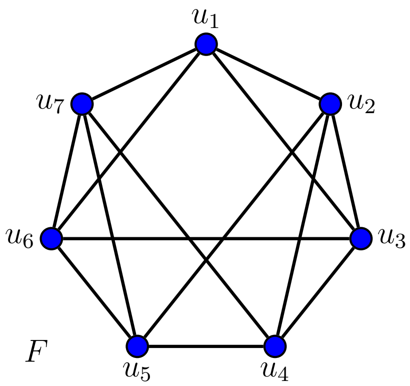
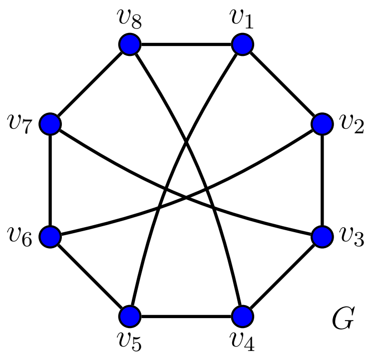
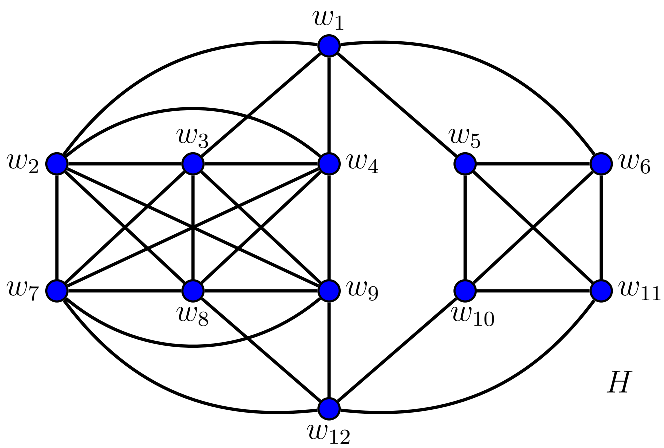
Graph theory

Mock Final Exam

Solutions







Exercise A

For graphs $\boldsymbol{G}$ and $\boldsymbol{H}$, write down the value of the largest $k \geq 0$ such that the graph is k-connected, and give a reason why it is not $(k+1)$-connected.

Solution for G:
$\boldsymbol{k = 3}$**:** G is not 4-connected since removing the three vertices $\{v_2, v_5, v_8\}$ disconnects v_1 from the rest of G.

Solution for H:
$\boldsymbol{k = 2}$**:** H is not 3-connected since removing the two vertices $\{w_1, w_{12}\}$ disconnects $H[\{w_2, w_3, w_4, w_7, w_8, w_9\}]$ from $H[\{w_5, w_6, w_{10}, w_{11}\}]$.

Exercise B

For graphs F and H, decide if the graph is Eulerian. Explain your answers.

Solution for F:

Yes: the graph F is 4-regular, and in particular every vertex of F has even degree, so F is Eulerian by the characterisation of connected Eulerian graphs. (*Alternatively:* Yes, the walk $u_1u_3u_6u_1u_2u_4u_7u_5u_2u_3u_4u_5u_6u_7u_1$ is a closed Euler trail in F.)

Solution for H:

No: the vertex w_1 has odd degree ($d(w_1) = 5$), so H is not Eulerian by the characterisation of connected Eulerian graphs.

Exercise C

For graphs G and H, find the clique number ω, and give a reason for why the clique number is $\geq \omega$.

Solution for G:

$\boldsymbol{\omega = 2}$**:** the two vertices $\{v_1, v_2\}$ span a K_2 in G, so the clique number is at least 2.

Solution for H:

$\boldsymbol{\omega = 6}$**:** the collection of six vertices $\{w_2, w_3, w_4, w_7, w_8, w_9\}$ spans a K_6 in H, so the clique number is at least 6.

Exercise D

For graphs $\boldsymbol{F}$ and $\boldsymbol{G}$, find the girth g, and give a reason for why the girth is $\leq g$.

Solution for F:	Solution for G:
$\boldsymbol{g = 3}$**:** there exists a cycle $u_1u_2u_3u_1$ of length 3 in F, so the girth is at most 3.	$\boldsymbol{g = 4}$**:** there exists a cycle $v_1v_2v_6v_5v_1$ of length 4 in G, so the girth is at most 4.

Exercise E

For graphs $\boldsymbol{F}$ and $\boldsymbol{G}$, find the edge chromatic number χ'. Explain your answers, giving reasons for both why the edge chromatic number is $\leq \chi'$ and why it is $\geq \chi'$.

[*Hint: If a graph K is r-regular, how would an admissible r-edge-colouring of K look like?*]

Solution for F:	Solution for G:
$\boldsymbol{\chi' = 5}$**.** We have $\chi' \leq 5$ by Vizing's Theorem, since $5 = \Delta(F) + 1$. (*Alternatively:* We have $\chi' \leq 5$, since picking different colours for the collections $\{u_1u_2, u_3u_4, u_5u_6\}$, $\{u_2u_3, u_4u_5, u_6u_7\}$, $\{u_1u_3, u_2u_4, u_5u_7\}$, $\{u_1u_6, u_2u_5, u_4u_7\}$ and $\{u_1u_7, u_3u_6\}$ of edges defines an admissible 5-edge-colouring for F.) We have $\chi' \geq 5$ since the edges of each colour have no vertices in common, implying that there are $\leq \frac{\lvert F \rvert}{2} < 4$ and therefore ≤ 3 edges of each colour; on the other hand, the graph F is 4-regular and so $e(F) = \frac{\lvert F \rvert d(F)}{2} = 14$, implying that we need $\geq \frac{14}{3} > 4$ and therefore ≥ 5 colours.	$\boldsymbol{\chi' = 3}$**.** We have $\chi' \leq 3$ since picking three different colours for the edges $\{v_1v_2, v_3v_4, v_5v_6, v_7v_8\}$, the edges $\{v_1v_8, v_2v_3, v_4v_5, v_6v_7\}$ and the edges $\{v_1v_5, v_2v_6, v_3v_7, v_4v_8\}$ defines an admissible 3-edge-colouring for G. We have $\chi' \geq 3$ since the three edges v_1v_2, v_1v_5 and v_1v_8 incident to the vertex v_1 must have distinct colours, implying that we need ≥ 3 colours.

Question 1

We say a graph G *decomposes into 2-paths* if $E(G) = \bigsqcup_{i=1}^{k} E(P^{(i)})$, where the subgraphs $P^{(1)}, \ldots, P^{(k)} \leq G$ are paths of length 2.

(a) Given $n \geq m \geq 1$, show that if K_m decomposes into 2-paths then so does K_n in the following cases:

- n is even and $m = n - 3$;
- n is odd and $m = n - 1$.

(b) Show that for any $n \geq 1$, the graph K_n decomposes into 2-paths if and only if $n \equiv 0$ or $1 \pmod 4$.

Solution:

(a) Suppose first that n is even and $m = n - 3$ (in particular, m is odd). Let the vertex set of K_n be $V(K_n) = \{v_1, \ldots, v_m, w_1, w_2, w_3\}$, so that the vertices $v_1, \ldots, v_m$ span a K_m, and let $P^{(1)}, \ldots, P^{(k)}$ be the 2-paths in the 2-path decomposition of K_m. The edges not covered by the paths in this decomposition are precisely $v_i w_j$ for $i = 1, \ldots, m$ and $j = 1, 2, 3$, as well as the edges w_1w_2, w_1w_3 and w_2w_3. We may then add the $3 \cdot \frac{m-1}{2}$ paths of length 2 of the form $v_{2i-1}w_jv_{2i}$ for $i = 1, \ldots, \frac{m-1}{2}$ and $j = 1, 2, 3$, along with the three 2-paths $v_mw_1w_2$, $v_mw_2w_3$ and $v_mw_3w_1$, to construct a 2-path decomposition of K_n, as required.

Suppose now that n is odd and $m = n-1$ (in particular, m is even). Let the vertex set of K_n be $V(K_n) = \{v_1, \ldots, v_m, w\}$, so that the vertices $v_1, \ldots, v_m$ span a K_m, and let $P^{(1)}, \ldots, P^{(k)}$ be the 2-paths in the 2-path decomposition of K_m. The edges not covered by the paths in this decomposition are precisely v_iw for $i = 1, \ldots, m$. We may then add the $3 \cdot \frac{m}{2}$ paths of length 2 of the form $v_{2i-1}wv_{2i}$ for $i = 1, \ldots, \frac{m}{2}$ to construct a 2-path decomposition of K_n, as required.

(b) Suppose first that $n \equiv 0$ or $1 \pmod 4$. We prove that K_n decomposes into 2-paths by induction on n; the base case, $n = 1$, is trivial since $E(K_1) = \varnothing$. So assume that $n > 1$. If $n \equiv 0 \pmod 4$, then n is even and we set $m := n - 3 \equiv 1 \pmod 4$; otherwise, $n \equiv 1 \pmod 4$ and in particular n is odd, and we set $m := n - 1 \equiv 0 \pmod 4$. In either case, by the inductive hypothesis, K_m decomposes into 2-paths, and hence, by part (a), K_n decomposes into 2-paths as well, as required.

Conversely, suppose that $n \not\equiv 0$ or $1 \pmod 4$. If $n \equiv 2 \pmod 4$, then both $\frac{n}{2}$ and $n-1$ are odd integers; otherwise, $n \equiv 3 \pmod 4$ and then both n and $\frac{n-1}{2}$ are odd integers. In either case, it follows that $e(K_n) = \binom{n}{2} = \frac{n(n-1)}{2}$ is odd, and so $E(K_n)$ cannot be expressed as a disjoint union of sets of cardinality 2. It follows that K_n does not decompose into two-paths, as required.

Question 2

Let $n \geq 1$, and let G be a bipartite graph with vertex classes W and M, such that $|W| = |M| = n$.

(a) Show that if $\delta(G) \geq \frac{n}{2}$, then G has a matching from W to M.

(b) For any integer δ with $1 \leq \delta < \frac{n}{2}$, show (by constructing an example) that if $\delta(G) = \delta$ then G does not need to have a matching.

Solution:

(a) By the Hall's Marriage Theorem, it is enough to show that (G, W) satisfies the Hall's Condition: that is, for each $A \subseteq W$, we have $|N_G(A)| \geq |A|$. Thus, let $A \subseteq W$. If $A = \varnothing$, then $|N_G(A)| = 0 = |A|$ and we're done. Thus, we may assume that $A \neq \varnothing$. Let $w \in A$; we then have

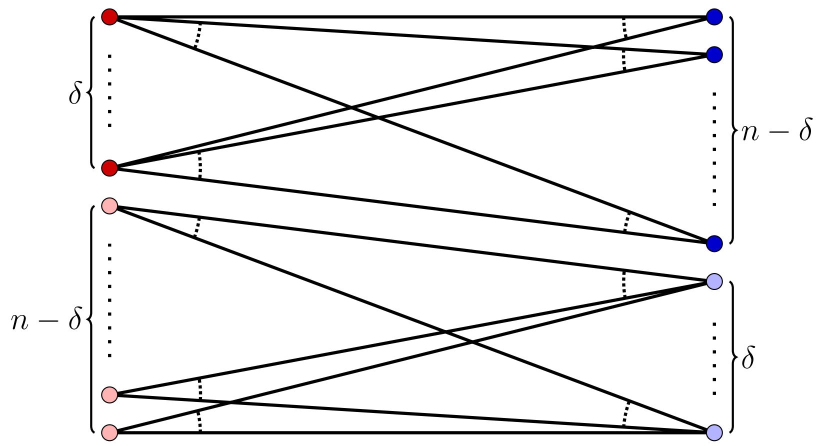
$$|N_G(A)| \geq |N_G(w)| = d_G(w) \geq \delta(G) \geq \frac{n}{2}.$$

Therefore, if $|A| \leq \frac{n}{2}$ then we have $|N_G(A)| \geq \frac{n}{2} \geq |A|$ and we're done, so we may assume that $|A| > \frac{n}{2}$.

We then claim that $N_G(A) = M$. Indeed, if there existed a vertex $v \in M \setminus N_G(A)$, then we would have $N_G(v) \subseteq W$ (since G is bipartite) and therefore $N_G(v) \subseteq W \setminus A$ (since $v \notin N_G(A)$), implying that we would have $d_G(v) \leq |W| - |A| < n - \frac{n}{2} = \frac{n}{2}$. But this would contradict the fact that $\delta(G) \geq \frac{n}{2}$; thus $N_G(A) = M$, as claimed.

But now we have $|N_G(A)| = |M| = n = |W| \geq |A|$. This shows that (G, W) satisfies Hall's Condition, as required.

(b) Consider the graph G constructed as a disjoint union of complete bipartite graphs $K_{\delta,n-\delta}$ and $K_{n-\delta,\delta}$, as follows:



Here the (light/dark) red and blue vertices form W and M, respectively. If $A \subseteq W$ is the set of light red vertices then $N_G(A)$ are precisely the light blue vertices, implying that $|N_G(A)| = \delta < n - \delta = |A|$. Thus, (G, W) does not satisfy the Hall's Condition, implying by the Hall's Marriage Theorem that G has no matchings from W to M. On the other hand, given $v \in G$, we have $d_G(v) = \delta$ if v is either light red or dark blue and $d_G(v) = n - \delta > \delta$ otherwise, implying that $\delta(G) = \delta$, as required.

Question 3

Let $n \geq k+1 \geq 2$, and let T be a tree of order $k+1$.

(a) Show that if G is a graph with $|G| = n$ and $\delta(G) \geq k$, then $T \leq G$.

(b) Show that $\mathrm{ex}(n;T) \leq (k-1)n - \binom{k}{2}$.

Solution:

(a) We first claim that we can list the vertices of T as $V(T) = \{v_0, \dots, v_k\}$ in such a way that $T[\{v_0, \dots, v_i\}]$ is a tree for all $i \in \{0, \dots, k\}$. Indeed, we can show this by induction on k: the base case, $k = 0$, is trivial. If T is a tree of order $k+1 \geq 2$, then we know (from Problem List 1) that T has a leaf, i.e. a vertex of degree 1. Let $v_k \in T$ be a leaf and let $T' = T - \{v_k\}$. Then no path in T between two vertices of T' passes through v_k (as $d_T(v_k) = 1$), implying that such a path is still in T'; therefore, since T is connected, it follows that T' is connected. Moreover, since T contains no cycles, neither does T', implying that T' is a tree of order k. Therefore, by the inductive hypothesis, we can list the vertices of T' as $V(T') = \{v_0, \dots, v_{k-1}\}$ in such a way that $T'[\{v_0, \dots, v_i\}]$ is a tree for all $i \in \{0, \dots, k-1\}$. It then follows that $T[\{v_0, \dots, v_i\}]$ is a tree for all $i \in \{0, \dots, k\}$, giving us the required list of vertices of T, as claimed.

Now we claim that $T \leq G$, i.e. that there exists an injective function $\varphi \colon V(T) \to V(G)$ such that if $v_i \sim_T v_j$ then $\varphi(v_i) \sim_G \varphi(v_j)$. Indeed, we can do this by inductively picking the vertices $\varphi(v_0), \dots, \varphi(v_k) \in G$, so that $\varphi|_{V(T_i)}$ realises $T_i := T[\{v_0, \dots, v_k\}]$ as a subgraph of G for all $i \in \{0, \dots, k\}$. Let $\varphi(v_0) \in G$ be any vertex (this clearly realises the single-vertex graph T_0 as a subgraph of G). Having defined $\varphi(v_0), \dots, \varphi(v_{i-1})$, consider the vertex $v_i \in T_i$. By construction, v_i is a leaf in T_i, and so there exists a unique $j \in \{0, \dots, i-1\}$ such that $v_i \sim_T v_j$. Note that we have $d_G(\varphi(v_j)) \geq \delta(G) \geq k \geq i$, implying that at least one vertex $w_i \in N_G(\varphi(v_j))$ does not appear among the $i-1$ vertices $\{\varphi(v_0), \dots, \varphi(v_{j-1}), \varphi(v_{j+1}), \dots, \varphi(v_{i-1})\}$. We then set $\varphi(v_i) := w_i$, so that $\varphi|_{\{v_0,\dots,v_i\}}$ is still an injective function realising T_i as a subgraph of G. Repeating this construction we eventually show that $T = T_k \leq G$, as required.

(b) We will show that this holds for all $n \geq k$ by induction on n. For the base case, note that if $n = k$ then we have

$$\mathrm{ex}(n;T) \leq e(K_n) = \binom{k}{2} = (k-1)k - \frac{(k-1)k}{2} = (k-1)n - \binom{k}{2},$$

as required. Now if a graph G of order $n \geq k+1$ is T-free, it follows from part (a) that $\delta(G) < k$, and therefore there exists some $v \in G$ such that $d_G(v) \leq k-1$. Then $H := G - \{v\}$ is still T-free and so $e(H) \leq \mathrm{ex}(n-1;T) \leq (k-1)(n-1) - \binom{k}{2}$ by the inductive hypothesis. It follows that $e(G) = e(H) + d_G(v) \leq (k-1)n - \binom{k}{2}$ for any T-free graph G of order n, and therefore $\mathrm{ex}(n;T) \leq (k-1)n - \binom{k}{2}$, as required.

Question 4

We say a graph G is *perfect* if $\chi(G[W]) = \omega(G[W])$ for all $W \subseteq V(G)$ (by convention, if $|G| = 0$ then $\omega(G) = \chi(G) = 0$).

(a) Show that G is perfect if and only if every non-empty induced subgraph H of G contains an independent set $A \subseteq V(H)$ such that $\omega(H - A) < \omega(H)$.

(b) Give an example (with justification) of a minimal imperfect graph, i.e. a graph G such that G is not perfect but any subgraph of G (apart from G itself) is perfect.

Solution:

(a) Suppose G is perfect, and let H be a non-empty induced subgraph of G. Then $\chi(H) = \omega(H)$ since G is perfect. Let $A \subseteq V(H)$ be one of the vertex classes in H (viewed as an $\omega(H)$-partite graph). Then $H - A$ is $(\omega(H)-1)$-partite, implying that $\omega(H) > \chi(H - A) = \omega(H - A)$ (since G is perfect and $H - A$ is an induced subgraph of G), as required.

Conversely, suppose that every non-empty induced subgraph H of G contains an independent set $A \subseteq V(H)$ such that $\omega(H - A) < \omega(H)$. Let $H \leq G$ be an induced subgraph; we claim that $\chi(H) = \omega(H)$. Indeed, construct the subsets $A_1, A_2, \ldots$ of $V(H)$ and induced subgraphs $H_0, H_1, H_2, \ldots$ of G inductively, as follows. Let $H_0 = H$. Having constructed H_i for some $i \geq 0$, we stop the construction if H_i has no vertices, and otherwise pick $A_{i+1} \subseteq V(H_i)$ such that $\omega(H_{i+1}) < \omega(H_i)$, where $H_{i+1} = H_i - A_{i+1}$. We then have $\omega(H) = \omega(H_0) > \omega(H_1) > \cdots$, implying that we have $\omega(H_r) = 0$ (i.e. H_r has no vertices, and so the construction terminates after r steps) for some $r \leq \omega(H)$.

Now since for each $i \in \{1, \ldots, r\}$ the subset $A_i \subseteq V(H_{i-1})$ is independent in H_{i-1} and since H_{i-1} is induced in H, it follows that A_i is independent in H as well. Thus $V(H) = \bigsqcup_{i=1}^{r} A_i$ with each A_i independent, implying that H is r-partite and therefore $\chi(H) \leq r \leq \omega(H)$. Since any r-partition of H can contain at most one vertex of any given complete subgraph $K \leq H$, it follows that $\chi(H) \geq \omega(H)$ and therefore $\chi(H) = \omega(H)$, as claimed. Thus G is perfect, as required.

(b) Consider $G = C_5$. We then have $\chi(G) > 2$ since G contains a cycle of odd length (i.e. itself) and therefore (by the characterisation of bipartite graphs) is not bipartite. On the other hand, G is triangle-free and therefore $\omega(G) \leq 2$. Thus $\chi(G) \neq \omega(G)$, implying that G is not perfect.

In order to show that G is *minimal* imperfect, it is enough to show that if $H \leq G$ is any subgraph of G apart from G itself, then $\chi(H) = \omega(H)$. But indeed, if H is such a subgraph then H has no cycles, and in particular no cycles of odd length, implying that $\chi(H) \leq 2$. Now if $\chi(H) = 0$, then H has no vertices and therefore $\omega(H) = 0 = \chi(H)$. If $\chi(H) = 1$, then H has no edges but has at least one vertex, implying that $\omega(H) = 1 = \chi(H)$. Finally, if $\chi(H) = 2$, then H has at least one edge; since G (and therefore H) has no triangles, it follows that $\omega(H) = 2 = \chi(H)$. Thus $\chi(H) = \omega(H)$ in any case, as required.

Question 5

(a) Let a graph G, subgraphs $G_1, G_2 \leq G$ and a vertex $v \in G$ be such that $V(G) = V(G_1) \cup V(G_2)$, $\{v\} = V(G_1) \cap V(G_2)$ and $E(G) = E(G_1) \sqcup E(G_2)$. Show that $x \cdot p_G(x) = p_{G_1}(x) \cdot p_{G_2}(x)$.

(b) Let $e \in E(G)$ be a bridge in a connected graph G, i.e. an edge such that $G - \{e\}$ is not connected. Show that $x \cdot p_G(x) = (x-1) \cdot p_{G-\{e\}}(x)$.

Solution:

(a) Let $x \geq 1$, and let Φ_1 and Φ_2 be the sets of admissible x-vertex-colourings of G_1 and G_2, respectively. Given $i \in \{1,2\}$ and $j \in [x]$, let $\Phi'_{i,j} = \{c \in \Phi_i \mid c(v) = j\}$. Then, by symmetry, we have $|\Phi'_{i,j}| = |\Phi'_{i,j'}|$ for $i \in \{1,2\}$ and any $j, j' \in [x]$ (as any $c \in \Phi'_{i,j}$ can be obtained from some $c' \in \Phi'_{i,j'}$ by swapping colours j and j', and vice versa). Since $\Phi_i = \bigsqcup_{j=1}^{x} \Phi'_{i,j}$ and $|\Phi_i| = p_{G_i}(x)$, it follows that $|\Phi'_{i,j}| = x^{-1} p_{G_i}(x)$.

Now define a function

$$\{\text{admissible colourings } c\colon V(G) \to [x]\} \to \bigsqcup_{j=1}^{x} \Phi'_{1,j} \times \Phi'_{2,j}$$

by sending $c \mapsto (c|_{V(G_1)}, c|_{V(G_2)})$. This function is well-defined since v is the only common vertex of G_1 and G_2, injective since $V(G) = V(G_1) \cup V(G_2)$, and surjective since all edges of G come from either G_1 or G_2. In particular, this map is a bijection. This implies that

$$\begin{aligned} p_G(x) &= \left| \bigsqcup_{j=1}^{x} \Phi'_{1,j} \times \Phi'_{2,j} \right| = \sum_{j=1}^{x} |\Phi'_{1,j}| \cdot |\Phi'_{2,j}| = x \cdot x^{-1} p_{G_1}(x) \cdot x^{-1} p_{G_2}(x) \\ &= \frac{p_{G_1}(x) p_{G_2}(x)}{x} \end{aligned}$$

and therefore $x \cdot p_G(x) = p_{G_1}(x) \cdot p_{G_2}(x)$, as required.

(b) Let G_1 and G_2 be the two connected components of $G - \{e\}$, and let $G' = G/e$ be the graph obtained from G by contracting the edge e. We may then identify $G_1, G_2 \leq G$ with subgraphs $G_1', G_2' \leq G'$, in such a way that $V(G') = V(G_1') \cup V(G_2')$, $V(G_1') \cap V(G_2') = \{v\}$ (where the vertex v comes from the endpoints of the edge e) and $E(G') = E(G_1') \sqcup E(G_2')$. It then follows from part (a) that $x \cdot p_{G'}(x) = p_{G_1'}(x) \cdot p_{G_2'}(x) = p_{G_1}(x) \cdot p_{G_2}(x)$.

On the other hand, an admissible x-vertex-colouring of $G - \{e\}$ corresponds to independent admissible x-vertex-colourings of G_1 and G_2, implying that $p_{G-\{e\}}(x) = p_{G_1}(x) \cdot p_{G_2}(x)$. Moreover, we know from the lectures that $p_G(x) = p_{G-\{e\}}(x) - p_{G'}(x)$. This implies that

$$\begin{aligned} p_G(x) &= p_{G_1}(x) \cdot p_{G_2}(x) - x^{-1} \cdot p_{G_1}(x) \cdot p_{G_2}(x) = \frac{x-1}{x} \cdot p_{G_1}(x) \cdot p_{G_2}(x) \\ &= \frac{x-1}{x} \cdot p_{G-\{e\}}(x) \end{aligned}$$

and therefore $x \cdot p_G(x) = (x-1) \cdot p_{G-\{e\}}(x)$, as required.