Graph theory

Mock Final Exam

The exam consists of two parts: "Exercises" and "Questions".
Please attempt ***ALL*** *Exercises and* ***THREE*** *Questions.*

Refer to any results you are using by name
(or state them if you don't remember their name).

Duration of the exam: 180 minutes.

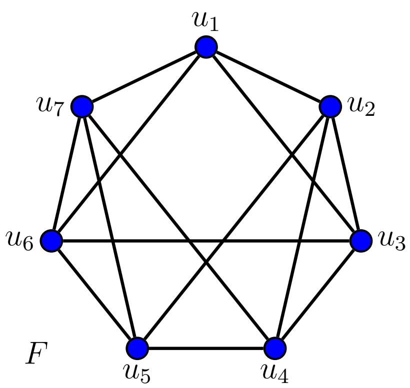
Exercises

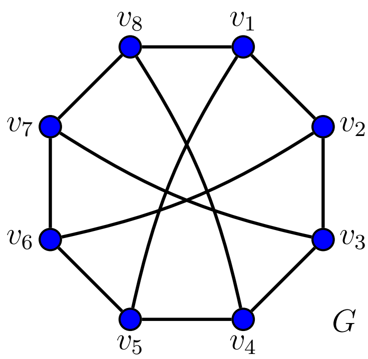
*Please attempt **ALL** Exercises (A, B, C, D and E) displayed below.*

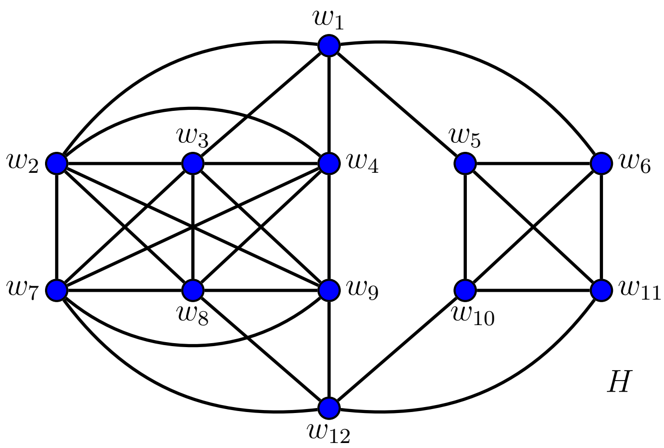
Wherever explanations are needed, please give precise *reasons*
(by explicitly writing down specific subgraphs, collections of vertices/edges, etc).

The Exercises are worth 9 points in total.

All of the exercises concern the following graphs:







Exercise A

For graphs G and H, write down the value of the largest $k \geq 0$ such that the graph is k-connected, and give a reason why it is not $(k+1)$-connected.

Solution for G:	Solution for H:

Exercise B

For graphs F and H, decide if the graph is Eulerian. Explain your answers.

Solution for F:	Solution for H:

Exercise C

For graphs *G* and *H*, find the clique number ω, and give a reason for why the clique number is $\geq \omega$.

Solution for G:	Solution for H:

Exercise D

For graphs *F* and *G*, find the girth g, and give a reason for why the girth is $\leq g$.

Solution for F:	Solution for G:

Exercise E

For graphs *F* and *G*, find the edge chromatic number χ'. Explain your answers, giving reasons for both why the edge chromatic number is $\leq \chi'$ and why it is $\geq \chi'$.

[*Hint: If a graph K is r-regular, how would an admissible r-edge-colouring of K look like?*]

Solution for F:	Solution for G:

Questions

*Please attempt **THREE** Questions (out of 5).*

Each Question is worth 7 points.

Question 1

We say a graph G *decomposes into 2-paths* if $E(G) = \bigsqcup_{i=1}^{k} E(P^{(i)})$, where the subgraphs $P^{(1)}, \dots, P^{(k)} \leq G$ are paths of length 2.

(a) Given $n \geq m \geq 1$, show that if K_m decomposes into 2-paths then so does K_n in the following cases:

- n is even and $m = n - 3$;
- n is odd and $m = n - 1$.

(b) Show that for any $n \geq 1$, the graph K_n decomposes into 2-paths if and only if $n \equiv 0$ or $1 \pmod 4$.

Question 2

Let $n \geq 1$, and let G be a bipartite graph with vertex classes W and M, such that $|W| = |M| = n$.

(a) Show that if $\delta(G) \geq \frac{n}{2}$, then G has a matching from W to M.

(b) For any integer δ with $1 \leq \delta < \frac{n}{2}$, show (by constructing an example) that if $\delta(G) = \delta$ then G does not need to have a matching.

Question 3

Let $n \geq k + 1 \geq 2$, and let T be a tree of order $k + 1$.

(a) Show that if G is a graph with $|G| = n$ and $\delta(G) \geq k$, then $T \leq G$.

(b) Show that $\mathrm{ex}(n; T) \leq (k-1)n - \binom{k}{2}$.

Question 4

We say a graph G is *perfect* if $\chi(G[W]) = \omega(G[W])$ for all $W \subseteq V(G)$ (by convention, if $|G| = 0$ then $\omega(G) = \chi(G) = 0$).

(a) Show that G is perfect if and only if every non-empty induced subgraph H of G contains an independent set $A \subseteq V(H)$ such that $\omega(H - A) < \omega(H)$.

(b) Give an example (with justification) of a minimal imperfect graph, i.e. a graph G such that G is not perfect but any subgraph of G (apart from G itself) is perfect.

Question 5

(a) Let a graph G, subgraphs $G_1, G_2 \leq G$ and a vertex $v \in G$ be such that $V(G) = V(G_1) \cup V(G_2)$, $\{v\} = V(G_1) \cap V(G_2)$ and $E(G) = E(G_1) \sqcup E(G_2)$. Show that $x \cdot p_G(x) = p_{G_1}(x) \cdot p_{G_2}(x)$.

(b) Let $e \in E(G)$ be a bridge in a connected graph G, i.e. an edge such that $G - \{e\}$ is not connected. Show that $x \cdot p_G(x) = (x-1) \cdot p_{G-\{e\}}(x)$.