Problem List 5 (Drawings and colourings)

Graph Theory, Winter Semester 2023/24, IM UWr

1.$^\circ$ Which Turán graphs $T_r(n)$ for $n \geq r \geq 2$ are planar? For each planar one, draw it. [*Hint: exactly* 10 *of them are planar.*]

2.$^\circ$ Fix $n \geq 3$. Show that if G is a bipartite planar graph with $|G| = n$, then $e(G) \leq 2n-4$. Can we have the equality $e(G) = 2n-4$ for some G?

3. Given a graph G, its *crossing number* $\mathrm{cr}(G)$ is the minimal number of crossings between edges in a "drawing" of G on the plane (formally, we may replace the condition "$\gamma_e((0,1)) \cap \gamma_f((0,1)) = \varnothing$ for all $e, f \in E(G)$ with $e \neq f$" in the definition of a drawing with the condition that $\sum_{\{e,f\} \subseteq E, e \neq f} |\gamma_e((0,1)) \cap \gamma_f((0,1))| \leq k$, and let $\mathrm{cr}(G)$ be the smallest k so that such a drawing exists). Thus, G is planar if and only if $\mathrm{cr}(G) = 0$.

   (a)$^\circ$ Compute $\mathrm{cr}(K_5)$, $\mathrm{cr}(K_{3,3})$ and $\mathrm{cr}(K_{3,4})$.

   (b)$^-$ If G is a graph with $|G| = n \geq 3$, show that $e(G) \leq 3n + \mathrm{cr}(G) - 6$.

   (c)$^\circ$ Using random graph methods, show that if G is a graph with $|G| = n$ and $e(G) = m \geq 4n$, then $\mathrm{cr}(G) > \frac{m^3}{64n^2}$.
   [*Hint: draw G on a plane with* $\mathrm{cr}(G)$ *crossings, and apply part (b) to $G[W]$, where $W \subseteq V(G)$ is a random subset containing each vertex with probability $\frac{4n}{m}$.*]

4.$^+$ Show that $K_{4,4}$ can be drawn on the torus $\mathbb{T}^2$. Can $K_{5,5}$ be drawn on $\mathbb{T}^2$?
[*Hint: use the Euler–Poincaré Formula and the fact that $K_{5,5}$ is bipartite.*]

5.$^\circ$ Let G be a graph with $|G| = n$ and $e(G) = m$, and suppose that $p_G(x) = x^n - mx^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$ is the chromatic polynomial of G. Show that $a_{n-2} = \binom{m}{2} - t$, where t is the number of triangles in G.

6. (*Kőnig's Edge Colouring Theorem.*) Let G be a bipartite graph with $e(G) > 0$.

   (a)$^\circ$ Show that G is a subgraph of a $\Delta(G)$-regular bipartite graph H.
   [*Hint: consider the graph obtained as follows: start with two disjoint copies of G, and add an edge between each pair of corresponding vertices of minimal degree.*]

   (b)$^\circ$ Show that $\chi'(H) = \Delta(H)$, and deduce that $\chi'(G) = \Delta(G)$.
   [*Hint: see Problem 1.8.*]

7. Given a graph G and an integer $x \geq 0$, let $p'_G(x)$ be the number of admissible x-edge-colourings of G.

   (a)$^\circ$ Show that p'_G is a polynomial.

   (b)$^-$ Assuming that $p'_G(x)$ has the form $x^{n'} - m'x^{n'-1} + \cdots$, give graph-theoretic interpretations of the numbers n' and m'.

The following problem is only relevant if the proof of Kuratowski's Theorem is covered in lectures at the end of the course. Otherwise, the problem is included here mostly for the interested students who have read Section 5.2 in the lecture notes.

8. Fill in the gaps in the proof of Kuratowski's Theorem given in the lectures:

   (a) Let G be a *minimal non-planar graph*—that is, a non-planar graph such that every subgraph $H \leq G$ with $H \neq G$ is planar. Explain (roughly) why G is 2-connected.

   (b) Let Q_1, Q_2, v, w and $H'_1, \ldots, H'_k$ be as in the proof of Kuratowski's Theorem given in the lectures. Show that no two interior H'_i overlap. Use this to give a rough explanation why there exists an interior H'_I containing vertices in both $Q_1 - \{v, w\}$ and $Q_2 - \{v, w\}$ that overlaps some exterior H'_O.
   [*Hint: explain why if this was not the case then G would be planar.*]

   (c) Show that one of the four configurations discussed at the end of the proof of Kuratowski's Theorem must appear in H'_I.
   [*Hint: split your argument into two cases, depending on whether or not we have* $V(H'_I) \cap V(C) \subseteq \{v, w, u_1, u_2\}$.]