Problem List 3 (Ramsey theory)

Graph Theory, Winter Semester 2023/24, IM UWr

1. (a)$^-$ By constructing a suitable colouring of K_8, show that $R(3,4) \geq 9$.

   (b)$^\circ$ Show that if $R(s-1,t)$ and $R(s,t-1)$ are both even, then we actually have $R(s,t) \leq R(s-1,t) + R(s,t-1) - 1$.

   (c)$^-$ Deduce that $R(3,4) = 9$.

2.$^+$ Consider the blue/orange colouring of K_{17} such that the edge ij is coloured blue if $i - j \equiv \pm 1, \pm 2, \pm 4$ or $\pm 8 \pmod{17}$, and orange otherwise. Show that with this colouring K_{17} has no monochromatic K_4, and therefore $R(4,4) > 17$.
[*Hint: in order to simplify your proof, start by showing that the map* $[17] \to [17]$ *sending* i *to* $3i$ *(modulo* 17*) swaps the blue edges with the orange edges.*]

3.$^-$ Show that $R(s,t) \leq \binom{s+t-2}{s-1}$ and that $R(s,t) \leq 2^{s+t-3}$ for all $s,t \geq 2$. Deduce that $R(s,s) = O(4^s)$.

4. Given two graphs G and H, we write $R(G,H)$ for the smallest $n \geq 2$ such that any blue/orange colouring of K_n has either a blue subgraph isomorphic to G, or an orange subgraph isomorphic to H.

   (a)$^-$ Why does $R(G,H)$ exist?

   (b)$^\circ$ Show that $R(K_{1,t}, K_{r+1}) = rt + 1$ for all $r,t \geq 1$.
   [*Hint: use Turán's Theorem.*]

   (c)$^\circ$ For $k \geq 1$, let I_k be the "set of k disjoint edges"—that is, a graph with $V(I_k) = \{v_i \mid i \in [k]\} \sqcup \{w_i \mid i \in [k]\}$ and $E(I_k) = \{v_i w_i \mid i \in [k]\}$. Show that $R(I_k, K_r) = 2k + r - 2$ for all $k \geq 1$ and $r \geq 2$.

   (d)$^\circ$ Show that $R(I_k, I_k) = 3k - 1$ for all $k \geq 1$.

5. Let $k \geq 1$.

   (a)$^+$ Show that every blue/orange colouring of the edges of $K_{2k-1,2k-1}$ contains a monochromatic tree of order $2k$ with two vertices of degree k.

   (b)$^-$ Give a blue/orange colouring of the edges of $K_{2k,2k}$ with no connected monochromatic subgraphs of order $2k+1$.

6. Given $k, s \geq 2$, we write $R_k(s)$ for the Ramsey number $R(\overbrace{s,\ldots,s}^{k})$.

   (a)$^-$ By exhibiting a suitable colouring of $K_{(s-1)^2}$, show that $R(s,s) = \Omega(s^2)$.

   (b)$^-$ Show that $R_k(s) = \Omega(s^k)$ for any fixed $k \geq 2$.

   (c)$^\circ$ Show that $R_k(s) \leq k^{ks}$ for all $k, s \geq 2$.

7. (a)$^{\circ}$ Show that $R_k(3) \leq k \cdot R_{k-1}(3)$ for all $k \geq 3$, and deduce that $R_k(3) \leq 3 \cdot k!$ for all $k \geq 2$.

   (b)$^{\circ}$ Let $x_1, \ldots, x_n \in \mathbb{R}^2$ be points such that no three of them lie on a straight line, where $n = 3 \cdot k!$ for some $k \geq 2$. Show that some three of these points form an angle $> \pi\left(1 - \frac{1}{k}\right)$.

   (c)$^{+}$ Show that if $G_1, \ldots, G_k \leq K_n$ are bipartite subgraphs and $E(K_n) = \bigcup_{i=1}^{k} E(G_i)$, then $n \leq 2^k$. Deduce that in the previous part of the problem we could take $n = 2^k + 1$ instead of $n = 3 \cdot k!$.

8.$^{\circ}$ Show that every sequence $(x_n)_{n=1}^{\infty}$ of real numbers has a monotone (that is, non-increasing or non-decreasing) subsequence.

9.$^{\circ}$ Let $g_1, \ldots, g_n \colon \mathbb{R} \to \mathbb{R}$ be bounded functions, and let $\varepsilon, \delta > 0$. Suppose that $f \colon \mathbb{R} \to \mathbb{R}$ is such that for all $x, y \in \mathbb{R}$ with $|f(x) - f(y)| > \delta$ we have $|g_i(x) - g_i(y)| > \varepsilon$ for some i. Show that f is bounded.

10. (*Ramsey's Theorem for hypergraphs*). Let $k \geq 2$.

   (a)$^{\circ}$ Let $s, t \geq k$. Show that there exists $n \geq k$ with the following property: given any blue/orange colouring of all k-element subsets of a set V with $|V| = n$, there exists either a subset $W_1 \subseteq V$ with $|W_1| = s$ all of whose k-element subsets are blue, or a subset $W_2 \subseteq V$ with $|W_2| = t$ all of whose k-element subsets are orange. [*Hint: letting* $R^{(k)}(s,t)$ *be the smallest such integer* n*, show that we have* $R^{(k)}(s,t) \leq R^{(k-1)}\left(R^{(k)}(s-1,t), R^{(k)}(s,t-1)\right) + 1$.]

   (b)$^{+}$ Show that for any blue/orange colouring of all k-element subsets of $\mathbb{N}$, there exists an infinite subset $W \subseteq \mathbb{N}$ all of whose k-element subsets have the same colour.

   (c)$^{-}$ Deduce that for any infinite subset $A \subseteq \mathbb{R}^2$, there exists an infinite subset $B \subseteq A$ such that either B is contained in a line, or no three points in B are collinear.

   (d)$^{+}$ Suppose we have a blue/orange colouring of all infinite subsets of $\mathbb{N}$. Is there always an infinite subset $W \subseteq \mathbb{N}$ all of whose infinite subsets have the same colour?
   [*Hint: could we have such a blue/orange colouring so that* A *and* $A \cup \{n\}$ *have different colours whenever* $n \notin A$*? Feel free to use the Axiom of Choice.*]