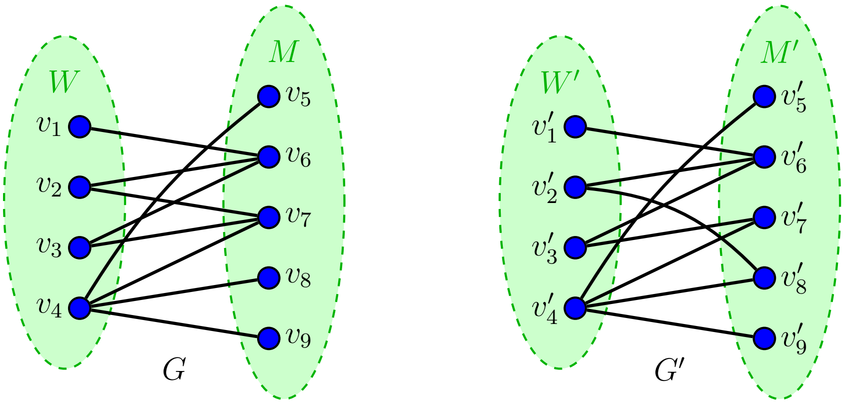
Graph theory

Class Test 2

Solutions

Exercise A

Consider the following bipartite graphs G and G':

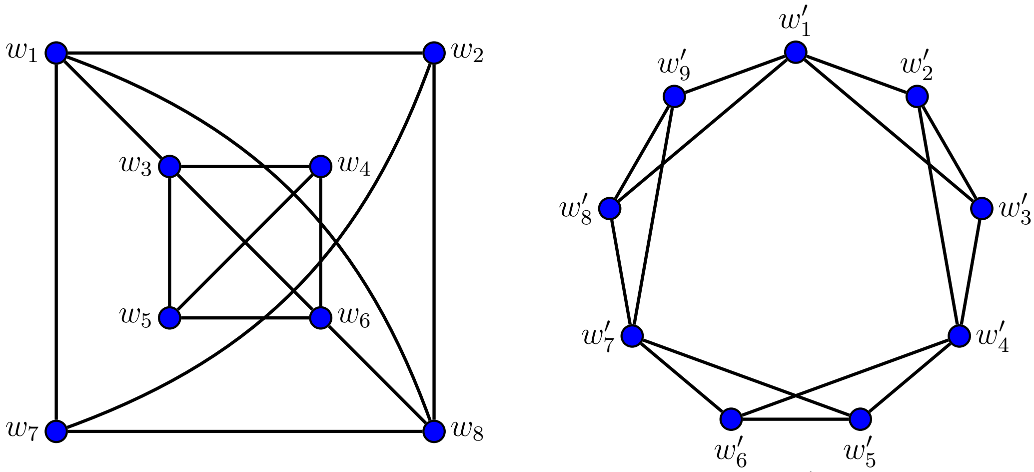


Decide whether or not each of these graphs has a matching between the given vertex classes. Explain your answers.

Solution for G (from W to M):	Solution for G' (from W' to M'):
No: For the subset $A = \{v_1, v_2, v_3\}$ of W we have $N_G(A) = \{v_6, v_7\}$ and therefore $$\lvert N_G(A)\rvert = 2 < 3 = \lvert A\rvert,$$ implying that G has no matchings from W to M by the Hall's Marriage Theorem.	**Yes:** for instance, $$\{v'_1v'_6, v'_2v'_8, v'_3v'_7, v'_4v'_5\}$$ is such a matching.

Exercise B

Consider the following graphs H and H':



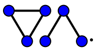
H H'

For each of these graphs, write down the value of the largest $k \geq 0$ such that the graph is k-edge-connected, and give a reason why it is not $(k+1)$-edge-connected.

Solution for H:
$\boldsymbol{k = 2}$**:** H is not 3-edge-connected since the graph $H - \{w_1w_3, w_6w_8\}$ consists of two connected components, namely $H[\{w_1, w_2, w_7, w_8\}] \cong K_4$ and $H[\{w_3, w_4, w_5, w_6\}] \cong K_4$.

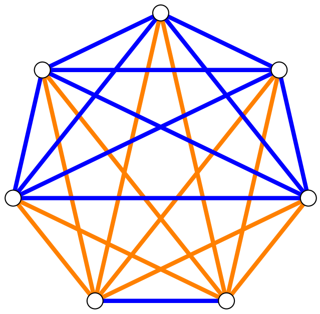
Solution for H':
$\boldsymbol{k = 3}$**:** H' is not 4-edge-connected since by removing the edges $\{w'_1w'_2, w'_2w'_3, w'_2w'_4\}$ we disconnect the vertex w'_2 from $H' - \{w'_2\}$.

Question 1

Show that $R(H, K_3) = 8$, where H is a disjoint union of K_3 and P_2, i.e. the graph .

[*Recall*: the Ramsey number $R(G, H)$ is the smallest integer $n \geq 1$ such that any blue/orange edge colouring of K_n contains a blue G or an orange H.]

Solution: To show that $R(H, K_3) > 7$, consider the edge colouring of K_7 in which the orange edges form $K_{2,5}$, i.e. the colouring



The orange edges form a bipartite subgraph, so in particular there is no orange cycle of odd length, and so no orange K_3. On the other hand, the blue edges form a graph consisting of two connected components $G_1 \cong K_2$ and $G_2 \cong K_5$. Since H has two connected components of order three, it follows that G_1 cannot contain any connected component of H, and G_2 can contain at most one component of H. Thus there is no blue H; this shows that $R(H, K_3) > 7$.

To show that $R(H, K_3) \leq 8$, consider a blue/orange edge colouring of K_8. Since $R(3, 3) = 6 \leq 8$, such a colouring contains a monochromatic triangle T. If T is orange then we are done, so suppose that T is blue, and let $G = K_8 - V(T) \cong K_5$.

Let E' be the set of blue edges in G. If E' contains two edges with the same endpoint, then these two edges together with T form a blue copy of H, and we're done. So suppose that all the edges in E' are *independent*, i.e. have no endpoints in common. This implies that $|E'| \leq \frac{|G|}{2} = \frac{5}{2}$ and so $|E'| \leq 2$.

Now the subgraph of G consisting of all five vertices and all $|E'|$ blue edges has $k = 5 - |E'|$ connected components. If we choose vertices $v_1, \ldots, v_k \in G$ by picking one vertex from each of these connected components, then $G[\{v_1, \ldots, v_k\}]$ is an orange subgraph of G isomorphic to K_k. Since we have $|E'| \leq 2$ and therefore $k = 5 - |E'| \geq 3$, it follows that we have vertices $v_1, v_2, v_3 \in G$ spanning an orange K_3, as required.

Thus in any edge colouring of K_8 we can find a blue H or an orange K_3, showing that $R(H, K_3) \leq 8$, as required.

Question 2

Given two graphs G and H with non-empty vertex sets, their *product* is the graph $G \cdot H$ with vertex set $V(G \cdot H) = V(G) \times V(H)$, such that two vertices $(v, v'), (w, w') \in G \cdot H$ are adjacent if and only if either $v \sim_G w$ and $v' \sim_H w'$, or $v = w$ and $v' \sim_H w'$, or $v \sim_G w$ and $v' = w'$. Show that for any graphs G and H we have $\alpha(G \cdot H) \leq R(\alpha(G) + 1, \alpha(H) + 1) - 1$. By considering $C_5 \cdot C_5$, show also that $\leq$ cannot be replaced with $<$.

[*Recall*: the independence number $\alpha(G)$ of G is the largest cardinality of an independent subset $A \subseteq V(G)$, i.e. a subset A such that no two vertices of A are adjacent.]

Solution: Let $A \subseteq V(G \cdot H)$ be an independent subset. Note that given two distinct non-adjacent vertices $(v, v'), (w, w') \in G \cdot H$, at least one of the following two alternatives holds: either $v \neq w$ and $v \nsim_G w$, or $v' \neq w'$ and $v' \nsim_H w'$. Now let $|A| = n$, so that $A = \{(v_1, w_1), \ldots, (v_n, w_n)\}$, and consider the following blue/orange edge colouring of K_n:

- colour $ij \in E(K_n)$ blue if $v_i \neq v_j$ and $v_i \nsim_G v_j$;
- colour $ij \in E(K_n)$ orange otherwise.

Note that since the subset A is independent, we have $w_i \neq w_j$ and $w_i \nsim_H w_j$ whenever ij is an orange edge in K_n.

Now suppose for contradiction that $n \geq R(\alpha(G) + 1, \alpha(H) + 1)$. Therefore, by Ramsey's Theorem, we can find either a blue $K_{\alpha(G)+1}$, or an orange $K_{\alpha(H)+1}$. However, if we had a blue $G' \cong K_{\alpha(G)+1}$, then we would have $v_i \neq v_j$ and $v_i \nsim_G v_j$ for all $i, j \in G'$, implying that $\{v_i \mid i \in G'\}$ would be an independent subset in G of cardinality $\alpha(G) + 1$; this contradicts the definition of $\alpha(G)$. Similarly, if we had an orange $H' \cong K_{\alpha(H)+1}$, then $\{w_i \mid i \in H'\}$ would be an independent subset in H of cardinality $\alpha(H) + 1$, contradicting the definition of $\alpha(H)$.

Thus we must have $n = |A| < R(\alpha(G) + 1, \alpha(H) + 1)$ for any independent subset $A \subseteq V(G \cdot H)$, and so $\alpha(G \cdot H) \leq R(\alpha(G) + 1, \alpha(H) + 1) - 1$, as required.

Finally, consider the graph $C_5 = v_1 v_2 v_3 v_4 v_5 v_1$. Any collection of three vertices of C_5 must contain a pair $\{v_i, v_{i+1}\}$ for some i (taking indices modulo 5), and so a pair of adjacent vertices. This shows that $\alpha(C_5) < 3$; but as $\{v_1, v_3\}$ is a pair of independent vertices in C_5, we have $\alpha(C_5) = 2$ and therefore

$$R(\alpha(C_5) + 1, \alpha(C_5) + 1) - 1 = R(3, 3) - 1 = 6 - 1 = 5.$$

On the other hand, consider $A = \{(v_1, v_2), (v_2, v_4), (v_3, v_1), (v_4, v_3), (v_5, v_5)\} \subseteq V(C_5 \cdot C_5)$, i.e. the set of all vertices of the form (v_i, v_{2i}) taking indices modulo 5. Given two distinct vertices $(v_i, v_{i'}), (v_j, v_{j'}) \in A$, we have either $i - j \equiv \pm 2 \pmod 5$, implying that $v_i \neq v_j$ and $v_i \nsim v_j$, or $i - j \equiv \pm 1 \pmod 5$ and therefore $i' - j' \equiv 2(i - j) \equiv \pm 2 \pmod 5$, implying that $v_{i'} \neq v_{j'}$ and $v_{i'} \nsim v_{j'}$. Thus we have $(v_i, v_{i'}) \nsim_{C_5 \cdot C_5} (v_j, v_{j'})$ in any case and therefore A is an independent subset of $C_5 \cdot C_5$ of cardinality 5. This shows that $\alpha(C_5 \cdot C_5) = 5 = R(\alpha(C_5) + 1, \alpha(C_5) + 1) - 1$, as required.

Question 3

Let $p \in (0,1)$ be a constant, and let $k\colon \mathbb{N} \to \mathbb{N}$ be such that $k(n) = o(n/\ln(n))$. By considering common neighbours of pairs of vertices in G, show that almost every $G \in \mathcal{G}(n,p)$ is $k(n)$-connected.

Solution: Suppose a graph G is not $k(n)$-connected. Then there exists a subset $A \subseteq V(G)$ with $|A| < k(n)$ and two vertices $v, w \in G$ that are in different connected components of $G - A$, implying in particular that there are no paths of the form vuw in $G - A$ and so $N_G(v) \cap N_G(w) \subseteq A$. In particular, we have $|N_G(v) \cap N_G(w)| < k(n)$, and so there exists a subset $U \subseteq V(G) \setminus \{v, w\}$ of cardinality $n - 1 - k(n)$ such that no vertex of U is adjacent to both v and w.

This implies that

$$\begin{aligned}
&\mathbb{P}(G \in \mathcal{G}(n,p) \text{ is not } k(n)\text{-connected}) \\
&\quad \le \mathbb{P}(\exists v \neq w \in G, \exists U \subseteq V(G) \setminus \{v,w\} \text{ as above}) \\
&\quad \le \sum_{v \neq w \in G} \left(\sum_{\substack{U \subseteq V(G)\setminus\{v,w\} \\ |U| = n-1-k(n)}} \mathbb{P}(\forall u \in U, v \nsim u \text{ or } w \nsim u) \right) \\
&\quad = \sum_{v \neq w \in G} \left(\sum_{\substack{U \subseteq V(G)\setminus\{v,w\} \\ |U| = n-1-k(n)}} \prod_{u \in U} \mathbb{P}(v \nsim u \text{ or } w \nsim u) \right) \\
&\quad = \sum_{v \neq w \in G} \left(\sum_{\substack{U \subseteq V(G)\setminus\{v,w\} \\ |U| = n-1-k(n)}} \prod_{u \in U} \left[1 - \mathbb{P}(uv \in E(G))\mathbb{P}(uw \in E(G))\right] \right) \\
&\quad = \sum_{v \neq w \in G} \left(\sum_{\substack{U \subseteq V(G)\setminus\{v,w\} \\ |U| = n-1-k(n)}} \prod_{u \in U} (1 - p^2) \right) \\
&\quad = \sum_{v \neq w \in G} \left(\sum_{\substack{U \subseteq V(G)\setminus\{v,w\} \\ |U| = n-1-k(n)}} (1 - p^2)^{n-1-k(n)} \right) \\
&\quad = \binom{n}{2} \cdot \binom{n-2}{n-1-k(n)} \cdot (1-p^2)^{n-1-k(n)} \\
&\quad = \binom{n}{2} \cdot \binom{n-2}{k(n)-1} \cdot (1-p^2)^{n-1-k(n)} \\
&\quad \le n^2 \cdot n^{k(n)-1} \cdot (1-p^2)^{n-1-k(n)} = \left(\frac{n}{1-p^2} \right)^{k(n)+1} \cdot (1-p^2)^n.
\end{aligned}$$

In particular, we have

$$\ln\left[\mathbb{P}(G \in \mathcal{G}(n,p) \text{ is not } k(n)\text{-connected})\right] \le (k(n)+1)\ln\left(\frac{n}{1-p^2}\right) + n\ln(1-p^2).$$

Now since $k(n) = o(n/\ln(n))$, we have $(k(n)+1)\ln\left(\frac{n}{1-p^2}\right) = o(n) = n \cdot o(1)$, implying that $\ln\big[\mathbb{P}(G \in \mathcal{G}(n,p)$ is not $k(n)$-connected$) \leq n \cdot [\ln(1-p^2) + o(1)]$. As we have $p > 0$ and therefore $\ln(1-p^2) < 0$, it follows that the latter expression tends to $-\infty$ and therefore $\mathbb{P}(G \in \mathcal{G}(n,p)$ is not $k(n)$-connected$) \to 0$ as $n \to \infty$, as required.