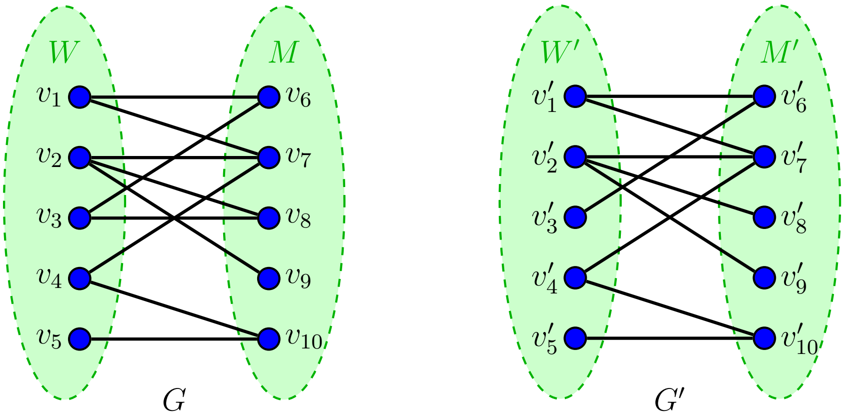
Graph theory

Mock Class Test 1

Solutions

Exercise A

Consider the following bipartite graphs G and G':

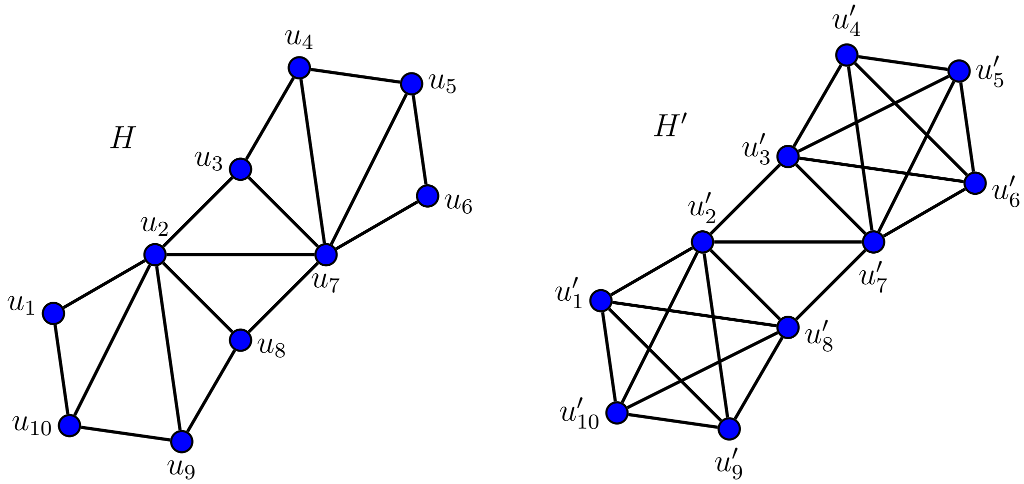


Decide whether or not each of these graphs has a matching between the given vertex classes. Explain your answers.

Solution for G (from W to M):	Solution for G' (from W' to M'):
Yes: for instance, the collection of edges $\{v_1v_6, v_2v_9, v_3v_8, v_4v_7, v_5v_{10}\}$ is a matching from W to M.	**No:** Hall's Condition is not satisfied for (G', W'): for instance, $A = \{v_1', v_3', v_4', v_5'\} \subseteq W$ has 4 vertices but $N_{G'}(A) = \{v_6', v_7', v_{10}'\}$ only has 3, so $|N_{G'}(A)| < |A|$. Therefore, by the Hall's Marriage Theorem, there are no matchings from W' to M' in G'.

Exercise B

Consider the following graphs H and H':



For each of these graphs, write down the largest $k \geq 0$ such that the graph is k-edge-connected, and give a reason why it is not $(k+1)$-edge-connected.

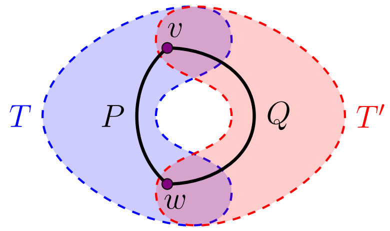
Solution for H:	Solution for H':
$\boldsymbol{k = 2}$**:** H is 2-edge-connected, but it is not 3-edge-connected since the removal of 2 edges $\{u_1u_2, u_1u_{10}\}$ disconnects the vertex u_1 from the remainder of H.	$\boldsymbol{k = 3}$**:** H' is 3-edge-connected, but not 4-edge-connected since the removal of 3 edges $\{u'_2u'_3, u'_2u'_7, u'_8u'_7\}$ disconnects $H'[\{u'_1, u'_2, u'_8, u'_9, u'_{10}\}]$ from the subgraph $H'[\{u'_3, u'_4, u'_5, u'_6, u'_7\}]$.

Question 1

Let G be a tree, and let $G_1, \ldots, G_r \leq G$ be a collection of $r \geq 2$ connected subgraphs such that $V(G_i) \cap V(G_j) \neq \varnothing$ for all i and j. Show that $\bigcap_{i=1}^{r} V(G_i) \neq \varnothing$.

Solution: Our proof proceeds in the following steps: showing that the intersection of two connected subgraphs of G is connected; showing the statement for $r = 3$; showing the statement for an arbitrary r.

We first claim that if $T, T' \leq G$ are connected subgraphs, then the intersection $T'' = T \cap T'$ (defined by $V(T'') = V(T) \cap V(T')$ and $E(T'') = E(T) \cap E(T')$) is connected as well. Indeed, suppose T'' is not connected, and let $v, w \in T''$ lie in different connected components of T''. Since T is connected, there exists a path $P = v \cdots w$ in T; suppose that v, w and P are chosen in such a way that P is as short as possible. It then follows that $V(P) \cap V(T'') = \{v, w\}$: indeed, if $P = v \cdots u \cdots w$ for some vertex $u \in T'' - \{v, w\}$, then u cannot be in the same connected component of T'' as both v and w, implying that P can be replaced by a subpath of the form $v \cdots u$ or $u \cdots w$; this contradicts the minimality of the length of P.



Now let $Q = w \cdots v$ be a path in T' (which exists since T' is connected); see the picture above. Note that since $P \leq T$, we have $V(P) \cap V(T') = V(P) \cap V(T'') = \{v, w\}$; since $Q \leq T'$, it follows that $V(P) \cap V(Q) = \{v, w\}$. Moreover, the edge vw (if it exists) cannot belong to both P and Q, since v and w lie in different connected components of $T \cap T'$; this implies that $P \cdot Q$, the concatenation of P and Q, is a cycle in G. But this contradicts the fact that G is a tree; thus T'' is connected, as claimed.

Now let $G_1, G_2, G_3 \leq G$ be connected subgraphs such that $V(G_i) \cap V(G_j) \neq \varnothing$ for all i and j. We claim that $V(G_1) \cap V(G_2) \cap V(G_3) \neq \varnothing$. Consider the union $G' = G_1 \cup G_2$ (defined by $V(G') = V(G_1) \cup V(G_2)$ and $E(G') = E(G_1) \cup E(G_2)$). Since there exists a vertex $u \in G_1 \cap G_2$ and since G_1 and G_2 are connected, it follows that u is in the same connected component of G' as any other vertex of G_i for $i \in \{1, 2\}$, implying that G' is connected. In particular, G' and G_3 are subtrees of G, implying by the above argument that $G' \cap G_3$ is connected. Thus, there exists a path $v_1 \cdots v_k$ in $G' \cap G_3$ from $v_1 \in G_1 \cap G_3$ to $v_k \in G_2 \cap G_3$ (we may pick such v_1 and v_k since $V(G_i) \cap V(G_3) \neq \varnothing$ for $i \in \{1, 2\}$). Now let ℓ be the largest such that $v_\ell \in G_1$. If $\ell = k$ then we have $v_k \in G_1 \cap (G_2 \cap G_3)$; otherwise, $v_\ell v_{\ell+1} \notin E(G_1)$ and therefore $v_\ell v_{\ell+1} \in E(G_2)$, implying that $v_\ell \in (G_1 \cap G_3) \cap G_2$. We thus have $V(G_1) \cap V(G_2) \cap V(G_3) \neq \varnothing$, as claimed.

We now show the general statement by induction on r. The base case, $r = 2$, is clear, and the case $r = 3$ is dealt with above, so suppose that $r \geq 4$. Consider the subgraphs $G'_1, \dots, G'_{r-1} \leq G$, where $G'_i = G_i \cap G_r$. Since each G_i (including G_r) is connected, it follows that each G'_i is connected; moreover, by the previous paragraph, we have $V(G'_i) \cap V(G'_j) = V(G_i) \cap V(G_j) \cap V(G_r) \neq \varnothing$ for all i and j. It then follows by the inductive hypothesis that we have $\bigcap_{i=1}^{r} V(G_r) = \bigcap_{i=1}^{r-1} V(G'_i) \neq \varnothing$, as required.

Question 2

Let $n \geq 1$. A *doubly stochastic matrix* is an $n \times n$ matrix with entries in $[0, 1]$ such that the sum of entries in each row and in each column is equal to 1. A *permutation matrix* is a doubly stochastic matrix with all entries equal to 0 or 1. Given a doubly stochastic matrix A, prove that there exist permutation matrices $P_1, \dots, P_k$ and real numbers $b_1, \dots, b_k \in [0, 1]$ such that $A = \sum_{i=1}^{k} b_i P_i$ and $\sum_{i=1}^{k} b_i = 1$.

Solution: Let m be the number of non-zero entries of A. We prove the claim by induction on m; the base case, $m = 0$, is clear. Suppose that $m \geq 1$.

Consider the bipartite graph G with vertex classes $R = \{r_1, \dots, r_n\}$ ("the rows") and $C = \{c_1, \dots, c_n\}$ ("the columns"), defined as follows: given $1 \leq i, j \leq n$, we have $r_i \sim c_j$ in G if and only if $A_{ij} \neq 0$. We claim that (G, R) satisfies the Hall's Condition. Indeed, let $R' \subseteq R$, and note that given any $r_i \in R'$ we have $\sum_{j=1}^{n} A_{ij} = \sum_{c_j \in N(r_i)} A_{ij} = \sum_{c_j \in N(R')} A_{ij}$ since $N(r_i) \subseteq N(R')$. In particular, we have

$$|R'| = \sum_{r_i \in R'} 1 = \sum_{r_i \in R'} \sum_{j=1}^{n} A_{ij} = \sum_{r_i \in R'} \sum_{c_j \in N(R')} A_{ij} = \sum_{c_j \in N(R')} \sum_{r_i \in R'} A_{ij}$$
$$\leq \sum_{c_j \in N(R')} \sum_{i=1}^{n} A_{ij} = \sum_{c_j \in N(R')} 1 = |N(R')|,$$

and therefore $|N(R')| \geq |R'|$, as required. Thus, by the Hall's Marriage Theorem, G has a matching $M = \{r_i c_{j_i} \mid 1 \leq i \leq n\}$ from R to C.

Now let P be the $n \times n$ matrix defined by $P_{ij} = 1$ if $j = j_i$ and $P_{ij} = 0$ otherwise. Then P has exactly one entry 1 in each row by definition, and therefore n entries 1 in total; since M is a matching, it follows that P has at most one entry 1 in each column, and thus exactly one entry 1 in each column. It follows that P is a permutation matrix.

Let $d = \min\{A_{ij_i} \mid 1 \leq i \leq n\}$, and note that $d > 0$ by the choice of M. If $d = 1$, then for $1 \leq i \leq n$ we have $A_{ij_i} = 1$, implying that $A_{ij} = 0$ whenever $j \neq j_i$ since $\sum_{j=1}^{n} A_{ij} = 1$. This shows that $A = P$ is a permutation matrix, and thus we are done (by taking $k = 1$, $P_1 = P$ and $b_1 = 1$). Thus, suppose that $d < 1$. Let $A' = A - dP$ and let $B = (1 - d)^{-1} A'$.

We claim that B is a doubly stochastic matrix. Indeed, we have $A'_{ij} = A_{ij_i} - d \geq 0$ if $j = j_i$ and $A'_{ij} = A_{ij} \geq 0$ otherwise, implying that $B_{ij} \geq 0$ for all i and j. Furthermore, since $\sum_{j=1}^{n} A_{ij} = 1 = \sum_{j=1}^{n} P_{ij}$ we have $\sum_{j=1}^{n} A'_{ij} = \sum_{j=1}^{n} A_{ij} - d \sum_{j=1}^{n} P_{ij} = 1 - d$ and therefore $\sum_{j=1}^{n} B_{ij} = (1 - d)^{-1} \sum_{j=1}^{n} A'_{ij} = 1$ for all $i \in \{1, \dots, n\}$; similarly, $\sum_{i=1}^{n} B_{ij} = 1$ for all $j \in \{1, \dots, n\}$. It also follows that $B_{ij} \leq 1$ for all i and j, and hence B is a doubly stochastic matrix, as required.

Moreover, we have $B_{ij} = 0$ whenever $A_{ij} = 0$, and there exists $i \in \{1, \dots, n\}$ such that $A_{ij_i} = d > 0$ but $B_{ij_i} = 0$. Therefore, B has $< m$ non-zero entries, implying (by the inductive hypothesis) that $B = \sum_{i=1}^{k'} b'_i P_i$ for some $b'_1, \dots, b'_{k'} \in [0, 1]$ with $\sum_{i=1}^{k'} b'_i = 1$ and some permutation matrices $P_1, \dots, P_{k'}$. Now let $k = k' + 1$, let

$b_i = (1-d)b'_i$ for $1 \le i \le k'$, let $b_k = d$ and let $P_k = P$. Then $P_1, \ldots, P_k$ are permutation matrices, $b_i \in [0,1]$ for all i, and we have

$$\sum_{i=1}^{k} b_i = (1-d)\left(\sum_{i=1}^{k-1} b'_i\right) + d = (1-d) + d = 1$$

and

$$\sum_{i=1}^{k} b_i P_i = (1-d)\left(\sum_{i=1}^{k-1} b'_i P_i\right) + dP = (1-d)B + dP = A' + dP = A,$$

as required. This completes the proof.

Question 3

Given a connected graph G, a vertex $v \in G$ and $r \geq 0$, we write V_r for the set of vertices $w \in G$ such that the shortest path from v to w has length exactly r, and we write $G_r := G[V_r]$. Show that there exists some $r \geq 0$ such that $\chi(G) \leq \chi(G_r) + \chi(G_{r+1})$. Moreover, for each $k \geq 1$, give an example of a connected graph G and a vertex $v \in G$ such that $\chi(G) = \chi(G_r) + \chi(G_{r+1})$ for $0 \leq r < k$.

Solution: Let $k = \max\{\chi(G_r) + \chi(G_{r+1}) \mid r \geq 0\}$. We then need to show that $\chi(G) \leq k$, that is, G is k-partite. In particular, we want to find a partition $V(G) = \bigsqcup_{i=1}^{k} V_i$ such that if $v_i \in V_i$, $v_j \in V_j$ and $v_i \sim v_j$, then $i \neq j$.

We construct the vertex classes $V_1, \dots, V_k$ by inductively (on r) assigning the vertices of G_r to these classes; in particular, we construct such an assignment in such a way that $V(G_r) \cap V_i \neq \varnothing$ for exactly $\chi(G_r)$ values of i. For $r = 0$, we have $G_0 = \{v\}$ and so we assign v to V_1; note also that $\chi(G_0) = 1$. Suppose that for some $r \geq 0$, we have assigned all the vertices of $G_0, \dots, G_r$ to the vertex classes. By the construction, we have $V(G_r) \cap V_i = \varnothing$ for exactly $k - \chi(G_r) \geq \chi(G_{r+1})$ values of i, and so there exist indices $1 \leq i_1 < \cdots < i_m \leq k$, where $m = \chi(G_{r+1})$, such that $V(G_r) \cap V_{i_j} = \varnothing$ for all j. Since G_{r+1} is m-partite, with vertex classes $U_1, \dots, U_m$, say, we can assign the vertices of G_{r+1} to the vertex classes so that $V(G_{r+1}) \cap V_{i_j} = U_j$ for $1 \leq j \leq m$ and $V(G_{r+1}) \cap V_i = \varnothing$ if $i \notin \{i_1, \dots, i_m\}$. Note that each U_j is non-empty (as G_{r+1} is not $(m-1)$-partite), implying that $V(G_{r+1}) \cap V_i \neq \varnothing$ for exactly $m = \chi(G_{r+1})$ values of i.

Now let $u \in V_i$, $w \in V_j$ and suppose that $u \sim w$ in G. We need to show that $i \neq j$. Let $r, s \geq 0$ be such that $u \in G_r$ and $w \in G_s$; without loss of generality, suppose that $s \geq r$.

We first claim that $s \leq r + 1$. Indeed, there exists a path $v_0 \cdots v_r$, where $v_0 = v$ and $v_r = u$. If $w = v_\ell$ for some ℓ, then $v_0 \cdots v_\ell$ is a path of length ℓ from v to w in G, implying that $s \leq \ell \leq r$; otherwise, $v_0 \cdots v_r w$ is a path of length $r + 1$, implying that $s \leq r + 1$. Thus $s \leq r + 1$ in either case, as claimed.

Now if $s = r$, then the fact that $v \sim w$ means that v and w lie in different vertex classes of the $\chi(G_r)$-partite graph G_r, implying by our construction that v and w lie in different vertex classes. If $s = r + 1$, it follows from the construction that no vertex of G_r lies in the same vertex class as a vertex of G_{r+1}. Thus we have $i \neq j$ in either case, as required. This shows that G is k-partite.

Finally, for the "moreover" part, let $k \geq 1$, let $G = P_{2k} = v_0 v_1 \cdots v_{2k}$, and let $v = v_k$. Then for each i, there exists a unique path from v to v_i in G, namely $v_k v_{k-1} \cdots v_i$ of length $k - i$ if $i \leq k$, and $v_k v_{k+1} \cdots v_i$ of length $i - k$ if $i > k$; this shows that $v_i \in G_{|i-k|}$ for all i. It follows that G_0 consists of a single vertex v_k, whereas for $1 \leq r \leq k$, the graph G_r has two vertices (v_{k-r} and v_{k+r}) and no edges; in particular, $\chi(G_r) = 1$ for $0 \leq r \leq k$. On the other hand, G is bipartite: if $V_{\text{even}} = \{v_0, v_2, \dots, v_{2k}\}$ and $V_{\text{odd}} = \{v_1, v_3, \dots, v_{2k-1}\}$, then the graphs $G[V_{\text{even}}]$ and $G[V_{\text{odd}}]$ have no edges. Since $e(G) = 2k > 0$, the graph G is not 1-partite and therefore $\chi(G) = 2$. Thus $\chi(G) = 1 + 1 = \chi(G_r) + \chi(G_{r+1})$ for $0 \leq r < k$, as required.