Problem List 4 (Random graphs, mainly)

Graph Theory, Winter Semester 2023/24, IM UWr

1. By colouring the vertices of a graph G blue/orange independently at random, show that $V(G)$ has a partition $V(G) = V_1 \sqcup V_2$ such that $e(G[V_1]) + e(G[V_2]) \leq \frac{1}{2}e(G)$. Give also a constructive proof of the same fact.

2. Show that for any $s, t \geq 2$ we have $R(s,t) \geq n - \binom{n}{s}p^{\binom{s}{2}} - \binom{n}{t}(1-p)^{\binom{t}{2}}$ for all $n \in \mathbb{N}$ and $p \in (0,1)$. By choosing $n = n(t)$ appropriately and taking $p = n^{-2/3}$, deduce that $R(4,t) = \Omega\left(\left(\frac{t}{\ln t}\right)^{3/2}\right)$.

3. Let G be a graph of order n, and let $\overline{G}$ be its complement.
   (a) Show that $\chi(G) \cdot \chi(\overline{G}) \geq n$.
   (b) Show that $\chi(G) + \chi(\overline{G}) \leq n + 1$.
   (c) Find all G for which $\chi(G) \cdot \chi(\overline{G}) = n$ and $\chi(G) + \chi(\overline{G}) = n + 1$.

4. Given a graph $G = (V, E)$, its *Mycielskian* $\mu(G)$ is a graph defined as follows. Let
$$V(\mu(G)) := V \sqcup \{u_v \mid v \in V\} \sqcup \{w\}$$
and
$$E(\mu(G)) := E \sqcup \{u_v v' \mid v, v' \in V, v \sim_G v'\} \sqcup \{u_v w \mid v \in V\},$$
so that $|\mu(G)| = 2|G| + 1$ and $e(\mu(G)) = 3e(G) + |G|$.
   (a) Show that if G is triangle-free then so is $\mu(G)$.
   (b) Show that $\chi(\mu(G)) = \chi(G) + 1$.
   (c) Deduce that for every $k \geq 2$, there exists a triangle-free graph G with $\chi(G) = k$.

5. Let $n \in \mathbb{N}$ players participate in a *tournament*, where each pair of players play a game and one of them beats the other (there are no draws).
   (a) For every $k \geq 1$, prove that there exists a tournament in which for every k players, some other player beats all k of them.
   (b) Construct such a tournament explicitly for $k = 2$.

6. Find a threshold function for $G \in \mathcal{G}(n,p)$ to contain a path of length 2.

7. A vertex v of a graph G is said to be *isolated* if $d_G(v) = 0$. Show that $\frac{\ln n}{n}$ is a threshold function for $G \in \mathcal{G}(n,p)$ to have no isolated vertices.

8. We define the *Rado graph* R as an infinite graph with $V(R) = \mathbb{Z}_{\geq 0} = \{0, 1, \ldots\}$, so that given non-negative integers $x < y$, we have $x \sim_R y$ if and only if $\lfloor \frac{y}{2^x} \rfloor$ is odd.

   (a) Show that R satisfies the *extension property*: given any finite disjoint subsets $U, W \subset V(R)$, there exists $v \in V(R) \setminus (U \cup W)$ such that $v \sim u$ for all $u \in U$ and $v \nsim w$ for all $w \in W$.

   (b) Let G, H be two infinite graphs, with $V(G)$ and $V(H)$ countable, satisfying the extension property. Show that $G \cong H$.

   (c) Given $p \in (0, 1)$, let $\mathcal{G}(\infty, p)$ be the probability space of all infinite graphs G with vertex set $\mathbb{Z}_{\geq 0}$, with each edge appearing independently at random with probability p. Show that $G \in \mathcal{G}(\infty, p)$ is isomorphic to R with probability one.