Problem List 6 (Algebraic methods)

1.[+] Let $s, t \geq 1$ with $(s, t) \neq (1, 1)$, and let A be the adjacency matrix of $K_{s,t}$. Find the eigenvalues of A and their multiplicities.
[*Hint: in order to find the possible eigenvalues, express A^3 in terms of A.*]

2.[○] Let G be a graph of order n with adjacency matrix A. Give graph-theoretic interpretations of the numbers $\frac{1}{2}\mathbf{x}^T A\mathbf{x}$, $\frac{1}{2}\mathbf{x}^T(A^2-A)\mathbf{x}$ and $\frac{1}{6}\operatorname{tr}(A^3)$, where $\mathbf{x} = (1, 1, \ldots, 1) \in \mathbb{R}^n$.

3. Let G be a connected graph with adjacency matrix A.

(a)[○] Show that if $-\Delta(G)$ is an eigenvalue of A then G is bipartite.

(b)[−] Show that if G is bipartite and λ is an eigenvalue of A, then $-\lambda$ is also an eigenvalue of A.

4.[○] Fix a constant $p \in (0, 1)$. Show that almost every $G \in \mathcal{G}(n, p)$ has diameter 2.

5.[−] By exhibiting an explicit isomorphism, show that the 3-regular Moore graph constructed during the lectures is isomorphic to the Petersen graph (both are displayed).
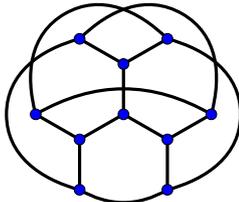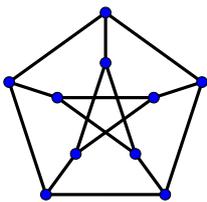
6. Let G be a graph with $|G| \geq 2$ such that any two distinct vertices $v, w \in G$ are contained in a unique triangle if $v \sim w$, and a unique 4-cycle if $v \nsim w$.

(a)[+] Show that G is k-regular for some $k \geq 1$ and that $|G| = \frac{k^2}{2} + 1$.

(b)[○] Show that $|G| \in \{3, 9, 99, 243, 6273, 494019\}$.