Problem List 1 (Structural properties)

Graph Theory, Winter Semester 2023/24, IM UWr

1.$^+$ Find (by "drawing" pictures representing graphs) all pairwise non-isomorphic graphs of order 4.

2.$^-$ For a graph G, define a relation $\approx$ on $V(G)$ by saying $v \approx w$ if and only if there exists a path in G with endpoints v and w. Show that $\approx$ is an equivalence relation—that is, show that $(\forall u \in G)(u \approx u)$, that $(\forall u, v \in G)(u \approx v \Rightarrow v \approx u)$, and that $(\forall u, v, w \in G)([u \approx v \land v \approx w] \Rightarrow u \approx w)$.

3.$^-$ Show that any graph of order at least 2 has two vertices of the same degree.

4. Given a graph G, define its *complement* $\overline{G}$ as a graph with vertices $V(\overline{G}) = V(G)$, such that given $v, w \in V(G)$ with $v \neq w$, we have $vw \in E(\overline{G})$ if and only if $vw \notin E(G)$.

   (a)$^-$ Show that if $G \cong \overline{G}$, then $|G| \equiv 0$ or $1 \pmod 4$.

   (b)$^+$ For any $n \geq 1$, construct a graph G of order $4n$ such that $G \cong \overline{G}$.
   [*Hint: take inspiration from the fact that P_3 has order 4 and $P_3 \cong \overline{P_3}$.*]

   (c)$^-$ Modify your construction to obtain a graph H of order $4n + 1$ such that $H \cong \overline{H}$.

5. (a)$^\circ$ Show that every connected graph G with $|G| \geq 1$ contains a vertex $v \in G$ such that $G - \{v\}$ is connected.
   [*Hint: pick v so that some connected component of $G - \{v\}$ is as big as possible.*]

   (b)$^\circ$ A connected graph with at least one vertex is called a *tree* if it has no cycles. Show that every tree with ≥ 2 vertices has a vertex of degree 1 (such a vertex is called a *leaf*).

   (c)$^\circ$ Deduce that if T is a tree then $e(T) = |T| - 1$.

   (d)$^+$ Let G be a graph with $|G| = n$. We say that a tuple $(d_G(v_1), \ldots, d_G(v_n))$, where $\{v_1, \ldots, v_n\} = V(G)$, is a *degree sequence* of G. Show that a given tuple $(d_1, \ldots, d_n)$ of integers, where $n \geq 2$, is a degree sequence of a tree if and only if $d_i \geq 1$ for all i and $\sum_{i=1}^{n} d_i = 2n - 2$.

6.$^+$ Let $G = (V, E)$ be a graph. Show that there exists a partition $V = A \sqcup B$ such that all vertices of $G[A]$ and of $G[B]$ have even degree.
[*Hint: consider what happens when we remove a vertex v of odd degree and "invert" adjacency between the neighbours of v.*]

7.$^+$ Suppose G is a graph that has no induced cycles of odd length—that is, for any $A \subseteq V(G)$, the graph $G[A]$ is not a cycle of odd length. Show that G is bipartite.

8.$^-$ Let G be a regular bipartite graph with vertex classes W and M, with $e(G) > 0$. Show that G contains a matching from W to M.

9.$^\circ$ Let $n \geq m \geq 1$. An $m \times n$ *Latin rectangle* is an $m \times n$ matrix with entries in $[n]$ such that each $i \in [n]$ appears exactly once in each row and at most once in each column. Show that any $m \times n$ Latin rectangle forms the first m rows of an $n \times n$ Latin rectangle.
[*Hint: use Hall's Marriage Theorem.*]

10. Let G be an infinite bipartite graph with (infinite) vertex classes W and M, and suppose that $|N_G(A)| \geq |A|$ for every $A \subseteq W$.

(a)$^+$ Show, by constructing an example, that such a graph G does not need to contain a matching from W to M.

(b)$^{++}$ Suppose that W is countable and $d_G(w) < \infty$ for all $w \in W$. Show that in this case G does contain a matching from W to M.
[*Hint: apply Hall's Marriage Theorem to finite subgraphs $G_1 \leq G_2 \leq \cdots$ of G.*]

11.$^\circ$ Show that any connected regular bipartite graph is 2-connected.

12. Let $k \geq 2$.

(a)$^+$ Give an example of a k-edge-connected graph that is not 2-connected. Is there an incomplete k-connected graph that is not 2-edge-connected?

(b)$^\circ$ Give an example of a graph G such that $G - \{v\}$ is not 2-edge-connected but $G - \{vw\}$ is k-edge-connected for some $v \in G$ and $w \in N_G(v)$.

13. Let G be an incomplete k-connected graph for some $k \geq 2$.

(a)$^+$ Show that for every $x \in G$ and every $U \subseteq V(G) \setminus \{x\}$ with $|U| \geq k$, there exists a collection of $(\{x\}, U)$-paths $P^{(1)}, \ldots, P^{(k)}$, where $P^{(i)} = xy_{i,1} \cdots y_{i,m_i}$, such that $y_{i,j} \neq y_{i',j'}$ for $(i,j) \neq (i',j')$ and such that $y_{i,j} \in U$ if and only if $j = m_i$.
[*Hint: add a vertex to G, connect it by edges to every vertex of U, show that the resulting graph is still k-connected, and use Menger's Theorem.*]

(b)$^+$ Show that if $|G| \geq 2k$ then G contains a cycle of length $\geq 2k$.

(c)$^+$ Show that every collection of k vertices in G is contained in a cycle.

14.$^+$ Let G be a k-edge-connected graph for some $k \geq 1$, and let $F \subseteq E(G)$ with $|F| = k$. Show that $G - F$ has at most two connected components.

15. Let G be an r-regular graph for some $r \geq 1$, and let $H = L_G$ be the line graph of G (appearing in the proof of the edge version of Menger's Theorem).

(a)$^-$ Show that H is regular.

(b)$^+$ Show that $L_H \cong G$ if and only if $r = 2$.