V^a. Note that this theorem inherits the stronger assumptions on V from Proposition 4.4.2. Its advantage is the allowance of large a when $V(x) \lesssim e^{\eta|x|}$ with small η. This is useful for instance when $V(x) \approx_g |x|^\alpha$.

Theorem 4.4.4. *Let V be an a.e. non-negative potential having an exponential growth* (4.4.16) *for some $\eta > 0$ and such that e^{-tL} has an exponential decay* (ED(δ)) *of order $\delta > 0$. Let $0 < a < \frac{\sqrt{\delta}}{\eta\sqrt{2d}}$, take $b > a$ and let c be the constant defined in* (4.4.14). *If*

$$K_c^a(V)(x) \lesssim_g 1 \qquad \textit{and} \qquad I^b(V)(x) \lesssim_g 1,$$

then R_V^a is bounded on L^1.

Proof. By duality it suffices to estimate the L^∞ norm of

$$\frac{1}{\Gamma(a)}\int_0^\infty e^{-tL}(V^a)t^{a-1}\,dt = \frac{1}{\Gamma(a)}\int_0^1 e^{-tL}(V^a)t^{a-1}\,dt + \frac{1}{\Gamma(a)}\int_1^\infty e^{-tL}(V^a)t^{a-1}\,dt =: L + G. \tag{4.4.32}$$

Using the bound $e^{\eta|x|} \lesssim e^{\frac{|x|^2}{4a}}$ and (4.4.5) from Proposition 4.4.1 we see that

$$L(x) \lesssim C(N),$$

whenever $|x| \leqslant N$. Then (4.4.13) together with (4.4.4) from Proposition 4.4.1 gives

$$\|L\|_\infty \lesssim 1.$$

The estimate

$$\|G\|_\infty \lesssim 1$$

is a straightforward consequence of our assumptions and Proposition 4.4.2. □

Proposition 4.4.1 and Proposition 4.4.3 allow us to improve Theorem 4.4.4 for potentials that grow at least as a constant times $|x|^2$. The improvement comes from the replacement of the condition $K_c^a(V)(x) \lesssim_g 1$ by $J^a(V)(x) \lesssim 1$. This is useful e.g. for potentials $V(x) = \beta^{|x|}$, $\beta > 1$, for which $K_c^a(V)$ may be unbounded.

Theorem 4.4.5. *Let $0 < a < \infty$ and let V be an a.e. non-negative potential which satisfies the estimate $c|x|^2 \lesssim_g V(x)$ for some $c > 0$. Assume that for all $\varepsilon > 0$ we have $V(x) \lesssim_\varepsilon e^{\varepsilon|x|^2}$. If*

$$J^a(V)(x) \lesssim_g \qquad \text{and} \qquad I^a(V)(x) \lesssim_g 1,$$

then R_V^a is bounded on L^1.

Proof. We use the splitting (4.4.32) again. The estimate $\|G\|_\infty \lesssim 1$ is a consequence of Proposition 4.4.3. Indeed, the assumption $V(x) \lesssim e^{\varepsilon|x|^2}$ with arbitrarily small $\varepsilon > 0$ implies that we can apply Proposition 4.4.3 with arbitrarily large $a > 0$. The bound $\|L\|_\infty \lesssim 1$ follows from the assumptions and Proposition 4.4.1 as in the proof of Theorem 4.4.4. □

As a corollary of Theorems 4.4.4 and 4.4.5 we obtain the L^1 boundedness of R_V^a for various classes of potentials. The corollary below is a restatement of Theorem 4.0.3 from the beginning of the chapter in the L^1 case.

Corollary 4.4.6. *Let $V\colon \mathbb{R}^d \to [0,\infty)$ be a function in L^∞_{loc}. Then in all the three cases*

1. *$V(x) \approx 1$ globally*
2. *For some $\alpha > 0$ we have $V(x) \approx |x|^\alpha$ globally*
3. *For some $\beta > 1$ we have $V(x) \approx \beta^{|x|}$ globally*

each of the Riesz transforms R_V^a, $a > 0$, is bounded on L^1.

Remark. Similarly to Corollary 4.3.6 the Euclidean norm $|\cdot|$ in (2) and (3) can be replaced by an arbitrary norm on $\mathbb{R}^d$.

Proof. In the proof implicit constants in $\lesssim$, $\gtrsim$, and $\approx$ do not depend on $x \in \mathbb{R}^d$ but may depend on $a > 0$, $\alpha > 0$ or $\beta > 1$.

Note that in all three cases the assumptions of Lemma 4.3.1 are satisfied so that the semigroup e^{-tL} satisfies (ED(δ)).

In case 1) we merely use (ED(δ)) and obtain

$$\frac{1}{\Gamma(a)} \int_0^\infty e^{-tL}(V^a)(x) t^{a-1}\, dt \lesssim \frac{1}{\Gamma(a)} \int_0^\infty \left\| e^{-tL}(\mathbb{1}) \right\|_\infty t^{a-1}\, dt \lesssim 1,$$

uniformly in $x \in \mathbb{R}^d$.

In the treatment of the remaining cases we will apply Theorem 4.4.4 in case 2) and Theorem 4.4.5 in case 3).

We start with case 2); the task is to check that the assumptions of Theorem 4.4.4 hold. Clearly (4.4.16) is true for any $\eta > 0$. In the proof of Corollary 4.3.6 we justified in (4.3.30) that $I^b(V)(x) \lesssim_g 1$ for any $b > 0$. Finally we need to control $K_c^a(V)(x)$. To this end we shall estimate $\sigma_x(s)$ from below. Let C, N, m and M be non-negative constants such that

$$m|x|^\alpha < V(x) < M|x|^\alpha \quad \text{for a.e.} \quad |x| > N$$

and

$$V(x) \leqslant C \quad \text{for a.e.} \quad |x| \leqslant N.$$

Take $|x| \geqslant N$ and assume that $|x - y| < \varepsilon |x| s^{1/\alpha}$, where $\varepsilon > 0$ is a constant to be determined in a moment. Then

$$|y| \leqslant |x| + |x - y| \leqslant |x|(1 + \varepsilon s^{1/\alpha})$$

so that for $|y| > N$ we have

$$V(y) \leqslant M|y|^\alpha \leqslant M|x|^\alpha \left(1 + \varepsilon s^{1/\alpha}\right)^\alpha \leqslant MA|x|^\alpha (1 + \varepsilon^\alpha s)$$

for some constant $A \geqslant 1$ depending only on α. On the other hand

$$V(x) \geqslant m|x|^\alpha$$

so taking ε such that $MA\varepsilon^\alpha = \frac{m}{2}$ we see that the inequality $|x - y| < \varepsilon |x| s^{1/\alpha}$ implies

$$V(y) \leqslant MA|x|^\alpha (1 + \varepsilon^\alpha s) \leqslant MA|x|^\alpha + \frac{sV(x)}{2} \leqslant \left(\frac{MA}{m} + \frac{s}{2}\right) V(x) \leqslant sV(x),$$

whenever s is large enough (independently of x). Thus we proved that $\sigma_x(s) \geqslant \varepsilon |x| s^{1/\alpha}$ for such s and a.e. $|x| \geqslant N$. Consequently,

$$K_c^a(V)(x) \lesssim_g 1 + \int_1^\infty e^{-c\varepsilon |x| s^{1/\alpha}} s^{a-1}\, ds \lesssim_g 1$$

for any $a, c > 0$ and an application of Theorem 4.4.4 completes the proof in case 2).

Finally we justify case 3). It is clear that $c|x|^2 \lesssim_g V(x) \lesssim e^{\varepsilon|x|^2}$ for some $c > 0$ and all $\varepsilon > 0$. Moreover, in the proof of Corollary 4.3.6 in (4.3.33) we justified that $I^a(V)(x) \lesssim_g 1$. Thus, in order to use Theorem 4.4.5 it remains to estimate $J^a(V)(x)$. Similarly, to case 2) we shall estimate $\sigma_x(s)$ from below. Let $M > 0$ be a constant such that $V(y) \leqslant M\beta^{|y|}$, for a.e. $y \in \mathbb{R}^d$ and let N, m be non-negative constants such that $m\beta^{|x|} < V(x)$ for a.e. $|x| \geqslant N$. Take $|x| \geqslant N$, $s \geqslant 1$ and assume that $|x-y| < \frac{1}{2}\log_\beta s$. Then we have $|y| \leqslant |x| + \frac{1}{2}\log_\beta s$, so that

$$V(y) \leqslant M s^{1/2}\beta^{|x|} \leqslant \frac{M}{m} s^{1/2} V(x) \leqslant sV(x),$$

for s large enough (independently of y and x). In other words we proved that $\sigma_x(s) \geqslant \frac{1}{2}\log_\beta s$ whenever $|x| \geqslant N$ and s is uniformly large enough. Consequently,

$$J^a(V)(x) \lesssim_g 1 + \int_1^\infty e^{-\frac{(\log_\beta s)^2}{32}} s^{a-1}\, ds \lesssim_g 1$$

for any $a > 0$ and an application of Theorem 4.4.5 completes the proof in case 3). □

We finish this section with improved results for Riesz transforms R_V^a in the range $0 < a < 1$. These results are not needed in the proof of Corollary 4.4.6, however they might be useful in other cases.

Using the L^1 boundedness of R_V^1 one may improve Proposition 4.4.2 in the range $0 \leqslant a \leqslant 1$.

Proposition 4.4.7. *Let $a \leqslant 1$ and assume that e^{-tL} satisfies* (ED(δ)) *with some $\delta > 0$. Then the estimate*

$$\int_1^\infty e^{-tL}(V^a)(x)\, t^{a-1} dt \lesssim 1 \tag{4.4.33}$$

holds uniformly in $x \in \mathbb{R}^d$.

Proof. Observe that for $a \leqslant 1$ we have

$$e^{-tL}(V^a)(x) \leqslant e^{-tL}(V)(x) + e^{-tL}(\mathbb{1})(x),$$

so that

$$\int_1^\infty e^{-tL}(V^a)(x)\, t^{a-1}\, dt \leqslant \int_1^\infty e^{-tL}(V)(x)\, t^{a-1}\, dt + \int_1^\infty e^{-tL}(\mathbb{1})(x)\, t^{a-1}\, dt. \tag{4.4.34}$$

From e.g. [2, Theorem 4.3] we see that the operator R_V^1 is bounded on L^1 which, by duality, means that the first integral in (4.4.34) is bounded independently of x. Boundedness of the second integral follows from (ED(δ)). □

Finally, combining Proposition 4.4.7 and Proposition 4.4.1 we obtain an improved version of Theorem 4.4.4 in the range $0 < a \leqslant 1$.

Theorem 4.4.8. *Let $0 < a \leqslant 1$ and let V be an a.e. non-negative potential which satisfies the growth estimate $V(x) \lesssim \exp\left(\frac{|x|^2}{4a}\right)$ and such that e^{-tL} has an exponential decay* (ED(δ)) *for some $\delta > 0$. If*

$$J^a(V)(x) \lesssim_g 1 \qquad \text{and} \qquad I^a(V)(x) \lesssim_g 1,$$

then R_V^a is bounded on L^1.

Proof. We use the splitting (4.4.32). The estimate $\|G\|_\infty \lesssim 1$ is an immediate consequence of Proposition 4.4.7. The bound $\|L\|_\infty \lesssim 1$ follows from the assumptions and Proposition 4.4.1 as in the proof of Theorem 4.4.4. □

Chapter 5

Dimension-free estimates for Riesz transforms associated with Schrödinger operators

In this chapter we investigate the same Riesz transforms R_V^a as in Chapter 4, see (4.0.1), but this time we aim at estimating their norm independently of the dimension d of the underlying space $\mathbb{R}^d$. In order to achieve the desired results we consider only potentials of the form

$$V(x) = V_1(x) + \cdots + V_d(x), \tag{5.0.1}$$

where each V_i acts only on the i-th coordinate of the argument x and has polynomial growth with the exponent not greater than 2, i.e. there are absolute constants m and M such that

$$m|x_i|^\alpha \leqslant V_i(x) \leqslant M|x_i|^\alpha \tag{5.0.2}$$

for some $0 < \alpha \leqslant 2$. This holds for example if $V_i(x) = x_i^2$ and $V(x) = |x|^2$, which results in the operator $L = -\frac{1}{2}\Delta + |x|^2$ called the harmonic oscillator. The Riesz transform $R_{|x|^2}^{1/2}$ associated with the harmonic oscillator is known to be bounded independently of the dimension, see [24, 28, 34], although only if $a = \frac{1}{2}$.

By the definition (4.0.1) of R_V^a and the positivity-preserving property of the semigroup e^{-tL} obtaining the L^∞ bounds for R_V^a amounts to estimating the value of $R_V^a(\mathbb{1})(x)$ independently of x and d, which in turn hints that the main part of the proof is estimating the semigroup applied to the constant function 1, i.e. $e^{-tL}(\mathbb{1})$. The particular structure of V (5.0.1) lets us write

$$L = \sum_{i=1}^{d} L_i, \quad \text{where } L_i = -\frac{1}{2}\frac{\partial^2}{\partial x_i^2} + V_i, \tag{5.0.3}$$

and, as a consequence, factorize the semigroup e^{-tL} in the following way

$$e^{-tL} = \prod_{i=1}^{d} e^{-tL_i} \quad \text{and hence} \quad e^{-tL}(\mathbb{1})(x) = \prod_{i=1}^{d} e^{-tL_i}(\mathbb{1})(x). \tag{5.0.4}$$

This is the key property allowing us to get estimates that does not depend on the dimension d.

The main result of the chapter is the following theorem.

Theorem 5.0.1. *Fix $\alpha > 0$ and let V given by* (5.0.1) *satisfy* (5.0.2)*. For $a > 0$ let the Riesz transform R_V^a be defined as in* (4.0.1)*. Then there is a constant $C > 0$ depending on m, M, and α and independent of the dimension d such that*

$$\|R_V^a f\|_\infty \leqslant C\|f\|_\infty, \quad f \in L^\infty.$$

As a by-product of our considerations we also obtain L^1 estimates for R_V^a, but only for a limited range of a. The reason for this is that we need to use concavity of the function x^a.

Theorem 5.0.2. *Fix $\alpha > 0$ and let V given by* (5.0.1) *satisfy* (5.0.2)*. For $a \leqslant 1$ let the Riesz transform R_V^a be defined as in* (4.0.1)*. Then there is a constant $C > 0$ depending on m, M, and α and independent of the dimension d such that*

$$\|R_V^a f\|_1 \leqslant C\|f\|_1, \quad f \in L^1.$$

As mentioned above, the proof of the theorems is based on the estimates of the semigroup e^{-tL}, thus similarly to the previous chapter we will extensively use the Feynman–Kac formula

$$e^{-tL} f(x) = \mathbb{E}_x \left[e^{-\int_0^t V(X_s)\, ds} f(X_t) \right], \quad f \in L^2. \tag{5.0.5}$$

It will let us obtain exponential estimates for the one-dimensional semigroups e^{-tL_i}, namely

$$e^{-tL_i}(\mathbb{1})(x) \leqslant e^{-c_N t V_i(x)} \quad \text{for } t \leqslant N,$$

which we will then combine, using the factorization property (5.0.4), into an estimate for the whole semigroup e^{-tL}. As mentioned in the introduction, it is noteworthy that the constant in front of the exponential in the above estimate is 1. This means that we can multiply one-dimensional bounds to estimate the full semigroup e^{-tL} without constants growing with the dimension. From that moment the proof will be similar to L^∞ and L^1 estimates of the operator R_V^a presented in Sections 4.3 and 4.4, in particular instead of estimating the L^1 norm of $R_V^a = V^a L^{-a}$ directly we estimate the L^∞ norm of the adjoint operator

$$(L^{-a} V^a) f(x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL}(V^a f)(x)\, t^{a-1}\, dt.$$

In this case the formula for the factorization of the semigroup becomes

$$e^{-tL}(V) = \sum_{i=1}^d e^{-tL}(V_i) = \sum_{i=1}^d e^{-tL^i}(\mathbb{1})\, e^{-tL_i}(V_i), \quad \text{where } L^i = L - L_i.$$

Since now we pursue dimension-free estimates, the notation $A \lesssim B$ means that $A \leqslant CB$ for some constant $C > 0$ which does not depend on the dimension d but may depend on a, α, m and M. If both $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$.

5.1 Definitions

In this chapter we consider the same operators as in Chapter 4 and thus we encourage the reader to consult Section 4.2, where we define the semigroup $e^{-tL}f$ for $f \in L^\infty$ and the Riesz transform R_V^a and then we present basic facts regarding these operators. In particular it follows that Theorem 5.0.1 may be rewritten as

Theorem 5.1.1. *Fix $\alpha > 0$ and let V given by (5.0.1) satisfy (5.0.2). For $a > 0$ let the Riesz transform R_V^a be defined as in (4.0.1). Then there is a constant $C > 0$ independent of the dimension d such that*

$$\|R_V^a(\mathbb{1})\|_\infty \leqslant C.$$

The only additional operators we need are the one-dimensional semigroups e^{-tL_i}, where

$$L_i = -\frac{1}{2}\frac{\partial^2}{\partial x_i^2} + V_i, \qquad i = 1, \dots, d.$$

For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $y \in \mathbb{R}$ and $f\colon \mathbb{R}^d \to \mathbb{R}$ let

$$f_{x^i}(y) = f(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_d).$$

Then for $i = 1, \dots, d$ we define

$$e^{-tL_i}f(x) := \mathbb{E}_{x_i}\left[e^{-\int_0^t V_i(X_s)\,ds} f_{x^i}(X_t^i)\right], \quad f \in L^\infty. \tag{5.1.1}$$

The expectation $\mathbb{E}_{x_i}$ is taken with regards to the Wiener measure of the standard one-dimensional Brownian motion $\{X_s^i\}_{s>0}$ starting at $x_i \in \mathbb{R}$.

As the next lemma shows, this is the definition that suits best our purpose of factorizing the semigroup e^{-tL} into one-dimensional factors e^{-tL_i}.

Lemma 5.1.2. *Fix d and let the d-dimensional semigroup e^{-tL} be given by (4.2.2) and the one-dimensional semigroup e^{-tL_i} by (5.1.1). Then for $f \in L^\infty$ we have*

$$e^{-tL}f(x) = \left(\left(\prod_{i=1}^d e^{-tL_i}\right) f\right)(x) \quad \text{and} \quad e^{-tL}(\mathbb{1})(x) = \prod_{i=1}^d \left(e^{-tL_i}(\mathbb{1})(x)\right). \tag{5.1.2}$$

Proof. We will prove by induction that for $k = 1, \dots, d$ we have

$$\left(\left(\prod_{i=1}^k e^{-tL_i}\right) f\right)(x) = \mathbb{E}_{(x_1,\dots,x_k)}\left[e^{-\int_0^t \sum_{i=1}^k V_i(X_s)\,ds} f(X_t^1, \dots, X_t^k, x_{k+1}, \dots, x_d)\right], \tag{5.1.3}$$

which justifies the first formula in (5.1.2) if we take $k = d$.

The case $k = 1$ is clear from the definition (5.1.1) of e^{-tL_1}. Now suppose that (5.1.3) holds. Then

$$\begin{aligned}
\left(\left(\prod_{i=1}^{k+1} e^{-tL_i}\right) f\right)(x) &= \mathbb{E}_{x_{k+1}}\left[e^{-\int_0^t V_{k+1}(X_s)\,ds}\left(\left(\prod_{i=1}^k e^{-tL_i}\right) f\right)_{x^{k+1}}(X_t^{k+1})\right] \\
&= \mathbb{E}_{x_{k+1}}\left[e^{-\int_0^t V_{k+1}(X_s)\,ds}\,\mathbb{E}_{(x_1,\dots,x_k)}\left[e^{-\int_0^t \sum_{i=1}^k V_i(X_s)\,ds} f(X_t^1, \dots, X_t^k, X_t^{k+1}, x_{k+2}, \dots, x_d)\right]\right] \\
&= \mathbb{E}_{(x_1,\dots,x_{k+1})}\left[e^{-\int_0^t \sum_{i=1}^{k+1} V_i(X_s)\,ds} f(X_t^1, \dots, X_t^{k+1}, x_{k+2}, \dots, x_d)\right].
\end{aligned}$$

Note that we can use the same Brownian motion in the inner and in the outer expected value since its coordinates are independent of each other and $V_i(X_s)$ depends only on X_s^i.

The second formula in (5.1.2) follows from the definitions of e^{-tL} and e^{-tL_i} and the fact that the coordinates of d-dimensional Brownian motion are independent. □

5.2 One-dimensional estimates

In this section we prove the aforementioned exponential decay of the one-dimensional semigroup which we will then combine to estimate the semigroup e^{-tL}.

Lemma 5.2.1. *For every $N > 0$ there is a constant $c_N > 0$ such that*

$$e^{-tL_i}(\mathbb{1})(x) \leqslant e^{-c_N t V_i(x)} \tag{5.2.1}$$

for all $x \in \mathbb{R}^d$ and $0 \leqslant t \leqslant N$. Moreover, if $|x_i| \leqslant 4$, then

$$e^{-tL_i}(\mathbb{1})(x) \leqslant e^{-c_N\left(t^{\frac{\alpha}{2}+1}+tV_i(x)\right)}, \quad t \leqslant N. \tag{5.2.2}$$

Proof. First we will show that (5.2.1) is satisfied for $0 \leqslant t \leqslant t_0$ for some t_0 and then we will extend the estimate to all $0 \leqslant t \leqslant N$.

We begin with the case $|x_i| \leqslant 4$. We will make use of the inequality

$$e^{-x} \leqslant 1 - x + \frac{x^2}{2}, \qquad x \geqslant 0. \tag{5.2.3}$$

The Feynman–Kac formula (5.1.1) together with (5.2.3) give

$$e^{-tL_i}(\mathbb{1})(x) \leqslant 1 - \mathbb{E}_{x_i}\left[\int_0^t V_i(X_s)\,ds\right] + \frac{1}{2}\mathbb{E}_{x_i}\left[\left(\int_0^t V_i(X_s)\,ds\right)^2\right]. \tag{5.2.4}$$

We need to estimate the first and the second expected value in the expression above. In order to do this we will need the fact that for any $a, b \geqslant 0$ and $\alpha > 0$ we have

$$(a+b)^\alpha \approx a^\alpha + b^\alpha \tag{5.2.5}$$

and an estimate for the moments of the standard normal distribution

$$\mathbb{E}\left|X_s^i\right|^\alpha \approx s^{\alpha/2}.$$

Let us begin by estimating $\mathbb{E}_{x_i}V_i(X_s)$ from below and assume without loss of generality that $x_i \geqslant 0$.

$$\mathbb{E}_{x_i}\left[V_i(X_s)\right] \gtrsim \mathbb{E}_0\left|X_s^i + x_i\right|^\alpha \geqslant \mathbb{E}_0\left[\mathbb{1}_{\{X_s^i \geqslant 0\}}(X_s^i + x_i)^\alpha\right] \approx s^{\alpha/2} + x_i^\alpha$$

Integrating this gives

$$\mathbb{E}_{x_i}\left[\int_0^t V_i(X_s)\,ds\right] \gtrsim t^{\frac{\alpha}{2}+1} + t x_i^\alpha.$$

Now we estimate the last term in (5.2.4) using Cauchy–Schwarz inequality.

$$\mathbb{E}_{x_i}\left[\left(\int_0^t V_i(X_s)\,ds\right)^2\right] \lesssim t\int_0^t \mathbb{E}_0\left[\left|X_s^i + x_i\right|^{2\alpha}\right] ds \lesssim t\int_0^t \mathbb{E}_0\left[\left|X_s^i\right|^{2\alpha} + x_i^{2\alpha}\right] ds$$
$$\approx t\int_0^t s^\alpha + x_i^{2\alpha}\,ds \approx t^{\alpha+2} + t^2 x_i^{2\alpha} \leqslant \left(t^{\frac{\alpha}{2}+1} + t x_i^\alpha\right)^2.$$

Plugging this into (5.2.4), recalling that $|x_i| \leqslant 4$, and choosing t_0 sufficiently small yields

$$e^{-tL_i}(\mathbb{1})(x) \leqslant 1 - c_1\left(t^{\frac{\alpha}{2}+1} + t x_i^\alpha\right) + c_2\left(t^{\frac{\alpha}{2}+1} + t x_i^\alpha\right)^2$$
$$\leqslant 1 - c\left(t^{\frac{\alpha}{2}+1} + t x_i^\alpha\right) \leqslant e^{-c\left(t^{\frac{\alpha}{2}+1} + t x_i^\alpha\right)}$$

which implies (5.2.1) and (5.2.2) for $t \leqslant t_0$.

The second case is when $|x_i| > 4$ and $tV_i(x) \leqslant 2A\log 5$, where $A = \frac{2^\alpha M}{m}$ with m and M as in (5.0.2). We will roughly show that then we have

$$\frac{d}{dt}e^{-tL_i}(\mathbb{1})(x) = -e^{-tL_i}(V_i)(x) \leqslant -cV_i(x). \tag{5.2.6}$$

However since the equality may not hold, we replace V_i with $V_i^n(x) = \min(V_i(x), n)$ for any $n > 0$, we establish (5.2.6) for V_i^n, then we prove (5.2.1) for V_i^n and finally we deduce (5.2.1) for V_i.

Recall that V_i satisfies $m|x_i|^\alpha \leqslant V_i(x) \leqslant M|x_i|^\alpha$ and take $x_i, y_i \in \mathbb{R}$ such that $|x_i - y_i| \leqslant \frac{|x_i|}{2}$. Then $\frac{|x_i|}{2} \leqslant |y_i| \leqslant 2|x_i|$ so that we have

$$V_i(y) \leqslant M|y_i|^\alpha \leqslant 2^\alpha M|x_i|^\alpha \leqslant \frac{2^\alpha M}{m}V_i(x) = AV_i(x)$$

and

$$V_i(y) \geqslant m|y_i|^\alpha \geqslant m\frac{|x_i|^\alpha}{2^\alpha} \geqslant \frac{m}{2^\alpha M}V_i(x) = \frac{1}{A}V_i(x)$$

We also calculate the probability that $\sup_{0\leqslant s\leqslant t}\left|X_s^i - x_i\right| \geqslant \frac{|x_i|}{2}$ using the reflection principle to get

$$\mathbb{P}\left(\sup_{0\leqslant s\leqslant t}\left|X_s^i - x_i\right| \geqslant \frac{|x_i|}{2}\right) \leqslant 4e^{-\frac{|x_i|^2}{8t}}. \tag{5.2.7}$$

Now, for $n > 0$, we define $V_i^n(x) = \min(V_i(x), n)$ and $L_i^n = -\frac{\Delta}{2} + V_i^n$ and use the Feynman–Kac formula and (5.2.7) to get

$$e^{-tL_i^n}(V_i^n)(x) = \mathbb{E}_{x_i}\left[e^{-\int_0^t V_i^n(X_s)\,ds}V_i^n(X_t)\right] \geqslant \mathbb{E}_{x_i}\left[e^{-\int_0^t V_i(X_s)\,ds}V_i^n(X_t)\right]$$
$$\geqslant \mathbb{P}\left(\forall_{0\leqslant s\leqslant t}\ \frac{V_i^n(x)}{A} \leqslant V_i^n(X_s)\ \text{ and }\ V_i(X_s) \leqslant AV_i(x)\right)\frac{V_i^n(x)}{A}e^{-AtV_i(x)}$$
$$\geqslant \mathbb{P}\left(\forall_{0\leqslant s\leqslant t}\ \frac{V_i(x)}{A} \leqslant V_i(X_s) \leqslant AV_i(x)\right)\frac{V_i^n(x)}{A}e^{-AtV_i(x)}$$
$$\geqslant \frac{V_i^n(x)}{A}\left(1 - 8e^{-\frac{|x_i|^2}{8t}}\right)e^{-2A^2\log 5}$$
$$\geqslant \frac{V_i^n(x)}{A}\left(1 - 8e^{-\frac{4^2}{8t_0}}\right)e^{-2A^2\log 5} \geqslant cV_i^n(x)$$

if t_0 is sufficiently small, which proves that

$$\frac{d}{dt}e^{-tL_i^n}(\mathbb{1})(x) = -e^{-tL_i^n}(V_i^n)(x) \leqslant -cV_i^n(x). \tag{5.2.8}$$

Differentiation is allowed here by the Leibniz integral rule. Now we show that this implies a version of (5.2.1) with V_i^n. Consider the function

$$f(t) = e^{-tL_i^n}(\mathbb{1})(x)\, e^{ctV_i^n(x)}.$$

If we differentiate it and use (5.2.8), we get

$$\begin{aligned} f'(t) &= \frac{d}{dt}e^{-tL_i^n}(\mathbb{1})(x)\, e^{ctV_i^n(x)} + cV_i^n(x)\, e^{-tL_i^n}(\mathbb{1})(x)\, e^{ctV_i^n(x)} \\ &\leqslant -cV_i^n(x)\, e^{ctV_i^n(x)} + cV_i^n(x)\, e^{ctV_i^n(x)} = 0. \end{aligned}$$

Since $f(0) = 1$, we conclude that

$$e^{-tL_i^n}(\mathbb{1})(x) \leqslant e^{-ctV_i^n(x)}.$$

Now we take the limit as n goes to infinity on both sides of the inequality. The left-hand side becomes

$$\lim_{n\to\infty} e^{-tL_i^n}(\mathbb{1})(x) = \lim_{n\to\infty} \mathbb{E}_{x_i}\left[e^{-\int_0^t V_i^n(X_s)\,ds}\right] = \mathbb{E}_{x_i}\left[e^{-\int_0^t V_i(X_s)\,ds}\right] = e^{-tL_i}(\mathbb{1})(x).$$

Passing with the limit under the integral sign is allowed since the integrand is dominated by the constant function $\mathbb{1}$ which is integrable. On the right-hand side we get

$$\lim_{n\to\infty} e^{-ctV_i^n(x)} = e^{-ctV_i(x)},$$

so altogether we get (5.2.1).

The last case to consider is $|x_i| > 4$ and $tV_i(x) > 2A\log 5$. We choose sufficiently small t_0 and use (5.2.7) to obtain

$$\begin{aligned} e^{-tL_i}(\mathbb{1})(x) &\leqslant e^{-\frac{tV_i(x)}{A}}\,\mathbb{P}\left(\forall_{0\leqslant s\leqslant t}\ \frac{V_i(x)}{A} \leqslant V_i(X_s)\right) + 1\cdot\mathbb{P}\left(\exists_{0\leqslant s\leqslant t}\ \frac{V_i(x)}{A} > V_i(X_s)\right) \\ &\leqslant e^{-\frac{tV_i(x)}{A}} + 4e^{-\frac{|x_i|^2}{8t}} \leqslant 5e^{-\frac{tV_i(x)}{A}} \leqslant e^{-\frac{tV_i(x)}{2A}}, \end{aligned} \tag{5.2.9}$$

which is (5.2.1). In the second-to-last inequality we used the assumption $\alpha \leqslant 2$.

Recall that we have just proved that

$$e^{-tL_i}(\mathbb{1})(x) \leqslant e^{-ctV_i(x)}$$

is satisfied for $t \leqslant t_0$ and $x \in \mathbb{R}^d$. If $N \leqslant t_0$, then the proof is finished, so suppose that $N > t_0$ and take $t \in [t_0, N]$. Then we have

$$e^{-tL_i}(\mathbb{1})(x) \leqslant e^{-t_0L_i}(\mathbb{1})(x) \leqslant e^{-ct_0V_i(x)} = e^{-c\frac{t_0}{t}tV_i(x)} \leqslant e^{-c\frac{t_0}{N}tV_i(x)} = e^{-c_NtV_i(x)}.$$

The inequality (5.2.2) can be extended to $t \in [0, N]$ in a very similar way. Suppose that $N > t$ and take $t \in [t_0, N]$. Then

$$e^{-tL_i}(\mathbb{1})(x) \leqslant e^{-c\left(t_0^{\frac{\alpha}{2}+1}+t_0V_i(x)\right)} \leqslant e^{-c\left(\left(\frac{t_0}{N}\right)^{\frac{\alpha}{2}+1}t^{\frac{\alpha}{2}+1}+\frac{t_0}{N}tV_i(x)\right)} = e^{-c_N\left(t^{\frac{\alpha}{2}+1}+tV_i(x)\right)}.$$

This finishes the proof. □

5.3 L^∞ dimension-free estimates

In this section we prove Theorem 5.1.1 using one-dimensional estimates from Lemmas 5.2.1 and 4.3.1. The latter result applied to each of the one-dimensional semigroups e^{-tL_i}, together with the factorization property (5.1.2), gives

$$e^{-tL}(\mathbb{1})(x) \leqslant e^{-t\delta d} \tag{5.3.1}$$

for $x \in \mathbb{R}^d$ and $t \geqslant N$, where $N > 0$ and $\delta > 0$ are universal constants.

First we estimate the upper part of the integral in (4.0.1), i.e. the integral from N to ∞, dividing the calculations into two cases depending on the value of a. If $a < 1$, then

$$\begin{aligned} V(x)^a \int_N^\infty e^{-tL}(\mathbb{1})(x)\, t^{a-1}\, dt &\leqslant V(x)^a\, e^{-\frac{N}{2}L}(\mathbb{1})(x) \int_N^\infty e^{-\frac{t}{2}\delta d}\, t^{a-1}\, dt \\ &\lesssim \frac{N^{a-1}}{\delta d} V(x)^a\, e^{-\frac{N}{2}L}(\mathbb{1})(x) \lesssim \frac{1}{d}\sum_{i=1}^d V_i(x)^a\, e^{-\frac{N}{2}L_i}(\mathbb{1})(x). \end{aligned}$$

In the last inequality we used the fact that

$$(x_1 + \dots + x_d)^a \leqslant x_1^a + \dots x_d^a$$

for $a \leqslant 1$ and $x_i \geqslant 0$.

If, on the other hand, $a \geqslant 1$, then

$$\begin{aligned} V(x)^a \int_N^\infty e^{-tL}(\mathbb{1})(x)\, t^{a-1}\, dt &\leqslant V(x)^a\, e^{-\frac{N}{2}L}(\mathbb{1})(x) \int_N^\infty e^{-\frac{t}{2}\delta d}\, t^{a-1}\, dt \\ &\lesssim \frac{1}{(\delta d)^a} V(x)^a\, e^{-\frac{N}{2}L}(\mathbb{1})(x) \lesssim \frac{1}{d}\sum_{i=1}^d V_i(x)^a\, e^{-\frac{N}{2}L_i}(\mathbb{1})(x). \end{aligned}$$

Here in the last inequality we used that fact that

$$(x_1 + \dots + x_d)^a \leqslant d^{a-1}\,(x_1^a + \dots + x_d^a)$$

for $a \geqslant 1$ and $x_i \geqslant 0$, which follows from Jensen's inequality or Hölder's inequality. Thus, we have reduced our problem to the one-dimensional case of estimating $V_i^a e^{-\frac{N}{2}L_i}(\mathbb{1})$, which may be done by invoking (5.2.1), namely

$$V_i(x)^a\, e^{-\frac{N}{2}L_i}(\mathbb{1})(x) \leqslant V_i(x)^a\, e^{-\frac{N}{2}c_N V_i(x)} \leqslant \left(\frac{2a}{N c_N e}\right)^a \tag{5.3.2}$$

Then we handle the lower part of the integral in (4.0.1). We estimate $e^{-tL}(\mathbb{1})(x)$ for $t \leqslant N$ independently of x and d by using (5.2.1) and the factorization property (5.1.2), which gives

$$e^{-tL}(\mathbb{1})(x) \leqslant e^{-c_N t V(x)},$$

and then integrate

$$V(x)^a \int_0^N e^{-tL}(\mathbb{1})(x)\, t^{a-1}\, dt \leqslant V(x)^a \int_0^\infty e^{-c_N t V(x)}\, t^{a-1}\, dt \approx c_N^{-a}.$$

This completes the proof of Theorem 5.1.1.

5.4 L^1 dimension-free estimates

In this section we will again use the one-dimensional estimates for the semigroups e^{-tL_i} to prove dimension-free estimates of the L^1 norm of R_V^a for $0 < a \leqslant 1$. The idea is the same as in Section 4.4, i.e. to estimate the L^∞ norm of the adjoint operator formally given by

$$(L^{-a}V^a)f(x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL}(V^a f)(x)\, t^{a-1}\, dt.$$

As before the positivity-preserving property of e^{-tL} lets us reduce the task to estimating the L^∞ norm of

$$L^{-a}(V^a)(x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL}(V^a)(x)\, t^{a-1}\, dt. \tag{5.4.1}$$

The question of well-definedness of the function $L^{-a}(V^a)(x)$ is addressed in Section 4.4. As in the L^∞ case we reformulate Theorem 5.0.2 in the following way

Theorem 5.4.1. *Fix* $\alpha > 0$ *and let* V *given by* (5.0.1) *satisfy* (5.0.2). *For* $a \leqslant 1$ *let the Riesz transform* R_V^a *be defined as in* (4.0.1). *Then there is a constant* $C > 0$ *independent of the dimension* d *such that*

$$\left\| L^{-a}(V^a) \right\|_\infty \leqslant C.$$

Before we move to the proof, we need two general results regarding the semigroup e^{-tL}. The first one is a factorization property for $e^{-tL}(V)$

$$e^{-tL}(V) = \sum_{i=1}^d e^{-tL}(V_i) = \sum_{i=1}^d e^{-tL^i}(\mathbb{1})\, e^{-tL_i}(V_i), \quad \text{where } L^i = L - L_i. \tag{5.4.2}$$

The second one is an estimate for $e^{-tL_i}(V_i^a)$

$$\begin{aligned} e^{-tL_i}(V_i^a)(x) &= \mathbb{E}_{x_i}\left[e^{-\int_0^t V_i(X_s)\,ds}\, V_i(X_t)^a\right] \\ &\lesssim \mathbb{E}_0\left[e^{-\int_0^t V_i(X_s+x)\,ds}\, V_i(X_t)^a\right] + \mathbb{E}_0\left[e^{-\int_0^t V_i(X_s+x)\,ds}\, V_i(x)^a\right] \\ &\lesssim \mathbb{E}_0\left[V_i(X_t)^a\right] + V_i(x)^a\, \mathbb{E}_{x_i}\left[e^{-\int_0^t V_i(X_s)\,ds}\right] \\ &\lesssim t^{\frac{a\alpha}{2}} + V_i(x)^a\, e^{-tL_i}(\mathbb{1})(x), \end{aligned} \tag{5.4.3}$$

valid for $t > 0$ and $x \in \mathbb{R}^d$. Here we used estimate (5.0.2) for V and (5.2.5). Now we are in position to prove Theorem 5.4.1

Proof of Theorem 5.4.1. We begin with the upper part of the integral in (4.0.1), i.e. the integral from N to ∞. Using subadditivity of the function x^a for $a \leqslant 1$, factorization (5.4.2), and (5.3.1) we obtain

$$\begin{aligned}\int_N^\infty e^{-tL}(V^a)(x)\,t^{a-1}\,dt &\leqslant \int_N^\infty \sum_{i=1}^d e^{-tL}(V_i^a)(x)\,t^{a-1}\,dt\\ &\leqslant \int_N^\infty \sum_{i=1}^d e^{-tL^i}(\mathbb{1})(x)\,e^{-tL_i}(V_i^a)(x)\,t^{a-1}\,dt\\ &\leqslant \int_N^\infty \sum_{i=1}^d e^{-t\delta(d-1)}e^{-tL_i}(V_i^a)(x)\,t^{a-1}\,dt\\ &\leqslant N^{a-1}\sum_{i=1}^d \int_N^\infty \left\|e^{-tL_i}(V_i^a)\right\|_\infty e^{-t\delta(d-1)}\,dt\\ &\lesssim \sum_{i=1}^d \left\|e^{-NL_i}(V_i^a)\right\|_\infty \int_N^\infty e^{-t\delta(d-1)}\,dt\\ &\lesssim \frac{1}{d-1}\sum_{i=1}^d \left\|e^{-NL_i}(V_i^a)\right\|_\infty.\end{aligned}$$

Then we use (5.4.3) and (5.2.1) and we estimate the resulting function similarly to (5.3.2).

To deal with the lower part we use the inequality

$$e^{-tL}(V^a) \leqslant e^{-tL}(V)^a, \quad a \leqslant 1,$$

which follows from Hölder's inequality. We use this and (5.4.2) to get

$$\int_0^N e^{-tL}(V^a)(x)\,t^{a-1}\,dt \leqslant \int_0^N \left(\sum_{i=1}^d e^{-tL^i}(\mathbb{1})(x)\,e^{-tL_i}(V_i)(x)\right)^a t^{a-1}\,dt.$$

Then we use (5.4.3) and obtain

$$\begin{aligned}\int_0^N e^{-tL}(V^a)(x)\,t^{a-1}\,dt &\lesssim \int_0^N \left(\sum_{i=1}^d e^{-tL^i}(\mathbb{1})(x)\,\left(t^{\frac{\alpha}{2}} + V_i(x)\,e^{-tL_i}(\mathbb{1})(x)\right)\right)^a t^{a-1}\,dt\\ &= \int_0^N \left(V(x)e^{-tL}(\mathbb{1})(x) + t^{\frac{\alpha}{2}}\sum_{i=1}^d e^{-tL^i}(\mathbb{1})(x)\right)^a t^{a-1}\,dt\\ &\leqslant \int_0^N V(x)^a e^{-tL}(\mathbb{1})(x)^a\,t^{a-1}\,dt + \int_0^N t^{\frac{a\alpha}{2}}\left(\sum_{i=1}^d e^{-tL^i}(\mathbb{1})(x)\right)^a t^{a-1}\,dt.\end{aligned}$$

To the first integral we apply (5.2.1) and factorization (5.1.2), which lets us estimate the first integral by a constant independent of x and the dimension d. To estimate the second integral we fix $x = (x_1, \ldots, x_d)$ and divide its coordinates x_j into those whose absolute value is greater than 4 and all others. Say there are k coordinates greater than 4 and $d-k$ not greater than 4. Then we consider three cases.

First we assume that $k = 0$ and apply (5.2.2) and (5.1.2) to get

$$\int_0^N t^{\frac{a\alpha}{2}}\left(\sum_{i=1}^d e^{-tL^i}(\mathbb{1})(x)\right)^a t^{a-1}\,dt \leqslant \int_0^N d^a e^{-a(d-1)c_N t^{\frac{\alpha}{2}+1}}\,t^{\frac{a\alpha}{2}}\,t^{a-1}\,dt \lesssim \frac{d^a}{d^a} = 1.$$

In the last inequality we used

$$\int_0^\infty e^{-At^\beta}\, t^\gamma\, dt = \frac{\Gamma\left(\frac{\gamma+1}{\beta}\right)}{\beta A^{\frac{\gamma+1}{\beta}}}, \tag{5.4.4}$$

with $A = a(d-1)c_N$, $\beta = \frac{\alpha}{2}+1$ and $\gamma = \frac{a\alpha}{2}+a-1$.

Then if $k = d$, we apply (5.2.1) and (5.1.2) and use the fact $V_i(x) \geqslant m \cdot 4^\alpha$ which gives

$$\begin{aligned}\int_0^N t^{\frac{a\alpha}{2}} \left(\sum_{i=1}^d e^{-tL^i}(\mathbb{1})(x)\right)^a t^{a-1}\, dt &\leqslant \int_0^N d^a e^{-4^\alpha mac_N t(d-1)}\, t^{\frac{a\alpha}{2}}\, t^{a-1}\, dt \\ &\lesssim \int_0^N d^a e^{-4^\alpha mac_N td}\, t^{a-1}\, dt \lesssim 1.\end{aligned}$$

The third case is when $0 < k < d$ in which the estimate is a mixture of the estimates for $k = 0$ and $k = d$. Observe that each $(d-1)$-element subsequence of $(x_1, \ldots, x_d)$ has at least $k-1$ elements greater than 4 and at least $d-k-1$ elements not greater than 4. By (5.2.1) and (5.2.2) this means that

$$\int_0^N t^{\frac{a\alpha}{2}} \left(\sum_{i=1}^d e^{-tL^i}(\mathbb{1})(x)\right)^a t^{a-1}\, dt \leqslant \int_0^N d^a e^{-4^\alpha mac_N t(k-1)}\, e^{-ac_N(d-k-1)t^{\frac{\alpha}{2}+1}} t^{\frac{a\alpha}{2}}\, t^{a-1}\, dt.$$

Then we use Hölder's inequality with $p = \frac{d-2}{k-1}$ ($p = \infty$ if $k = 1$) and $q = \frac{d-2}{d-k-1}$ ($q = \infty$ if $k = d-1$) to the functions $e^{-4^\alpha mac_N t(k-1)}$ and $e^{-ac_N(d-k-1)t^{\frac{\alpha}{2}+1}} t^{\frac{a\alpha}{2}}$ with respect to the measure $t^{a-1}\, dt$ which yields

$$\begin{aligned} d^a &\int_0^N e^{-4^\alpha mac_N t(k-1)} \cdot e^{-ac_N(d-k-1)t^{\frac{\alpha}{2}+1}} t^{\frac{a\alpha}{2}} \cdot t^{a-1}\, dt \\ &\lesssim d^a \left(\int_0^N e^{-4^\alpha mac_N t(d-2)}\, t^{a-1}\, dt\right)^{1/p} \left(\int_0^N e^{-ac_N(d-2)t^{\frac{\alpha}{2}+1}} t^{\frac{a\alpha}{2}}\, t^{a-1}\, dt\right)^{1/q} \\ &\lesssim d^a \left(\frac{1}{d^a}\right)^{1/p} \left(\frac{1}{d^a}\right)^{1/q} = 1.\end{aligned}$$

Again, in the last inequality we used (5.4.4). $\square$

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