Graph theory

Mock Class Test 1

The class test consists of two parts: "Exercises" and "Questions".
Please attempt ***BOTH*** *Exercises and* ***TWO*** *Questions.*

Refer to any results you are using by name
(or state them if you don't remember their name).

Duration of the class test: 90 minutes.

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Exercises

Please attempt ***BOTH*** *Exercises (A and B) displayed below.*

Wherever explanations are needed, please give precise *reasons*
(by explicitly writing down specific subgraphs, collections of vertices/edges, etc).

The Exercises are worth 6 points in total.

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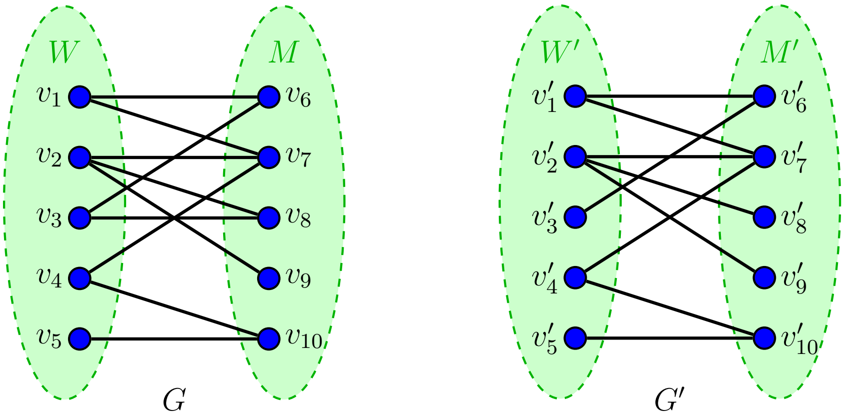
Exercise A

Consider the following bipartite graphs G and G':

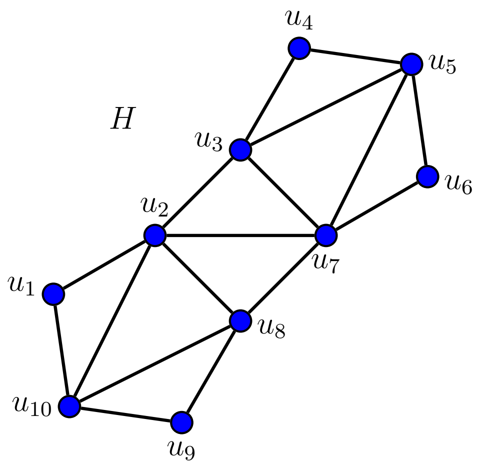


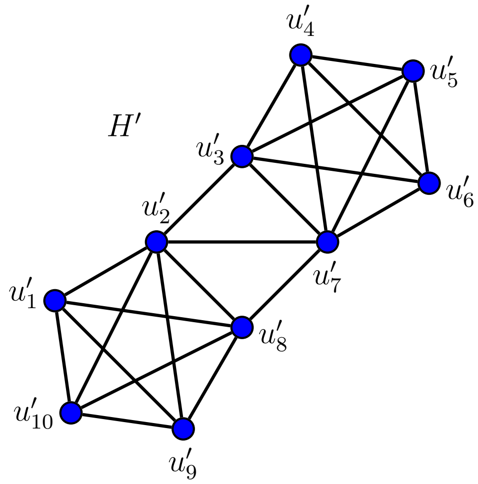
Decide whether or not each of these graphs has a matching between the given vertex classes. Explain your answers.

Solution for G (from W to M):	Solution for G' (from W' to M'):

Exercise B

Consider the following graphs H and H':





For each of these graphs, write down the largest $k \geq 0$ such that the graph is k-edge-connected, and give a reason why it is not $(k+1)$-edge-connected.

Solution for H:

Solution for H':

Questions

*Please attempt **TWO** Questions (out of 3).*

Each Question is worth 7 points.

Question 1

Let G be a tree, and let $G_1, \dots, G_r \leq G$ be a collection of $r \geq 2$ connected subgraphs such that $V(G_i) \cap V(G_j) \neq \varnothing$ for all i and j. Show that $\bigcap_{i=1}^{r} V(G_i) \neq \varnothing$.

Question 2

Let $n \geq 1$. A *doubly stochastic matrix* is an $n \times n$ matrix with entries in $[0,1]$ such that the sum of entries in each row and in each column is equal to 1. A *permutation matrix* is a doubly stochastic matrix with all entries equal to 0 or 1. Given a doubly stochastic matrix A, prove that there exist permutation matrices $P_1, \dots, P_k$ and real numbers $c_1, \dots, c_k \in [0,1]$ such that $A = \sum_{i=1}^{k} c_i P_i$ and $\sum_{i=1}^{k} c_i = 1$.

Question 3

Given a connected graph G, a vertex $v \in G$ and $r \geq 0$, we write V_r for the set of vertices $w \in G$ such that the shortest path from v to w has length exactly r, and we write $G_r := G[V_r]$. Show that there exists some $r \geq 0$ such that $\chi(G) \leq \chi(G_r) + \chi(G_{r+1})$. Moreover, for each $k \geq 0$, give an example of a connected graph G and a vertex $v \in G$ such that $\chi(G) = \chi(G_r) + \chi(G_{r+1})$ for $0 \leq r < k$.