Problem List 2 (Extremal problems, mainly)

GRAPH THEORY, WINTER SEMESTER 2023/24, IM UWR

1.$^+$ Let T be a tree, and let φ be an *automorphism* of T, i.e. a bijection $\varphi\colon V(T) \to V(T)$ such that $v \sim w$ if and only if $\varphi(v) \sim \varphi(w)$. Show that either $\varphi(v) = v$ for some $v \in T$, or $\varphi(v) = w$ and $\varphi(w) = v$ for some $vw \in E(T)$.

2.$^+$ Recall that a (possibly infinite) *tree* is a (possibly infinite) connected graph with at least one vertex and no cycles. Show that every infinite connected graph G has a spanning tree, i.e. a tree $T \leq G$ such that $V(T) = V(G)$.
[*Hint: feel free to use the Axiom of Choice in the form of Zorn's Lemma.*]

3.$^\circ$ Prove that an incomplete regular graph of order n cannot contain a complete subgraph of order $> \frac{n}{2}$.

4.$^\circ$ Let $n \geq 1$, and let $x_1, \ldots, x_{3n} \in \mathbb{R}^2$ be points such that $\|x_i - x_j\| \leq 1$ for all i and j. Prove that $\|x_i - x_j\| > 1/\sqrt{2}$ for at most $3n^2$ pairs (i, j) with $i < j$.
[*Hint: can four of these points have all pairwise distances greater than* $1/\sqrt{2}$*?*]

5. Let G be a graph with $n \geq r + 2 \geq 4$ vertices and $t_r(n) + 1$ edges.

   (a)$^\circ$ Show that for every p with $r + 1 \leq p \leq n$, G has a subgraph H with $|H| = p$ and $e(H) = t_r(p) + 1$.

   (b)$^-$ Deduce that $K_{r+2} - \{e\} \leq G$, where $e \in E(K_{r+2})$.

6.$^\circ$ For any integers $n \geq t \geq 1$, construct a graph G with $|G| = n$, $\Delta(G) = t - 1$, and $e(G) = \lfloor \frac{n(t-1)}{2} \rfloor$.

7. Let $p \geq 2$ be a prime number, and let $\mathbb{F}_p$ be the field with p elements. Consider the *projective plane over* $\mathbb{F}_p$, that is, $\mathbb{F}_p P^2 = (\mathbb{F}_p^3 \setminus \{(0,0,0)\})/\simeq$, where $\simeq$ is defined by setting $(x, y, z) \simeq (\lambda x, \lambda y, \lambda z)$ for $\lambda \in \mathbb{F}_p \setminus \{0\}$; we write $(x : y : z)$ for the equivalence class of $(x, y, z) \in \mathbb{F}_p^3$ in $\mathbb{F}_p P^2$. Let G be a graph with $V(G) = \mathbb{F}_p P^2$ such that $(x : y : z) \sim_G (x' : y' : z')$ if and only if $(x : y : z) \neq (x' : y' : z')$ and $xx' + yy' + zz' = 0$.

   (a)$^\circ$ Show that G is C_4-free.
   [*Hint:* $\mathbb{F}_p$ *is a field, so you can use everything you know about vector spaces.*]

   (b)$^\circ$ Compute the degree of any vertex $(x : y : z) \in V(G)$ (consider two cases, depending on whether or not $x^2 + y^2 + z^2 = 0$).

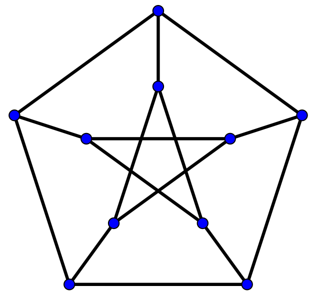
   (c)$^\circ$ Compute $|G|$ and give a positive lower bound on $\frac{e(G)^2}{|G|^3}$.

   (d)$^-$ Bertrand's Postulate, proved by P. L. Chebyshev, states that for any integer $n \geq 2$ there exists a prime number p with $n < p < 2n$. Use this to show that $\mathrm{ex}(n; C_4) = \Omega(n\sqrt{n})$.

8.$^+$ Show that $\mathrm{ex}(n; K_{t,t}) \leq \frac{1}{2} z_t(n)$ for all $n \geq t \geq 1$.

9.$^\circ$ Let G be an infinite graph. For $n \geq 2$, define $x_n = \max\{e(H)/\binom{n}{2} \mid H \leq G, |H| = n\}$. Show that the sequence $(x_n)_{n=2}^{\infty}$ is non-increasing, and deduce that it converges.

10. Determine the chromatic number of the following graphs.



(a)$^-$ P_n and C_n for any n.

(b)$^-$ The *Petersen graph*, displayed on the right.

(c)$^\circ$ The *join* K of graphs G and H, defined by setting $V(K) := V(G) \sqcup V(H)$ and $E(K) := E(G) \sqcup E(H) \sqcup \{vw \mid v \in G, w \in H\}$, where $\chi(G) = r$ and $\chi(H) = s$.

11. Let G be a graph with $e(G) \geq 1$, and let $r = \chi(G)$.

(a)$^-$ Explain why $\mathrm{ex}(n; G) \geq \mathrm{ex}(n; K_r)$ for all $n \geq r$.

(b)$^\circ$ Show that if $\mathrm{ex}(n; G) = \mathrm{ex}(n; K_r)$ for some $n \geq r$, then there exists $e \in E(G)$ such that $G - \{e\}$ is $(r-1)$-partite.

12.$^\circ$ For each $r \geq 3$, construct a K_r-free graph G with $\chi(G) = r$.
[*Hint: remove a certain collection of* $2r-1$ *edges from* K_{2r-1}.]

13. (a)$^\circ$ For each $d \in \{1, 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$, construct an infinite graph G_d with $\mathrm{ud}(G_d) = d$.

(b)$^\circ$ Given an infinite graph G, define its *lower density* as

$$\ell\mathrm{d}(G) := \lim_{n\to\infty} \min\left\{e(H)/\tbinom{n}{2} \;\middle|\; H \text{ is an induced subgraph of } G \text{ of order } n\right\}.$$

Why does $\ell\mathrm{d}(G)$ exist? Which values can $\ell\mathrm{d}(G)$ take?

14. (a)$^\circ$ Let G be a connected graph of order $n \geq 1$, and let $k < n$ be such that for any $v, w \in G$ with $v \neq w$ and $v \nsim w$ we have $d(v) + d(w) \geq k$. Show that $P_k \leq G$.

(b)$^+$ Show that if G is P_k-free then $e(G) \leq \frac{|G|(k-1)}{2}$. Hence compute $\mathrm{ex}(nk, P_k)$ for all $n, k \geq 1$.

15.$^\circ$ For every $k \geq 2$, give an example of a k-connected graph of order ≥ 3 that is not Hamiltonian. What is the smallest possible order of such a graph?
[*Hint: see Problem 1.13.*]

16.$^\circ$ Is the Petersen graph (see Problem 2.10) Hamiltonian?

17.$^\circ$ Show that a connected graph G has an Euler trail if and only if it has ≤ 2 vertices of odd degree.