This is a note on the proof presented at ćwiczenia on 11.10.2023 of the theorem that each finite connected graph G admits a spanning tree (i.e. a subgraph $T \leq G$ that is a tree such that $V(T) = V(G)$).

The proof goes via induction on $|V(G)|$.

The claim holds when $|G| = 1$ trivially.

Assume that G is a graph such that $|V(G)| > 1$ and the claim holds for all connected graphs of order smaller than $|V(G)|$. Pick a vertex $v \in V(G)$. Let $G_1, \ldots, G_k$ be the connected components of $G - v$. For each $1 \leq i \leq k$ choose, using the inductive assumption, a spanning tree T_i of G_i.

For each $1 \leq i \leq k$ choose one edge $e_i \in E(G)$ that connects v to a vertex of G_i. (It exists by the following argument. Pick any vertex $x \in V(G_i)$. By connectedness of G there exists a path $v, x_1, \ldots, x_{m-1}, x$ in G connecting the vertex v with x. Then the path $x_1, \ldots, x$ is a path in $G - v$, so it is contained in the connected component of $G - v$ containing x, namely G_i. Therefore $\{v, x_1\}$ is an example of such an edge.) Define T by $V(T) = V(G)$ and $E(T) = \bigcup_{1 \leq i \leq k} E(T_i) \cup \{e_i\}$.

We show below that T is a tree.

T is connected. Follows from the facts that each of the T_i is connected and for each i the edge e_i connects v with a vertex of T_i.

T is acyclic. Consider a cycle $x_1 \ldots x_m$ in T. If there is no $j \in \{1, \ldots, m\}$ such that $x_j = v$, then all of the x_j belong to one connected component G_i of $G - v$, so they form a cycle in T_i, which is a contradiction. Otherwise, assume without loss of generality that $x_1 = v$. Then $x_2, \ldots, x_m$ form a path in $G - v$, therefore all of them belong to one of the components G_i. In particular, both $\{v, x_2\}$ and $\{x_m, v\}$ are edges in T that connect v with a vertex of G_i, so they are equal to e_i, in particular $x_m = x_2$, which is a contradiction.