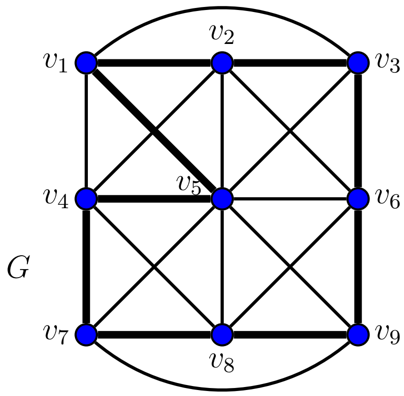
Graph theory

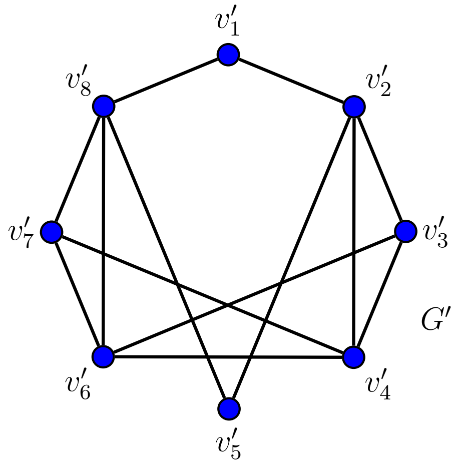
Mock Class Test 2

Solutions

Exercise A

Consider the following graphs G and G':





Decide whether or not each of these graphs is Hamiltonian. Explain your answers.

Solution for G:

Yes: for instance,

$$v_1 v_2 v_3 v_6 v_9 v_8 v_7 v_4 v_5 v_1$$

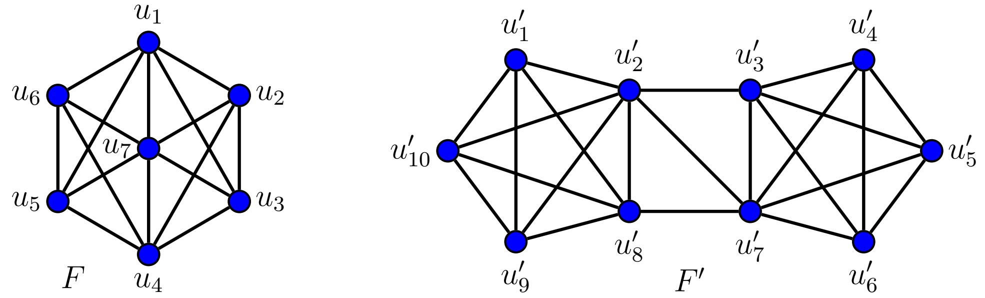
(drawn as thick edges in the picture) is a Hamilton cycle in G.

Solution for G':

No: any cycle passing through v'_1 (respectively v'_5) must contain $v'_8 v'_1 v'_2$ (respectively $v'_8 v'_5 v'_2$) as a subpath. But since any cycle passes through v'_8 and v'_2 at most once, it follows that the only cycle containing both $v'_8 v'_1 v'_2$ and $v'_8 v'_5 v'_2$ as subpaths (and therefore the only cycle passing through both v'_1 and v'_5) is $v'_8 v'_1 v'_2 v'_5 v'_8$, which has length $4 < |G'|$.

Exercise B

Consider the following graphs F and F':



Find the independence number α of each of these graphs, and explain your answers (giving reasons for both why the independence number is $\geq \alpha$ and why it is $\leq \alpha$).

Solution for F:

$\boldsymbol{\alpha(F) = 2}$**:** we have $\alpha(F) \geq 2$ since, for instance, the subset $\{u_1, u_4\}$ is independent.

We have $\alpha(F) \leq 2$ since the subgraphs $F[\{u_1, u_2, u_3\}] \cong K_3$ and $F[\{u_4, u_5, u_6, u_7\}] \cong K_4$ are complete, so any independent subset of F can contain at most one vertex from each of those subgraphs.

Solution for F':

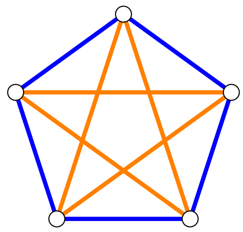
$\boldsymbol{\alpha(F') = 2}$**:** we have $\alpha(F') \geq 2$ since, for instance, the subset $\{u'_1, u'_3\}$ is independent.

We have $\alpha(F) \leq 2$ since the subgraphs $F'[\{u'_1, u'_2, u'_8, u'_9, u'_{10}\}] \cong K_5$ and $F'[\{u'_3, u'_4, u'_5, u'_6, u'_7\}] \cong K_5$ are complete, so any independent subset of F' can contain at most one vertex from each of those subgraphs.

Question 1

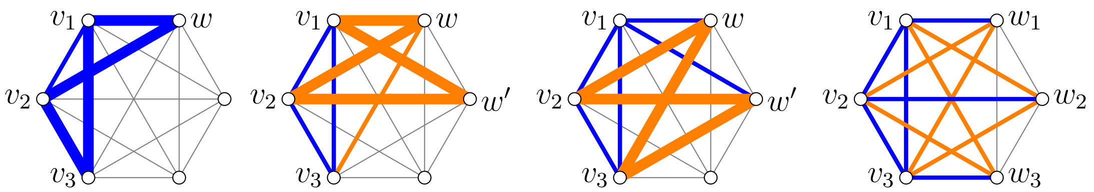
Show that $R(C_4, C_4) = 6$.

Solution: Consider the following blue/orange edge-colouring of K_5:



Then the edges of each colour forms a subgraph isomorphic to C_5, and C_5 has no subgraphs isomorphic to C_4. Thus this edge-colouring of K_5 contains no monochromatic 4-cycles, showing that $R(C_4, C_4) > 5$.

Conversely, suppose we have a blue/orange edge-colouring of K_6; we aim to find a monochromatic C_4. Since $R(3,3) = 6$, we can find a monochromatic triangle $K_6[T] \leq K_6$: without loss of generality, it is blue. Let $T = \{v_1, v_2, v_3\}$, and given a vertex $w \in K_6 - T$, let $n(w)$ be the number of blue edges among the edges wv_1, wv_2 and wv_3. The following pictures represent the argument below:



(a) $n(w) \geq 2$; (b) $n(w) = 0$; (c) $u_w = u_{w'} = v_1$; (d) otherwise.

If $n(w) \geq 2$ for some $w \in K_6 - T$ then we may assume, without loss of generality, that the edges wv_1 and wv_2 are blue (a). Then $wv_1v_3v_2w$ is a blue C_4, as required.

Thus, we may assume that $n(w) \leq 1$ for all $w \in K_6 - T$. If $n(w) = 0$ for some $w \in K_6 - T$, pick any $w' \in K_6 - (T \cup \{w\})$. Then $n(w') \leq 1$ and therefore, without loss of generality, the edges $w'v_1$ and $w'v_2$ are orange (b). Since the edges wv_1, wv_2 and wv_3 are all orange, the cycle $wv_1w'v_2w$ is an orange C_4, as required.

Thus, we may assume that $n(w) = 1$ for all $w \in K_6 - T$, and so for each $w \in K_6 - T$ there exists a unique vertex $u_w \in T$ such that the edge wu_w is blue. If $u_w = u_{w'}$ for some $w \neq w'$—without loss of generality, $u_w = u_{w'} = v_1$ (c)—then the edges wv_2, wv_3, $w'v_2$ and $w'v_3$ are all orange and so $wv_2w'v_3w$ is an orange C_4, as required.

Thus, we may assume that $u_w \neq u_{w'}$ for $w \neq w'$, and so we may enumerate the vertices $V(K_6 - T) = \{w_1, w_2, w_3\}$ so that the edge w_iv_j is blue if $i = j$ and orange otherwise (d). Now if at least one of the edges of $K_6 - T$ is blue—without loss of generality, w_1w_2 is blue—then $w_1w_2v_2v_1w_1$ is a blue C_4, as required. Otherwise, all three edges of $K_6 - T$ are orange, and so $w_1v_2w_3w_2w_1$ is an orange C_4, as required.

Question 2

Let $n \geq 2$, and let G_n be a graph with $V(G_n) = \{(a,b) \in \mathbb{N}^2 \mid 1 \leq a < b \leq n\}$, such that $(a,b) \sim_{G_n} (a',b')$ if and only if either $a = b'$ or $b = a'$. By using the Multicolour Ramsey's Theorem, show that $\chi(G_n) \to \infty$ as $n \to \infty$.

Solution: Given $k \in \mathbb{N}_{\geq 2}$, consider the Ramsey number $N(k) = R(\overbrace{3,\ldots,3}^{k})$, which exists by the Multicolour Ramsey's Theorem. It is then enough to show that $\chi(G_n) > k$ whenever $n \geq N(k)$.

Let $n \in \mathbb{N}$ with $n \geq N(k)$, and suppose for contradiction that $\chi(G_n) \leq k$. Thus, there exists an admissible k-colouring $c\colon V(G_n) \to [k]$. We define a k-edge-colouring $C\colon E(K_n) \to [k]$ as follows: given $v = (a,b) \in V(G_n)$, we set $C(ab) = c(v)$.

Since $n \geq N(k)$, it follows by the definition of $N(k)$ that there exists a monochromatic triangle in $T \leq K_n$ of colour $i \in [k]$ with respect to C. Let $V(T) = \{a,b,c\}$, ordered so that $a < b < c$. Then, in particular, we have $c(a,b) = C(ab) = i$ and $c(b,c) = C(bc) = i$. However, the vertices (a,b) and (b,c) are adjacent in G_n, contradicting the fact that the k-colouring c is admissible.

Thus $\chi(G_n) > k$, as required.

Question 3

Let $p, \varepsilon \in (0,1)$ be constants. Using the fact that $\ln(t!) = t\ln(t) - t + o(t)$, show that $\Delta(G) < (ep+\varepsilon)n$ for almost every $G \in \mathcal{G}(n,p)$ (where $e = 2.718\ldots$).

Solution: Let $c = ep + \varepsilon$ and $t = \lceil cn \rceil$, and let X be the number of subgraphs of $G \in \mathcal{G}(n,p)$ isomorphic to $K_{1,t}$; note that we have $\Delta(G) < t$ if and only if $X = 0$. Suppose, without loss of generality, that $c < 1$ (otherwise the result is trivial). Now given a subset $U \subseteq V(G)$ with $|U| = t$ and a vertex $v \in G - U$, the probability that G contains a $K_{1,t}$ with vertex classes $\{v\}$ and U is p^t. There are $\binom{n}{t}$ ways to choose U and, having chosen U, there are $n-t$ ways to choose v, implying that

$$\mathbb{E}X = \binom{n}{t} \cdot (n-t) \cdot p^t = \frac{n!p^t}{t!(n-t-1)!}.$$

Now since $\ln(t!) - \ln((t-1)!) = \ln(t) = o(t)$ and $t - 1 < cn \leq t$, we have an approximation

$$\ln(t!) = cn\ln(cn) - cn + o(n) = cn\ln(c) + cn\ln(n) - cn + o(n);$$

similarly, since $\ln((n-t+1)!) - \ln((n-t-1)!) \leq 2\ln(n-t+1) = o(n)$ and $n - t - 1 < (1-c)n < n - t + 1$, we have

$$\begin{aligned}\ln((n-t-1)!) &= (1-c)n\ln((1-c)n) - (1-c)n + o(n) \\ &= (1-c)n\ln(1-c) + (1-c)n\ln(n) - (1-c)n + o(n).\end{aligned}$$

Combining with the fact that $t\ln(p) = cn\ln(p) + O(1)$, this implies that

$$\begin{aligned}\ln(\mathbb{E}X) &= n\ln(n) - n + cn\ln(p) - cn\ln(c) - cn\ln(n) + cn \\ &\qquad - (1-c)n\ln(1-c) - (1-c)n\ln(n) + (1-c)n + o(n) \\ &= [1 - c - (1-c)]n\ln(n) - [1 - c - (1-c)]n + cn\ln(p) \\ &\qquad - cn\ln(c) - (1-c)n\ln(1-c) + o(n) \\ &= n\left[-c\ln(c/p) - (1-c)\ln(1-c) + o(1)\right].\end{aligned}$$

Finally, note that $\ln(c/p) = \ln(e + \frac{\varepsilon}{p}) > 1$, implying that $\ln(\mathbb{E}X) < -n \cdot f(c)$ for n large enough, where $f(x) = x + (1-x)\ln(1-x)$. The function f is differentiable on $(0,1)$, with derivative

$$f'(x) = 1 + (-1) \cdot \left[1 \cdot \ln(1-x) + (1-x) \cdot \frac{1}{1-x}\right] = -\ln(1-x) > 0,$$

implying that f is increasing on $(0,1)$ and therefore $f(c) > f(0) = 0$. This shows that $\ln(\mathbb{E}X) \to -\infty$ and therefore $\mathbb{E}X \to 0$ as $n \to \infty$. Therefore, by Markov's Inequality, we have

$$\mathbb{P}(X > 0) = \mathbb{P}(X \geq 1) \leq \mathbb{E}X \to 0$$

as $n \to \infty$, implying that almost every $G \in \mathcal{G}(n,p)$ has no subgraphs isomorphic to $K_{1,t}$ and therefore satisfies $\Delta(G) < t$, as required.